Drop formation of a power-law fluid on a thin film coating a vertical fibre

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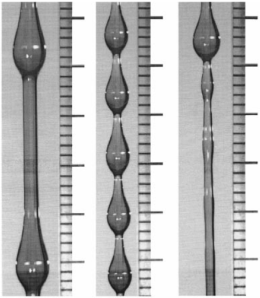
Motivation

Manufacture of polymeric and optical fibres.
The coating fluid is often non-Newtonian.

Shear-thinning
Duprat, Ruyer-Quil & Giorgiutti-Dauphiné

Phys. Fluids 2009



Newtonian

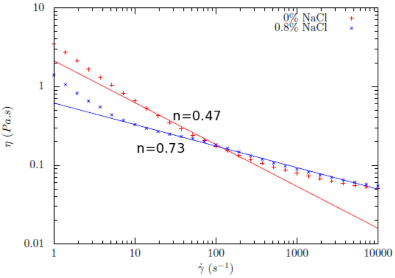
Kliakhandler, Davis & Bankoff JFM 2001

Governing equations

Constitutive equation

Power-law viscosity: $\mu = \beta \left| \frac{\partial u}{\partial y} \right|^{n-1}$

Xanthan solutions



Boulogne et al, Private Communication

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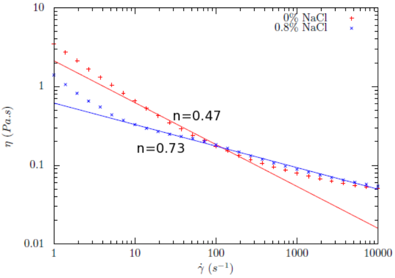

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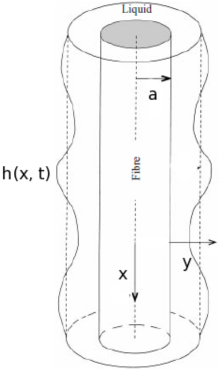
Geometry
Axisymmetric



Governing equations

Lubrication framework

Momentum: $$0 = -\frac{\mathrm{d}p}{\mathrm{d}x} + \rho g + \frac{\partial \sigma_{xy}}{\partial y}$$

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Volume flux: $$Q = \beta^{-\frac{1}{n}} \frac{n}{2n+1} \left(\rho g - \frac{\mathrm{d}p}{\mathrm{d}x} \right)^{\frac{1}{n}} h^{(2+\frac{1}{n})}$$

Note: $(\cdot)^{\frac{1}{n}} = \mathrm{sign}(\cdot)|\cdot|^{\frac{1}{n}}$

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Mass conservation: $$h_t + Q_x = 0$$

Governing equations

Non-dimensionalisation

Lengthscales:

- Fibre radius, a, in x direction.
- Initial film thickness, h_0, in y direction.

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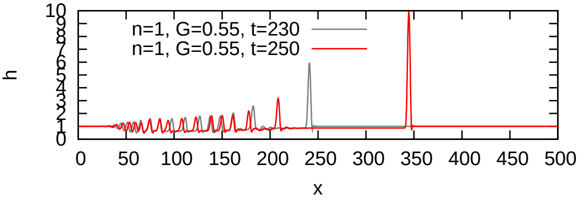
$$\boxed{h_t + \left(h^{2+\frac{1}{n}}\left(G + (h + h_{xx})_x\right)^{\frac{1}{n}}\right)_x = 0}$$

where Bond number $G = \frac{\rho g a^3}{\gamma h_0}$.

Time-dependent numerical simulations

Periodic forcing at inlet: $\omega = 1$

G small (thicker film):



Ever growing large pulses

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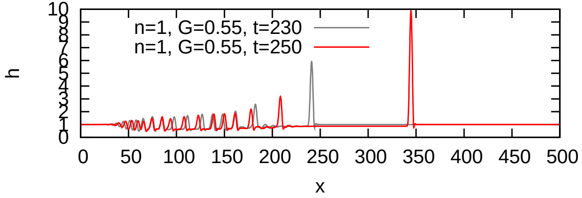

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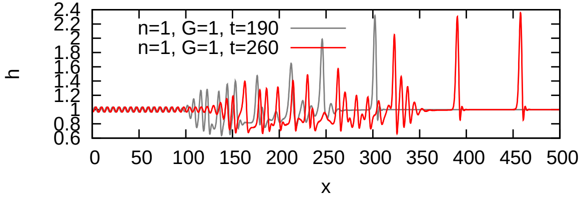

Solitary waves of similar amplitude and speed

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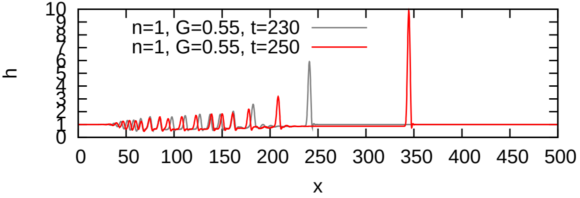

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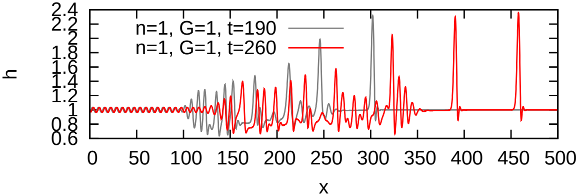

Solitary waves of similar amplitude and speed

This talk: Solitary waves? When? Properties?

