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GIFFEN GOODS AND THEIR REFLEXION PROPERTY*

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The following 'reflexion property' of Giffen behaviour is proved: the two-good direct utility function (DUF) obtained by reversing the sign of two-good indirect utility function (IUF) displays Giffen behaviour with respect to one of the two goods if and only if the IUF itself displays Giffen behaviour with respect to the *other* good. A particular IUF showing Giffen behaviour for both goods (in non-overlapping regions of the price space) is constructed and the reflexion property is verified. The example IUF is extended to more than two goods, and Giffenity is verified in this case.

1 Introduction

Ever since the pioneering work of Wold and Jureen (1953), an interesting challenge has been lurking in the outer reaches of consumer theory. This challenge is to specify a two-good utility function that, over part of its domain, predicts Giffen behaviour (i.e. an increase in the quantity demanded of a good in response to an increase in its own price), while meeting some reasonable conditions. These conditions are (i) that the function be defined in closed form and is continuous and twice differentiable over the entire positive quadrant, (ii) that it be monotonically increasing in both arguments and globally quasi-concave, and (iii) that it have Marshallian demand functions that are also expressible in closed form as explicit functions of the price variables.

There has been a recent revival of interest in the Giffen phenomenon, as evidenced by the book *New Insights into the Theory of Giffen Goods* edited by Heijman and von Mouche (2011). A number of authors have sought to meet the above challenge at varying levels of completeness, e.g. Moffatt (2002), Sørenson (2007) and Doi *et al.* (2009).

More recently, Moffatt (2011) has set out to meet the challenge from the standpoint of the indirect utility function (IUF), an advantage of this approach being that a closed-form representation of the Marshallian demand functions is guaranteed through the application of Roy's identity. We develop this approach further in Section 3 of this paper, following some

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preliminaries in Section 2. Necessary and sufficient conditions for Giffen behaviour in the context of a two-good IUF are derived in Section 3.1. The focal point of the paper is a theorem, proved in Section 3.2, that links an IUF to its 'mirror' direct utility function (DUF), the latter being defined in terms of the former by a simple change of sign. This theorem gives rise to a 'reflexion property' of Giffen goods: the mirror DUF obtained by reversing the sign of an IUF displays Giffen behaviour with respect to one of the two goods if and only if the IUF itself displays Giffen behaviour with respect to the *other* good. This theorem is potentially very useful in the search for Giffen behaviour, since some of the problems that are known to arise when starting with a DUF can be overcome by finding an IUF which gives rise to Giffen behaviour and then simply reversing the sign. It also provides one means for the verification of Giffenity: if a DUF is proposed which claims Giffenity, the claim can be easily checked by reversing the sign, applying Roy's identity, and then examining the properties of the Marshallian demands.

In Section 3.3, this reflexion property of Giffen goods is illustrated through the example of a specific two-good utility function proposed by Moffatt (2011) that satisfies all the reasonable conditions specified above. Lest it be supposed that all demand features have this reflexion property, it is further shown from this example that the demand feature of inferiority does *not* in general have this property. The reflexion property of Giffen goods is therefore non-trivial.

It is furthermore possible to extend this two-good IUF to the case of more than two goods. This is demonstrated in Section 4, and it is verified that for this special choice of IUF, all goods have the Giffen property in non-overlapping regions of the price space.

2 Preliminary Considerations

Let $U(x, y)$ be the (direct) utility function for a two-good situation with quantities (x, y). We assume that $U(x, y)$ satisfies the basic axioms of consumer theory, namely the monotonicity conditions

$$U_x \equiv \frac{\partial U}{\partial x} > 0 \qquad U_y \equiv \frac{\partial U}{\partial y} > 0 \tag{1}$$

and the condition that the contours $U(x, y) = \text{const.}$ be convex to the origin (i.e. that U be quasi-concave), a condition that may be expressed in the 'bordered Hessian' form (Takayama, 1985, chap. 1):

