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$$\nabla^2 f = \rho \qquad \equiv \qquad \delta I = 0, \quad I = \int \tfrac{1}{2} |\nabla f|^2 + \rho f$$

so

$$K_{ij} f_j = r_i$$

with

$$K_{ij} = \int \nabla \phi_i \cdot \nabla \phi_j \qquad r_i = \int \rho \phi_i$$

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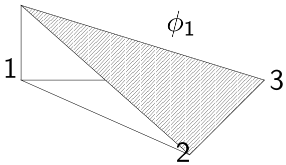
This time – Finite Elements, part 2

Details in 2D with linear triangular elements

Consider one triangle
$|\nabla\phi_1| = 1/h_1$



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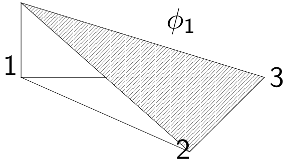

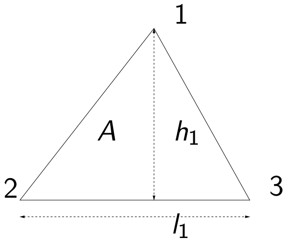

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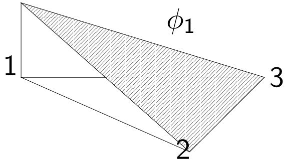

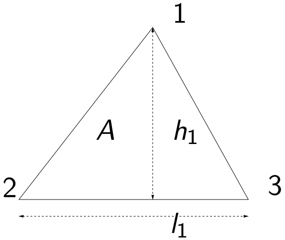

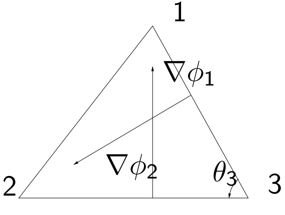

$$K_{11} = \int \nabla\phi_1 \cdot \nabla\phi_1 = \frac{A}{h_1^2} \quad K_{12} = \int \nabla\phi_1 \cdot \nabla\phi_2 = -\frac{A\cos\theta_3}{h_1 h_2}$$

further manipulations

$$h_1 = \ell_2 \sin\theta_3 \quad \text{and} \quad h_2 = \ell_1 \sin\theta_3.$$

and

$$A = \tfrac{1}{2}\ell_1\ell_2 \sin\theta_3.$$

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$$K_{11} + K_{12} + K_{13} = 0.$$

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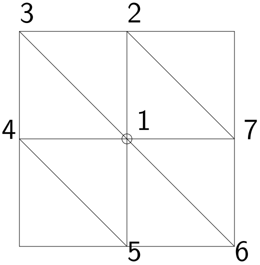
Because

$$\nabla\phi_1 \cdot \nabla\left(\phi_1 + \phi_2 + \phi_3 \equiv 1\right) \equiv 0.$$

Assembling contributions from different triangles

Special grid:



For the 123-triangle,

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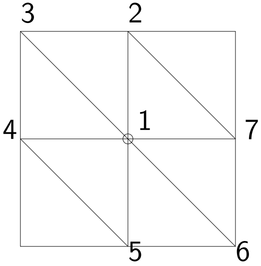

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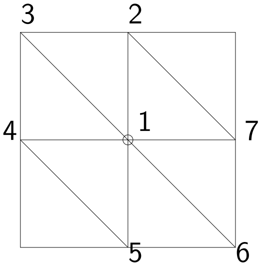

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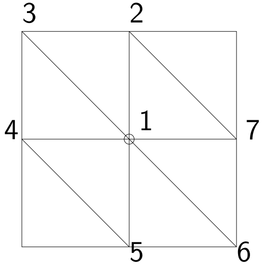

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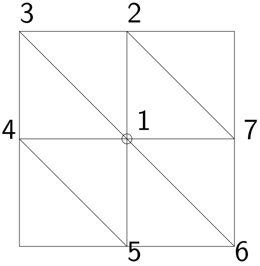

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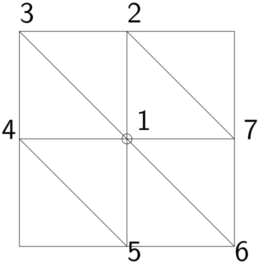

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Assembling from all triangles

$$K_{11} = 4, \quad K_{12} = K_{14} = K_{15} = K_{17} = -1, \quad K_{13} = K_{16} = 0.$$

Forcing

$$r_i = \int \rho \phi_i = \frac{1}{3} A \rho_i$$

by linear variation of ϕ_i on each of the 6 triangle (so $A = 3h^2$).

