VORTEX DYNAMICS: THE LEGACY OF HELMHOLTZ AND KELVIN

Keith Moffatt

Trinity College, Cambridge, CB2 1TQ, UK
hkm2@damtp.cam.ac.u

Abstract. The year 2007 will mark the centenary of the death of William Thomson (Lord Kelvin), one of the great nineteenth-century pioneers of vortex dynamics. Kelvin was inspired by Hermann von Helmholtz's [7] famous paper "Ueber Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen", translated by P. G. Tait and published in English [17] under the title "On Integrals of the Hydrodynamical Equations, which Express Vortex-motion". Kelvin conceived his "Vortex theory of Atoms" (1867–1875) on the basis that, since vortex lines are frozen in the flow of an ideal fluid, their topology should be invariant. We now know that this invariance is encapsulated in the conservation of helicity in suitably defined Lagrangian fluid subdomains. Kelvin's efforts were thwarted by the realisation that all but the very simplest three-dimensional vortex structures are dynamically unstable, and his vortex theory of atoms perished in consequence before the dawn of the twentieth century. The course of scientific history might have been very different if Kelvin had formulated his theory in terms of magnetic flux tubes in a perfectly conducting fluid, instead of vortex tubes in an ideal fluid; for in this case, stable knotted structures, of just the kind that Kelvin envisaged, do exist, and their spectrum of characteristic frequencies can be readily defined. This introductory lecture will review some aspects of these seminal contributions of Helmholtz and Kelvin, in the light of current knowledge.

Keywords: Knotted vortex tubes, vortex filaments, magnetohydrodynamics, magnetic flux tubes

1. The fluid dynamical origins of knot theory and topology

The origins of vortex dynamics lie in the seminal work of Hermann von Helmholtz [7], who (i) introduced the concepts of vortex line and vortex filament (the fluid bounded by the vortex lines passing through the points of an "infinitely small closed curve"), (ii) derived the vorticity equation for an ideal incompressible fluid, and (iii) demonstrated that vortex lines are

A.V. Borisov et al. (eds.), IUTAM Symposium on Hamiltonian Dynamics, Vortex Structures, Turbulence, 1–10.

transported with the fluid with intensification proportional to the stretching of its constituent line-elements. This work provided the basis for the bold, though ultimately erroneous, "vortex atom" conjecture of William Thomson (Lord Kelvin) [21, 22], Professor of Natural Philosophy at the University of Glasgow, who sought to explain the structure and spectra of atoms of all the known elements in terms of knotted and linked vortex filaments in a hypothetical background ideal fluid "ether" permeating the universe. It was this conjecture that led Peter Guthrie Tait, Kelvin's opposite number at the nearby University of Edinburgh, to develop techniques for the classification of knots of low crossing number (the minimum number of double points in any plane projection of a knot) [18–20] and thus to sow the seeds for the development of topology as a recognisable branch of modern mathematics. These developments of the period 1858–1885 have been discussed in depth by Epple [6], who conveys well the excitement and drama of this remarkable phase of Victorian science.

2. Tait's role in attracting Kelvin's interest

Helmholtz's work became more widely known when it was republished in English translation by Tait [17], who indicates in a concluding paragraph that his version "does not pretend to be an exact translation" but, following revisions that had been made by Helmholtz, "may be accepted as representing the spirit of the original". Tait had made this translation as soon as he received the German version in 1858, and, stimulated by Helmholtz's concluding remarks concerning the behaviour of vortex rings of small cross section, developed a technique for the experimental demonstration of vortex ring propagation, and of the "leap-frogging" of vortex rings propagating in succession along a common axis of symmetry. Although Kelvin had known of Helmholtz's work in 1858, it was only when Tait, in his Edinburgh laboratory in 1867, showed him his vortex ring demonstration that he was in turn stimulated to undertake his own extensive studies in vortex dynamics.

