Particles impacting on a granular bed

or dropping pebbles on the beach

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Experimental observations

Penetration depth δ of spheres diameter D dropping height H

$$\frac{\delta}{D} = \left(\frac{\rho_{\mathrm{sphere}}}{\rho_{\mathrm{bed}}}\right)^{\alpha} \left(\frac{H}{D}\right)^{\beta}$$

with variously $(\alpha, \beta) = (\frac{1}{2}, \frac{1}{3})$, $(\frac{2}{5}, \frac{2}{5})$, $(\frac{1}{3}, \frac{1}{3})$, $(\frac{1}{2}, \frac{1}{2})$.

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Power-laws uncertain in experiments.

Governing equation

Resistive force - two parts   e.g. Pacheco-Vázquez *et. al.* (2011) PRL

- fluid-like inertial part (form drag)
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Mathematical study of this equation

Non-dimensionalize

$$\ddot{z} = k - \dot{z}^2 - z$$

$k \approx 0.2$.

Initial conditions

$$z(0) = 0 \quad \text{and} \quad \dot{z}(0) = V_0$$

$V_0 \approx$1–10.

Direct integration

$$\ddot{z} = k - \dot{z}^2 - z$$

Introduce (Riccati transformation)

$$z = \ln x, \quad \text{so} \quad \dot{z} = \frac{\dot{x}}{x} \quad \text{and} \quad \ddot{z} = \frac{\ddot{x}}{x} - \frac{\dot{x}^2}{x^2}$$

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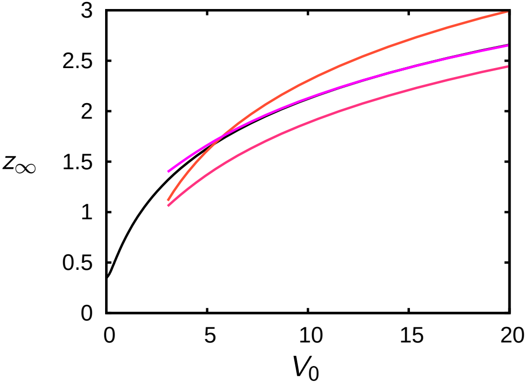
Thence first integral, and second.

Penetration depth

$$z_\infty \sim \ln V_0 - \ln\sqrt{\ln V_0} + \frac{1}{2\ln V_0}\left(\ln\sqrt{\ln V_0} + k + \tfrac{1}{2}\right)$$



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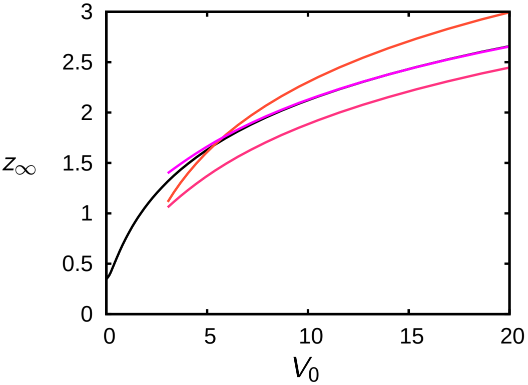
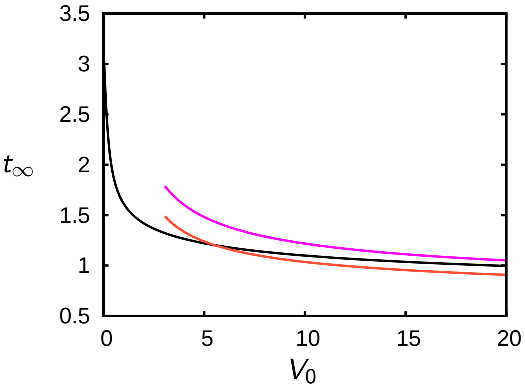


– not very power-law like.

Stopping time

$$t_\infty \sim \frac{\pi}{2\sqrt{\ln V_0}}\left[1+\frac{1}{2\ln V_0}\left(\ln\sqrt{\ln V_0}+k-\tfrac{1}{2}+\ln 2\right)\right]$$



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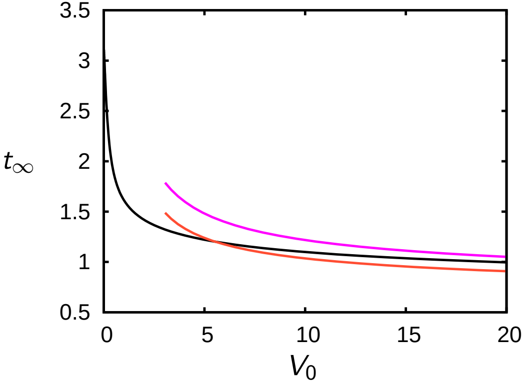

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Fast initial phase

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$$\text{slow time} \quad \tau = V_0 t$$

Expand

$$z(t, V_0) \sim \zeta_1(\tau) + \frac{1}{V_0^2}\zeta_2(\tau)$$

Initial Fast phase

First approximation

$$\zeta_{1\tau\tau} = -\zeta_{1\tau}^2, \quad \zeta_1(0) = 0, \quad \zeta_{1\tau} = 1$$

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$$\zeta_{1\tau} = \frac{1}{1+\tau}, \quad \zeta_1 = \ln(1+\tau).$$

Correction

$$\zeta_2 = \tfrac{1}{6}(1+\tau)^2\left(-\ln(1+\tau) + k + \tfrac{5}{6}\right) + \tfrac{1}{3}(k + \tfrac{1}{3})/(1+\tau) - \tfrac{1}{2}k - \tfrac{1}{4}.$$

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Expand final stopping phase

$$z(t) = \ln V_0 + Z_1(T) + \frac{1}{\ln V_0} Z_2(T)$$

with $T = \sqrt{\ln V_0}\, t$

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$$Z_2 = \int_0^T (\cot s - \cot T)\sin^2 s \left(\ln\sqrt{\ln V_0} + k - \ln(\sin s)\right)\, ds$$

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The reasons why

Early times, $\dot{z}^2$ drag term dominates

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solution

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