Stationary solitary waves

Governing equations

In the frame of the solitary waves traveling with speed c:

$$(G + (h + h_{xx})_x) = \frac{\left(c(h-1) + G^{\frac{1}{n}}\right)^n}{h^{2n+1}}$$

$$h \to 1, \quad \text{as} \quad x \to \pm\infty$$

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Numerically construct the stationary solitary waves.

- Integrate from $x = -\infty$ to $x = 0$,
  and from $x = +\infty$ to $x = 0$.
- Hence need starting conditions at $x = \pm\infty$.

Stationary solitary waves

Initial conditions for numerics

At $x = \pm\infty$: $\quad h \sim 1 + \tilde{h}$ with $\tilde{h} \ll 1$.

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Linearised equation:

$$\tilde{h}''' + \tilde{h}' - A\tilde{h} = 0$$

where $A = nG^{1-1/n}c - (2n+1)G > 0$.

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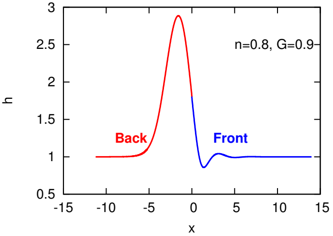
Three solutions of exponential form:

- $\tilde{h}_1 = a_1 e^{m_1 x}$
  m_1 real and positive.
  Use in 'Back' (1 DoF).
- $\tilde{h}_{2,3} = a_{2,3} e^{m_{2,3} x}$
  $m_{2,3}$ complex conjugates with negative real parts.
  Use in 'Front' (2 DoF).



For fixed G:

1. Shoot from Back, with $a_2 = a_3 = 0$. Stop when $h'' = 0$, $h' < 0$.

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Stationary solitary waves

Numerical construction

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3. Vary the phase of $a_{2,3}$ in Front to match h.

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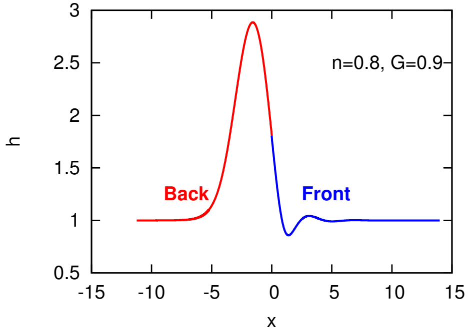
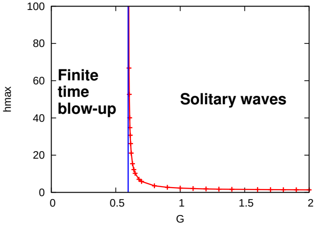
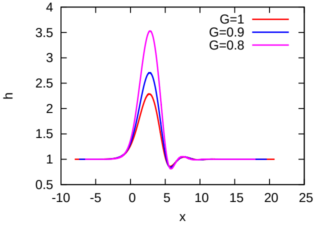


Stationary solitary waves

Results: $n = 1$ Kalliadasis & Chang, J. Fluid Mech. 1994





- No stationary solitary waves for $G < G_0$.
- As $G \downarrow G_{0+}$, $h_{\max} \to \infty$.

- Width of the 'Main Body' independent of G.

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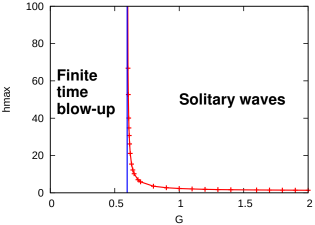

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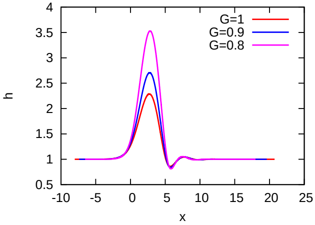


- Width of the 'Main Body' independent of G.

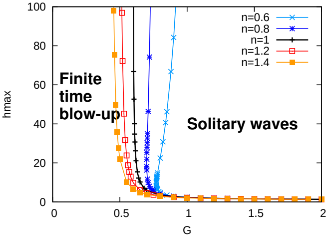
Agreement with experiment Quéré, Europhys. Lett. 1990:

- Critical h_c to observe disturbance $\propto a^3$.
- $G = \frac{\rho g a^3}{\sigma h_0} \Rightarrow h_c \propto a^3$ at $G = G_0$.

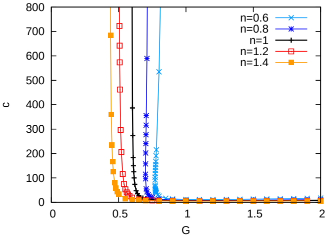
Stationary solitary waves

Results: various n

h_{max} vs G



c vs G



- Two branches of solutions for $n < 1$.

