Part II continued – more details on general issues

Last time – Finite Differences

Higher orders – central, 1-sided, non-equispaced

Compact 4th order Poisson solver

Upwinding

Grids – non-Cartesian, stretched, staggered

Conservative

Part II continued – more details on general issues

Last time – Finite Differences

Higher orders – central, 1-sided, non-equispaced

Compact 4th order Poisson solver

Upwinding

Grids – non-Cartesian, stretched, staggered

Conservative

This time – Finite Elements

Finite Elements

Good for engineering problems with complex geometries

– 'just' need to triangulate domain

Good for elliptic, OK for parabolic, poor for hyperbolic

Good for accuracy & conservative

Finite Elements

Good for engineering problems with complex geometries

– 'just' need to triangulate domain

Good for elliptic, OK for parabolic, poor for hyperbolic

Good for accuracy & conservative

Poor difficult programming on unstructured grid

Poor no efficient Poisson solver on unstructured grid

Poor difficult presenting results on unstructured grid

Use packages, do not program yourself

Finite Elements = Two ideas

1. Simple representation for unknown function over the finite element
   – not point data of FD

Finite Elements = Two ideas

1. Simple representation for unknown function over the finite element
   – not point data of FD

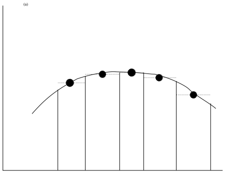
2. Weak formulation of the governing equations
   – variational statement

Representations in 1D

a. Constant elements

$$f(x) = f_i$$

in $x_{i-1} \leq x < x_i$

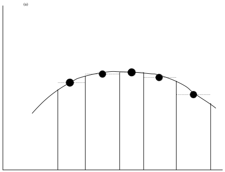


Representations in 1D

a. Constant elements

$$f(x) = f_i$$

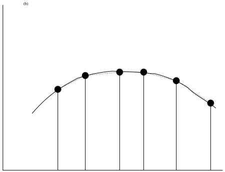
in $x_{i-1} \leq x < x_i$



b. Linear elements

$$f(x) = f_{i-1}\frac{x_i - x}{x_i - x_{i-1}} + f_i\frac{x - x_{i-1}}{x_i - x_{i-1}}$$

in $x_{i-1} \leq x < x_i$



More representations in 1D

First map element to unit interval

$$x(\xi) = x_{i-1} + (x_i - x_{i-1})\xi \quad \text{for } 0 \le \xi \le 1$$

More representations in 1D

First map element to unit interval

$$x(\xi) = x_{i-1} + (x_i - x_{i-1})\xi \quad \text{for } 0 \le \xi \le 1$$

c. Quadratic elements

$$f(x) = f_{i-1}(1-\xi)(1-2\xi) + f_{i-\frac{1}{2}} 4\xi(1-\xi) + f_i \xi(2\xi - 1)$$

NB: f' discontinuous at boundaries

More representations in 1D

First map element to unit interval

$$x(\xi) = x_{i-1} + (x_i - x_{i-1})\xi \quad \text{for } 0 \le \xi \le 1$$

c. Quadratic elements

$$f(x) = f_{i-1}(1-\xi)(1-2\xi) + f_{i-\frac{1}{2}}4\xi(1-\xi) + f_i\xi(2\xi-1)$$

NB: f' discontinuous at boundaries

d. Cubic elements

Obvious generalisation, but better:

$$\begin{aligned} f(x) \;\; &= \;\; f_{i-1}(1-\xi)^2(1+2\xi) + f'_{i-1}(1-\xi)^2\xi \\ & \quad + f_i\xi^2(3-2\xi) + f'_i\xi^2(1-\xi), \end{aligned}$$

Now only f'' discontinuous at boundaries – see splines later

basis functions

In all cases, write:

$$f(x) = \sum f_i \phi_i(x)$$

f_i amplitudes $\phi_i(x)$ basis functions, nonzero only in a few elements

basis functions

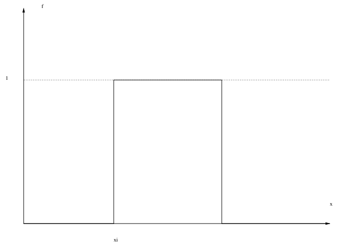
In all cases, write:

$$f(x) = \sum f_i \phi_i(x)$$

f_i amplitudes $\phi_i(x)$ basis functions, nonzero only in a few elements

For the constant elements, the basis functions are

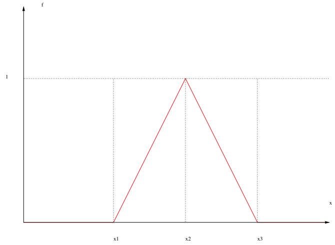
$$\phi_i(x) = \begin{cases} 1 & \text{in} \quad x_{i-1} \leq x < x_i \\ 0 & \text{otherwise.} \end{cases}$$