The second paragraph of Helmholtz's paper (in Tait's translation) deserves comment. He writes:

Yet Euler [Histoire de l'Académie des Sciences de Berlin 1755, p. 292] has distinctly pointed out that there are cases of fluid motion in which no velocity-potential exists, — for instance, the rotation of a fluid about an axis when every element has the same angular velocity. Among the forces which can produce such motions may be named magnetic attractions upon a fluid conducting electric currents, and particularly friction, whether among the elements of the fluid or against fixed bodies. The effect of fluid friction has not hitherto been mathematically defined; yet it is very great, and, except in the case of indefinitely small oscillations, produces most marked differences between theory and fact. The difficulty of defining this effect, and of finding expressions for *its measurement, mainly consisted in the fact that no idea had been formed of the species of motion which friction produces in fluids. Hence it appeared to me to be of importance to investigate the species of motion for which there is no velocity-potential.*

The mention of what amounts to the rotationality of the Lorentz force (*magnetic attractions upon a fluid conducting electric currents*) here shows remarkable foresight, as does recognition of the crucial role of internal friction (i.e. viscosity). It is evident however that Helmholtz was unaware of the epic work of Stokes [15,16] in which the effects of viscosity in a fluid continuum had been analysed in considerable detail. Tait adds a footnote to his translation in which he gently draws attention to this omission:

A portion of the contents of the paper had been anticipated by Professor Stokes in various excellent papers in the Cambridge Philosophical Transactions; but the discovery of the nature and motions of vortex-filaments is entirely novel, and of great consequence.

3. The analogy between vorticity and current as source fields

I was myself a student at the University of Edinburgh from 1953 to 1957 in the (then) Tait Institute for Mathematical Physics, and I recall seeing demonstrations with the "vortex ring generator" (sometimes known as a "Kelvin box" though perhaps more appropriately described as a "Tait box") in connexion with the third-year course on theoretical hydrodynamics given by Robin Schlapp that I attended exactly 50 years ago. The traditional style of presentation of this material, with Lamb's *Hydrodynamics* as the one and only recommended treatise, had been well maintained and cultivated since the time of Kelvin and Tait. We were taught a parallel course on Electromagnetism by Nicholas Kemmer (successor in 1953 to Max Born in the Edinburgh Chair of Natural Philosophy), in which context the name of James Clerk Maxwell, born and schooled in Edinburgh, and later first Cavendish Professor of Experimental Physics at the University of Cambridge (1871–1879), was equally venerated. The fact that the relationship between vortex filaments in fluid mechanics and the velocity field to which they gave rise (via the Biot–Savart Law) is the same as that between currents in conducting wires (i.e. "current filaments") and the magnetic field to which *they* give rise had been noted by Helmholtz and was equally familiar to Kelvin, who was in regular correspondence with Maxwell on this and related topics. We now know, as I shall discuss below, that such interdisciplinary analogies admit powerful exploitation in a manner that was not recognised until the development of magnetohydrodynamics nearly a century later. I propose to argue that, had Kelvin conceived of the ether as a perfectly conducting fluid medium supporting a tangle of magnetic flux tubes rather than as an ideal (inviscid) medium supporting a tangle of vortex filaments, then his theory would have been much more robust, and the development of natural philosophy (i.e. physics) in the early twentieth century might have followed a very different course.

4. The (imperfect) analogy between vorticity and magnetic field

The curious thing is that the basic principles underlying magnetohydrodynamics (MHD) were already known by the mid-nineteenth century, well before Maxwell introduced the "displacement current" that was needed to guarantee charge conservation; this is neglected in MHD, current $\mathbf{j}$ being assumed instantaneously related to magnetic field $\mathbf{B}$ by Ampère's Law: $\mathbf{j} = \operatorname{curl}\mathbf{B}$ (in "Alfvén units" for which $\mathbf{B}$ has the dimensions of a velocity). When combined with Faraday's Law of Induction, and Ohm's Law in a medium of resistivity η moving with velocity $\mathbf{v}$, this yields the well-known "induction equation" for the evolution of magnetic field:

