Computational Methods in Fluid Mechanics

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Course structure

Three parts:

- Simple Navier-Stokes problem by simple method
  - accuracy, stability, pressure

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  – discretisation, time-stepping, linear algebra

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  – accuracy, stability, pressure

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- Collection of special topics

  – demo FreeFem, hyperbolic, free surfaces, fast Poisson,

  wavelets, particle methods

1. The driven cavity

Incompressible Navier-Stokes

$$\nabla \cdot \mathbf{u} = 0,$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u},$$

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2D, $L \times L$-box

$$\mathbf{u} = 0 \quad \text{on } y = 0 \text{ and } 0 < x < L, \text{ and on } x = 0 \text{ or } L \text{ and } 0 < y < L,$$

$$\text{and} \quad \mathbf{u} = (U(x), 0) \quad \text{on } y = L \text{ and } 0 < x < L.$$

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To find the force on the lid

$$F = \int_0^L \mu \left. \frac{\partial u}{\partial y} \right|_{y=L} dx$$

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  - $Re \ll 1$ must resolve fast diffusion of vorticity,
  - $Re \gg 1$ must resolve thin boundary layers,
  - we study $Re = 10$.

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- $\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}$ – second order hyperbolic

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  IC $\phi(x,0)$ and $\phi_t(x,0)$ and

  BC at both ends either ϕ or ϕ_x or mixed.

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- $\frac{\partial\phi}{\partial t} = D\frac{\partial^2\phi}{\partial x^2}$ – Diffusion equation, parabolic

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- Naming from quadratic forms

  $ax^2 + bxy + cy^2 + dx + fy + g = 0$

  $a\frac{\partial^2\phi}{\partial x^2} + b\frac{\partial^2\phi}{\partial x\partial y} + c\frac{\partial^2\phi}{\partial y^2} + d\frac{\partial\phi}{\partial x} + e\frac{\partial\phi}{\partial y} + f\phi = 0$

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- Numerically
  - hyperbolic – tough
  - elliptic – costly
  - parabolic – safest

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- Therefore we take $\mathbf{u} = (U_0 \sin^2 \pi x/L, 0)$

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Scale u on U_0, x and y on L, t on L/U_0 and p on ρU_0^2. Then

$$Re = \frac{\text{inertial terms } \rho U_0^2/L}{\text{viscous terms } \mu U_0/L^2} = \frac{U_0 L}{\nu}.$$

The non-dimensionalised problem

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with BCs

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Finally the force, scaled by μU_0

$$F = \int_0^1 \left.\frac{\partial u}{\partial y}\right|_{y=1} dx.$$

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EJH recommends IVP, linear.

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Methods for relaxing to SS $\equiv$ pseudo time-stepping.

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- Find the ∇p that ensures $\nabla \cdot \mathbf{u} = 0$
  – primitive variable formulation
- Eliminate p by forming the vorticity equation
  – streamfunction-vorticity formulation

2. Streamfunction-vorticity formulation

Automatically satisfy constraint $\nabla \cdot \mathbf{u} = 0$
by using the streamfunction representation $\psi(x, y)$

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}.$$

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In 2D flow vorticity is

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\nabla^2 \psi.$$

Vorticity equation

Take curl of momentum equation to eliminate p

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$\rightarrow$ sides $=$ streamline $\quad \rightarrow \psi = 0$.

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BC2: tangential velocity
$\frac{\partial \psi}{\partial y} = \sin^2 \pi x$ on top $y = 1$, $0 < x < 1$
$\frac{\partial \psi}{\partial y} = 0$ on bottom $y = 0$, $0 < x < 1$
$\frac{\partial \psi}{\partial x} = 0$ on sides $x = 0$ and 1, $0 < y < 1$

Solve as decoupled pair

1. At each *t* given ω, find ψ:

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$\rightarrow$ not quite decoupled.