Basis functions for linear elements

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & \text{in} \quad x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & \text{in} \quad x_i \le x \le x_{i+1} \\ 0 & \text{otherwise,} \end{cases}$$

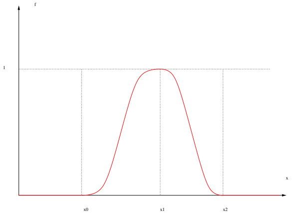
with obvious modifications for the end elements.

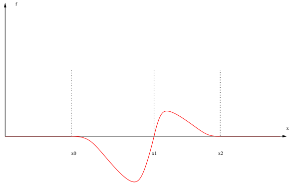


Basis functions for cubic elements

$$\phi_i(x) = \begin{cases} \frac{(x_{i+1}-x)^2(x_{i+1}+2x-3x_i)}{(x_{i+1}-x_i)^3} & \text{in} \quad x_i \leq x < x_{i+1} \\ \frac{(x-x_{i-1})^2(3x_i-2x-x_{i-1})}{(x_i-x_{i-1})^3} & \text{in} \quad x_{i-1} \leq x < x_i \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{\phi}_i(x) = \begin{cases} \frac{(x-x_i)(x_{i+1}-x)^2}{(x_{i+1}-x_i)^2} & \text{in} \quad x_i \leq x < x_{i+1} \\ \frac{(x-x_i)(x-x_{i-1})^2}{(x_i-x_{i-1})^2} & \text{in} \quad x_{i-1} \leq x < x_i \\ 0 & \text{otherwise.} \end{cases}$$





Representations in 2D

Mostly triangles, sometimes rectangles

Representations in 2D

Mostly triangles, sometimes rectangles

a. Constant elements

$$f(x) = f_i \quad \text{in each triangle } i.$$

Representations in 2D

Mostly triangles, sometimes rectangles

a. Constant elements

$$f(x) = f_i \quad \text{in each triangle } i.$$

b. Linear elements Need $\ell_{12}(\mathbf{x})$ vanishing on two vertices, unity on third

$$\ell_{12}(x, y) = \frac{(x - x_1)(y_2 - y_1) - (x_2 - x_1)(y - y_1)}{(x_3 - x_1)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_1)}.$$

Representations in 2D

Mostly triangles, sometimes rectangles

a. Constant elements

$$f(x) = f_i \quad \text{in each triangle } i.$$

b. Linear elements Need $\ell_{12}(\mathbf{x})$ vanishing on two vertices, unity on third

$$\ell_{12}(x, y) = \frac{(x - x_1)(y_2 - y_1) - (x_2 - x_1)(y - y_1)}{(x_3 - x_1)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_1)}.$$

Then

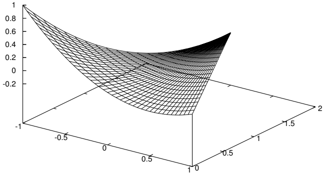
$$f(\mathbf{x}) = f_1\ell_{23}(\mathbf{x}) + f_2\ell_{31}(\mathbf{x}) + f_3\ell_{12}(\mathbf{x}).$$

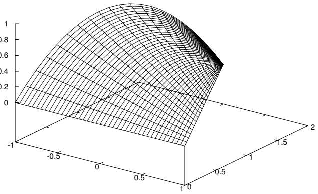
Representation continuous over domain

more representations in 2D

c. Quadratic elements Values at vertices and mid-points

$$
\begin{aligned}
f(\mathbf{x}) &= f_1 \ell_{23}(\mathbf{x})(2\ell_{23}(\mathbf{x}) - 1) \\
&\quad + f_2 \ell_{31}(\mathbf{x})(2\ell_{31}(\mathbf{x}) - 1) \\
&\quad + f_3 \ell_{12}(\mathbf{x})(2\ell_{12}(\mathbf{x}) - 1) \\
&\quad + f_{23} 4\ell_{12}(\mathbf{x})\ell_{31}(\mathbf{x}) + f_{31} 4\ell_{23}(\mathbf{x})\ell_{12}(\mathbf{x}) + f_{12} 4\ell_{31}(\mathbf{x})\ell_{23}(\mathbf{x}).
\end{aligned}
$$





more representations in 2D

d. Cubic elements

Cubic in 2D has 10 degrees of freedom:
1 constant + 2 linear + 3 quadratic + 4 cubic.

more representations in 2D

d. Cubic elements

Cubic in 2D has 10 degrees of freedom:
1 constant + 2 linear + 3 quadratic + 4 cubic.

