Lecture 16: Satellite–disc / Particle–disc interaction

16.1. Satellite–disc torques

The y component of the force on the disc, per unit x, at location x, is

$$\begin{aligned}
&\Delta p_y \quad \times \quad \text{surface density} \quad \times \quad \text{encounter rate} \\
&= \frac{(CGM_\mathrm{s})^2}{2S^3x^5} \cdot \Sigma \cdot |Sx| \\
&\propto x^{-4}\,\mathrm{sgn}(x).
\end{aligned}$$

The torque per unit radius is the same $\times r_0$. The satellite experiences an equal and opposite torque.

- The effect is of second order in M_s
- A similar result is obtained for the response of a fluid disc, in which case density waves are excited
- The x^{-4} divergence is moderated within $|x| \lesssim H$
- Angular-momentum transport is outward
- The gravitational satellite–disc interaction is effectively 'repulsive'!
- The one-sided torque leads to gap opening if M_s is large enough and ν small enough
- Asymmetries between inner and outer torques lead to a net torque on the satellite and to migration (usually inwards)

16.2. Drag forces on particles

The behaviour of solid particles such as dust grains in a gaseous disc is of fundamental importance in the processes of planet formation. Observations of discs around young stars often reveal more about the dust than the gas, even though the dust may constitute only about 1% of the total mass.

A particle of mass m with velocity $\dot{\mathbf{x}}$ moving in a gas with velocity $\mathbf{u}(\mathbf{x}, t)$ experiences a relative wind velocity $\mathbf{u} - \dot{\mathbf{x}}$ and a drag force

$$\mathbf{F} = k(\mathbf{u} - \dot{\mathbf{x}}).$$

For subsonic relative motion ($|\mathbf{u} - \dot{\mathbf{x}}| \ll v_\mathrm{s}$), the coefficient k can be regarded as independent of the relative velocity if either

- the size of the particle is small compared to the mean free path of the gas (the kinetic regime, *Epstein drag*), or
- the Reynolds number of the relative motion is small, resulting in a laminar fluid flow around it (the laminar hydrodynamic regime, *Stokes drag*).

The acceleration of the particle due to drag can then be written as

$$\frac{\mathbf{F}}{m} = \gamma(\mathbf{u} - \dot{\mathbf{x}}) = \frac{\mathbf{u} - \dot{\mathbf{x}}}{t_\mathrm{s}},$$

where $t_\mathrm{s} = m/k$ is the *stopping time* and $\gamma = 1/t_\mathrm{s}$. The stopping time is an increasing function of particle size.

(For larger particles and faster relative motion, k is an increasing function of the relative speed.)

16.3. Radial drift

In the local approximation, the equation of motion of a solid particle is

$$\begin{aligned}
\ddot{x} - 2\Omega\dot{y} &= 2\Omega S x + \gamma(u_x - \dot{x}), \\
\ddot{y} + 2\Omega\dot{x} &= \gamma(u_y - \dot{y}), \\
\ddot{z} &= -\Omega_z^2 z + \gamma(u_z - \dot{z}).
\end{aligned}$$

If we take the gas velocity to be the orbital shear flow $\mathbf{u} = -Sx\,\mathbf{e}_y$, then the equations are linear. Both the horizontal (epicyclic) and vertical oscillations of the particle are damped.

Exercise: Show that the general solution of these equations under this assumption involves an epicyclic oscillation with an amplitude that decays $\propto e^{-\gamma t}$, while the vertical motion behaves as a damped harmonic oscillator. Therefore particles tend to settle into circular orbits in the midplane.

Now allow for a small departure from the orbital shear flow, so that

$$\mathbf{u} = [-Sx + v_y(x)]\ \mathbf{e}_y.$$

The radial force balance for the gas associates the zonal flow $v_y(x)$ with a radial pressure gradient. In a 2D model,

$$-2\Omega v_y = -\frac{1}{\Sigma}\frac{\partial P}{\partial x}.$$

Write the particle velocity as

$$\dot{\mathbf{x}} = -Sx\,\mathbf{e}_y + \mathbf{w}.$$

Then the equation of motion of the particle is

$$\begin{aligned}
\dot{w}_x - 2\Omega w_y &= \gamma(-w_x),\\
\dot{w}_y + (2\Omega - S)w_x &= \gamma(v_y - w_y),\\
\dot{w}_z &= -\Omega_z^2 z + \gamma(-w_z).
\end{aligned}$$

To the extent that v_y can be treated as a constant, a steady solution for the dust velocity is given by $w_z = z = 0$ and

$$-2\Omega w_y = -\gamma w_x, \qquad (2\Omega - S)w_x = \gamma(v_y - w_y),$$

which gives

$$w_x = \left(\frac{2\Omega\gamma}{\gamma^2 + \Omega_r^2}\right) v_y, \qquad w_y = \left(\frac{\gamma^2}{\gamma^2 + \Omega_r^2}\right) v_y.$$

In terms of the dimensionless *Stokes number*

$$\mathrm{St} = \Omega_r t_\mathrm{s} = \frac{\Omega_r}{\gamma},$$

we have

$$w_x = \left(\frac{\mathrm{St}}{1 + \mathrm{St}^2}\right)\frac{1}{\Omega_r}\frac{1}{\Sigma}\frac{\partial P}{\partial x}, \qquad w_y = \frac{v_y}{1 + \mathrm{St}^2}.$$

Particles with $\mathrm{St} \ll 1$ (typically sub-cm grains) nearly follow the azimuthal motion of the gas, while larger particles with $\mathrm{St} \gg 1$ move independently. In either case the drag causes a radial drift up the pressure gradient. The drift speed is maximized for $\mathrm{St} = 1$.