$$C(x, y) = \frac{1}{U_x^3}\begin{vmatrix} 0 & U_x & U_y \\ U_x & U_{xx} & U_{xy} \\ U_y & U_{xy} & U_{yy} \end{vmatrix} = \frac{AU_x + BU_y}{U_x^3} > 0 \tag{2}$$

where

$$A(x, y) = U_y U_{xy} - U_x U_{yy} \qquad B(x, y) = U_x U_{xy} - U_y U_{xx} \tag{3}$$

and suffixes denote partial differentiation. Note that the 'convexity function' $C(x, y)$, as defined here, is invariant under replacement of $U(x, y)$ by any function of the form $\hat{U}(x, y) = F(U(x, y))$: it is a property of the geometry of the contour map, but not of the values of U on the various contours. The curvature $\kappa(x, y)$ of the contour $U = \text{const.}$ passing through the point (x, y) is related to $C(x, y)$ by $\kappa(x, y) = (1 + U_y^2/U_x^2)^{-3/2} C(x, y)$. The radius of curvature is $R = \kappa^{-1}$.

3 Giffen Goods and the Reflexion Theorem

3.1 Giffenity in Terms of the IUF

Following Moffatt (2011), let us now consider the phenomenon of Giffenity from the viewpoint of an IUF $V(p, q)$, a function of the prices (p, q) conjugate to (x, y), with normalized budget constraint

$$px + qy = 1 \tag{4}$$

The function $V(p, q)$ must satisfy the negative gradient conditions $V_p < 0$, $V_q < 0$, and, by analogy with (2), the convexity condition

$$C(p, q) = \frac{A(p, q)V_p + B(p, q)V_q}{V_p^3} > 0 \tag{5}$$

an inequality that must be satisfied at all points of the positive (p, q) quadrant (see, for example, Suen, 1992).

The advantage of this approach is that the Marshallian demands are given explicitly by Roy's identity

$$x(p, q) = \frac{V_p}{pV_p + qV_q} \qquad y(p, q) = \frac{V_q}{pV_p + qV_q} \tag{6}$$

If $x(p, q)$ is an increasing function of its own price p (at constant q) in some region $\mathcal{G}_1$ of the (p, q) plane, then good 1 is a Giffen good in $\mathcal{G}_1$. The condition for Giffenity of good 1 is simply that $\partial x/\partial p$ be positive, or equivalently that $\partial(x^{-1})/\partial p = -x^{-2}\partial x/\partial p$ be negative, i.e. that

$$G(p, q) \equiv \frac{\partial}{\partial p}\frac{1}{x} = \frac{V_p^2 + q(V_p V_{pq} - V_q V_{pp})}{V_p^2} = 1 + \frac{qB(p, q)}{V_p^2} < 0 \tag{7}$$

This Giffenity condition admits simple geometric interpretation. With reference to Fig. 1, let A be a point with coordinates (P, Q) on the contour

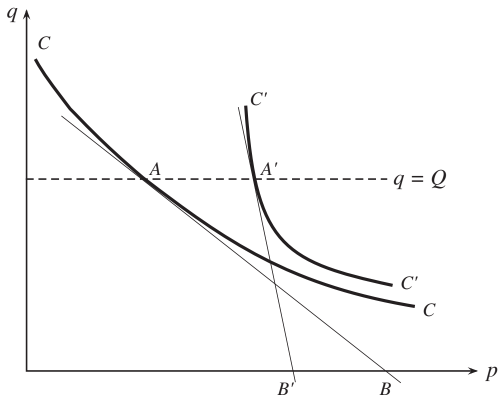

FIG. 1. The Curves C and C' are Contours $V = c$ and $V = c'$ of an IUF $V(p, q)$. The Line $q = Q$ Intersects C at A, C' at A'. AB is Tangent to C and $A'B'$ is Tangent to C'. As A Moves to the Right across Intermediate Contours towards A', B Moves to the Left towards B', a Symptom of Giffenity of Good 1

