Note on the polytropic model and the gamma function

The gamma function is defined for $p > 0$ by

$$\Gamma(p) = \int_0^\infty x^{p-1}\, e^{-x}\, dx$$

and satisfies $\Gamma(p+1) = p\Gamma(p)$. We have $\Gamma(p) = (p-1)!$ if p is a positive integer.

Make the substitution $x = y^2$:

$$\Gamma(p) = \int_0^\infty 2y^{2p-1}\, e^{-y^2}\, dy. \tag{1}$$

Consider the integral

$$I_p = \int_{-1}^{1} (1-x^2)^p\, dx, \qquad p \geqslant 0.$$

Make the substitution $x = \cos\theta$:

$$I_p = \int_{-\pi/2}^{\pi/2} \cos^{2p+1}\theta\, d\theta.$$

Multiply both sides by a certain integral of the type (1) to make a double integral in polar coordinates:

$$\begin{aligned} I_p \int_0^r 2r^{2p+2}\, e^{-r^2}\, dr &= \int_{-\pi/2}^{\pi/2} \cos^{2p+1}\theta\, d\theta \int_0^r 2r^{2p+2}\, e^{-r^2}\, dr \\ &= \int_{-\pi/2}^{\pi/2} \int_0^\infty 2(r\cos\theta)^{2p+1}\, e^{-r^2}\, r\, dr\, d\theta. \end{aligned}$$

Transform from polar to Cartesian coordinates:

$$\begin{aligned} I_p \int_0^r 2r^{2p+2}\, e^{-r^2}\, dr &= \int_{-\infty}^\infty \int_0^\infty 2x^{2p+1}\, e^{-(x^2+y^2)}\, dx\, dy \\ &= \int_{-\infty}^\infty e^{-y^2}\, dy \int_0^\infty 2x^{2p+1}\, e^{-x^2}\, dx. \end{aligned}$$

Use equation (1) to obtain

$$I_p = \frac{\sqrt{\pi}\,\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}. \tag{2}$$

The polytropic disc model is of the form

$$\tilde{\rho} = C_\rho \left(1 - \frac{\tilde{z}^2}{a^2}\right)^n, \qquad \tilde{p} = C_p \left(1 - \frac{\tilde{z}^2}{a^2}\right)^{n+1},$$

where C_ρ, C_p and a are to be determined.

To satisfy $d\tilde{p}/d\rho = -\tilde{\rho}\tilde{z}$, we require

$$2(n+1)C_p = a^2 C_\rho.$$

To satisfy the normalization conditions $\int \tilde{\rho}\, d\tilde{z} = \int \tilde{p}\, d\tilde{z} = 1$, we require

$$I_n a C_\rho = I_{n+1} a C_p = 1.$$

Thus

$$a^2 = 2(n+1)\frac{C_p}{C_\rho} = 2(n+1)\frac{I_n}{I_{n+1}} = 2(n+1)\frac{\Gamma(n+1)}{\Gamma(n+2)}\frac{\Gamma(n+\frac{5}{2})}{\Gamma(n+\frac{3}{2})} = 2(n+\tfrac{3}{2}) = 2n+3.$$

The solution is

$$C_\rho = \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\frac{1}{\sqrt{(2n+3)\pi}}, \qquad C_p = \frac{\Gamma(n+\frac{5}{2})}{\Gamma(n+2)}\frac{1}{\sqrt{(2n+3)\pi}},$$

Please send any comments and corrections to `gio10@cam.ac.uk`