Resumé: Driven cavity, ψ–ω formulation

Poisson problem: $\nabla^2\psi = -\omega$, SOR

Vorticity evolution: $\frac{\partial\omega}{\partial t} = -\frac{\partial(\omega,\psi)}{\partial(x,y)} + \frac{1}{Re}\nabla^2\omega$

BC for ω

Timestep instability $\rightarrow \Delta t = \frac{1}{5} Re \Delta x^2$

Check $O(\Delta x^2)$ accuracy

Results, force on lid

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But boundary condition on p?

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$$\frac{\partial u}{\partial t} + \quad u \quad \frac{\partial u}{\partial x} + v \quad \frac{\partial u}{\partial y} \quad = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\quad \frac{\partial^2 u}{\partial x^2} + \quad \frac{\partial^2 u}{\partial y^2} \right)$$

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NB: pressure arbitrary to additive constant

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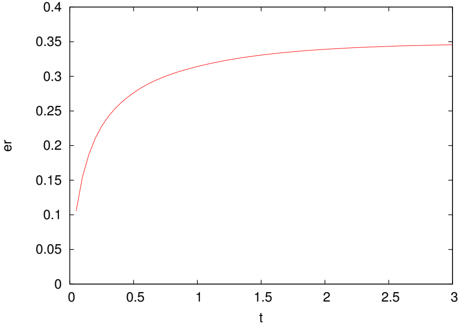
Error is satisfying $\nabla \cdot \mathbf{u} = 0$ ($N = 20$) as function of t



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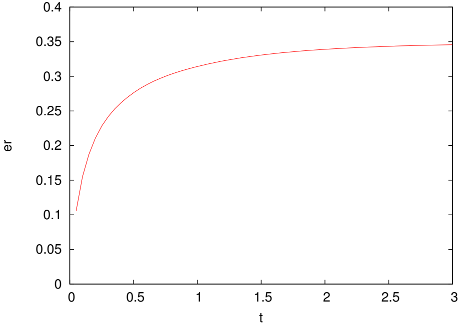

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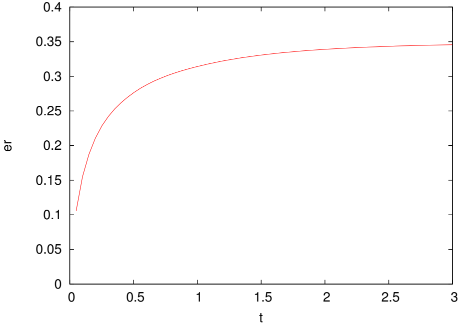

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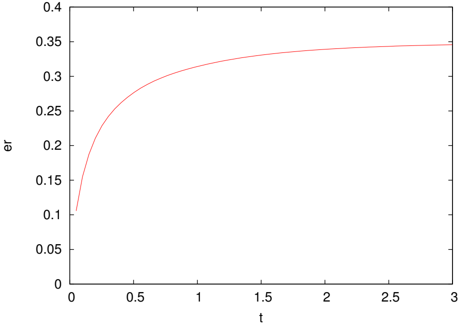

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Pressure equation assumes $\nabla \cdot \mathbf{u}^n = 0$, and does not correct if untrue, so error accumulates.

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In space of all $\mathbf{u}(\mathbf{x}, t)$,

solution constrained to surface S where $\nabla \cdot \mathbf{u} = 0$

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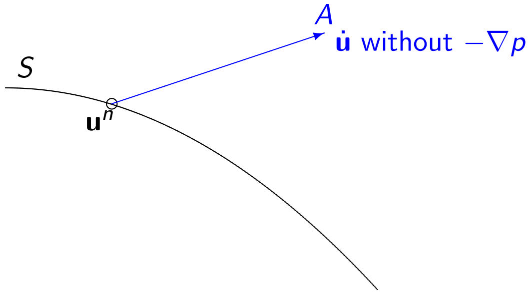

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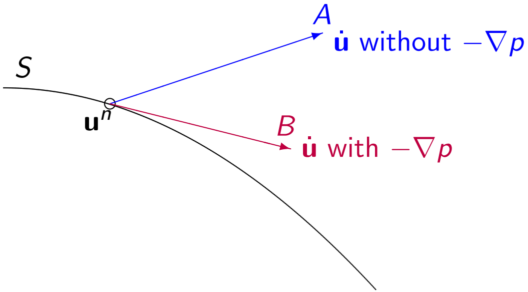

