## Lecture 6: Time-dependent accretion

We return to the diffusion equation governing the spreading of a Keplerian disc,

$$
\frac{\partial \Sigma}{\partial t}=\frac{3}{r} \frac{\partial}{\partial r}\left[r^{1 / 2} \frac{\partial}{\partial r}\left(r^{1 / 2} \bar{\nu} \Sigma\right)\right] .
$$

Time-dependent solutions illustrate the mechanics of accretion.

### 6.1. Linear diffusion equation

The linear case, in which $\bar{\nu}=\bar{\nu}(r)$, can be treated using Green's function.
Let $\Delta\left(r, r_{0}, t\right)$ be the solution $\mathscr{M}(r, t)=2 \pi r \Sigma(r, t)$ of the diffusion equation with the initial condition $\mathscr{M}(r, 0)=\delta\left(r-r_{0}\right)$, i.e. a very narrow ring of radius $r_{0}$ and unit mass. Then the solution for any initial condition $\mathscr{M}(r, 0)=\mathscr{M}_{0}(r)$ and time $t>0$ is

$$
\mathscr{M}(r, t)=\int_{0}^{\infty} \Delta\left(r, r_{0}, t\right) \mathscr{M}_{0}\left(r_{0}\right) d r_{0}
$$

by linear superposition.
It is possible to calculate $\Delta\left(r, r_{0}, t\right)$ in terms of Bessel functions for any power law $\bar{\nu} \propto r^{a}$ and any boundary conditions. These functions become elementary if $a=(1+4 n) /(1+2 n)$ for some integer $n$. The easiest special case for illustration is $\bar{\nu} \propto r$. Let

$$
\bar{\nu}=A r, \quad y=\sqrt{\frac{4 r}{3 A}} \quad(\propto h), \quad g=r^{1 / 2} \bar{\nu} \Sigma=A r^{3 / 2} \Sigma=\frac{A r^{1 / 2} \mathscr{M}}{2 \pi} \quad(\propto \mathscr{G})
$$

to obtain the classical diffusion equation (see Example 1.4),

$$
\frac{\partial g}{\partial t}=\frac{\partial^{2} g}{\partial y^{2}}
$$

The fundamental solution of the diffusion equation, corresponding to the initial condition $g(y, 0)=$ $\delta(y)$ and no boundary conditions, is

$$
g(y, t)=\frac{1}{\sqrt{4 \pi t}} \exp \left(-\frac{y^{2}}{4 t}\right)
$$

The solution representing a spreading ring with zero torque $(g=0)$ at the inner boundary $r=r_{\text {in }}$ (corresponding to $y=y_{\text {in }}$ ) and with initial radius $r_{0}>r_{\text {in }}$ (corresponding to $y=y_{0}$ ) is therefore

$$
g(y, t)=\frac{C}{\sqrt{4 \pi t}}\left\{\exp \left[-\frac{\left(y-y_{0}\right)^{2}}{4 t}\right]-\exp \left[-\frac{\left(y+y_{0}-2 y_{\text {in }}\right)^{2}}{4 t}\right]\right\}
$$

This uses a superposition of translations of the fundamental solution to construct a solution satisfying the boundary condition by the method of images.

The initial condition

$$
\mathscr{M}=\delta\left(r-r_{0}\right) \quad \Rightarrow \quad g=\frac{A r_{0}^{1 / 2}}{2 \pi} \delta\left(y-y_{0}\right) \frac{1}{\sqrt{3 A r_{0}}}
$$

(the last factor coming from $d y / d r$ at $r=r_{0}$ ) gives $2 \pi C=\sqrt{A / 3}$, so Green's function is

$$
\Delta\left(r, r_{0}, t\right)=\frac{1}{\sqrt{12 \pi A r t}}\left\{\exp \left[-\frac{\left(\sqrt{r}-\sqrt{r_{0}}\right)^{2}}{3 A t}\right]-\exp \left[-\frac{\left(\sqrt{r}+\sqrt{r_{0}}-2 \sqrt{r_{\text {in }}}\right)^{2}}{3 A t}\right]\right\}
$$




Radial distributions of mass (left) and angular momentum (right) at times $t=0.001,0.01,0.1,1$ for a spreading ring with viscosity $\bar{\nu}=A r$, in units such that $A=1$ and $r_{0}=1$.