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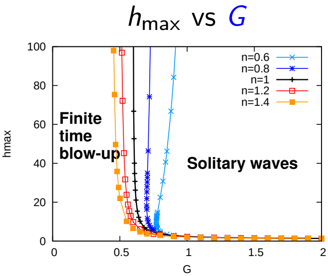

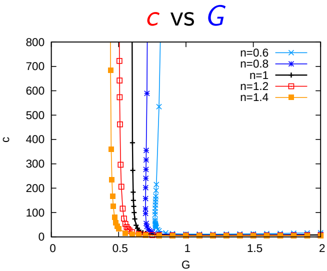

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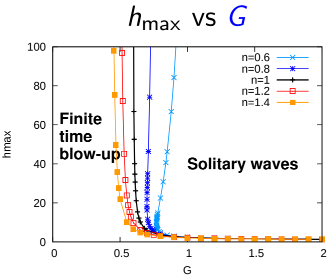

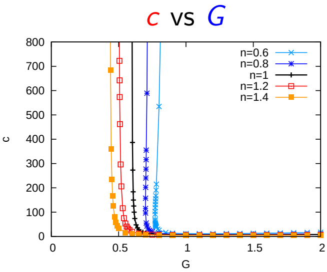

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Look at large fast stationary solitary waves close to G_0.

Large fast solitary waves

Pulse divided into 3 regions:

- 'Main body' region: h big, $x \sim O(1)$.

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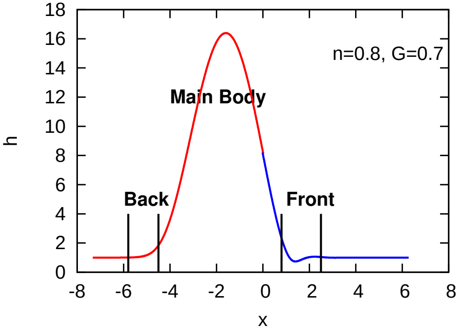

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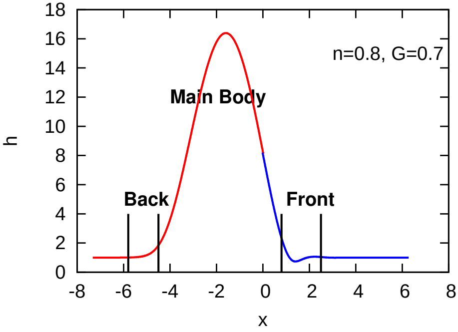

Asymptotic analysis for each region, and match. Very complicated!

Large fast solitary waves

Main body region: leading order

h big, $x \sim O(1)$

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$$h = \frac{1}{2} h_{\max} (1 - \cos x) \quad \text{in} \quad 0 \leq x \leq 2\pi.$$

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For matching,

$$h \sim \frac{1}{4} h_{\text{max}} (x - x_0)^2,$$

with $x_0 = 0$ at the Back and $x_0 = 2\pi$ at the Front.

Large fast solitary waves

Transition regions: leading order

$h \sim O(1)$, x small

$$\left(G + (h + \mathbf{h_{xx}})_{\mathbf{x}}\right) = \frac{\left(c(\mathbf{h} - \mathbf{1}) + G^{\frac{1}{n}}\right)^{n}}{\mathbf{h^{2n+1}}}$$

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Modified Bretherton equation:

$$h_{\xi\xi\xi} = \frac{(h-1)^n}{h^{2n+1}} \quad \text{with} \quad \xi = c^{n/3}(x - x_0).$$

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Solutions towards 'Main Body'

$$h \sim \tfrac{1}{2}P_\pm \xi^2 + Q\xi + R_\pm \quad \text{as} \quad \xi \to \pm\infty$$

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Use 1 DoF to redefine origin so $Q = 0$.

DoF at Back $1 - 1(Q = 0) = 0$: P_+ and R_+ uniquely determined.

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Large fast solitary waves

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$$P_- = P_+$$

So now P_- unique and hence R_- unique.

Large fast solitary waves

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DoF at Back $1 - 1(Q = 0) = 0$: P_+ and R_+ uniquely determined.
DoF at Front $2 - 1(Q = 0) = 1$ in P_- and R_-.

Main body: $h \sim \frac{1}{4} h_{\text{max}}(x - x_0)^2$ near $x_0 = 0,\ 2\pi$.

Transition regions: $h \sim \frac{1}{2} P_\pm \xi^2 + R_\pm$ as $\xi \to \pm\infty$.

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$$P_- = P_+$$

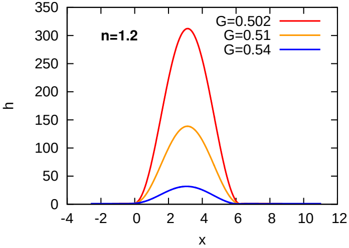
So now P_- unique and hence R_- unique.

$$\frac{1}{2} P(\xi = c^{n/3}(x - x_0))^2 = \frac{1}{4} h_{\text{max}}(x - x_0)^2$$

$$\boxed{h_{\text{max}} = 2Pc^{2n/3}}$$

Note: capillary pressure in the main body $p = \frac{1}{2} h_{max} = Pc^{2n/3}$.