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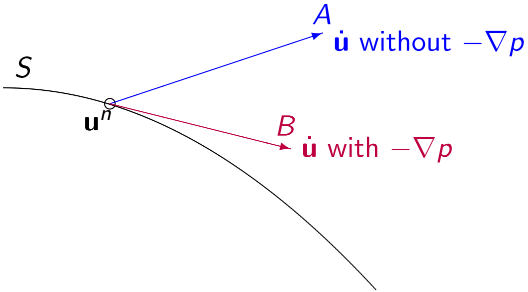
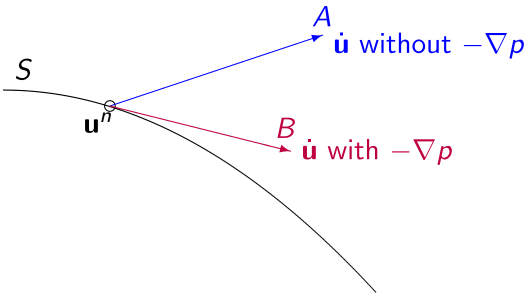


Role of ∇p is to project out component of $\partial \mathbf{u}/\partial t$ normal to surface.

. . . incompressibility as a constraint

Forward time stepping $O(\Delta t) \rightarrow$ slow drift away from surface



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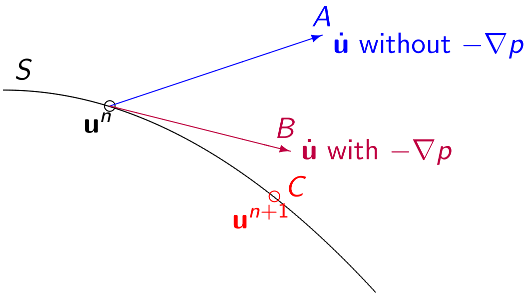

Avoid slow accumulation of errors by *projecting* $\mathbf{u}$ *back onto surface at* C

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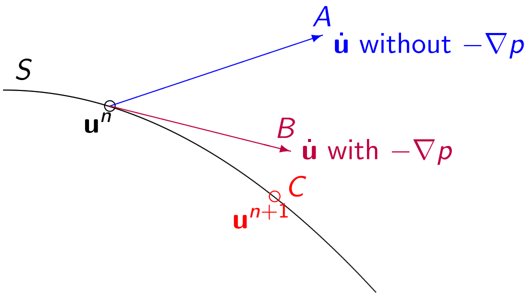

Avoid slow accumulation of errors by *projecting* $\mathbf{u}$ *back onto surface at* C

Implemented by split time step.

3.4 Split time step

First part (no ∇p)

$$\mathbf{u}^* = \mathbf{u}^n + \Delta t \left(-\mathbf{u}^n \cdot \boldsymbol{\nabla} \mathbf{u}^n + \frac{1}{Re} \nabla^2 \mathbf{u}^n \right).$$

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and then evaluate $\mathbf{u}^{n+1}$.

First part (no ∇p)

At interior points

$$
\begin{aligned}
u^*_{ij} &= u^n_{ij} \\
&+ \Delta t \left(-u^n_{ij} \frac{u^n_{i+1j} - u^n_{i-1j}}{2\Delta x} - v^n_{ij} \frac{u^n_{ij+1} - u^n_{ij-1}}{2\Delta x} \right) \\
&+ \frac{\Delta t}{Re \Delta x^2} \begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix} u^n_{ij},
\end{aligned}
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and a similar expression for v^*_{ij}.

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Several algorithms for the projection step.

Projection step – algorithm 2

$$\frac{\Delta t}{\Delta x^2}\begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix} p_{i\,j} = \frac{u^*_{i+1\,j} - u^*_{i-1\,j}}{2\Delta x} + \frac{v^*_{i\,j+1} - v^*_{i\,j-1}}{2\Delta x}$$

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Now with our central differencing

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\left.\frac{\partial u^{n+1}}{\partial x}\right|_{i\,j} &= \frac{u^{n+1}_{i+1\,j} - u^{n+1}_{i-1\,j}}{2\Delta x} \\
&= \frac{\left(u^{*}_{i+1\,j} - \Delta t\,\frac{p_{i+2\,j} - p_{i\,j}}{2\Delta x}\right) - \left(u^{*}_{i-1\,j} - \Delta t\,\frac{p_{i\,j} - p_{i-2\,j}}{2\Delta x}\right)}{2\Delta x},
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and similarly for $\partial v^{n+1}/\partial y$.
Hence pressure should satisfy (recall $f'' \neq (f')'$)

$$\frac{\Delta t}{4\Delta x^2}\begin{pmatrix} & & 1 & & \\ & & 0 & & \\ 1 & 0 & -4 & 0 & 1 \\ & & 0 & & \\ & & 1 & & \end{pmatrix} p_{i\,j} = \left(\frac{u^*_{i+1\,j} - u^*_{i-1\,j}}{2\Delta x} + \frac{v^*_{i\,j+1} - v^*_{i\,j-1}}{2\Delta x}\right).$$