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}. \tag{1}$$

This bears an obvious superficial similarity to the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \operatorname{curl}(\mathbf{u} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega} \tag{2}$$

in a non-conducting medium of kinematic viscosity ν, superficial because whereas $\boldsymbol{\omega}$ is related to $\mathbf{u}$ in (2) by $\boldsymbol{\omega} = \operatorname{curl}\mathbf{u}$, $\mathbf{B}$ bears no such relation to the transporting velocity field $\mathbf{v}$ in (1). This imperfection in the analogy between $\mathbf{B}$ and $\boldsymbol{\omega}$ does not however vitiate an important conclusion: just as (2) implies that the $\boldsymbol{\omega}$-lines (i.e. vortex lines) are transported with the fluid when $\nu = 0$, so (1) implies that the $\mathbf{B}$-lines (i.e. Faraday's magnetic lines of force) are so transported when $\eta = 0$. Thus, conservation of topology of the $\mathbf{B}$-field in a perfectly conducting fluid could have provided an equally good starting point for Kelvin (rather than conservation of topology of the $\boldsymbol{\omega}$-field in an inviscid fluid) in formulating a theory of the structure and spectra of atoms, and indeed a more plausible one since, as was recognised early in the twentieth century, atoms do involve microscopic current circuits (conventionally pictured as electrons orbiting in their various shells around a nucleus) and their associated magnetic fields.

5. The long-delayed development of magnetohydrodynamics

Thus all the principles were available in the 1860s for such a complementary approach, but Kelvin's preoccupation was with vortices, while on the electromagnetic front, Maxwell's preoccupation was with providing a unified theory of electricity and magnetism. MHD was a subject waiting to be discovered, but it was not until the work of Alfven [1] that the subject was in the event developed to the point at which the crucial “frozen-in” property of the magnetic field in a perfectly conducting fluid was finally recognised. Soon after this, the analogy between vorticity and magnetic field referred to above was recognised and exploited by Batchelor [4] in a first investigation of the effect of turbulence on a random magnetic field. The explosive development of MHD in the 1950s and 1960s was greatly stimulated by technological problems associated with controlled thermonuclear fusion, as well as with an expanding recognition of its vital role in understanding fundamental processes in astrophysics and geophysics.

6. Helicity: the bridge between fluid mechanics and topology

Kelvin’s vision of the role of knotted or linked vortex tubes in a hypothetical ether was largely qualitative in character. He correctly perceived that knots and linkages would be conserved by virtue of the frozen-in property of vortex lines, but he had no quantitative measure of such knottedness or linkage. The simplest such quantitative measure for any localised vorticity distribution is in fact provided by its helicity, the integrated scalar product of the vorticity field $\boldsymbol{\omega}$ and the velocity $\mathbf{u}$ to which it gives rise:

$$H = \int \mathbf{u} \cdot \boldsymbol{\omega} dV. \tag{3}$$

This quantity is an invariant of the Euler equations, either for an incompressible fluid or for a compressible fluid under the barotropic condition that pressure p is a function of density ρ alone: $p = p(\rho)$ [8, 13]. For the prototype linkage of two vortex tubes of circulation κ_1 and κ_2 (each having no internal twist), centred on unknotted but possibly linked closed curves C_1 and C_2, the helicity may be easily evaluated in the form

$$H = \pm 2n\kappa_1\kappa_2, \tag{4}$$

where the plus or minus sign is chosen according as whether the linkage is right- or left-handed, and n is an integer, actually the Gauss linking number of C_1 and C_2. It is here that the link between topology and fluid dynamics is at its most transparent.

7. Knotted vortex tubes

For a single vortex tube T of circulation κ whose axis C is in the form of a knot of type K, the situation is more subtle. The helicity in this case is given by

$$H = \kappa^2(Wr + Tw), \tag{5}$$

where Wr and Tw are respectively the writhe of C and twist of T [12]. The writhe is given by a double integral round C analogous to the Gauss integral, and admits interpretation as the sum of the (signed) crossings of the knot averaged over all projections. The twist can be decomposed in the form

$$Tw = \frac{1}{2\pi}\left(\int \tau(s)ds + N\right), \tag{6}$$

where $\tau(s)$ is the torsion of C as a function of arc-length s, and N represents the intrinsic twist of vortex lines around the axis C as they traverse the circuit round the tube (an integer if these vortex lines are closed curves). If the vortex tube is deformed through any configuration that instantaneously contains an inflexion point, then N jumps by an integer at this instant, but the jump is compensated by an equal and opposite jump in the total torsion, so that Tw varies in a continuous manner [12]. As shown by Calugareanu [5] in a purely geometric context, and as generalised to higher dimension by White [23], the sum [5] is indeed constant under arbitrary deformation of the tube.