Original profile:



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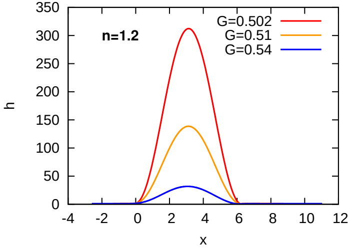
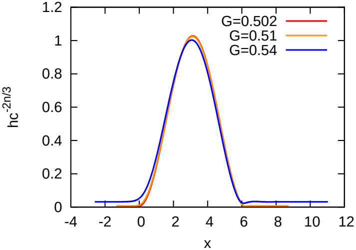


Main body:
height scaled by $c^{-2n/3}$



Large fast solitary waves

Checking scalings

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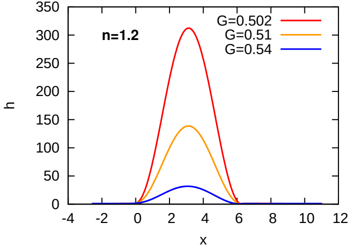

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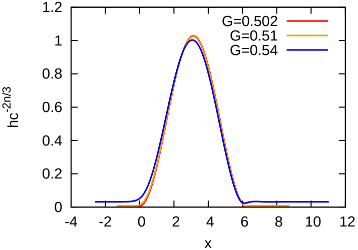
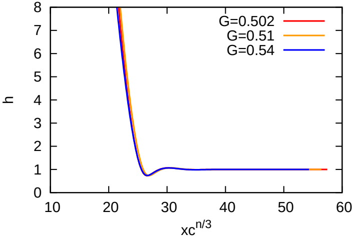


'Front' transition region:
width scaled by $c^{n/3}$



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Critical G

So far have $h_{\mathrm{max}}(c)$. G yet to appear

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- Different apparent film thickness, $R_{\pm}$, at 'Back' and 'Front'.

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- Different apparent film thickness, $R_\pm$, at 'Back' and 'Front'.

Need 1st correction of Main Body: $h \sim c^{2n/3} h_0 + h_2$

$$(\mathbf{G} + (\mathbf{h} + \mathbf{h_{xx}})_\mathbf{x}) = \frac{\left(c(h-1) + G^{\frac{1}{n}}\right)^n}{h^{2n+1}}$$

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So far have $h_{\max}(c)$. G yet to appear

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Matching gives critical G_0:

$$\boxed{G_0 = (R_+ - R_-)/2\pi}$$

Finding $R_\pm$ accurately

Numerics

$$h''' = \frac{(h-1)^n}{h^{2n+1}}$$

$$h \sim \frac{1}{2}P\xi^2 + R_\pm + S\xi^{1-2n} + T\xi^{-1-2n} + \dots$$

with $S = \frac{2^{n+1}}{(1-2n)(-2n)(-1-2n)P^{n+1}}$, $T = \frac{2^{n+2}((n+1)R_\pm + n)}{P^{n+2}(-1-2n)(-2-2n)(-3-2n)}$.

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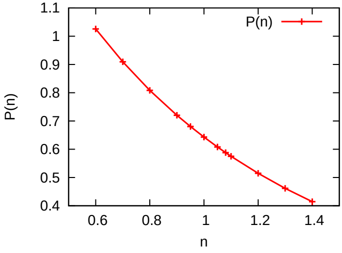
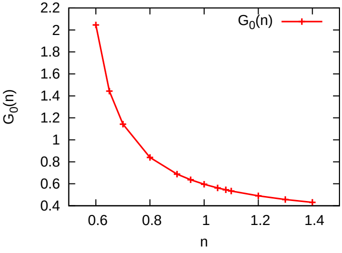
Least-square-fit[100:150]: $P(n) \pm 0.00001$, $R_\pm(n) \pm 0.001$.

$P(n)$



$G_0(n) = (R_+ - R_-)/2\pi$



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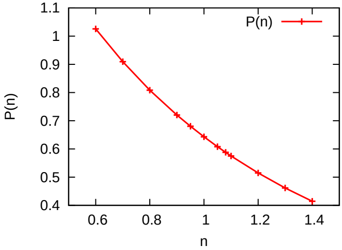

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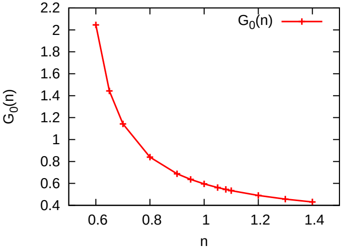

Also flow stops in finite distance – see after end.

So far have $h_{\max}(c)$ and critical G_0. Yet to find $G(c)$.

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Solution

$$P^{n+1} h_3 = \frac{(n+1)\sin x}{n(2n+1)(1-\cos x)^n} - \frac{(n+(n+1)\cos x)\sin x}{(2n+1)(2n-1)(1-\cos x)^n} + \frac{(n-1)(n+(n+1)\cos x)}{(2n+1)(2n-1)} \int_\pi^x \frac{1}{(1-\cos t)^{n-1}}\, dt - G_1 x$$

Large fast solitary waves

c as a function of G

Near $x = x_0$

$$h_3 \sim S(x - x_0)^{1-2n} + D_{\pm} - G_1 x + \dots$$

- The singular term matches the same in transition regions.

