

Lecture 9: Thermal instability / Hydrodynamics of the shearing sheet

9.1. Thermal instability

So far, we have assumed a balance between heating and cooling: $\frac{9}{4}\bar{\nu}\Sigma\Omega^2 = \mathscr{H} = \mathscr{C} = 2F_{\mathrm{s}}$.

Now relax this assumption, but assume that $\alpha \ll 1$ so that $t_{\mathrm{dyn}} \ll t_{\mathrm{th}} \ll t_{\mathrm{visc}}$. Consider behaviour on the timescale t_{th}; we can then assume that the disc is hydrostatic and that the surface density does not evolve.

By solving the equations of vertical structure *except* thermal balance, we can calculate $\mathscr{H}$ and $\mathscr{C}$ as functions of $(\Sigma, \bar{\nu}\Sigma)$. In fact $\mathscr{H}$ depends only on $\bar{\nu}\Sigma$. The equation of thermal balance $\mathscr{H} = \mathscr{C}$ defines a curve in the $(\Sigma, \bar{\nu}\Sigma)$ plane.

Along the equilibrium curve, $d\mathscr{H} = d\mathscr{C}$ and $d(\bar{\nu}\Sigma) = \beta\bar{\nu}\,d\Sigma$, where $\beta = \left(\frac{\partial\ln(\bar{\nu}\Sigma)}{\partial\ln\Sigma}\right)_r$:

$$\frac{d\mathscr{H}}{d(\bar{\nu}\Sigma)}\,d(\bar{\nu}\Sigma) = \frac{\partial\mathscr{C}}{\partial\Sigma}\,d\Sigma + \frac{\partial\mathscr{C}}{\partial(\bar{\nu}\Sigma)}\,d(\bar{\nu}\Sigma)$$
$$\frac{d\mathscr{H}}{d(\bar{\nu}\Sigma)} = \frac{1}{\beta\bar{\nu}}\frac{\partial\mathscr{C}}{\partial\Sigma} + \frac{\partial\mathscr{C}}{\partial(\bar{\nu}\Sigma)}.$$

The internal energy content of disc per unit area is $\sim P \sim (\Omega/\alpha)\bar{\nu}\Sigma$. If some heat is added, $\bar{\nu}\Sigma$ increases but Σ is fixed on the timescale t_{th}. The system is thermally unstable if the excess heating outweighs the excess cooling, i.e. if

$$\frac{d\mathscr{H}}{d(\bar{\nu}\Sigma)} > \frac{\partial\mathscr{C}}{\partial(\bar{\nu}\Sigma)}, \qquad \text{i.e. if} \qquad \frac{1}{\beta\bar{\nu}}\frac{\partial\mathscr{C}}{\partial\Sigma} > 0.$$

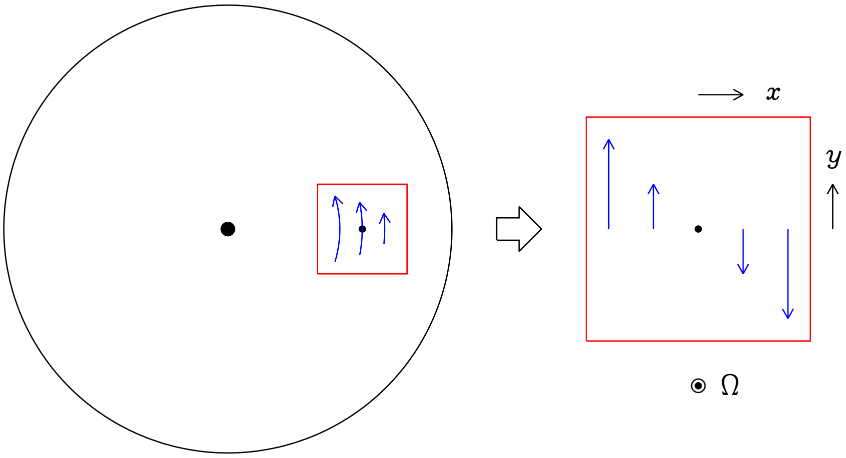
In practice $\partial\mathscr{C}/\partial\Sigma < 0$ (because, at fixed $\bar{\nu}\Sigma$, $\Sigma \propto 1/\bar{\nu} \propto 1/(\alpha T)$, and $\mathscr{C}$ generally increases with T), so thermal instability occurs (like viscous instability) when $\beta < 0$. Thermal instability then dominates (as its timescale is shorter).