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Programming challenge

On more general unstructured grids need

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Can use list of triangles to assemble sparse matrix K_{ij}.

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Require residual to be OG all N basis functions

$$\langle A(u) - f, \phi_j \rangle = 0 \quad \text{all } j$$

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i.e.

$$\int A(u) = \int f$$

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part of Navier-Stokes

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Substitute FE representation $u = \sum u_i(t)\phi_i(x)$

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i.e.

$$M_{ij}\dot{u}_i = -K_{ij}u_i$$

with 'Mass' $\quad M_{ij} = \langle \phi_i, \phi_j \rangle \quad$ and 'Stiffness' $\quad K_{ij} = \langle \nabla \phi_i, \nabla \phi_j \rangle$

b. In 1D

Using linear elements on equal intervals h

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$$M_{ij} = \begin{cases} \frac{2}{3}h & i = j \\ \frac{1}{6}h & i = j \pm 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad K_{ij} = \begin{cases} 2/h & i = j \\ -1/h & i = j \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

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Hence

$$h\left(\tfrac{1}{6}\dot{u}_{i-1} + \tfrac{2}{3}\dot{u}_i + \tfrac{1}{6}\dot{u}_{i+1}\right) = \frac{1}{h}\left(u_{i-1} - 2u_i + u_{i+1}\right).$$

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Remark Time step this “semi-discretised” form with any FD (NOT FE) algorithm, e.g.

$$u_i^{n+1} = u_i^n + \Delta t \dot{u}_i$$

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Assemble contributions to M and K from different triangles

$$M_{ij} = \begin{cases} \frac{1}{12} h^2 & i = j \\ \frac{1}{24} h^2 & i \neq j, \end{cases} \qquad K_{ij} \quad \text{as before}$$

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So

$$\frac{1}{2}h^2\left(\dot{u}_1 + \frac{1}{6}(\dot{u}_2 + \dot{u}_3 + \dot{u}_4 + \dot{u}_5 + \dot{u}_6 + \dot{u}_7)\right) = u_2 + u_4 + u_5 + u_7 - 4u_1$$

with linear problem to find $\dot{u}_i$

Navier-Stokes

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a. Weak formulation

Use FE representation

$$\mathbf{u}(\mathbf{x},t) = \sum_i \mathbf{u}_i(t)\phi_i(\mathbf{x}),$$
$$p(\mathbf{x},t) = \sum_i p_i(t)\psi_i(\mathbf{x}),$$

Need different ϕ_i and ψ_i.

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Need different ϕ_i and ψ_i.

Galerkin

$$\left\langle \rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla p - \mu\nabla^2\mathbf{u}, \phi_j \right\rangle = 0 \quad \text{all } \phi_j,$$

and incompressibility constraint

$$\langle \nabla\cdot\mathbf{u}, \psi_j\rangle = 0 \quad \text{all } \psi_j.$$

Integration by parts

Integration by parts

$$\rho\left(M_{ij}\dot{\mathbf{u}}_j + Q_{ijk}\mathbf{u}_j\mathbf{u}_k\right) = -B_{ji}p_j - \mu K_{ij}\mathbf{u}_j,$$

and

$$-B_{ij}\mathbf{u}_j = 0,$$

with mass M and stiffness K as before, and two new coupling matrices

$$Q_{ijk} = \langle \phi_i \nabla \phi_j, \phi_k \rangle \quad \text{and} \quad B_{ij} = \langle \nabla \psi_i, \phi_j \rangle = -\langle \psi_i, \nabla \phi_j \rangle.$$