Large fast solitary waves

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Finally we have found the relationship between c and G

$$\boxed{G = G_0 + c^{-(2n-1)n/3} G_1}$$

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Plot $c^{-(2n-1)n/3}$ vs G

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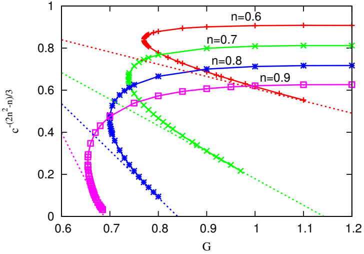
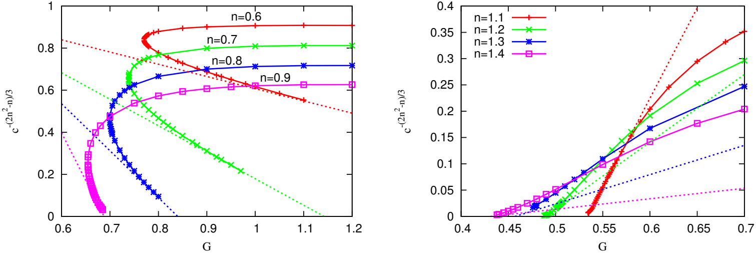


Large fast solitary waves

Results

$$G = G_0 + c^{-(2n-1)n/3} G_1$$

Plot $c^{-(2n-1)n/3}$ vs G



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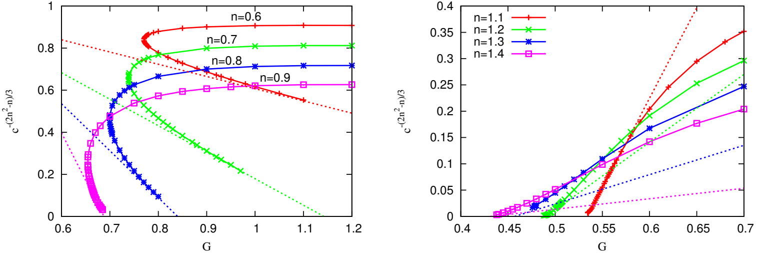

- When $n < 1$, $G_1 < 0$. Negative slope at G_0.

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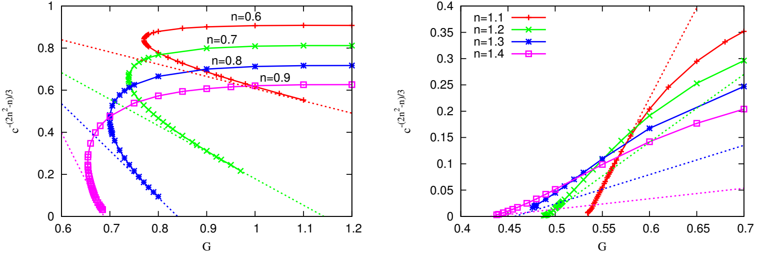

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- When $n > 1$, $G_1 > 0$. Positive slope at G_0.

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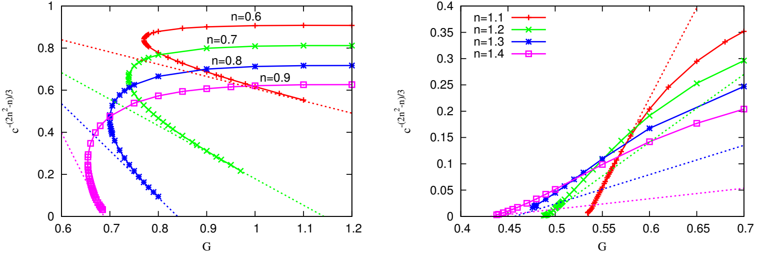

- When $n < 1$, $G_1 < 0$. Negative slope at G_0.
- When $n > 1$, $G_1 > 0$. Positive slope at G_0.
- When $n = 1$, $G_1 = 0$. No relationship between G and c yet.

More terms

transition regions

With scaling $\xi = c^{n/3}(x - x_0)$,

$$h_{\xi\xi\xi} = \frac{(h-1)^n}{h^{2n+1}} - c^{-2n/3} h_\xi - c^{-n} G + c^{-1} \frac{n(h-1)^{n-1} G^{1/n}}{h^{2n+1}} + \dots$$

More terms

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Expand h as

$$h \sim h_0 + c^{-2n/3} h_2 + c^{-n} h_3 + c^{-1} h_4 + c^{-4n/3} h_5 + \dots$$

More terms

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$$h_0''' = \frac{(h-1)^n}{h^{2n+1}}, \qquad h_0 \sim \frac{P}{2}\xi^2 + \mathbf{R}_\pm + Sx^{1-2n}$$

$$h_2''' = \frac{(h_0-1)^{n-1}\left(-(n+1)h_0 + (2n+1)\right)}{h_0^{2n+2}} h_2 - \mathbf{h_0'}, \qquad h_2 \sim -\frac{P}{4!}\xi^4 + \frac{a_{2\pm}}{2}\xi^2 + \mathbf{c_{2\pm}} + k_2\xi^{3-2n}$$