9.2. Outbursts

We have seen that a radiative disc with gas pressure and Thomson opacity has $\bar{\nu}\Sigma \propto r\Sigma^{5/3}$ and is viscously and thermally stable. For cooler discs undergoing H ionization, the graph of $\bar{\nu}\Sigma$ versus Σ can involve an ‘S curve’, leading to instability and limit-cycle behaviour, which explains the outbursts in many cataclysmic variables, X-ray binaries and other systems.

9.3. Hydrodynamics of the shearing sheet

Recall the local view of an astrophysical disc: a linear shear flow $\mathbf{u}_0 = -Sx\,\mathbf{e}_y$ in a frame rotating with $\boldsymbol{\Omega} = \Omega\,\mathbf{e}_z$. Here Ω and $S = -r\,d\Omega/dr$ are evaluated at the reference radius r_0.



The model is either horizontally unbounded or equipped with (modified) periodic boundary conditions (see later). Possible treatments of the vertical structure are:

- ignore z completely (2D shearing sheet)
- neglect vertical gravity: homogeneous in z
- include vertical gravity: isothermal, uniform, polytropic, radiative, etc. models

9.4. Homogeneous incompressible fluid

Consider a 3D model, unbounded or periodic in (x, y, z), with a uniform kinematic viscosity ν.

The equation of motion is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{u} = -\boldsymbol{\nabla}\Phi_\mathrm{t} - \frac{1}{\rho}\boldsymbol{\nabla}p + \nu\nabla^2\mathbf{u},$$

subject to the incompressibility condition

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0.$$

The basic state is $\mathbf{u} = \mathbf{u}_0 = -Sx\,\mathbf{e}_y$, with hydrostatic pressure $p = p_0(z)$. There is a uniform viscous stress, but it has no divergence and causes no accretion flow.

Introduce perturbations (not necessarily small):

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}(\mathbf{x}, t), \qquad p = p_0 + \rho\psi(\mathbf{x}, t).$$

Then

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{u}_0\cdot\boldsymbol{\nabla}\mathbf{v} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{u}_0 + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} + 2\boldsymbol{\Omega}\times\mathbf{v} = -\boldsymbol{\nabla}\psi + \nu\nabla^2\mathbf{v},$$

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0.$$

In components:

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right) v_x - 2\Omega v_y = -\frac{\partial \psi}{\partial x} + \nu\nabla^2 v_x,$$
$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right) v_y + (2\Omega - S) v_x = -\frac{\partial \psi}{\partial y} + \nu\nabla^2 v_y,$$
$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right) v_z = -\frac{\partial \psi}{\partial z} + \nu\nabla^2 v_z.$$

Consider a plane-wave solution in the form of a *shearing wave* :

$$\mathbf{v}(\mathbf{x},t) = \mathrm{Re}\left\{\tilde{\mathbf{v}}(t)\,\exp[i\mathbf{k}(t)\cdot\mathbf{x}]\right\},$$
$$\psi(\mathbf{x},t) = \mathrm{Re}\left\{\tilde{\psi}(t)\,\exp[i\mathbf{k}(t)\cdot\mathbf{x}]\right\},$$

with time-dependent wavevector $\mathbf{k}(t)$. Then

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y}\right)\mathbf{v} = \mathrm{Re}\left\{\left[\frac{d\tilde{\mathbf{v}}}{dt} + \left(i\frac{d\mathbf{k}}{dt}\cdot\mathbf{x} - Sx\,ik_y\right)\tilde{\mathbf{v}}\right]\exp[i\mathbf{k}(t)\cdot\mathbf{x}]\right\}.$$

If we choose

$$\frac{d\mathbf{k}}{dt} = Sk_y\,\mathbf{e}_x,$$

then two terms cancel and we are left with $\dfrac{d\tilde{\mathbf{v}}}{dt}$.