$C: V(p, q) = c$. On this contour, $dV \equiv V_p dp + V_q dq = 0$, so the gradient of the tangent to the contour at A is $m(P, Q) = [dq/dp]_{V=\text{cst.}} = -V_p/V_q$. The equation of this tangent is therefore

$$q - Q = m(P, Q)(p - P) \tag{8}$$

and this intersects the p-axis at the point B with coordinates $(g, 0)$, where

$$g(P, Q) = P - \frac{Q}{m(P, Q)} = \frac{PV_P + QV_Q}{V_P} = \frac{1}{x(P, Q)} \tag{9}$$

If x increases as a function of P, then g decreases, i.e. as A moves to the right across adjacent contours, B moves to the left on the p-axis, as illustrated in Fig. 1. This condition is evidently satisfied if adjacent contours converge towards each other sufficiently rapidly in the downward direction.

3.2 *Giffenity in Terms of the DUF*

We may now seek a similar criterion for Giffenity in terms of a DUF $U(x, y)$. Consistent with (7) above, we first *define* $G(X, Y)$ by the equation

$$G(X, Y) = \frac{U_X^2 + Y(U_X U_{XY} - U_Y U_{XX})}{U_X^2} = 1 + \frac{YB(X, Y)}{U_X^2} \tag{10}$$

Now, much as before, consider the tangent at the point A (coordinates (X, Y)) of a contour $U(x, y) = c$ (see Fig. 2); this meets the x-axis at the point B (coordinates $(g, 0)$), where now (cf. equation (9))

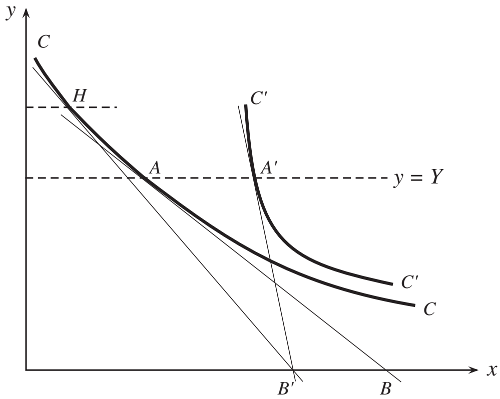

FIG. 2. The Same as Fig. 1, but now in the (x, y)-plane. $B'H$ is now the Tangent from B' to C, and it is Evident that, as a Result of the Convexity of C, H is above A on C, i.e. $y_H > y_A\ (= y_{A'})$

$$g(X, Y) = X + Y\frac{U_Y}{U_X} \tag{11}$$

If this tangent is the budget line

$$px + qy = 1\,(= pX + qY) \tag{12}$$

with gradient $m_b = -p/q$, then this gradient is equal to $-U_X/U_Y$ at the point (X, Y), i.e.

$$pU_Y - qU_X = 0 \tag{13}$$

Consider now what happens if we increase the price q of good 2, keeping the price p of good 1 (and the budget) fixed; this merely changes the gradient m_b while keeping $g = 1/p$ fixed. Thus

$$dg \equiv \frac{\partial g}{\partial X}dX + \frac{\partial g}{\partial Y}dY = 0 \tag{14}$$

Also, from (4), the point of contact (X, Y) of the budget line varies according to

$$pdX + qdY + Ydq = 0 \tag{15}$$

Hence, eliminating dX, (14) and (15) give

$$\frac{dY}{dq} = \frac{Y\partial g/\partial X}{p\partial g/\partial Y - q\partial g/\partial X} \tag{16}$$

Now, from (11), we have

$$\frac{\partial g}{\partial X} = 1 + Y\frac{U_X U_{XY} - U_Y U_{XX}}{U_X^2} = G(X, Y) \tag{17}$$

and, using (2),

$$\frac{\partial g}{\partial Y} = \frac{U_Y}{U_X} + Y\frac{U_X U_{YY} - U_Y U_{XY}}{U_X^2} = \frac{U_Y(U_X^2 G(X,Y) - YU_Y^2 C(X,Y))}{U_X^3} \tag{18}$$

