## Lecture 15: Satellite-disc interaction

The interaction of an orbiting companion (moon, planet, star, black hole, etc.) with a disc is one of the most important problems in the theory of astrophysical discs. For such massive satellites (as opposed to dust grains or planetesimals) the interaction is predominantly gravitational rather than hydrodynamic. The gravity of the satellite perturbs nearby orbital motion in the disc and excites waves and other disturbances. Angular momentum and energy are exchanged, leading to orbital evolution of the satellite, e.g. inward or outward radial migration of a planet in a protoplanetary disc.

### 15.1. Excitation of epicyclic motion by a satellite

We consider the dynamics of test particles in the $x y$ plane, using the local approximation:

$$
\begin{aligned}
& \ddot{x}-2 \Omega \dot{y}=2 \Omega S x-\frac{\partial \Psi}{\partial x}, \\
& \ddot{y}+2 \Omega \dot{x}=-\frac{\partial \Psi}{\partial y} .
\end{aligned}
$$

For a satellite of mass $M_{\mathrm{s}}$ on a circular orbit at the reference radius ( $x_{\mathrm{s}}=y_{\mathrm{s}}=0$ ), the potential of the satellite is

$$
\Psi=-\frac{G M_{\mathrm{s}}}{\sqrt{x^{2}+y^{2}}}
$$

The general solution in the absence of $\Psi$ (recall $\S 3.1$ ) is

$$
\begin{aligned}
& x=x_{0}+\operatorname{Re}\left(A e^{-i \Omega_{r} t}\right) \\
& y=y_{0}-S x_{0} t+\operatorname{Re}\left(\frac{2 \Omega A}{i \Omega_{r}} e^{-i \Omega_{r} t}\right) .
\end{aligned}
$$

This involves an epicyclic oscillation of complex amplitude $A$ around a guiding centre that follows a circular orbit $\left(x_{0}, y_{0}-S x_{0} t\right)$.

To express $x_{0}$ and $A$ in terms of position and velocity:

$$
\begin{aligned}
& x=x_{0}+\operatorname{Re}\left(A e^{-i \Omega_{r} t}\right), \\
& \dot{x}=\operatorname{Re}\left(-i \Omega_{r} A e^{-i \Omega_{r} t}\right)=\Omega_{r} \operatorname{Im}\left(A e^{-i \Omega_{r} t}\right), \\
& \dot{y}=-S x_{0}-2 \Omega \operatorname{Re}\left(A e^{-i \Omega_{r} t}\right) .
\end{aligned}
$$

So the canonical $y$-momentum (per unit mass) is

$$
p_{y}=\dot{y}+2 \Omega x=(2 \Omega-S) x_{0}=\frac{\Omega_{r}^{2}}{2 \Omega} x_{0}
$$

and we can find the epicyclic amplitude from

$$
\begin{aligned}
A e^{-i \Omega_{r} t} & =\operatorname{Re}\left(A e^{-i \Omega_{r} t}\right)+i \operatorname{Im}\left(A e^{-i \Omega_{r} t}\right) \\
& =-\frac{(\dot{y}+S x)}{(2 \Omega-S)}+\frac{i \dot{x}}{\Omega_{r}} \\
A & =\left[-\frac{2 \Omega}{\Omega_{r}^{2}}(\dot{y}+S x)+\frac{i \dot{x}}{\Omega_{r}}\right] e^{i \Omega_{r} t} .
\end{aligned}
$$

The specific energy is

$$
\varepsilon=\frac{1}{2}\left(\dot{x}^{2}+\dot{y}^{2}\right)-\Omega S x^{2} .
$$

Now

$$
\begin{aligned}
\Omega_{r} A & =\left[-\frac{2 \Omega}{\Omega_{r}}(\dot{y}+S x)+i \dot{x}\right] e^{i \Omega_{r} t} \\
\Omega_{r}^{2}|A|^{2} & =\dot{x}^{2}+\frac{4 \Omega^{2}}{\Omega_{r}^{2}}(\dot{y}+S x)^{2} \\
& =2 \varepsilon-\dot{y}^{2}+2 \Omega S x^{2}+\frac{4 \Omega^{2}}{\Omega_{r}^{2}}(\dot{y}+S x)^{2} \\
& =2 \varepsilon+\frac{2 \Omega S}{\Omega_{r}^{2}} \dot{y}^{2}+\frac{8 \Omega^{2} S}{\Omega_{r}^{2}} \dot{y} x+\frac{8 \Omega^{3} S}{\Omega_{r}^{2}} x^{2} \\
& =2 \varepsilon+\frac{2 \Omega S}{\Omega_{r}^{2}}(\dot{y}+2 \Omega x)^{2},
\end{aligned}
$$

so

$$
\varepsilon=\frac{1}{2} \Omega_{r}^{2}|A|^{2}-\frac{\Omega S}{\Omega_{r}^{2}} p_{y}^{2}=\text { constant } .
$$

In the presence of a satellite potential, we have instead

$$
\begin{aligned}
\dot{p}_{y} & =-\frac{\partial \Psi}{\partial y}, \\
\varepsilon+\Psi & =\text { constant }, \\
\dot{A} & =\left[-\frac{2 \Omega}{\Omega_{r}^{2}}(\ddot{y}+S \dot{x})+\frac{i \ddot{x}}{\Omega_{r}}-\frac{2 i \Omega}{\Omega_{r}}(\dot{y}+S x)-\dot{x}\right] e^{i \Omega_{r} t}, \\
& =\left[-\frac{2 \Omega}{\Omega_{r}^{2}}(\ddot{y}+2 \Omega \dot{x})+\frac{i}{\Omega_{r}}(\ddot{x}-2 \Omega \dot{y}-2 \Omega S x)\right] e^{i \Omega_{r} t}, \\
& =\left(\frac{2 \Omega}{\Omega_{r}^{2}} \frac{\partial \Psi}{\partial y}-\frac{i}{\Omega_{r}} \frac{\partial \Psi}{\partial x}\right) e^{i \Omega_{r} t} .
\end{aligned}
$$