The inner boundary condition is that the torque vanishes at $r_{\text {in }}=0.1$.
Angular momentum is transported outwards and taken up by a diminishing fraction of the initial mass moving to larger and larger radii.

In the limit of large time, $t \gg r_{0} / A$, we obtain

$$
\begin{aligned}
\Delta & \approx \frac{1}{\sqrt{12 \pi A r t}}\left\{\left[1-\frac{\left(\sqrt{r}-\sqrt{r_{0}}\right)^{2}}{3 A t}\right]-\left[1-\frac{\left(\sqrt{r}+\sqrt{r_{0}}-2 \sqrt{r_{\text {in }}}\right)^{2}}{3 A t}\right]\right\} \\
2 \pi r \Sigma & \approx \frac{1}{\sqrt{12 \pi A r t}} \frac{1}{3 A t}\left(2 \sqrt{r}-2 \sqrt{r_{\text {in }}}\right)\left(2 \sqrt{r_{0}}-2 \sqrt{r_{\text {in }}}\right) \\
\bar{\nu} \Sigma & \approx \frac{1}{3 \pi} \frac{\left(\sqrt{r_{0}}-\sqrt{r_{\text {in }}}\right)}{\sqrt{3 \pi A t^{3}}}\left(1-\sqrt{\frac{r_{\text {in }}}{r}}\right),
\end{aligned}
$$

which looks like the solution for a steady disc, but with declining accretion rate $\left(\sqrt{r_{0}}-\sqrt{r_{\text {in }}}\right) / \sqrt{3 \pi A t^{3}}$.
Exercise: Show that, in the joint limit of large radius, $r \gg r_{0}, r_{\text {in }}$, and large time, $t \gg r_{0} / A$, we obtain a similarity solution (see later) of the form

$$
\Delta \approx \frac{2\left(\sqrt{r_{0}}-\sqrt{r_{\text {in }}}\right) e^{-r / 3 A t}}{\sqrt{\pi(3 A t)^{3}}}
$$

The error function $\operatorname{erf}(x)$ and the complementary error function $\operatorname{erfc}(x)$ are defined by

$$
\operatorname{erf}(x)=\frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-\xi^{2}} d \xi=\frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-\xi^{2}} d \xi, \quad \operatorname{erfc}(x)=1-\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-\xi^{2}} d \xi
$$

We have $\operatorname{erf}(x) \sim(2 / \sqrt{\pi}) x$ for $|x| \ll 1$ and $\operatorname{erfc}(x) \sim(1 / \sqrt{\pi}) x^{-1} e^{-x^{2}}$ for $x \gg 1$.


The mass remaining in the disc at time $t$ is (substitute $\sqrt{r}=\sqrt{r_{0}}+\sqrt{3 A t} \xi$ in the first term and $\sqrt{r}=2 \sqrt{r_{\text {in }}}-\sqrt{r_{0}}+\sqrt{3 A t} \xi$ in the second term, giving $\left.d r=2 \sqrt{r} \sqrt{3 A t} d \xi\right)$

$$
\int_{r_{\text {in }}}^{\infty} \Delta\left(r, r_{0}, t\right) d r=\frac{1}{\sqrt{\pi}} \int_{-\xi_{\text {in }}}^{\infty} e^{-\xi^{2}} d \xi-\frac{1}{\sqrt{\pi}} \int_{\xi_{\text {in }}}^{\infty} e^{-\xi^{2}} d \xi=\frac{1}{\sqrt{\pi}} \int_{-\xi_{\text {in }}}^{\xi_{\text {in }}} e^{-\xi^{2}} d \xi=\operatorname{erf}\left(\xi_{\text {in }}\right),
$$

where $\xi_{\text {in }}=\left(\sqrt{r_{0}}-\sqrt{r_{\text {in }}}\right) / \sqrt{3 A t}$. This declines $\propto t^{-1 / 2}$ for large $t$.
The radial mass flux is

$$
\mathscr{F}=-6 \pi r^{1 / 2} \frac{\partial}{\partial r}\left(r^{1 / 2} \bar{\nu} \Sigma\right)=-6 \pi \sqrt{\frac{1}{3 A}} \frac{\partial g}{\partial y},
$$

giving an accretion rate at $r_{\text {in }}$ :

$$
\dot{M}_{\mathrm{in}}(t)=-\mathscr{F}\left(r_{\mathrm{in}}, t\right)=\frac{1}{\sqrt{\pi}} \frac{\xi_{\text {in }}}{t} \exp \left(-\xi_{\mathrm{in}}^{2}\right) .
$$