Thus (16) gives

$$\frac{dY}{dq} = \frac{YU_X^3 G(X,Y)}{-pYU_Y^3 C + U_X^2(pU_Y - qU_X)G(X,Y)} = \frac{-U_X^3 G(X,Y)}{pU_Y^3 C(X,Y)} \tag{19}$$

using (2) and (13). It follows immediately (using (1), (2) and $p > 0$) that

$$G(X,Y) < 0 \Leftrightarrow dY/dq > 0 \tag{20}$$

Thus $G(X, Y) < 0$ is now a *necessary and sufficient* condition for Giffenity of good 2. Thus, if Giffenity is claimed for any DUF $U(x, y)$, the claim can be immediately checked by evaluating the associated ‘Giffen function’ $G(x, y)$ given by (10).

This result also admits simple geometrical interpretation—see Fig. 2, which shows two adjacent contours C and C' with tangents AB, $A'B'$, and with $Y_A = Y_{A'}$; the Giffenity condition $G < 0$ means that B' is to the left of B, as shown. As B moves to the left along the x-axis towards B', the point of contact of the tangent from B to C moves up the curve from A by virtue of the convexity condition, and reaches the point H when B reaches B'. When B reaches B' therefore, $X_H < X_A$ and $Y_H > Y_A = Y_{A'}$. This geometrical argument, which implicitly uses both conditions $C > 0$ and $G < 0$, indeed confirms Giffenity in good 2. The argument is reversible, confirming that the conditions $Y_H > Y_{A'}$ and $C > 0$ together imply that $G < 0$.

The above result motivates introduction of the concept of a ‘mirror’ pair of utility functions (direct and indirect) $\{U(x, y), V(p, q)\}$ related by the substitutions $x \leftrightarrow p$, $y \leftrightarrow q$ and a simple change of sign:

$$V(p,q) = -U(p,q) \qquad \text{or equivalently} \quad U(x,y) = -V(x,y) \tag{21}$$

so that the monotonicity conditions are compatible, and the contour maps of U and V identical.[1] For such a mirror pair, we have in effect proved the following:

[1]Needless to say, the mirror function of a DUF $U(x, y)$ should not be confused with its conventional dual IUF $V_{\text{dual}}(p, q) = U(x(p, q), y(p, q))$.

Reflexion Theorem:[2] Let $U(x, y)$ be a two-good DUF satisfying the basic axioms of consumer theory, and let $V(p, q) = -U(p, q)$ be the mirror IUF. Then good 1 associated with $V(p, q)$ is a Giffen good in a region $\mathcal{G}_1$ of the positive (p, q) quadrant if and only if good 2 associated with $U(x, y)$ is a Giffen good in the corresponding region $\mathcal{G}_1$ of the positive (x, y) quadrant.

The Reflexion Theorem also provides an alternative and perhaps more convincing test for Giffenity of a DUF $U(x, y)$: consider the mirror IUF $V(p, q)$ and compute the Marshallian demands as given by (6). Giffenity of good 1 for $V(p, q)$ then implies corresponding Giffenity of good 2 for $U(x, y)$.

Of course, if neither good is a Giffen good, then the Marshallian demand functions for both goods slope downwards (i.e. satisfy the 'law of demand'). A corollary of the Reflexion Theorem is that if the law of demand is satisfied (in the two-good situation) by a DUF $U(x, y)$, then it is also satisfied by the mirror IUF $V(p, q) = -U(p, q)$.