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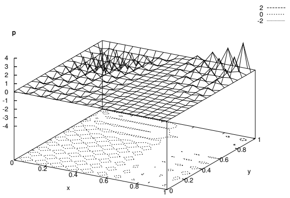

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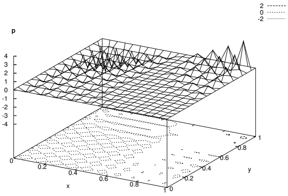
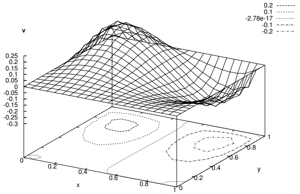


– effect on velocity v



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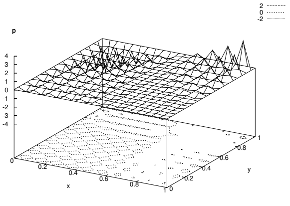

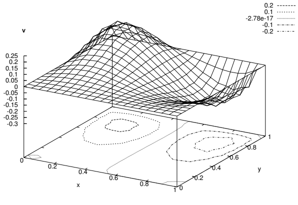

Two uncoupled solutions for pressure on odd/even $i + j$

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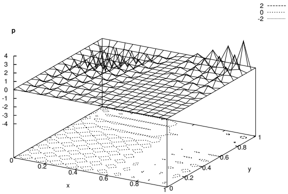
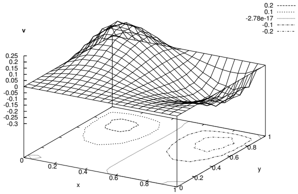
Pressure – effect on velocity v





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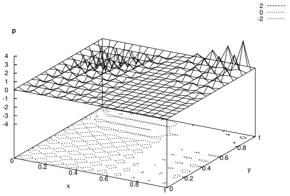

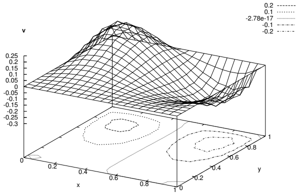

Two uncoupled solutions for pressure on odd/even $i + j$

Spurious mode $+ \; - \; + \; - \; +$ has no ∇p

Also errors 4 times larger from wide span molecule

3.6 Staggered grid – algorithm 4

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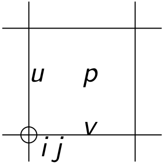
New idea – a staggered grid with different variables at different locations



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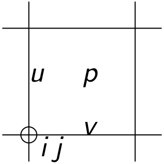

Write $u_{i\,j+\frac{1}{2}}$, $v_{i+\frac{1}{2}\,j}$ and $p_{i+\frac{1}{2}\,j+\frac{1}{2}}$

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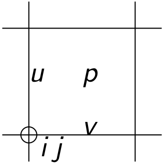

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Good for central differencing

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Boundaries coincide with mass flow BC

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$$u_{0\,j+\frac{1}{2}} = u_{N\,j+\frac{1}{2}} = 0$$
$$v_{i+\frac{1}{2}\,0} = v_{i+\frac{1}{2}\,N} = 0$$

for $j = 0 \to N - 1$ and $i = 0 \to N - 1$ respectively

Boundary conditions

Boundaries coincide with mass flow BC

$$u_{0\,j+\frac{1}{2}} = u_{N\,j+\frac{1}{2}} = 0$$
$$v_{i+\frac{1}{2}\,0} = v_{i+\frac{1}{2}\,N} = 0$$

for $j = 0 \to N-1$ and $i = 0 \to N-1$ respectively

The tangential component of velocity is held half a grid block away

Boundary conditions

Boundaries coincide with mass flow BC

$$u_{0\,j+\frac{1}{2}} = u_{N\,j+\frac{1}{2}} = 0$$
$$v_{i+\frac{1}{2}\,0} = v_{i+\frac{1}{2}\,N} = 0$$

for $j = 0 \rightarrow N-1$ and $i = 0 \rightarrow N-1$ respectively

The tangential component of velocity is held half a grid block away
Use one point outside

