## Lecture 4: Evolution of an accretion disc

### 4.1. Conservation of mass and angular momentum

The evolution of an accretion disc is regulated by the conservation of mass and angular momentum. These are embodied in the 3D equations of fluid dynamics, which we reduce to a 1D form by integration.

The equation of mass conservation is

$$
\frac{\partial \rho}{\partial t}+\boldsymbol{\nabla} \cdot(\rho \mathbf{u})=0
$$

where $\rho$ is the mass density and $\mathbf{u}$ is the velocity. In cylindrical polar coordinates,

$$
\frac{\partial \rho}{\partial t}+\frac{1}{r} \frac{\partial}{\partial r}\left(r \rho u_{r}\right)+\frac{1}{r} \frac{\partial}{\partial \phi}\left(\rho u_{\phi}\right)+\frac{\partial}{\partial z}\left(\rho u_{z}\right)=0 .
$$

Integrate this equation over the cylinder $\mathscr{C}_{r}$ of radius $r$ :

$$
\int_{\mathscr{C}_{r}} \cdot d A=\int_{-\infty}^{\infty} \int_{0}^{2 \pi} \cdot r d \phi d z
$$

Assuming no loss or gain through the vertical boundaries, we obtain

$$
\frac{\partial \mathscr{M}}{\partial t}+\frac{\partial \mathscr{F}}{\partial r}=0
$$

where

$$
\mathscr{M}(r, t)=\int_{\mathscr{C}_{r}} \rho d A
$$

is the 1D mass density (mass per unit radius) and

$$
\mathscr{F}(r, t)=\int_{\mathscr{C}_{r}} \rho u_{r} d A
$$

is the radial mass flux. Accretion corresponds to radial inflow $(\mathscr{F}<0)$.
The equation of angular-momentum conservation comes from the equation of motion, which we write in the form

$$
\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}=-\boldsymbol{\nabla} \Phi+\frac{1}{\rho} \boldsymbol{\nabla} \cdot \mathbf{T}
$$

where the symmetric stress tensor T accounts for momentum transport due to the collective effects of the fluid (pressure, viscosity, magnetic fields, self-gravity, turbulence, etc.). Combine this with the equation of mass conservation to obtain the equation of momentum conservation,

$$
\frac{\partial}{\partial t}(\rho \mathbf{u})+\boldsymbol{\nabla} \cdot(\rho \mathbf{u u}-\mathrm{T})=-\rho \boldsymbol{\nabla} \Phi .
$$

Here $\Phi$ is the external gravitational potential in which the disc orbits, which will usually be dominated by the central object.

Assuming that $\Phi$ is axisymmetric, the $\phi$-component of this equation, multiplied by $r$, is

$$
\frac{\partial}{\partial t}\left(\rho r u_{\phi}\right)+\frac{1}{r} \frac{\partial}{\partial r}\left[r^{2}\left(\rho u_{r} u_{\phi}-T_{r \phi}\right)\right]+\frac{1}{r} \frac{\partial}{\partial \phi}\left[r\left(\rho u_{\phi}^{2}-T_{\phi \phi}\right)\right]+\frac{\partial}{\partial z}\left[r\left(\rho u_{z} u_{\phi}-T_{z \phi}\right)\right]=0
$$

Integrate this equation over the cylinder $\mathscr{C}_{r}$ of radius $r$, again assuming no loss or gain through the vertical boundaries, and assuming (to be examined later) that $r u_{\phi}=h(r)$ from orbital dynamics:

$$
\frac{\partial}{\partial t}(\mathscr{M} h)+\frac{\partial}{\partial r}(\mathscr{F} h+\mathscr{G})=0
$$

where

$$
\mathscr{G}(r, t)=-\int_{\mathscr{C}_{r}} r T_{r \phi} d A
$$

The radial flux of angular momentum, $\mathscr{F} h+\mathscr{G}$, is the sum of two parts:

- advection of orbital advection ( $\mathscr{F} h$ ) by the accretion flow
- an internal torque $(\mathscr{G})$ due to collective effects