b. Time integration

Time step semi-discretised form with any FD algorithm

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Incompressible by projection split step

$$\begin{aligned} \mathbf{u}^* &= \mathbf{u}_i^n + \Delta t \,(\dot{\mathbf{u}}_i^n \quad \text{without the } p \text{ term})\,, \\ \mathbf{u}^{n+1} &= \mathbf{u}^* + \Delta t \,(\dot{\mathbf{u}}_i^n \quad \text{with just the } p \text{ term})\,, \end{aligned}$$

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with p chosen so the incompressibility at the end of the step

$$B\mathbf{u}^{n+1} = 0.$$

Problems with pressure – Locking

Consider triangles with velocity linear and pressure constant

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Then

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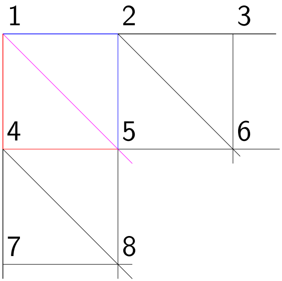
$$\oint_{\Delta_j} u_n = 0,$$

i.e. no net volume flux out of triangle Δ_j.

. . . locking

Consider top corner, with $\mathbf{u} = 0$ on boundary (74123).



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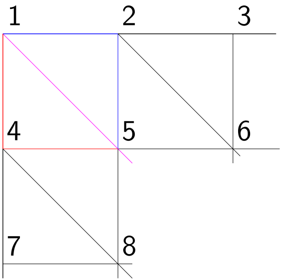

For triangle 145,

...locking

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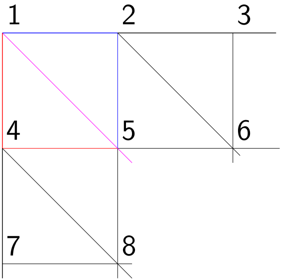

For triangle 145,
flux in over edge 45 is $\frac{1}{2} h v_5$,

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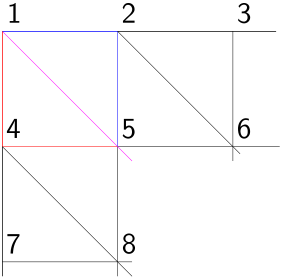

For triangle 145,
flux in over edge 45 is $\frac{1}{2}hv_5$, flux out over edge 15 is $\frac{1}{2}h(u_5 + v_5)$

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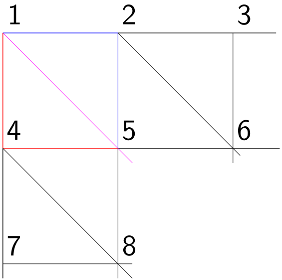

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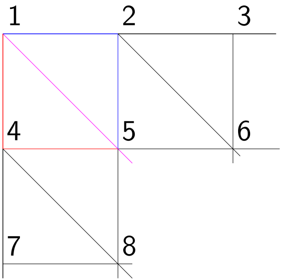

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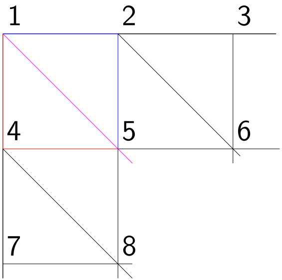

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Then $\mathbf{u}_6 = 0$ and $\mathbf{u}_8 = 0$, so $\mathbf{u} \equiv 0$.