8. Magnetic helicity and the lower bound on magnetic energy

In consequence of the analogy (albeit imperfect) between vorticity and magnetic field, there is an analogous topological invariant of a magnetic field $\mathbf{B}$ in a perfectly conducting fluid, namely the magnetic helicity

$$H_M = \int \mathbf{A}\cdot\mathbf{B}\,dV \tag{7}$$

where $\mathbf{A}$ is a vector potential for $\mathbf{B}$: $\mathbf{A} = \mathrm{curl}\,\mathbf{B}$ and note that the integral (7) is gauge-invariant provided the normal component of $\mathbf{B}$ vanishes on the boundary of the fluid domain). This invariant was discovered by Woltjer [24], but its topological interpretation was not recognised until some years later [8]. This invariant provides an important lower bound on the magnetic energy

$$M = \int \mathbf{B}^2/2\,dV, \tag{8}$$

namely [3]

$$M \geqslant q|H_M|, \tag{9}$$

where q is a constant (with the dimensions of $(\text{length})^{-1}$), which depends only on the domain topology, geometry and scale. There is no corresponding lower bound for the kinetic energy associated with a vorticity field in an ideal fluid, and it is here that there is great advantage in switching attention to the magnetic problem.

9. Magnetic relaxation

Let us then conceive of a perfectly conducting incompressible fluid contained in a fixed domain $\varDelta$ with surface S, containing a magnetic field $\mathbf{B}_0(\mathbf{x})$ of nonzero magnetic helicity, the fluid being at rest at time $t = 0$. In general, the associated Lorentz force $\mathbf{j} \times \mathbf{B}$ is rotational, and the fluid will move under the action of this force; as it moves, it transports the magnetic field, whose topology is conserved. If we suppose that the fluid has nonzero viscosity, then, for so long as the fluid is in motion, energy (magnetic M plus kinetic K) is dissipated through the agency of viscosity, and is therefore monotonic decreasing; it is however constrained by the inequality (9), which implies that ultimately $M + K$ tends to a constant, and so the rate of dissipation of energy tends to zero. It is at least reasonable then to conjecture that the velocity field must tend to zero identically in $\varDelta$, and that we must arrive at an equilibrium state that is stable within the framework of perfect conductivity because magnetic energy is then minimal under frozen-field perturbations; this magnetostatic equilibrium is described by the force balance

$$\mathbf{j} \times \mathbf{B} = \nabla p, \tag{10}$$

where p is the fluid pressure. The asymptotic field $\mathbf{B}$ results from deformation of $\mathbf{B}_0(\mathbf{x})$ by a velocity field $\mathbf{v}(\mathbf{x}, t)$ which dissipates a finite amount of energy over the whole time interval $0 < t < \infty$ in this sense, it may be said to be "topologically accessible" from $\mathbf{B}_0$. This process has been described in detail by Moffatt [9]. One important feature is that, in general, tangential discontinuities of $\mathbf{B}$ (i.e. current sheets) may develop during the relaxation process. The prototype configuration for which this happens is that consisting of two unknotted, untwisted, linked magnetic flux tubes which, under relaxation, contract in length and expand in cross section (volume being conserved) until they make contact on an open surface which is then necessarily such a surface of tangential discontinuity. Actually, in this situation, one tube then spreads round the other, the ultimate magnetostatic equilibrium being axisymmetric and the current sheet (asymptotically) a torus.