This means

$$k_x = k_{x0} + Sk_y t, \qquad k_y = \text{constant}, \qquad k_z = \text{constant}.$$

Tilting of the wavefronts by the shear flow, and

Dual shear flow in Fourier space:

Furthermore, the nonlinear term vanishes:

$$\begin{aligned}
\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} &= \mathrm{Re}\left[\tilde{\mathbf{v}}\,e^{i\mathbf{k}\cdot\mathbf{x}}\right]\cdot\boldsymbol{\nabla}\mathrm{Re}\left[\tilde{\mathbf{v}}\,e^{i\mathbf{k}\cdot\mathbf{x}}\right] \\
&= \mathrm{Re}\left[\mathbf{k}\cdot\tilde{\mathbf{v}}\,e^{i\mathbf{k}\cdot\mathbf{x}}\right]\mathrm{Re}\left[i\tilde{\mathbf{v}}\,e^{i\mathbf{k}\cdot\mathbf{x}}\right] \\
&= 0,
\end{aligned}$$

because $\boldsymbol{\nabla}\cdot\mathbf{v}=0$ implies $i\mathbf{k}\cdot\tilde{\mathbf{v}}=0$. (This is a special result for an incompressible fluid. Note also that the nonlinear term does not vanish for a superposition of shearing waves.)

The amplitude equations for a shearing wave are

$$\begin{aligned}
\frac{d\tilde{v}_x}{dt}-2\Omega\tilde{v}_y &= -ik_x\tilde{\psi}-\nu k^2\tilde{v}_x, \\
\frac{d\tilde{v}_y}{dt}+(2\Omega-S)\tilde{v}_x &= -ik_y\tilde{\psi}-\nu k^2\tilde{v}_y, \\
\frac{d\tilde{v}_z}{dt} &= -ik_z\tilde{\psi}-\nu k^2\tilde{v}_z, \\
i\mathbf{k}\cdot\tilde{\mathbf{v}} &= 0,
\end{aligned}$$

with $k^2=|\mathbf{k}|^2$.

The viscous terms can be taken care of by a viscous decay factor

$$\begin{aligned}
E_\nu(t) &= \exp\left(-\int\nu k^2\,dt\right) \\
&= \exp\left\{-\nu\left[(k_{x0}^2+k_y^2+k_z^2)t+Sk_{x0}k_yt^2+\frac{1}{3}S^2k_y^2t^3\right]\right\}.
\end{aligned}$$

The decay is faster than exponential if $k_y\neq 0$.

Write $\tilde{\mathbf{v}}=E_\nu(t)\hat{\mathbf{v}}(t)$ and $\tilde{\psi}=E_\nu(t)\hat{\psi}(t)$ to eliminate the ν terms in the amplitude equations. Then eliminate variables in favour of $\hat{v}_x$ to obtain (see Example 2.1)

$$\frac{d^2}{dt^2}\left(k^2\hat{v}_x\right)+\Omega_r^2k_z^2\hat{v}_x=0,$$

where $\Omega_r^2=2\Omega(2\Omega-S)$ is the square of the epicyclic frequency in the local approximation.

Summary of outcomes (see Example 2.1):

- Stable if $\Omega_r^2>0$: $|\hat{\mathbf{v}}|^2$ oscillates if $k_y=0$, or decays algebraically if $k_y\neq 0$.
- Unstable if $\Omega_r^2<0$: $|\hat{\mathbf{v}}|^2$ grows exponentially if $k_y=0$, or grows algebraically if $k_y\neq 0$.

When $\nu>0$, E_ν kills off any algebraic growth for $k_y\neq 0$. But axisymmetric disturbances ($k_y=0$) of sufficiently large scale grow exponentially.

We conclude that a rotating shear flow is linearly stable when $\Omega_r^2>0$, but unstable when $\Omega_r^2<0$.

This agrees with the stability of circular test-particle orbits. It also agrees with *Rayleigh's criterion* for the linear stability of a cylindrical shear flow $\mathbf{u}=r\Omega(r)\,\mathbf{e}_\phi$ to axisymmetric perturbations: the flow is unstable if the specific angular momentum $|r^2\Omega|$ decreases outwards.

The case $\Omega_r^2=0$ (either a non-rotating shear flow or one with uniform specific angular momentum) is marginally Rayleigh-stable and allows algebraic growth in the absence of viscosity.