Can fit f and ∇f at vertices, plus value in centre = the 'bubble'.

more representations in 2D

d. Cubic elements

Cubic in 2D has 10 degrees of freedom:
1 constant + 2 linear + 3 quadratic + 4 cubic.

Can fit f and ∇f at vertices, plus value in centre = the 'bubble'.

e. Basis functions

In all cases, write:

$$f(x) = \sum f_i \phi_i(x)$$

more representations in 2D

d. Cubic elements

Cubic in 2D has 10 degrees of freedom:
1 constant + 2 linear + 3 quadratic + 4 cubic.

Can fit f and ∇f at vertices, plus value in centre = the 'bubble'.

e. Basis functions

In all cases, write:

$$f(x) = \sum f_i \phi_i(x)$$

For linear elements, ϕ_i is non-zero at only one vertex, vanishing on opposite sides of triangles, to form a several-sided pyramid.

more representations in 2D

d. Cubic elements

Cubic in 2D has 10 degrees of freedom:
1 constant + 2 linear + 3 quadratic + 4 cubic.

Can fit f and ∇f at vertices, plus value in centre = the 'bubble'.

e. Basis functions

In all cases, write:

$$f(x) = \sum f_i \phi_i(x)$$

For linear elements, ϕ_i is non-zero at only one vertex, vanishing on opposite sides of triangles, to form a several-sided pyramid.

Local nature $\rightarrow$ sparse coupling matrices for PDEs

more representations in 2D

f. Rectangles

more representations in 2D

f. Rectangles

Obvious constant elements

more representations in 2D

f. Rectangles

Obvious constant elements

Bilinear, taking values at vertices

$$f(\mathbf{x}) = f_1\xi\eta + f_2(1-\xi)\eta + f_3\xi(1-\eta) + f_4(1-\xi)(1-\eta).$$

Continuous over domain.

more representations in 2D

f. Rectangles

Obvious constant elements

Bilinear, taking values at vertices

$$f(\mathbf{x}) = f_1\xi\eta + f_2(1-\xi)\eta + f_3\xi(1-\eta) + f_4(1-\xi)(1-\eta).$$

Continuous over domain.

Biquadratic – sum of 9 terms, each product of quadratic in separate coordinates, taking values at vertices and midpoints.

more representations in 2D

f. Rectangles

Obvious constant elements

Bilinear, taking values at vertices

$$f(\mathbf{x}) = f_1\xi\eta + f_2(1-\xi)\eta + f_3\xi(1-\eta) + f_4(1-\xi)(1-\eta).$$

Continuous over domain.

Biquadratic – sum of 9 terms, each product of quadratic in separate coordinates, taking values at vertices and midpoints.

Continuous and continuous tangential derivative at boundaries.

Variational statement of Poisson problem

$$\nabla^2 f = \rho \quad \text{in volume } V$$

with boundary condition, say $\quad f = g \quad$ on surface S,

with $\rho(\mathbf{x})$ and $g(\mathbf{x})$ given.

Variational statement of Poisson problem

$$\nabla^2 f = \rho \quad \text{in volume } V$$

$$\text{with boundary condition, say} \quad f = g \quad \text{on surface } S,$$

with $\rho(\mathbf{x})$ and $g(\mathbf{x})$ given.

Rayleigh-Ritz variational formulation:

Variational statement of Poisson problem

$$\nabla^2 f = \rho \quad \text{in volume } V$$

$$\text{with boundary condition, say} \quad f = g \quad \text{on surface } S,$$

with $\rho(\mathbf{x})$ and $g(\mathbf{x})$ given.

Rayleigh-Ritz variational formulation: out of all those functions $f(\mathbf{x})$ that satisfy BCs, the one that minimises

$$I(f) = \int_V \left(\frac{1}{2} |\nabla f|^2 + \rho f \right) dV$$

also satisfies the Poisson problem.