In a featureless protoplanetary disc model in which P is a monotonically decreasing function of r, particles with $\mathrm{St} \sim 1$ drift into the central star on a timescale $\sim (r/H)^2$ times the orbital timescale, i.e. less than 1000 yr from 1 AU:

$$|w_x| \sim \frac{1}{r\Omega}\frac{P}{\Sigma} \sim \frac{c_\mathrm{s}^2}{r\Omega}$$

$$\frac{r}{|w_x|} \sim \frac{r^2\Omega}{H^2\Omega^2} \sim \left(\frac{r}{H}\right)^2 \frac{1}{\Omega}.$$

If the disc has structures such as zonal flows, vortices or spiral density waves in which local pressure maxima occur, then particles may be trapped in them, enhancing the processes of planet formation.

16.4. Trapping of dust in a vortex

Recall the flow inside an equilibrium elliptical vortex patch of aspect ratio r and strength ζ_0 (see §10.4):

$$\mathbf{u} = A\left(\frac{y}{r}, -rx\right), \qquad A = \frac{S}{r-1}, \qquad \frac{\zeta_0}{S} = -\frac{(r+1)}{r(r-1)}.$$

The equation of motion of a particle inside the vortex is

$$\begin{aligned} \ddot{x} - 2\Omega\dot{y} &= 2\Omega S x - \gamma(\dot{x} - u_x), \\ \ddot{y} + 2\Omega\dot{x} &= -\gamma(\dot{y} - u_y). \end{aligned}$$

This is a linear system with solutions $x, y \propto e^{\lambda t}$:

$$\begin{pmatrix} \lambda^2 + \gamma\lambda - 2\Omega S & -2\Omega\lambda - \gamma A r^{-1} \\ 2\Omega\lambda + \gamma A r & \lambda^2 + \gamma\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{gathered} \left(\lambda^2 + \gamma\lambda - 2\Omega S\right)\left(\lambda^2 + \gamma\lambda\right) + \left(2\Omega\lambda + \gamma A r^{-1}\right)\left(2\Omega\lambda + \gamma A r\right) = 0 \\ \lambda^4 + 2\gamma\lambda^3 + \left(\gamma^2 + \Omega_r^2\right)\lambda^2 - 2\Omega\zeta_0\gamma\lambda + \gamma^2 A^2 = 0. \end{gathered}$$

The solutions all decay provided that $\mathrm{Re}(\lambda) < 0$ for all four roots of the quartic equation.

Assume that $\Omega_r^2 > 0$; otherwise we have orbital and hydrodynamic instability.

- In the limit of large γ (small St, small particles), the product of roots is $O(\gamma^2)$ and the sum of roots is $O(\gamma^1)$. Two roots are $O(\gamma^1)$ and two are $O(\gamma^0)$. The first scaling gives the balance $\lambda^4 + 2\gamma\lambda^3 + \gamma^2\lambda^2 \sim 0$, i.e. $\lambda \sim -\gamma$ (twice). These solutions decay. The second scaling gives the balance $\gamma^2\lambda^2 + \gamma^2 A^2 \sim 0$, i.e. $\lambda \sim \pm iA$. Expanding further, with $\lambda \sim \pm iA + c\gamma^{-1}$, gives, at $O(\gamma^1)$, $2(\pm iA)^3 + 2(\pm iA)c - 2\Omega\zeta_0(\pm iA) = 0$, i.e. $c = A^2 + \Omega\zeta_0$. These solutions decay if $\Omega\zeta_0 < -A^2$.

- In the limit of small γ (large St, large particles), the product of roots is $O(\gamma^2)$ and the sum of roots is $O(\gamma^1)$. Two roots are $O(\gamma^0)$ and two are $O(\gamma^1)$. The first scaling gives the balance $\lambda^4 + \Omega_r^2\lambda^2 \sim 0$, i.e. $\lambda \sim \pm i\Omega_r$. Expanding further, with $\lambda \sim \pm i\Omega_r + c\gamma$, gives, at $O(\gamma^1)$, $4(\pm i\Omega_r)^3 c + 2(\pm i\Omega_r)^3 + 2\Omega_r^2(\pm i\Omega_r)c - 2\Omega\zeta_0(\pm i\Omega_r) = 0$, i.e. $c = -1 - (\Omega\zeta_0/\Omega_r^2)$. These solutions decay if $\Omega\zeta_0 > -\Omega_r^2$. The second scaling gives the balance $\Omega_r^2\lambda^2 - 2\Omega\zeta_0\gamma\lambda + \gamma^2 A^2 = 0$, i.e. $\lambda \sim (\Omega\zeta_0 \pm \sqrt{(\Omega\zeta_0)^2 - \Omega_r^2 A^2})(\gamma/\Omega_r^2)$. These solutions decay if $\Omega\zeta_0 < 0$.

For all roots to decay in both limits, we require $-\Omega_r^2 < \Omega\zeta_0 < -A^2$. For a Keplerian disc this translates into anticyclonic vortices with $r > 3$. Vortices of such shapes trap particles of all sizes.

Exercise: Use the Routh–Hurwitz stability criteria (§14.1) to verify this conclusion.

Please send any comments and corrections to `gio10@cam.ac.uk`