$$h_3''' = \frac{(h_0-1)^{n-1}(-(n+1)h_0 + (2n+1))}{h_0^{2n+2}} h_3 - \mathbf{G_0}, \qquad h_3 \sim -\frac{G_0}{3!}\xi^3 + \frac{a_{3\pm}}{2}\xi^2 + \mathbf{c_{3\pm}}$$

$$h_4''' = \frac{(h_0-1)^{n-1}(-(n+1)h_0 + (2n+1))}{h_0^{2n+2}} h_4 + \frac{n(h_0-1)^{n-1} G_0^{1/n}}{h_0^{2n+1}}, \qquad h_4 \sim \frac{1}{2} a_{4\pm}\xi^2 + c_{4\pm}$$

More terms

main body

With $h = c^{2n/3} H$,

$$(H + H_{xx})_x = -c^{-2n/3} G + c^{-(2n+1)n/3} \frac{(1 - \frac{c^{-2n/3}}{H} + \frac{G^{1/n}(c^{-1-2n/3})}{H})^n}{H^{n+1}}.$$

More terms

main body

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Expand H as

$$H \sim H_0 + c^{-2n/3} H_2 + c^{-(2n+1)n/3} H_3 + c^{-n} H_4 + c^{-1} H_5 + c^{-4n/3} H_6 + \dots$$

and G as

$$G \sim G_0 + G_1 c^{-(2n-1)n/3} + G_2 c^{-2n/3} + \dots$$

More terms

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and G as

$$G \sim G_0 + G_1 c^{-(2n-1)n/3} + G_2 c^{-2n/3} + \dots$$

$$H_0' + H_0''' = 0, \qquad H_0 = P(1 - \cos x)$$

$$H_2' + H_2''' = -G_0 \qquad H_2 = G_0 (\sin x - x) + A_2 + C_2 \cos x$$

$$H_3' + H_3''' = -G_1 + \frac{1}{P^{n+1}(1 - \cos x)^{n+1}}, \qquad H_3 = Sx^{1-2n} + D_{\pm} - G_1 x + k_2 x^{3-2n}$$

$$H_4' + H_4''' = 0, \qquad H_4 = A_4 + B_4 \sin x + C_4 \cos x$$

$$H_5' + H_5''' = 0, \qquad H_5 = A_5 + B_5 \sin x + C_5 \cos x$$

$$H_6' + H_6''' = -G_2, \qquad H_6 = G_2(\sin x - x) + A_6 + B_6 \sin x + C_6 \cos x$$

More terms

Matching: transition regions

Transition regions=

$$
\begin{array}{rllllll}
 & h_0 & h_2 & h_3 & h_4 & h_5 & \\
c^{\frac{2n}{3}}\Big[& \frac{P}{2}x^2 & -\frac{P}{4!}x^4 & & & +\frac{P}{6!}x^6 & +\dots\Big] \\
+c^0\Big[& R_\pm & +\frac{a_2}{2}x^2 & -\frac{G_0}{3!}x^3 & & -\frac{a_2}{4!}x^4 & +\dots\Big] \\
+c^{-\frac{2n^2}{3}+\frac{n}{3}}\Big[& Sx^{1-2n} & +k_2x^{3-2n} & & & +k_3x^{5-2n} & +\dots\Big] \\
+c^{-\frac{n}{3}}\Big[& & & +\frac{a_3}{2}x^2 & & & +\dots\Big] \\
+c^{\frac{2n}{3}-1}\Big[& & & & \frac{a_4}{2}x^2 & & +\dots\Big] \\
+c^{-\frac{2n}{3}}\Big[& & c_{2\pm} & & & +\frac{a_5}{2}x^2 & +\dots\Big]
\end{array}
$$

More terms

Matching: main body region

Main body=

$$
\begin{aligned}
c^{\frac{2n}{3}} \quad & \left[\tfrac{P}{2}x^2 - \tfrac{P}{4!}x^4 + \tfrac{P}{6!}x^6 + \dots\right] \\
+c^{0} \quad & \left[-G_0 x_0 + A_2 + C_2 - \tfrac{C_2}{2}x^2 - \tfrac{G_0}{3!}x^3 + \tfrac{C_2}{4!}x^4 + \dots\right] \\
+c^{-\frac{2n^2}{3}+\frac{n}{3}} \quad & \left[S x^{1-2n} - G_1 x_0 + D_{\pm} + k_2 x^{3-2n} + k_3 x^{5-2n} + \dots\right] \\
+c^{-\frac{n}{3}} \quad & \left[A_4 + C_4 - \tfrac{C_4}{2}x^2 + \dots\right] \\
+c^{\frac{2n}{3}-1} \quad & \left[A_5 + C_5 - \tfrac{C_5}{2}x^2\right] \\
+c^{-\frac{2n}{3}} \quad & \left[-G_2 x_0 + A_6 + C_6 - \tfrac{C_6}{2}x^2 - \tfrac{G_2+B_6}{3!}x^3 + \dots\right]
\end{aligned}
$$