Substitute FE representation

$$f(\mathbf{x}) = \sum f_i \phi_i(\mathbf{x})$$

Substitute FE representation

$$f(\mathbf{x}) = \sum f_i \phi_i(\mathbf{x})$$

Then

$$I(f) = \frac{1}{2} \sum_{ij} f_i f_j \underbrace{\int \nabla\phi_i \cdot \nabla\phi_j}_{\text{global stiffness } K_{ij}} + \sum_i f_i \underbrace{\int \rho\phi_i}_{\text{forcing } r_i}$$

Substitute FE representation

$$f(\mathbf{x}) = \sum f_i \phi_i(\mathbf{x})$$

Then

$$I(f) = \frac{1}{2} \sum_{ij} f_i f_j \underbrace{\int \nabla\phi_i \cdot \nabla\phi_j}_{\text{global stiffness } K_{ij}} + \sum_i f_i \underbrace{\int \rho\phi_i}_{\text{forcing } r_i}$$

Minimise over f_i

$$K_{ij} f_j + r_i = 0.$$

Substitute FE representation

$$f(\mathbf{x}) = \sum f_i \phi_i(\mathbf{x})$$

Then

$$I(f) = \frac{1}{2} \sum_{ij} f_i f_j \underbrace{\int \nabla\phi_i \cdot \nabla\phi_j}_{\text{global stiffness } K_{ij}} + \sum_i f_i \underbrace{\int \rho\phi_i}_{\text{forcing } r_i}$$

Minimise over f_i

$$K_{ij} f_j + r_i = 0.$$

With these f_j, the f satisfies

$$-\int \nabla f \cdot \nabla\phi_i = \int \rho\phi_i \quad \text{for all } i,$$

Substitute FE representation

$$f(\mathbf{x}) = \sum f_i \phi_i(\mathbf{x})$$

Then

$$I(f) = \frac{1}{2} \sum_{ij} f_i f_j \underbrace{\int \nabla\phi_i \cdot \nabla\phi_j}_{\text{global stiffness } K_{ij}} + \sum_i f_i \underbrace{\int \rho\phi_i}_{\text{forcing } r_i}$$

Minimise over f_i

$$K_{ij} f_j + r_i = 0.$$

With these f_j, the f satisfies

$$-\int \nabla f \cdot \nabla\phi_i = \int \rho\phi_i \quad \text{for all } i,$$

i.e. satisfy PDE in all (finite) ϕ_i directions.

Substitute FE representation

$$f(\mathbf{x}) = \sum f_i \phi_i(\mathbf{x})$$

Then

$$I(f) = \frac{1}{2} \sum_{ij} f_i f_j \underbrace{\int \nabla\phi_i \cdot \nabla\phi_j}_{\text{global stiffness } K_{ij}} + \sum_i f_i \underbrace{\int \rho\phi_i}_{\text{forcing } r_i}$$

Minimise over f_i

$$K_{ij} f_j + r_i = 0.$$

With these f_j, the f satisfies

$$-\int \nabla f \cdot \nabla\phi_i = \int \rho\phi_i \quad \text{for all } i,$$

i.e. satisfy PDE in all (finite) ϕ_i directions.
The weak formulation of the PDE (f can be non-C^2)

Details in 1D

$$\frac{d^2 f}{dx^2} = \rho \quad \text{in } a < x < b, \quad \text{with } f(a) = A \text{ and } f(b) = B,$$

where $\rho(x), A$ and B given.

Details in 1D

$$\frac{d^2 f}{dx^2} = \rho \quad \text{in } a < x < b, \quad \text{with } f(a) = A \text{ and } f(b) = B,$$

where $\rho(x), A$ and B given.

Divide $[a, b]$ into N equal segments $h = (b - a)/N$.

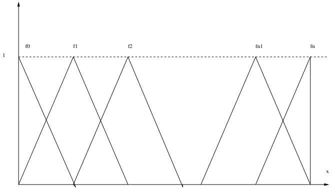
Details in 1D

$$\frac{d^2 f}{dx^2} = \rho \quad \text{in } a < x < b, \quad \text{with } f(a) = A \text{ and } f(b) = B,$$

where $\rho(x), A$ and B given.

Divide $[a, b]$ into N equal segments $h = (b - a)/N$.

Use linear finite elements with basis functions



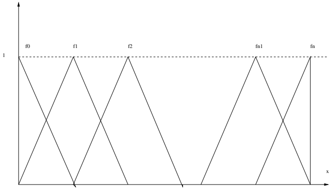
Details in 1D

$$\frac{d^2 f}{dx^2} = \rho \quad \text{in } a < x < b, \quad \text{with } f(a) = A \text{ and } f(b) = B,$$

where $\rho(x), A$ and B given.