10. Relaxation of knotted flux tubes

A flux tube of volume V, carrying magnetic flux $\varPhi$ (the analogue of κ) and knotted in the form of a knot of type K, has magnetic helicity the analogue of (5), i.e.

$$H_M = h\varPhi^2, \tag{11}$$

where $h = Wr + Tw$ is the conserved writhe-plus-twist of the tube. This tube will relax under the procedure outlined above to a minimum energy state of magnetostatic equilibrium, in which the minimum energy $M_{\min}$ is determined by the three characteristic properties of the initial field that are conserved during relaxation, namely Φ, V, and h; on dimensional grounds, this relationship must take the form

$$M_{\min} = m_K(h)\Phi^2 V^{-1/3}, \tag{12}$$

where $m_K(h)$ is a dimensionless function of the dimensionless helicity parameter h, whose form is determined solely by the knot type K [11]. Moreover, this state, being stable, will be characterised by a spectrum of real frequencies ω_n, which, again on dimensional grounds, are given by

$$\omega_n = \Omega_{Kn}(h)\Phi V^{-1}, \tag{13}$$

where the $\Omega_{Kn}(h)$ $(n = 1, 2, 3, \ldots)$ are again dimensionless functions of h, determined solely by the knot type K. I suspect that it was just such relations as (12) and (13) that Kelvin was seeking in relation to knotted vortex tubes. He was unsuccessful because there is no known relaxation procedure in three dimensions analogous to that described above that conserves *vorticity* topology and minimises *kinetic* energy.

11. The analogous Euler flows

There is nevertheless a second analogy (and this time it is perfect!) which is an extension of the analogy already recognised by Helmholtz and Kelvin, and touched on in §3 above. This is the analogy between $\mathbf{B}$ and $\mathbf{u}$ (and consequently between $\mathbf{j} = \operatorname{curl}\mathbf{B}$, and $\boldsymbol{\omega} = \operatorname{curl}\mathbf{u}$). The analogue of (10) is then

$$\mathbf{u} \times \omega = \nabla H, \tag{14}$$

where $H = p_0 - p$, for some constant p_0. Equation (14) may be immediately recognised as the steady form of the Euler equation with H the total head. Thus, to each magnetostatic equilibrium satisfying (10), there corresponds a steady Euler flow, obtained by simply replacing $\mathbf{B}$ by $\mathbf{u}$, $\mathbf{j}$ by $\boldsymbol{\omega}$, and p by $p_0 - H$. Note here that, through this analogy, a magnetic flux tube corresponds not to a vortex tube in the Euler flow, but to a streamtube! So a knotted flux tube corresponds to a knotted streamtube, a somewhat curious concept within the context of the Euler equations. However, although the analogy is perfect as far as the steady state is concerned, it does not extend to the stability of the steady state: stability of the minimum energy knotted flux configurations does not imply stability of the analogous Euler flows. The reason is that under perturbation of the magnetostatic equilibrium, the $\mathbf{B}$-field must be frozen in the fluid, whereas under perturbation of the Euler flow satisfying the time-dependent Euler equation, it is not the "analogous" $\mathbf{u}$-field, but rather the $\boldsymbol{\omega}$-field, that is frozen in the fluid. This subtle distinction completely changes the stability criterion for steady states [10]. One should in fact expect all the analogous Euler flows to be in general unstable if only because they will generally contain vortex sheets (the analogue of the current sheets referred to above) and these will be generically subject to Kelvin–Helmholtz instability. It has in fact been shown by Rouchon [14] that steady Euler flows that are nontrivially three-dimensional fail to satisfy Arnold's [2] sufficient condition for stability: the constant-energy trajectories on the "isovortical" folium through a fixed point in the space of divergence-free velocity fields of finite energy are in general hyperbolic in character, so that the perturbed flow is not constrained by conservation of energy to remain near the fixed point. This does not imply instability, but it makes it very likely!

12. Conclusions

Kelvin was frustrated in his vortex ambitions on two accounts: first in failing to find steady non-axisymmetric solutions of the Euler equations having knotted vortex lines; and second in being unable to demonstrate the stability of even the simplest vortex ring configurations. His investigations of the 1870s and 1880s laid the basis for many subsequent investigations of problems of vortex structure and stability that remain very much alive today; but his initial concept of the "vortex atom" failed to gain ground because of these two fundamental barriers to progress. If instead one adopts the complementary scenario of magnetic flux tubes in a perfectly conducting fluid, then the natural technique of magnetic relaxation, as described above, leads in principle to stable equilibria of magnetic flux tubes knotted in an arbitrary manner. The actual realisation of the relaxation process, and the determination of the frequency spectra of these stable equilibria, present computational challenges that should be within the power of current super-computers. I hope that someone may soon be able to rise to these challenges, and thus revive the vision and spirit of the great nineteenth-century pioneers of the subject of this Symposium.