### 4.2. Diffusion equation for mass evolution

Since $h$ depends only on $r$, our two 1D conservation equations are

$$
\frac{\partial \mathscr{M}}{\partial t}+\frac{\partial \mathscr{F}}{\partial r}=0, \quad \frac{\partial \mathscr{M}}{\partial t} h+\frac{\partial}{\partial r}(\mathscr{F} h+\mathscr{G})=0
$$

Eliminate $\mathscr{M}$ to obtain

$$
\mathscr{F} \frac{d h}{d r}+\frac{\partial \mathscr{G}}{\partial r}=0
$$

which determines $\mathscr{F}$ instantaneously. Therefore $\mathscr{M}$ evolves according to

$$
\frac{\partial \mathscr{M}}{\partial t}=\frac{\partial}{\partial r}\left[\left(\frac{d h}{d r}\right)^{-1} \frac{\partial \mathscr{G}}{\partial r}\right] .
$$

The physical interpretation of this analysis is as follows. Since the motion is assumed to be dominated by circular orbital motion in the midplane, the specific angular momentum of a fluid element determines its orbital radius through the function $h(r)$ (increasing for stable orbits). Any radial transport of angular momentum $(\mathscr{G})$ implies a radial transport of mass $(\mathscr{F})$. Therefore the evolution of the mass distribution of the disc is governed by the transport of angular momentum.

A more usual notation refers instead to the surface density $\Sigma(r, t)$, the mean radial velocity $\bar{u}_{r}(r, t)$ and the mean effective kinematic viscosity $\bar{\nu}(r, t)$, related via

$$
\mathscr{M}=2 \pi r \Sigma, \quad \mathscr{F}=2 \pi r \Sigma \bar{u}_{r}, \quad \mathscr{G}=-2 \pi \bar{\nu} \Sigma r^{3} \frac{d \Omega}{d r}
$$

These can be defined by

$$
\Sigma=\int_{-\infty}^{\infty}\langle\rho\rangle d z, \quad \Sigma \bar{u}_{r}=\int_{-\infty}^{\infty}\left\langle\rho u_{r}\right\rangle d z, \quad \bar{\nu} \Sigma r \frac{d \Omega}{d r}=\int_{-\infty}^{\infty}\left\langle T_{r \phi}\right\rangle d z,
$$

where

$$
\langle\cdot\rangle=\frac{1}{2 \pi} \int_{0}^{2 \pi} \cdot d \phi
$$

is an azimuthal average (if required). Here the internal torque is being represented as if it were a viscous torque resulting from the orbital shear.
In these variables, we obtain

$$
\frac{\partial \Sigma}{\partial t}=\frac{1}{r} \frac{\partial}{\partial r}\left[\left(\frac{d h}{d r}\right)^{-1} \frac{\partial}{\partial r}\left(-\bar{\nu} \Sigma r^{3} \frac{d \Omega}{d r}\right)\right] .
$$

For a Keplerian disc, with $\Omega \propto r^{-3 / 2}$ and $h=r^{2} \Omega \propto r^{1 / 2}$, this equation simplifies to

$$
\frac{\partial \Sigma}{\partial t}=\frac{3}{r} \frac{\partial}{\partial r}\left[r^{1 / 2} \frac{\partial}{\partial r}\left(r^{1 / 2} \bar{\nu} \Sigma\right)\right],
$$

which is a diffusion equation for the surface density.
Exercise: Using specific angular momentum $h(r)$ as a spatial variable instead of $r$, show that

$$
\mathscr{F}=-\frac{\partial \mathscr{G}}{\partial h}, \quad \frac{\partial \mathscr{M}}{\partial t}=\frac{d h}{d r} \frac{\partial^{2} \mathscr{G}}{\partial h^{2}} .
$$

If $\bar{\nu}$ depends on $r$ only, show that we obtain a diffusion equation in the form

$$
\frac{\partial \mathscr{G}}{\partial t}=\left(-\bar{\nu} r^{2} \frac{d \Omega}{d r} \frac{d h}{d r}\right) \frac{\partial^{2} \mathscr{G}}{\partial h^{2}} .
$$

A narrow ring spreads diffusively because viscous or frictional processes transport angular momentum outwards from the more rapidly rotating inner part to the less rapidly rotating outer part. The inner part loses angular momentum and spreads inwards, while the outer part gains angular momentum and spreads outwards.