3.3 Example

In a recent article (Moffatt, 2011), we have constructed the following IUF (symmetric in p and q), and verified that it exhibits a region of Giffenity:

$$V(p,q) = \frac{S(p,q) - p - q + 2\lambda(1-\lambda)}{2(1-\lambda^2)} \tag{22}$$

where

$$S(p,q) = \sqrt{[p+q-2\lambda(1-\lambda)]^2 - 4(1-\lambda^2)[pq-(1-\lambda)^2]} \tag{23}$$

Here, λ is a parameter which can be varied between zero and one, and the region of Giffenity appears when λ is greater than approximately 0.7. Figure 3 shows the contour map of the IUF (22) when $\lambda = 0.8$. The essential feature is that the indifference curves (i.e. the contours, all hyperbolic in form) become more strongly curved with distance along the diagonal from the origin as far as the point $(\lambda^{-1}, \lambda^{-1})$, here (1.25, 1.25). (The parameter λ represents the rate at which the curvature increases on this portion of the diagonal. Beyond the point $(\lambda^{-1}, \lambda^{-1})$ on the diagonal, the curvature decreases, and there is no Giffenity in the region in which $\{x > \lambda^{-1}, y > \lambda^{-1}\}$.)

Tangents to two adjacent contours at the same value of q (= 1.5) are shown in Fig. 3. As in Fig. 1, these tangents intersect above the axis $q = 0$, an indication of Giffenity of good 1, as explained in Section 3.1. The region $\mathcal{G}_1$ of Giffenity of good 1 is the region inside the dashed loop above the diagonal.

[2]This theorem is related to a theorem of Kohli (1985), who established a relationship between the 'demand function' for good 1, $\partial x/\partial p|_{q=\text{cst.}}$, and the 'inverse demand function' for good 2, $\partial q/\partial y|_{x=\text{cst.}}$, for given DUF $U(x, y)$; the novel element here is the introduction of the mirror IUF, with corresponding reinterpretation of Kohli's result.

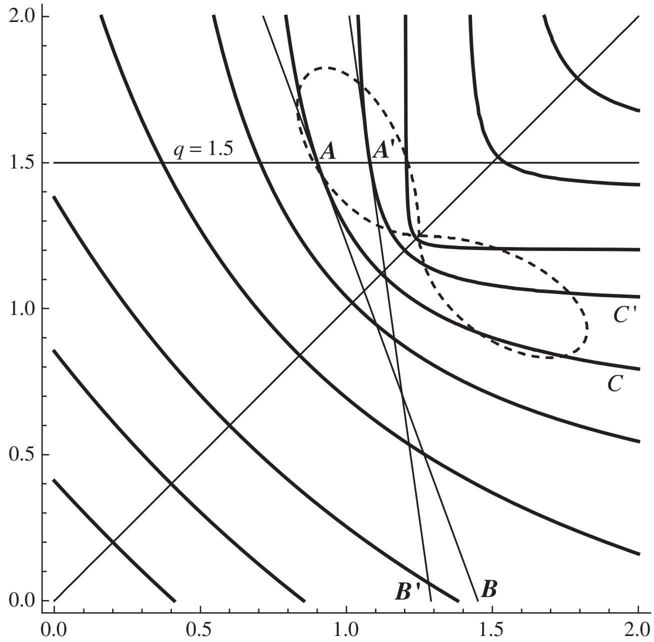

FIG. 3. Contours of the IUF $V(p, q)$, Given by (22) with $\lambda = 0.8$; the Giffen Regions $\mathcal{G}_1(p,q)$ and $\mathcal{G}_2(p,q)$ are the Interiors of the Closed Curves (Dashed) above and below the Diagonal respectively. These Regions do not Overlap, but their Boundaries (on which Neither Good is Giffen) Make Contact at the Singular Point (1.25, 1.25). Tangents to Adjacent Contours C and C' at the Same Value of q (= 1.5) at the Points A and A' (Both in $\mathcal{G}_1(p,q)$) Intersect above the p-axis (Just as in Fig. 1) indicating that Good 1 is indeed a Giffen Good

By the symmetry of the IUF (22), the region $\mathcal{G}_2$ of Giffenity of good 2 is obtained by reflexion in the diagonal $p = q$. Note that $\mathcal{G}_1$ and $\mathcal{G}_2$ do not overlap, consistent with the fact that, for given (p, q), at most one of the two goods can be a Giffen good.