This does indeed integrate to give (note that $\xi_{\text {in }} \propto t^{-1 / 2}$, so $d t / t=-2 d \xi_{\text {in }} / \xi_{\text {in }}$ )

$$
\int_{0}^{t} \dot{M}_{\text {in }}\left(t^{\prime}\right) d t^{\prime}=\operatorname{erfc}\left(\xi_{\text {in }}\right)
$$

The angular momentum remaining in the disc is

$$
\begin{aligned}
& \int_{r_{\text {in }}}^{\infty} \sqrt{G M r} \Delta\left(r, r_{0}, t\right) d r \\
& =\frac{\sqrt{G M}}{\sqrt{\pi}} \int_{-\xi_{\text {in }}}^{\infty}\left(\sqrt{r_{0}}+\sqrt{3 A t} \xi\right) e^{-\xi^{2}} d \xi-\frac{1}{\sqrt{\pi}} \int_{\xi_{\text {in }}}^{\infty}\left(2 \sqrt{r_{\text {in }}}-\sqrt{r_{0}}+\sqrt{3 A t} \xi\right) e^{-\xi^{2}} d \xi \\
& =\frac{\sqrt{G M}}{\sqrt{\pi}} \int_{-\infty}^{\xi_{\text {in }}}\left(\sqrt{r_{0}}-\sqrt{3 A t} \xi\right) e^{-\xi^{2}} d \xi-\frac{1}{\sqrt{\pi}} \int_{\xi_{\text {in }}}^{\infty}\left(2 \sqrt{r_{\text {in }}}-\sqrt{r_{0}}+\sqrt{3 A t} \xi\right) e^{-\xi^{2}} d \xi \\
& =h_{0}-h_{\text {in }} \operatorname{erfc}\left(\xi_{\text {in }}\right),
\end{aligned}
$$

i.e. the initial angular momentum minus that accreted through $r_{\text {in }}$.

### 6.2. Nonlinear diffusion equation

Consider a power-law viscosity

$$
\bar{\nu}=A r^{a} \Sigma^{b}, \quad A=\text { constant },
$$

and let $r_{\text {in }} \rightarrow 0$.
The problem is then scale-free and admits special algebraic similarity solutions (see Examples 1.5 and 1.6). These are generally attractors for solutions of the initial-value problem. Unlike the linear case, these solutions may have free boundaries beyond which the density vanishes. If the torque vanishes at the origin, then the total angular momentum is conserved, while the total mass of the disc declines as a power-law in time.

The conserved angular momentum is

$$
\sqrt{G M} C, \quad C=\int \sqrt{r} \Sigma 2 \pi r d r
$$

Dimensional analysis gives

$$
[A]=M^{-b} L^{2-a+2 b} T^{-1}, \quad[C]=M L^{1 / 2}
$$

Using $C, A$ and $t$ (the time elapsed since the formation of the disc), we can construct a time-dependent characteristic length-scale $R(t)$ given by

$$
R^{2-a+(5 / 2) b}=C^{b} A t .
$$

Thus

$$
R \propto t^{2 /(4-2 a+5 b)}, \quad M_{\mathrm{disc}} \propto \frac{C}{R^{1 / 2}} \propto t^{-1 /(4-2 a+5 b)} .
$$

A spreading ring with a power-law viscosity tends towards the similarity solution long after the ring has spread to the inner boundary and forgotten its initial radius.

Exercise: Verify that the linear diffusion equation with $\bar{\nu}=A r^{a}(a<2)$ admits similarity solutions

$$
\Sigma \propto R^{-5 / 2} \xi^{-a} \exp \left[-\frac{\xi^{2-a}}{(2-a)^{2}}\right]
$$

(with conserved angular momentum) and

$$
\Sigma \propto R^{-2} \xi^{-(1 / 2)-a} \exp \left[-\frac{\xi^{2-a}}{(2-a)^{2}}\right]
$$

(with conserved mass), where the similarity variable is $\xi=r / R(t)$ with $R^{2-a}=3 A t$.