## [FIGURE]

### 4.3. Evolution of orbital energy

Recall that $d \varepsilon=\Omega d h$ for circular orbits.
Consider

$$
\begin{aligned}
\frac{\partial}{\partial t}(\mathscr{M} \varepsilon)+\frac{\partial}{\partial r}(\mathscr{F} \varepsilon) & =\varepsilon\left(\frac{\partial \mathscr{M}}{\partial t}+\frac{\partial \mathscr{F}}{\partial r}\right)+\mathscr{F} \frac{d \varepsilon}{d r} \\
& =0+\mathscr{F} \Omega \frac{d h}{d r} \\
& =-\Omega \frac{\partial \mathscr{G}}{\partial r} .
\end{aligned}
$$

Therefore

$$
\frac{\partial}{\partial t}(\mathscr{M} \varepsilon)+\frac{\partial}{\partial r}(\mathscr{F} \varepsilon+\mathscr{G} \Omega)=\mathscr{G} \frac{d \Omega}{d r}
$$

Here $\mathscr{G} \Omega$ is a radial energy flux associated with the internal torque. The RHS of this equation is minus the rate of dissipation of orbital energy per unit radius.

In the more usual notation (dividing through by $2 \pi r$ to get quantities per unit area), this equation becomes

$$
\frac{\partial}{\partial t}(\Sigma \varepsilon)+\frac{1}{r} \frac{\partial}{\partial r}\left(r \Sigma \bar{u}_{r} \varepsilon-\bar{\nu} \Sigma r^{3} \frac{d \Omega}{d r} \Omega\right)=-\bar{\nu} \Sigma r^{2}\left(\frac{d \Omega}{d r}\right)^{2}
$$

Assuming that the dissipated energy is converted into heat and lost locally by blackbody radiation, the surface temperature $T_{\mathrm{s}}(r, t)$ of the disc is given by

$$
2 \sigma T_{\mathrm{s}}^{4}=\bar{\nu} \Sigma r^{2}\left(\frac{d \Omega}{d r}\right)^{2}
$$

More generally this defines the effective temperature $T_{\text {eff }}(r, t)$.

### 4.4. Viscosity

Possible contributors to the stress $T_{r \phi}$ include:

- magnetic fields: $\frac{B_{r} B_{\phi}}{\mu_{0}}$ (for sufficiently ionized discs)
- self-gravity: $-\frac{g_{r} g_{\phi}}{4 \pi}$ (for sufficiently massive discs)
- fluctuating velocities due to waves, instabilities or turbulence: $-\left\langle\rho u_{r}^{\prime} u_{\phi}^{\prime}\right\rangle$
- true viscous stress: $-\rho \nu r \frac{d \Omega}{d r}$ (rarely significant)

Consideration of angular-momentum transport processes and the local vertical structure of the disc (see later) leads plausibly to a relation of the form

$$
\bar{\nu}=\bar{\nu}(r, \Sigma) \quad \text { (e.g. double power law). }
$$

- If $\bar{\nu}=\bar{\nu}(r)$ only, the diffusion equation is linear
- If $\bar{\nu}=\bar{\nu}(r, \Sigma)$, the diffusion equation is nonlinear

Exercise: Suppose that non-zero fluxes of mass and angular momentum through the vertical boundaries are permitted. Show that, if $S(r, t)$ and $T(r, t)$ are the rates at which mass and angular momentum are added to the disc per unit area (azimuthally averaged, if necessary), then the diffusion equation is modified to

$$
\frac{\partial \Sigma}{\partial t}=\frac{1}{r} \frac{\partial}{\partial r}\left\{\left(\frac{d h}{d r}\right)^{-1}\left[\frac{\partial}{\partial r}\left(-\bar{\nu} \Sigma r^{3} \frac{d \Omega}{d r}\right)+r(S h-T)\right]\right\}+S .
$$

