Lecture 12: Outcome of gravitational instability

12.1. Outcome of gravitational instability

If $Q < 1$, there is an axisymmetric GI that grows exponentially. The disc tends to break up into rings.

If $1 < Q \lesssim 2$, there can be a weaker, non-axisymmetric GI involving substantial transient growth. The disc tends to form spiral waves or clumps.

Since $Q \propto v_\mathrm{s} \propto \sqrt{T}$, thermostatic regulation is possible. If Q falls below about 2, instability occurs, producing motion that is dissipated through shocks and viscosity, heating the disc and raising Q.

Two possible outcomes of the GI are:

- *fragmentation*: formation of gravitationally bound objects (moonlets, planets, stars, etc.);
- *gravitational turbulence*: sustained activity of non-axisymmetric density waves while the disc maintains $1 < Q \lesssim 2$.

Efficient cooling promotes fragmentation.

The typical horizontal scale of the GI is a few times the scaleheight H. e.g. the fastest-growing mode of the axisymmetric GI has a radial wavelength $2\pi/k_x = 2\pi Q(v_\mathrm{s}/\Omega_r) \sim 2\pi Q\sqrt{\gamma}H$.

The ratio of the ('local') disc mass to the central mass for a Keplerian disc can be estimated as

$$\frac{\pi r^2\Sigma}{M} = \frac{1}{Q}\frac{v_\mathrm{s}\Omega r^2}{GM} = \frac{\sqrt{\gamma}}{Q}\frac{c_\mathrm{s}}{r\Omega} \sim \frac{H}{r}$$

when $Q \sim 1$. The typical mass of fragments formed by the GI is then a few times $(H/r)^3 M$. The GI can potentially form giant planets in the outer parts of sufficiently massive protoplanetary discs, or stars in the outer parts of discs in active galactic nuclei.

12.2. Thermal balance in a Keplerian disc with gravitational turbulence

Gravitational turbulence produces outward angular-momentum transport, which can be quantitifed by a dimensionless viscosity parameter α.

Suppose the disc has a constant cooling timescale $\tau = \beta/\Omega$, where β is dimensionless. The cooling rate per unit area is then

$$\mathscr{C} = \frac{P}{\gamma - 1}\frac{1}{\tau},$$

while the heating rate per unit area is

$$\mathscr{H} = \frac{9}{4}\alpha P\Omega.$$

Equating these gives a relation between α, β and γ (also valid for processes other than the GI):

$$\frac{9}{4}\alpha\beta(\gamma - 1) = 1.$$

Numerical simulations of the GI show fragmentation for $\beta \lesssim 4$ and turbulence for $\beta \gtrsim 4$, implying a maximum α of about $1/(9(\gamma-1))$.

What is the mean effective viscosity $\bar{\nu}$ in a self-gravitating Keplerian disc with $Q \approx 1$?

$$Q = \frac{v_{\mathrm{s}}\Omega}{\pi G \Sigma} \qquad \Rightarrow \qquad \bar{\nu} = \frac{\alpha c_{\mathrm{s}}^2}{\Omega} = \frac{\alpha v_{\mathrm{s}}^2}{\gamma \Omega} = \frac{\alpha Q^2 (\pi G \Sigma)^2}{\gamma \Omega^3}.$$

If $\beta =$ constant then $\alpha =$ constant and $\bar{\nu} \propto \Sigma^2 r^{9/2}$.