The much-neglected Second Normal Stress Difference

John Hinch

CMS-DAMTP, University of Cambridge

September 24, 2020

What are normal stresses?

In simple shear $\mathbf{u} = (\gamma y, 0, 0)$, the stress tensor is

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{pmatrix},$$

with off-diagonal tangential viscous dissipative stresses,
and diagonal normal stresses, which do no work.

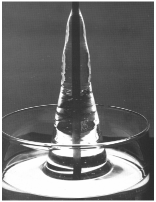
In a Newtonian viscous fluid, the normal stresses are equal $\sigma_{xx} = \sigma_{yy} = \sigma_{zz}$ and equal to (negative) pressure.

In a visco-elastic fluid, the normal stresses are not equal. We consider their differences

$$N_1 = \sigma_{xx} - \sigma_{yy}, \quad N_2 = \sigma_{yy} - \sigma_{zz}.$$

N_1 (dominant for polymers) is a tension in the streamlines.

Effect of N_1 on flow – tension in the streamlines

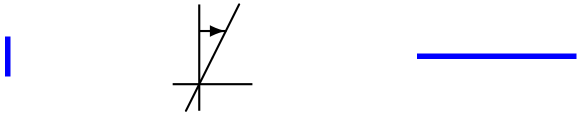


- Rod climbing
- Secondary flows
- Migration of particles to centre of pipe flow
- Stabilisation of jets
- Purely-elastic Taylor-Couette instability
- Co-extrusion instability

The origin of N_1

For polymers and microstructures of fibres:

Fibres in shear flow become stretched and aligned

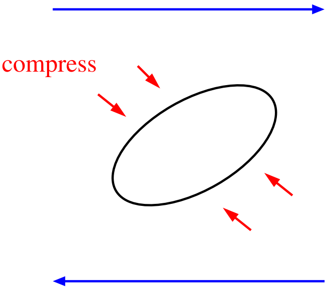


Thus tension in the streamlines, N_1.

And no N_2 from fibres – why much neglected.

The origin of N_2

Need a thick microstructure that can be compressed, e.g. droplets in an emulsion



Then spin by vorticity to align with flow gives $N_2 = \sigma_{yy} - \sigma_{zz} < 0$.

$N_2 < 0$ is tension in the vortex lines.

Associated non-affine behaviour
– remark for rheologists only

Thick microstructures, unlike thin fibres, strain with reduced efficiency, $\theta < 100\%$.

So deform with

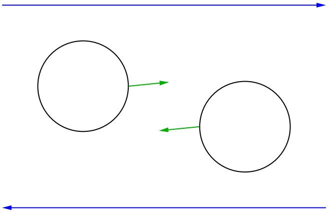
$$\nabla \mathbf{u} \to \tfrac{1}{2}\left(\nabla \mathbf{u} - \nabla \mathbf{u}^T\right) + \theta \tfrac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right)$$

This gives a second normal stress difference and shear-thinning.

$$N_2 \propto -\frac{\theta(1-\theta)\gamma^2}{1+(1-\theta^2)\gamma^2}, \quad \mu_p \propto \frac{\theta}{1+(1-\theta^2)\gamma^2},$$

Another origin of N_2

In non-Brownian suspensions, particles impact in the x-direction, leading to a pressure, $\sigma_{xx} < 0$.



When concentrated $\phi > 20\%$, they also impact layer above and below, leading to a similar pressure $\sigma_{yy} \approx \sigma_{xx}$ (force-chains at 45° to flow), so $N_1 \approx 0$.

Easier to pass in z-direction, so $\sigma_{zz} \approx 0$, so $N_2 = \sigma_{yy} - \sigma_{zz} < 0$.

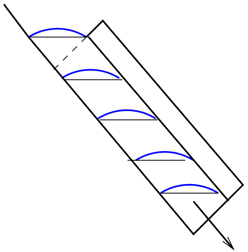
Boyer, Pouliquen & Guazzelli (2017)

Effect of N_2 on flow – tension in the vortex lines

- Bowing of interface in Tanner tilted channel
- Longitudinal vortices in granular chute flow
- Negative rod-climbing
- Edge instability in rheometers
- Lopsided de-wetting on a vertical fibre

Tanner's tilted trough

Inclined V-shaped open channel Kuo & Tanner 1974



Higher shear-rate in centre

$\rightarrow$ higher tension in vortex lines in centre,

$\rightarrow$ pull fluid to centre

$\rightarrow$ surface bows up

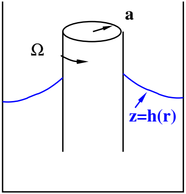
Same mechanism for longitudinal vortices in granular chute flow?

