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time consuming transform and nonlinear terms
Sometimes FAST transform + less modes needed → competitive

Two ideas - as in FE

Spectral representation

$$u(x,t) = \sum^{N} \hat{u}_n(t)\phi_n(x)$$

with amplitudes $u_n(t)$ and basis functions $\phi_n(x)$, e.g. Fourier

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Galerkin approximation "weighted residuals". For PDE

$$A(u) = f$$

require residue to be orthogonal to each ϕ_m:

$$\langle A(u) - f, \phi_m \rangle = 0 \quad \text{for} \quad m = 1, \ldots, N$$

Local vs Global

E.g. for Fourier

$$u(x) = \int e^{ikx} \hat{u}(k)\, dk \qquad \hat{u}(k) = \frac{1}{2\pi} \int e^{-ikx} u(x)\, dx$$

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Poisson problem

$$\frac{d^2 u}{dx^2} = \rho \quad \text{expensive global problem in real space}$$

$$-k^2 \hat{u} = \hat{\rho} \quad \text{local in Fourier space}$$

Local/Global continued

Nonlinear terms and spatially vary coefficients

$$u(x)v(x) \quad \text{local in real space}$$

$$\widehat{uv}(k) = \frac{1}{2\pi}\int_{l+m=k} \hat{u}(l)\hat{v}(m) \quad \text{global in Fourier}$$

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Navier-Stokes has both local & global in real or Fourier – need compromise

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combines Fourier and real space operations

Evaluate the nonlinear term in real space, and in Fourier space evaluate derivatives and invert the Poisson problem.

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Alternative method of satisfying PDE at collocation points rather than in Galerkin projection.

Choice of spectral basis function $\phi_n(x)$

1. complete
2. orthogonal for some weight w

$$\langle \phi_n \phi_m \rangle = \int \phi_n \phi_m w(x)\, dx = N_n \delta_{nm}$$

3. smooth
4. fast convergence
5. FAST transform
6. satisfy boundary conditions

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Strongly recommend

- Fully periodic $\rightarrow$ Fourier, $e^{in\theta}$
- Finite interval $\rightarrow$ Chebyshev $T_n(\cos\theta) = \cos n\theta$

Chebyshev polynomials

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$$T_0(x) = 1, \qquad T_1(x) = x, \qquad T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x, \qquad T_4(x) = 8x^4 - 8x^2 + 1$$

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$$(1-x^2)\,T_n'' - x\,T_n' + n^2\,T_n = 0$$
$$T_{n+1} = 2xT_n - T_{n-1}$$
$$2\,T_n = \tfrac{1}{n+1}\,T_{n+1}' - \tfrac{1}{n-1}\,T_{n-1}'$$

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– awkward $\frac{1}{2}a_0$ if use sines and cosines.

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E.g.

$$f(\theta) = \sum_{m=-\infty}^{\infty} \frac{1}{(\theta - 2\pi m)^2 + a^2} \quad \to \quad \hat{f}_n = \frac{\pi}{a} e^{-|n|a}$$

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– convergence controlled by singularity of $f(\theta)$ in complex θ-plane

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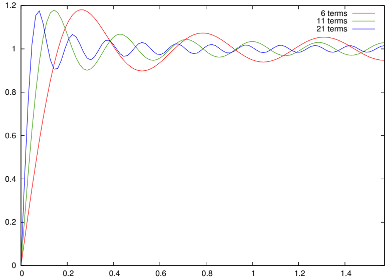

with point-wise convergence
but 14% overshoot within $\frac{1}{N}$ of discontinuity

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Hence function $|\, x \,|$ on $-1 < x < 1$
becomes fully 2π periodic in $-\pi < \theta < 0$

Discrete Fourier Transform (DFT)

Odd $N = 2M + 1$.

Equi-spaced collocation points $\theta_j = \frac{2\pi j}{N}$ for $j = 1, \ldots, N$

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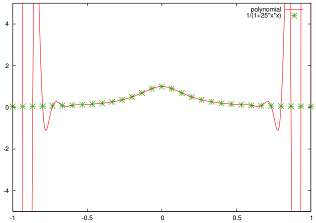
Runge phenomenon

Fitting polynomial through equi-spaced points can be badly wrong in between fitting points.



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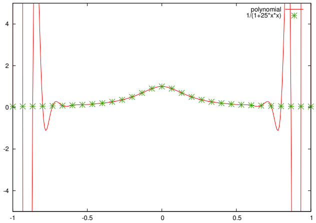

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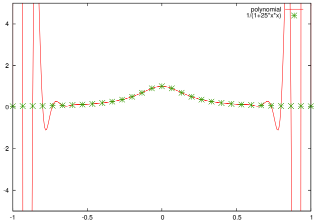

However DFT well behaved, because effectively Chebyshev polynomials fitted at points $x_j = \cos(\pi j/N)$ – crowed at ends.

Aliasing

– counter rotating wagon wheels in strobe light

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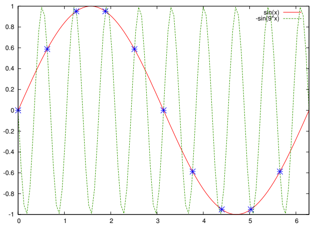
E.g. $N = 10$ equispaced points cannot distinguish between $\sin\theta$ and $-\sin 9\theta$



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of exact Fourier modes $\hat{f}_n$ in $n > M$

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In 3D throw away $\frac{19}{27}$ of the modes.

Fast Fourier Transform

DFT calculation for $n = -\frac{1}{2}N, \ldots, \frac{1}{2}N$

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looks like N coefficients $\times$ sum of N terms $= N^2$ operations.

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which is 2 lots of DFT on $\frac{1}{2}N$ points $2(\frac{1}{2}N)^2 = \frac{1}{2}N^2$ operations

Fast Fourier Transform

DFT calculation for $n = -\frac{1}{2}N, \ldots, \frac{1}{2}N$

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Program: identify even/odd at each 2^n-level $n = 1, \ldots, K$,
i.e. binary representation of j

Orzsag speed up in two dimensions

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looks line MN terms to sum at MN points (x_i, y_j)

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Also FFT speed up

Differential Matrix

To differentiate data with exponential accuracy

$$f(\theta_j) \quad \xrightarrow{\text{transform}} \quad \tilde{f}_n \quad \xrightarrow{\text{differentiate}} \quad n\tilde{f}_n \quad \xrightarrow{\text{transform}} \quad f'(\theta_j)$$

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&\text{Npts} \to \qquad\qquad\qquad\qquad \to \text{error} \quad N^{-N}
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NB $D^{(2)} \neq DD$

Navier-Stokes

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u}$$

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with $\widehat{\mathbf{u} \cdot \nabla \mathbf{u}}$ by pseudo-spectral real space evaluation

Boundary conditions

If homogeneous BCs, recombine to satisfy BCs

$$\phi_{2n} = T_{2n} - T_0 \quad \text{and} \quad \phi_{2n-1} = T_{2n-1} - T_1$$

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$1/N^2$ crowding of $x_j = \cos\theta_j$ near ± 1
→ stability if $\Delta t < D/N^4$

Bridging the gap

Local

Global

Finite Elements

FE h^p

Finite Differences

point data

Spectral

whole interval

Bridging the gap

Local

Global

Finite Elements

FE h^p

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point data

Spectral

whole interval

Splines

global points

Wavelets

local waves