Divide $[a, b]$ into N equal segments $h = (b - a)/N$.

Use linear finite elements with basis functions



Unknown $f(x)$ represented (BCs built in)

$$f(x) = A\phi_0(x) + B\phi_N(x) + \sum_{i=1}^{N-1} f_i \phi_i(x)$$

more details in 1D

At interior pts

$$K_{ij} = \int \nabla\phi_i \cdot \nabla\phi_j = \begin{cases} 2/h & \text{if } i = j, \\ -1/h & \text{if } i = j \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

by $\nabla\phi_i = 0, +1/h, -1/h, 0$

more details in 1D

At interior pts

$$K_{ij} = \int \nabla\phi_i \cdot \nabla\phi_j = \begin{cases} 2/h & \text{if } i = j, \\ -1/h & \text{if } i = j \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

by $\nabla\phi_i = 0, +1/h, -1/h, 0$
Take given $\rho(x)$ to be piecewise constant, then forcing

$$r_i = \int \rho(x)\phi_i = h\rho_i.$$

more details in 1D

At interior pts

$$K_{ij} = \int \nabla\phi_i \cdot \nabla\phi_j = \begin{cases} 2/h & \text{if } i = j, \\ -1/h & \text{if } i = j \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

by $\nabla\phi_i = 0, +1/h, -1/h, 0$
Take given $\rho(x)$ to be piecewise constant, then forcing

$$r_i = \int \rho(x)\phi_i = h\rho_i.$$

So equation governing unknown amplitudes f_i becomes

$$\frac{1}{h}\left(-f_{i-1} + 2f_i - f_{i+1}\right) + h\rho_i = 0 \quad \text{for } i = 1, 2, \ldots, N-1,$$

more details in 1D

At interior pts

$$K_{ij} = \int \nabla\phi_i \cdot \nabla\phi_j = \begin{cases} 2/h & \text{if } i = j, \\ -1/h & \text{if } i = j \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

by $\nabla\phi_i = 0, +1/h, -1/h, 0$
Take given $\rho(x)$ to be piecewise constant, then forcing

$$r_i = \int \rho(x)\phi_i = h\rho_i.$$

So equation governing unknown amplitudes f_i becomes

$$\frac{1}{h}(-f_{i-1} + 2f_i - f_{i+1}) + h\rho_i = 0 \quad \text{for } i = 1, 2, \ldots, N-1,$$

– same for the point values in the finite difference approach.

more details in 1D

Remark If evaluate r_i more accurately

$$r_i = \int \rho(x)\phi_i(x) = \rho_i + \frac{h^3}{12}\rho_i'' + O(h^5).$$

So obtain f_i to $O(h^4)$.

more details in 1D

Remark If evaluate r_i more accurately

$$r_i = \int \rho(x)\phi_i(x) = \rho_i + \frac{h^3}{12}\rho_i'' + O(h^5).$$

So obtain f_i to $O(h^4)$.
Yet $f(x)$ still only $O(h^2)$ in interior of elements.

more details in 1D

Remark If evaluate r_i more accurately

$$r_i = \int \rho(x)\phi_i(x) = \rho_i + \frac{h^3}{12}\rho_i'' + O(h^5).$$

So obtain f_i to $O(h^4)$.
Yet $f(x)$ still only $O(h^2)$ in interior of elements.

Remark For non-equispaced intervals, obtain

$$\frac{1}{h_{i-\frac{1}{2}}}\left(-f_{i-1} + f_i\right) + \frac{1}{h_{i+\frac{1}{2}}}\left(f_i - f_{i+1}\right) + \frac{h_{i-\frac{1}{2}} + h_{i+\frac{1}{2}}}{2}\rho_i = 0.$$

more details in 1D

Remark If evaluate r_i more accurately

$$r_i = \int \rho(x)\phi_i(x) = \rho_i + \frac{h^3}{12}\rho_i'' + O(h^5).$$

So obtain f_i to $O(h^4)$.
Yet $f(x)$ still only $O(h^2)$ in interior of elements.

Remark For non-equispaced intervals, obtain

$$\frac{1}{h_{i-\frac{1}{2}}}\left(-f_{i-1} + f_i\right) + \frac{1}{h_{i+\frac{1}{2}}}\left(f_i - f_{i+1}\right) + \frac{h_{i-\frac{1}{2}} + h_{i+\frac{1}{2}}}{2}\rho_i = 0.$$

i.e. FE approach naturally conservative.