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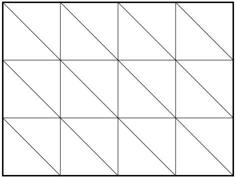
But on a 4×3 grid



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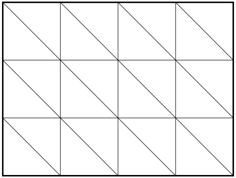

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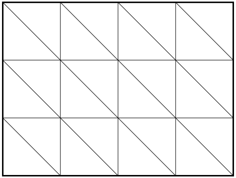

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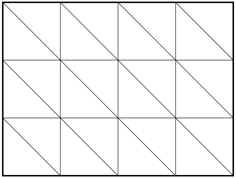

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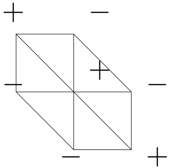
Create more u & v with bubble functions (vanish on boundaries of elements), or reduce number of pressure

Spurious pressure modes

if have p linear over triangle



As in Algorithm 2 of driven cavity, above pressure drives no flow

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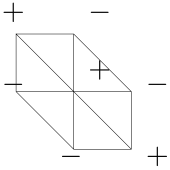

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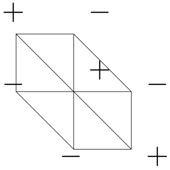

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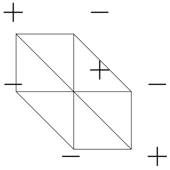

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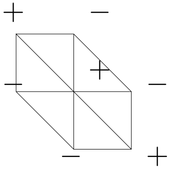

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Weak formulation

$$B_{ij} \mathbf{u}_j - \beta h^2 p_i = 0.$$

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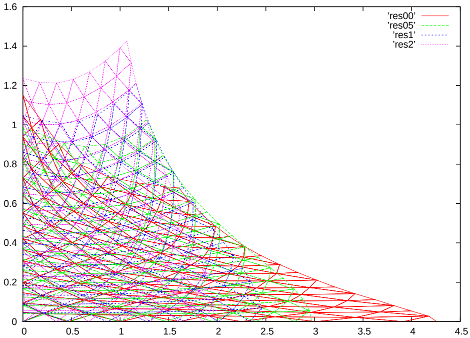

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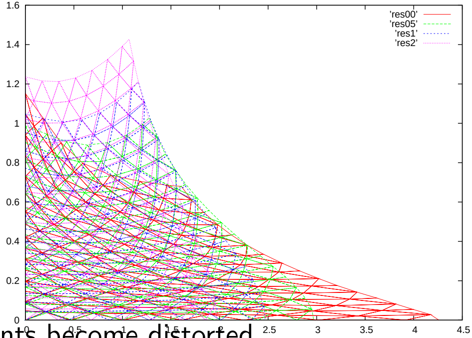

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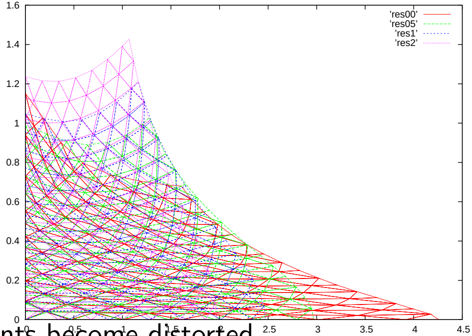

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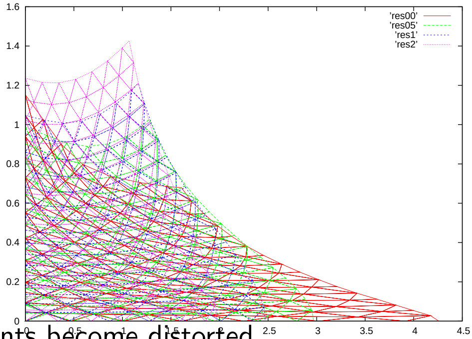

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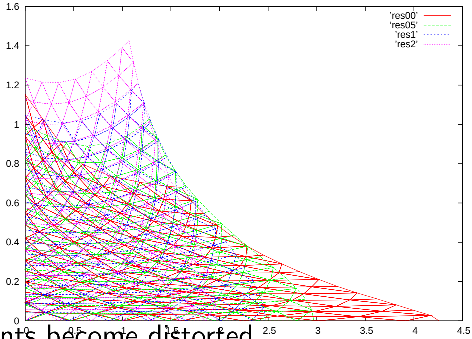

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- ALE – somewhere between Lagrangian and Eulerian.