More terms: matching two regions

At c^0:

$$G_0 = (R_+ - R_-)/2\pi$$

At $c^{-(2n^2-n)/3}$:

$$G_1 = -(D_+ - D_-)/2\pi$$

At c^{-1}:

$$G_2 = (c_{2+} - c_{2-})/2\pi$$

Hence,

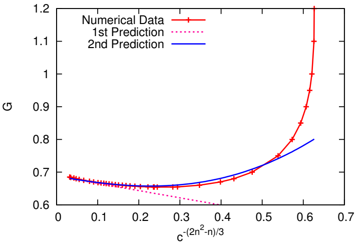
$$\boxed{G = G_0 + G_1 c^{-(2n-1)n/3} + G_2 c^{-2n/3}}$$

More terms: Results

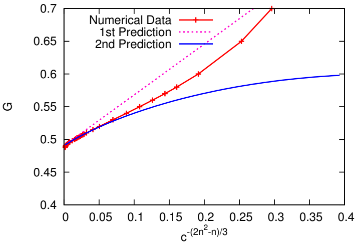
$$G = G_0 + G_1 c^{-(2n-1)n/3} + G_2 c^{-2n/3}$$

Plot G vs $c^{-(2n-1)n/3}$

$n = 0.9$



$n = 1.2$



More terms: Results

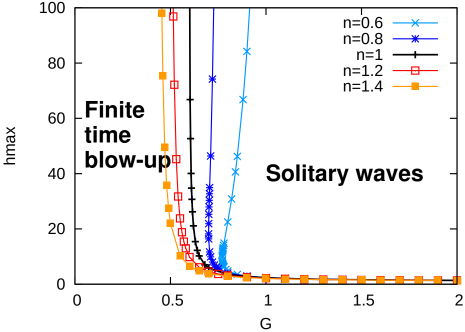
$$\boxed{G = G_0 + G_1 c^{-(2n^2-n)/3} + G_2 c^{-2n/3}}$$

When $n = 1$, $G_1 = 0$, so

$$G = G_0 + G_2 c^{-2/3}$$

Need even more terms for Newtonian $n = 1$ – see beyond end.

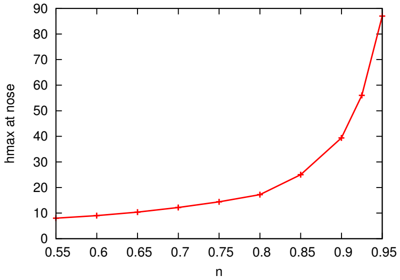
Two branches for $n < 1$

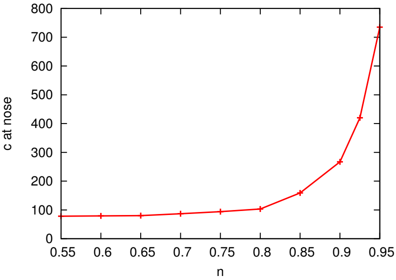


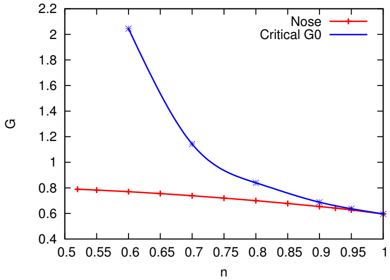
Upper branch is unstable – solutions either blow up or decay to lower branch.

Hence there is a maximum size of stable solitary for shear-thinning fluids.

Maximum solitary wave of shear-thinning fluids







Future Work

- Newtonian fluid $n = 1$ $\surd$
- What happens at big G? $\surd$
- Finite flow domain for shear-thinning fluids $\surd$
- Comparison with experimental data.
- Relax the thin film approximation?
- Normal stress effect.

Transition regions=

$$
\begin{array}{rlllll}
 & h_0 & h_2 & h_3 & h_4 & \\
c^{2/3}\Big[& \frac{P}{2}x^2 & -\frac{P}{4!}x^4 & & +\frac{P}{6!}x^6 & +\dots\Big] \\
+c^{0}\Big[& R_\pm & +\frac{a_2}{2}x^2 & -\frac{G_0}{3!}x^3 & -\frac{a_2}{4!}x^4 & +\dots\Big] \\
+c^{-1/3}\Big[& -\frac{2}{3P^2x} & & +\frac{a_3}{2}x^2 & +\frac{11}{1080P^2}x^3 & +\dots\Big] \\
+c^{-2/3}\Big[& & +c_{2\pm} & & +\frac{a_4}{2}x^2 & +\dots\Big] \\
+c^{-1}\log c\Big[& & & +\frac{4G_0}{9P^3} & & +\dots\Big] \\
+c^{-1}\Big[& \frac{2(1+2R_\pm)}{15P^3x^3} & +\frac{8R_\pm+4+20a_2}{15P^3x} & +\frac{4G_0}{3P^3}\log x + c_{3\pm} & & +\dots\Big]
\end{array}
$$