### 15.2. Linear perturbation theory

We can calculate the change $\Delta A$ induced by $\Psi$ using linear perturbation theory.
The basic state is an unperturbed circular orbit $(A=0)$ at radial separation $x_{0}$ from the satellite:

$$
x=x_{0}=\text { constant }, \quad y=-S x_{0} t .
$$

Then, with $\Psi=-G M_{\mathrm{s}}\left(x^{2}+y^{2}\right)^{-1 / 2}$,

$$
\begin{aligned}
\dot{A} & =\left(\frac{2 \Omega}{\Omega_{r}^{2}} \frac{\partial \Psi}{\partial y}-\frac{i}{\Omega_{r}} \frac{\partial \Psi}{\partial x}\right) e^{i \Omega_{r} t} \\
& =G M_{\mathrm{s}}\left(x^{2}+y^{2}\right)^{-3 / 2}\left(\frac{2 \Omega y}{\Omega_{r}^{2}}-\frac{i x}{\Omega_{r}}\right) e^{i \Omega_{r} t} \\
& \approx-i \frac{G M_{\mathrm{s}}}{\Omega_{r} x_{0}^{2}}\left(1+S^{2} t^{2}\right)^{-3 / 2}\left(1-i \frac{2 \Omega}{\Omega_{r}} S t\right) e^{i \Omega_{r} t}
\end{aligned}
$$

giving (the real part of the integral vanishes by symmetry)

$$
\begin{aligned}
\Delta A & =\int_{-\infty}^{\infty} \dot{A} d t \\
& =-i \frac{G M_{\mathrm{s}}}{\Omega_{r} x_{0}^{2}} \int_{-\infty}^{\infty}\left(1+S^{2} t^{2}\right)^{-3 / 2}\left(\cos \Omega_{r} t+\frac{2 \Omega}{\Omega_{r}} S t \sin \Omega_{r} t\right) d t
\end{aligned}
$$

Let

$$
f(k)=\int_{-\infty}^{\infty}\left(1+x^{2}\right)^{-3 / 2} \cos k x d x=2 k K_{1}(k) \quad(k>0)
$$

where $K_{1}$ is a modified Bessel function. $[f(k)$ decreases monotonically from 2 to 0 as $k$ increases from 0 to $\infty$.] Then

$$
\Delta A=-i C \frac{G M_{\mathrm{s}}}{\Omega_{r} S x_{0}^{2}}
$$

where

$$
C=f\left(\frac{\Omega_{r}}{S}\right)-\frac{2 \Omega}{\Omega_{r}} f^{\prime}\left(\frac{\Omega_{r}}{S}\right)
$$

is a function of $q$ only. For Keplerian orbits $\left(\Omega_{r} / S=2 / 3\right), C \approx 3.36$.
So the gravitational encounter of a test particle with the satellite excites an epicyclic oscillation at first order.

Long before and after the encounter, $\Psi \rightarrow 0$. Since $\varepsilon+\Psi$ is exactly conserved, $\Delta \varepsilon=0$ in the encounter. But

$$
\varepsilon=\frac{1}{2} \Omega_{r}^{2}|A|^{2}-\frac{\Omega S}{\Omega_{r}^{2}} p_{y}^{2}
$$

so

$$
\Delta\left(p_{y}^{2}\right)=\frac{\Omega_{r}^{4}}{2 \Omega S} \Delta\left(|A|^{2}\right)
$$

Assume a circular orbit before the encounter:

$$
A=0, \quad p_{y}=\frac{\Omega_{r}^{2}}{2 \Omega} x_{0}
$$

Then, after the encounter,

$$
\begin{aligned}
A & \approx-i C \frac{G M_{\mathrm{s}}}{\Omega_{r} S x_{0}^{2}} \\
\Delta\left(p_{y}^{2}\right) & \approx 2 \frac{\Omega_{r}^{2}}{2 \Omega} x_{0} \Delta p_{y}=\frac{\Omega_{r}^{4}}{2 \Omega S}\left(C \frac{G M_{\mathrm{s}}}{\Omega_{r} S x_{0}^{2}}\right)^{2}
\end{aligned}
$$

SO

$$
\Delta p_{y}=\frac{\left(C G M_{\mathrm{s}}\right)^{2}}{2 S^{3} x_{0}^{5}}
$$

correct to second order.
Some irreversibility or dissipation is implicit in assuming the initial orbit to be circular.


### 15.3. Impulse approximation

A simplified version of the calculation treats the interaction of the test particle with the satellite as a impulsive two-body interaction at the point of closest approach.
[FIGURE]

We estimate

$$
\begin{aligned}
& \left.\Delta v_{\perp} \approx \frac{G M_{\mathrm{s}}}{x_{0}^{2}} \frac{1}{S} \quad \text { (acceleration } \times \text { time }\right) \\
& \Delta\left(v_{\perp}^{2}\right)+\Delta\left(v_{\|}^{2}\right)=0 \quad(\text { conservation of energy) } \\
& \left(\frac{G M_{\mathrm{s}}}{S x_{0}^{2}}\right)^{2}+2 S x_{0} \Delta v_{\|} \approx 0 \\
& \Delta v_{\|} \approx-\frac{\left(G M_{\mathrm{s}}\right)^{2}}{2 S^{3} x_{0}^{5}},
\end{aligned}
$$

which is the same result but lacking the dimensionless factor $C^{2} \approx 11.3$.