References

1. Alfven H. (1942) On the existence of electromagnetic-hydromagnetic waves, *Arkiv. F. Mat. Fysik*, **29B**(2), (7 pp.).
2. Arnol'd V. (1966) Sur la géometrie différentielle des groupes de Lie de dimension infinite et ses applications à l'hydrodynamique des fluides parfaits, *Ann. Inst. Fourier Grenoble*, **16**, 319–361.
3. Arnol'd V. (1974) The asymptotic Hopf invariant and its applications *(in Russian). Proc. Summer School in Diff. Eqns.*, Erevan, Armenian SSR Acad. Sci, 1974. (*English translation:* 1986, *Sel. Math. Sov.* , **5**, 327–345).
4. Batchelor G. K. (1950) On the spontaneous magnetic field in a conducting liquid in turbulent motion, *Proc. Roy. Soc. A*, **201**, 405–416.
5. Călugăreanu G. (1961) Sur les classes d'isotopie des nœuds tridimensionels et leurs invariants, *Czechoslovak Math. J.*, **11**, 588.

6. Epple M. (1998) Topology, matter, and space, I: Topological notions in 19th-century natural philosophy, *Arch. Hist. Exact Sci.*, **52**, 297–392.
7. Helmholtz H. von (1858) Über Integrale der hydrodynamischen Gleichungen, welche der Wirbelbewegung entsprechen, *J. für die reine und angewandte Mathematik*, **55**, 25–55.
8. Moffatt H. K. (1969) The degree of knottedness of tangled vortex line, *J. Fluid Mech.*, **35**, 117–129.
9. Moffatt H. K. (1985) Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. Part 1, Fundamentals, *J. Fluid Mech.*, **159**, 359–378.
10. Moffatt H. K. (1986) Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. Part 2, Stability considerations, *J. Fluid Mech.*, **166**, 359–378.
11. Moffatt H. K. (1990) The energy spectrum of knots and links, *Nature*, **347**, p. 367–369 [see also *News and Views*, p. 332].
12. Moffatt H. K. and Ricca R. (1992) Helicity and the Călugăreanu Invariant, *Proc. R. Soc. Lond. A*, **439**, 411–429.
13. Moreau J.-J. (1961) Constants d'un ilôt tourbillonaire en fluide parfait barotrope, *Comptes Rendus, Acad. des Sciences*, **252**, 28–103.
14. Rouchon P. (1991) On the Arnol'd stability criterion for steady-state flows of an ideal fluid, *Eur. J. Mech. B/Fluids*, **10**, 651–661.
15. Stokes G. G. (1845) On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids, *Proc Camb. Phil. Soc.*
16. Stokes G. G. (1850) On the effect of the internal friction of fluids on the motion of pendulums, *Proc Camb. Phil. Soc.*
17. Tait P. G. (1867) Translation of (Helmholtz 1858): On the integrals of the hydrodynamical equations, which express vortex-motion, *Phil.Mag.*, **33**, 485–512.
18. Tait P. G. (1877) On knots, *Trans. Roy. Soc. Edin.*, **28**, 273–317.
19. Tait P. G. (1884) On knots: Part II, *Trans. Roy. Soc. Edin.*, **32**, 327–339.
20. Tait P. G. (1885) On knots: Part III, *Trans. Roy. Soc. Edin.*, **32**, 493–506.
21. Thomson W. (Lord Kelvin) (1867) On vortex atoms, *Proc. Roy. Soc. Edin.*, **6**, 94–105.
22. Thomson W. (Lord Kelvin) On vortex motion, *Trans. Roy. Soc. Edin.*, 1869, **25**, 217–260.
23. White J. H. (1969) Self-linking and the Gauss integral in higher dimensions, *Am. J. Math.*, **91**, 693–728.
24. Woltjer L. (1958) A theorem on force-free magnetic fields, *Proc. Natl. Acad. Sci.*, **44**, 489–491.