$$v_{-\frac{1}{2}\,j} = -v_{\frac{1}{2}\,j} \quad \text{and} \quad v_{N+\frac{1}{2}\,j} = -v_{N-\frac{1}{2}\,j}$$
$$u_{i\,-\frac{1}{2}} = -u_{i\,\frac{1}{2}} \quad \text{and} \quad u_{i\,N+\frac{1}{2}} = 2\sin^2(i * \Delta x) - u_{i\,N-\frac{1}{2}}$$

for $j = 1 \rightarrow N-1$ and $i = 1 \rightarrow N-1$ respectively

Momentum equation at $i\,j+\frac{1}{2}$

First part of split time step (without pressure)

$$u^{*}_{i\,j+\frac{1}{2}} = u^{n}_{i\,j+\frac{1}{2}}$$

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First part of split time step (without pressure)

$$u^*_{i\,j+\frac{1}{2}} = u^n_{i\,j+\frac{1}{2}} - \Delta t\, u^n_{i\,j+\frac{1}{2}} \frac{u^n_{i+1\,j+\frac{1}{2}} - u^n_{i-1\,j+\frac{1}{2}}}{2\Delta x}$$

Momentum equation at $i\,j+\frac{1}{2}$

First part of split time step (without pressure)

$$
\begin{aligned}
u^*_{i\,j+\frac{1}{2}} &= u^n_{i\,j+\frac{1}{2}} \\
&- \Delta t\, u^n_{i\,j+\frac{1}{2}} \frac{u^n_{i+1\,j+\frac{1}{2}} - u^n_{i-1\,j+\frac{1}{2}}}{2\Delta x} \\
&- \Delta t \frac{1}{4}\left(v^n_{i+\frac{1}{2}\,j} + v^n_{i-\frac{1}{2}\,j} + v^n_{i+\frac{1}{2}\,j+1} + v^n_{i-\frac{1}{2}\,j+1}\right) \frac{u^n_{i\,j+\frac{3}{2}} - u^n_{i\,j-\frac{1}{2}}}{2\Delta x}
\end{aligned}
$$

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&+ \frac{\Delta t}{Re \Delta x^2} \begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix} u^n_{i\,j+\frac{1}{2}},
\end{aligned}
$$

Incompressibility at $i+\frac{1}{2}\,j+\frac{1}{2}$

Compact

$$\frac{\Delta t}{\Delta x^2}\begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix} p_{i+\frac{1}{2}\,j+\frac{1}{2}} = \frac{u^*_{i+1\,j+\frac{1}{2}} - u^*_{i\,j+\frac{1}{2}}}{\Delta x} + \frac{v^*_{i+\frac{1}{2}\,j+1} - v^*_{i+\frac{1}{2}\,j}}{\Delta x}.$$

Incompressibility at $i + \frac{1}{2}\, j + \frac{1}{2}$

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$$\frac{\Delta t}{\Delta x^2} \begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix} p_{i+\frac{1}{2}\, j+\frac{1}{2}} = \frac{u^*_{i+1\, j+\frac{1}{2}} - u^*_{i\, j+\frac{1}{2}}}{\Delta x} + \frac{v^*_{i+\frac{1}{2}\, j+1} - v^*_{i+\frac{1}{2}\, j}}{\Delta x}.$$

Pressure boundary condition at $O(\Delta x^2)$

$$p_{-\frac{1}{2}\, j+\frac{1}{2}} = p_{\frac{1}{2}\, j+\frac{1}{2}} + \frac{1}{Re} \left(\frac{-u_{3\, j+\frac{1}{2}} + 4u_{2,j+\frac{1}{2}} - 5u_{1\, j+\frac{1}{2}} + 2u_{0\, j+\frac{1}{2}}}{\Delta x} \right),$$

on left boundary and similar others

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on left boundary and similar others

NB: On boundary need to advance normal component of $\mathbf{u}^n$ to nonzero $\mathbf{u}^*$ and the apply pressure projection to $\mathbf{u}^{n+1}$ back to zero, to avoid erroneously making $\partial p/\partial n = 0$

3.7 Results for algorithm 4

First check consistency of accuracy.

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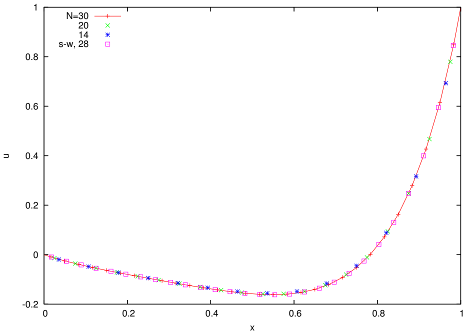
Steady horizontal velocity at $x = \frac{1}{2}$



$Re = 10$ and $N = 14$, 20 and 30.
Also result from $\psi - \omega$ formulation at $N = 28$.

