Mathematical Tripos Part II(B) E.J. Hinch
Fluid Dynamics (B20) October 1998

Example Sheet 2

1. A force is applied to a cube at its centre in a direction normal to one flat surface. Using reversibility in space, show that the cube moves in the direction of the applied force, also without rotating. Now using linearity, deduce that in all orientations a cube of uniform density sediments vertically without rotating. [*Hint: resolve force into components.*]

[** What of a tetrahedron, an ellipsoid and a helix? **]

2. If the strain-rate tensor $\mathbf{e}(\mathbf{x})$ vanishes throughout a connected region, show that the flow is rigid body motion. [*Hint: first show* $\partial^2 u_1/\partial x_2 \partial x_3 \equiv 0$.]

Show that if the surface traction is specified on a bounding surface, then the Stokes flow in the interior is unique to within the addition of a rigid body motion.

3. Derive the Stokes flow outside a rotating rigid sphere

$$\mathbf{u}(\mathbf{x}) = \boldsymbol{\omega} \wedge \mathbf{x} \frac{a^3}{r^3} \qquad \text{and} \quad p = 0.$$

Show that the couple exerted on the sphere is $-8\pi\mu a^3 \boldsymbol{\omega}$.

4. If $\boldsymbol{\phi}(\mathbf{x})$ is a vector harmonic function, i.e. $\nabla^2 \boldsymbol{\phi} = 0$, show that

$$\mathbf{u} = 2\boldsymbol{\phi} - \nabla(\boldsymbol{\phi}\cdot\mathbf{x}) \qquad \text{and} \quad p = -2\mu \nabla\cdot\boldsymbol{\phi}$$

satisfy the Stokes equation.

5. Find upper and lower bounds for the couple on a tetrahedron rotating about its centre in a viscous fluid.

6. A spherical annulus of incompressible viscous liquid occupies the region $R_1(t) < r < R_2(t)$ between two free surfaces on which pressures (normal traction) $P_1(t)$ and $P_2(t)$ are applied. The resulting flow is spherically symmetric. Show (neglecting inertia and surface tension)

$$\frac{d}{dt}\left(R_1^3\right) = \frac{\pi(P_1 - P_2)}{\mu V} R_1^3 \left(R_1^3 + 3V/4\pi\right),$$

where V is the constant volume of the liquid. [*Hint:* $\sigma_{rr} = -p + 2\mu \partial u/\partial r$ *in this flow.*]

Show that if $P_1 - P_2$ is maintained positive and constant, then R_1 becomes infinite in a finite time. What happens if $P_1 - P_2$ is maintained negative and constant.

7. Fluid is contained in the region $-\alpha < \theta < \alpha$ between two rigid hinged plates. Thus the velocity components in plane polar coordinates satisfy

$$u_r = 0, \qquad u_\theta = \mp \omega r \qquad \text{on} \quad \theta = \pm\alpha.$$

Neglecting inertia forces, show that a solution to the Stokes problem may be found in the form

$$\psi = \tfrac{1}{2}\omega r^2 g(\theta)$$

and find the function $g(\theta)$. Deduce the pressure field $p(r, \theta)$. Discuss the limitations of the model.

8. Viscous fluid is contained between two planes $y = \pm a$ and a two-dimensional flow with streamfunction $\psi(x, y)$ is generated by some agency (e.g. a rotating cylinder) near $x = y = 0$. It is required to find the form of the flow field for large positive x. Find the general solution of $\nabla^4\psi = 0$ of the form

$$\psi = f(y)e^{-kx} \qquad \mathrm{Re}\, k > 0,$$

for which $f(y)$ is an even function of y, and hence show that k is determined by the equation

$$2ka + \sin 2ka = 0.$$

Show that this equation as no real roots. The equation has complex roots, that with the smallest real part being $2ka = 4.2 \pm 2.3i$. Sketch the streamlines of the flow.

9. A rigid sphere of radius a falls through a fluid of viscosity μ under gravity towards a horizontal rigid plane. Use lubrication theory to show that, when the minimum gap h_0 is very small, the speed of approach of the sphere is

$$h_0 W / 6\pi\mu a^2,$$

where W is the weight of the sphere corrected for buoyancy.

10. Oil is forced by a pressure difference Δp through the narrow gap between two parallel circular cylinders of radius a with axes $2a + b$ apart. Show that, provided $b \ll a$ and $\rho b^3 \Delta p \ll \mu^2 a$, the volume flux is approximately

$$\frac{2b^{5/2}\Delta p}{9\pi a^{1/2}\mu}$$

when the cylinders are fixed.

Show also that when the two cylinders rotate with angular velocities Ω_1 and Ω_2 in opposite directions, the change in the volume flux is

$$\tfrac{2}{3}ab(\Omega_1 + \Omega_2).$$

11. A viscous fluid coats the outer surface of a cylinder of radius a which rotates with angular velocity Ω about its axis which is horizontal. The angle θ is measured from the horizontal on the rising side. Show that the volume flux per unit length $Q(\theta, t)$ is related to the thickness $h(\theta, t)$ of the fluid layer by

$$Q = \Omega a h - \frac{g}{3\nu} h^3 \cos\theta,$$

and deduce an evolution equation for $h(\theta, t)$.

Consider now the possibility of a steady state with $Q = \text{const}$, $h = h(\theta)$. Show that a steady solution with $h(\theta)$ continuous and 2π-periodic exists only if

$$\Omega a > (9Q^2 g / 4\nu)^{1/3}.$$

12. A drop spreads on a horizontal table in a region confined between two parallel vertical walls. Assuming that the drop has become a thin layer and that there is no variation in the direction perpendicular to the walls, find how the drops spreads. [It is not possible to integrate the volume in closed form.]

Please notify all errors to E.J.Hinch@damtp.cam.ac.uk.