Main body =

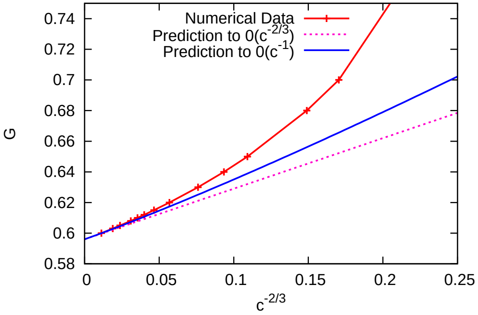
$$
\begin{aligned}
& c^{2/3} && \left[\tfrac{P}{2}x^2 - \tfrac{P}{4!}x^4 + \tfrac{P}{6!}x^6 + \dots\right] \\
& +c^{0} && \left[-G_0 x_0 + A_2 + C_2 - \tfrac{C_2}{2}x^2 - \tfrac{G_0}{3!}x^3 + \tfrac{C_2}{4!}x^4 + \dots\right] \\
& +c^{-1/3} && \left[-\tfrac{2}{3P^2x} + (A_3 + C_3) + (\tfrac{1}{18P^2} + B_3)x - \tfrac{C_3}{2}x^2 + (\tfrac{1}{1080P^2} - \tfrac{B_3}{3!})x^3 \dots\right. \\
& +c^{-2/3} && \left[-G_2 x_0 + A_4 + C_4 + B_4 x - \tfrac{C_4}{2}x^2 - \tfrac{G_2}{3!}x^3 + \dots\right] \\
& +c^{-1}\log c && \left[A_5 + C_5 - \tfrac{C_5}{2}x^2 + \dots\right] \\
& +c^{-1} && \left[\tfrac{2(1+2R_\pm)}{15P^3x^3} + \tfrac{4(1+2A_2-3C_2)}{15P^3x} + \tfrac{4G_0}{3P^3}\log x - G_3 x_0 + A_6 + C_6 \dots\right]
\end{aligned}
$$

$n = 1$ Newtonian fluid, even more terms

Results

At c^0: $G_0 = (R_+ - R_-)/2\pi$
At $c^{-2/3}$: $G_2 = (c_{2+} - c_{2-})/2\pi$
At c^{-1}: $G_3 = (c_{3+} - c_{3-})/2\pi$
Hence,

$$G = G_0 + G_2 c^{-2/3} + G_3 c^{-1}$$



Big G

$$h \sim 1 + \frac{1}{G} h_1 \quad c \sim \left(2 + \frac{1}{n}\right) G^{\frac{1}{n}} + c_1 G^{\frac{1}{n}-1}$$

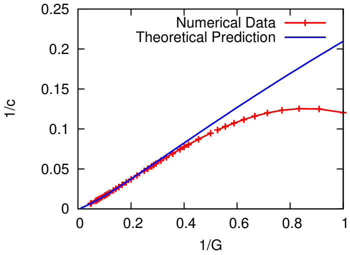
where h_1 satisfies the nonlinear equation

$$h_1' + h_1''' = n c_1 h_1 + h_1^2 \left(-n(2n+1) + \frac{n(n-1)}{2}\left(2 + \frac{1}{n}\right)^2\right)$$

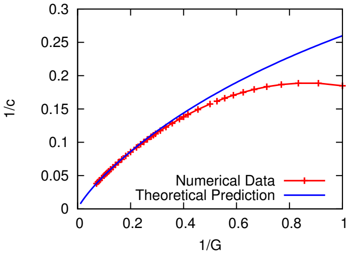
This equation can be solved numerically to give the value of c_1 for different values of n.

Big G results

$n = 0.8$



$n = 1.2$



Finite flow domain for shear-thinning fluids

$$h''' = \frac{(h-1)^n}{h^{2n+1}}$$

Integrating from $\pm\infty$ where $h \sim 1 + \tilde{h}$ ($\tilde{h} \ll 1$), $\tilde{h}$ satisfies:

$$\tilde{h}''' = \tilde{h}^n. \Leftarrow \text{ No exponential solutions for } n \neq 1.$$

Solution at 'Back'

$$\tilde{h} = A(\xi - \xi_0)^{\frac{3}{1-n}}, \quad n < 1$$

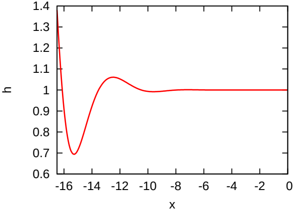
$\tilde{h}$ becomes 0 at a finite distance.

While viscosity thins as $\gamma \to \infty$ it thickens as $\gamma \to 0$,
and so flow stops in a finite distance.

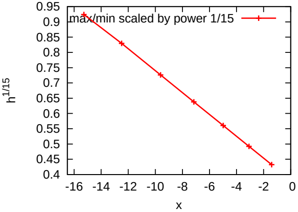
Finite flow domain for shear-thinning fluids

Solution at 'Front' ($n = 0.8$)

Decaying nonlinear oscillations

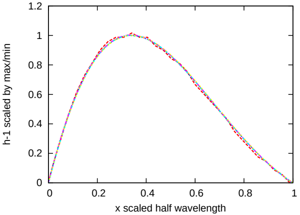


$h_{\max}^{3/(1-n)}$ vs $x_{\max}$



Decays to zero in finite distance

Each half-cycle normalised by maximum and by wavelength



Universal shape