Negative rod-climbing

Standard analysis

Beavers & Joseph 1975

$$h(r) = \frac{1}{\rho g} \left(\tfrac{1}{4} N_1 + N_2 \right) \frac{a^2}{r^2}$$



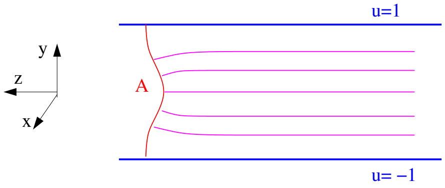
For polymers, $N_1 > 0$ and $N_2 \approx 0$, so climbs $h > 0$,
by tension in streamlines

For concentrated non-Brownian suspensions, $N_1 \approx 0$ and $N_2 < 0$,
so $h < 0$ dips (negative climb)
by tension in the vertical vortex lines

Boyer, Pouliquen & Guazzelli 2017

Edge instability in rheometer

At edge of plate-plate rheometer, top plate coming towards you, bottom away. Perturb liquid interface, in at A.



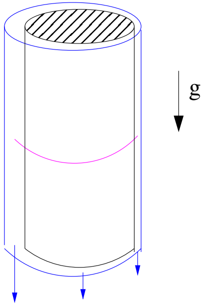
Contours of constant u must meet interface at 90°.

Crowding of contours at A,
$\rightarrow$ increase shear-rate at A,
$\rightarrow$ higher tension in vortex lines at A,
$\rightarrow$ pulls A further into liquid.

Hemmingway & Fielding 2017, Tanner 1993

Lopsided de-wetting of coating on a vertical fibre

Boulogne, Pauchard & Giorgiutti-Dauphiné 2012



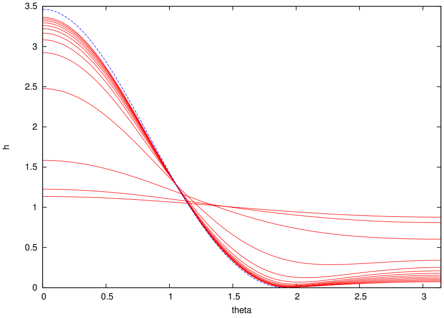
Thicker side
$\rightarrow$ higher shear-rate
$\rightarrow$ higher tension in vortex lines
$\rightarrow$ pulls round to make thicker

Lopsided de-wetting of coating on a vertical fibre

Lubrication equations for thin coating.
Case no z-variations, $h(\theta, t)$

$$h_t + \left(h^3(h_{\theta\theta} + h + Bh^2)_\theta\right)_\theta = 0, \quad \text{where} \quad B = \frac{\Psi_2 g^2}{8\mu\nu^2 a^2}.$$

h



θ

$h(\theta, t)$ at $t = 2^n$
$n = -2, \ldots, 11,$
for $B = 0.5$.

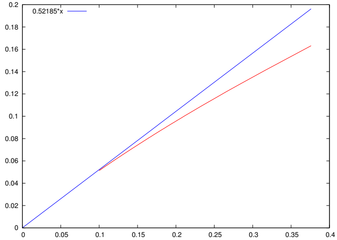
Dotted blue is a steady state which wets only $0 \le \theta \le 1.9071$

Lopsided de-wetting of coating on a vertical fibre

Draining of small region to right

Small region drains as $t^{-1/4}$

$h(\pi)$



$t^{-1/4}$

$$h(\pi) \sim \frac{1+\cos L}{t^{1/4}} \left(\frac{K((\pi - L)\cos L + \sin L}{4Q\sin^5 L} \right)^{1/4}$$

cf P.S.Hammond 1983

Effect of N_2 on flow – tension in the vortex lines

- Bowing of interface in Tanner tilted channel
- Longitudinal vortices in granular chute flow
- Negative rod-climbing
- Edge instability in rheometers
- Lopsided de-wetting on a vertical fibre

N_2 should not have been neglected!

And Anthony . . .

The last problem combines

- Practical application of mathematics - L'Oréal
- Non-Newtonian fluid mechanics
- Surface tension, if not Marangoni
- Asymptotic analysis

Four aspects of Anthony's pioneering work of the 1950s.

Happy birthday Anthony