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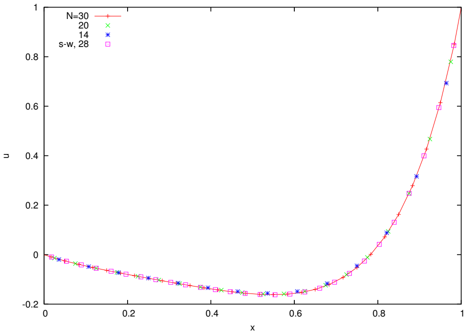
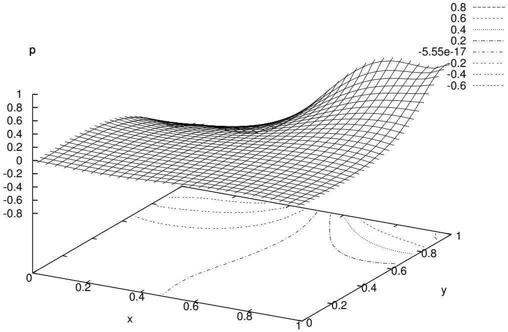


$Re = 10$ and $N = 14$, 20 and 30.
Also result from $\psi - \omega$ formulation at $N = 28$.

VERY IMPORTANT – agrees

. . . results

Pressure



$N = 30$

. . . results

Force on lid

$$F = \sum_{i=1}^{N-1} \frac{u_{i\,N+\frac{1}{2}} - u_{i\,N-\frac{1}{2}}}{\Delta x} \times \Delta x + O(\Delta x^2).$$

. . . results

Force on lid

$$F = \sum_{i=1}^{N-1} \frac{u_{i\,N+\frac{1}{2}} - u_{i\,N-\frac{1}{2}}}{\Delta x} \times \Delta x + O(\Delta x^2).$$

Similarly the viscous force on the bottom

. . . results

Force on lid

$$F = \sum_{i=1}^{N-1} \frac{u_{i\,N+\frac{1}{2}} - u_{i\,N-\frac{1}{2}}}{\Delta x} \times \Delta x + O(\Delta x^2).$$

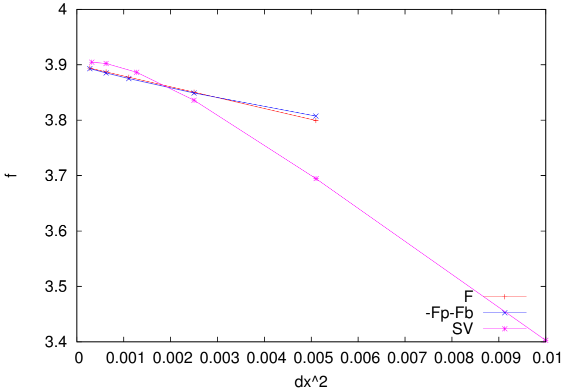
Similarly the viscous force on the bottom

Pressure force on sides from

$$\sum_{j=0}^{N-1} \frac{1}{2} \left(-p_{-\frac{1}{2}\,j+\frac{1}{2}} - p_{\frac{1}{2}\,j+\frac{1}{2}} + p_{N-\frac{1}{2}\,j+\frac{1}{2}} + p_{N+\frac{1}{2}\,j+\frac{1}{2}} \right) \times \Delta x + O(\Delta x^2).$$

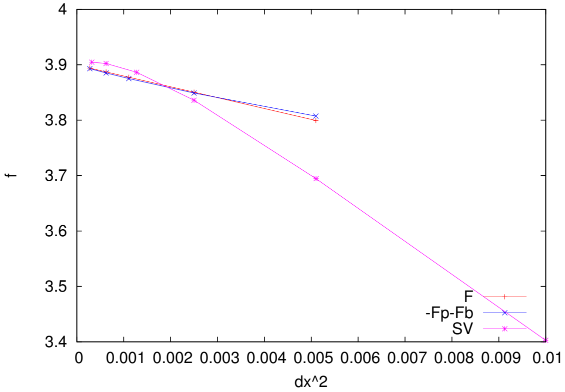
. . . results

Steady force at $Re = 10$, as function of Δx^2



. . . results

Steady force at $Re = 10$, as function of Δx^2



$$F = 3.8998 \pm 0.0002$$

and force on bottom is -0.254