The mirror DUF is given by

$$U(x, y) = \frac{-S(x, y) + x + y - 2\lambda(1-\lambda)}{2(1-\lambda^2)} \tag{24}$$

where now

$$S(x, y) = \sqrt{[x + y - 2\lambda(1-\lambda)]^2 - 4(1-\lambda^2)[xy - (1-\lambda)^2]} \tag{25}$$

The contour map of the DUF (24) again for $\lambda = 0.8$ is of course the same as in Fig. 3, but is redrawn again in Fig. 4 with a different purpose. As in Fig. 2, we here draw tangents to two adjacent contours from a point B' on the axis $q = 0$; these represent budget lines before and after a rise in the price q of good

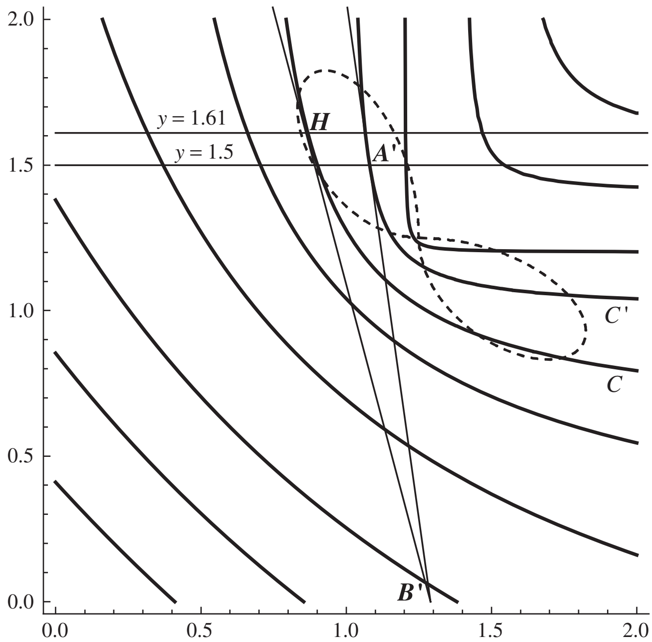

FIG. 4. Contour Map of DUF (24) with $\lambda = 0.8$. Tangents $B'A'$, $B'H$ are Drawn from the Point B' on the p-axis to Two Adjacent Contours C and C'; as the Budget line Rotates Anti-clockwise about B', the Point of Tangency Rises from A' to H (cf. Fig. 2), indicating that Good 2 is a Giffen Good. The Regions of Giffenity are Identical with those of Fig. 3

2. It is evident that the budget line on the left has a higher point of tangency ($y_H = 1.61$) than that on the right ($y_{A'} = 1.5$), indicating Giffenity of good 2. The region of Giffenity of good 2 in Fig. 4 is identical with that of good 1 in Fig. 3, as indeed it must be by virtue of the Reflexion Theorem.

The Marshallian demand $x(p, q)$ associated with the IUF $V(p, q)$ is given by Roy's identity (6), and is shown in Fig. 5. This clearly indicates the region of Giffenity in which x increases as a function of p, at constant q. For the mirror DUF $U(x, y)$, by virtue of the reflexion theorem, y increases as a function of q in the corresponding region.

We should note here that by no means all demand properties satisfy this type of reflexion principle. For example, if good 1 is an inferior good (i.e. a good whose demand falls when the normalized prices of *both* goods fall in equal proportion) for an IUF $V(p, q)$ in a region $\mathcal{R}$ of the (p, q) plane, and if good 2 is an inferior good for the mirror DUF $U(x, y)$ in a region $\mathcal{R}'$ of the (x, y), these two regions $\mathcal{R}$ and $\mathcal{R}'$ may overlap but are not identical. Figure 6 shows the region $\mathcal{R}$ (shaded) for the IUF (22) in a portion of the positive quadrant, and the region $\mathcal{R}'$ for the mirror DUF (bounded by the dashed lines); these regions do indeed overlap but neither completely overlaps

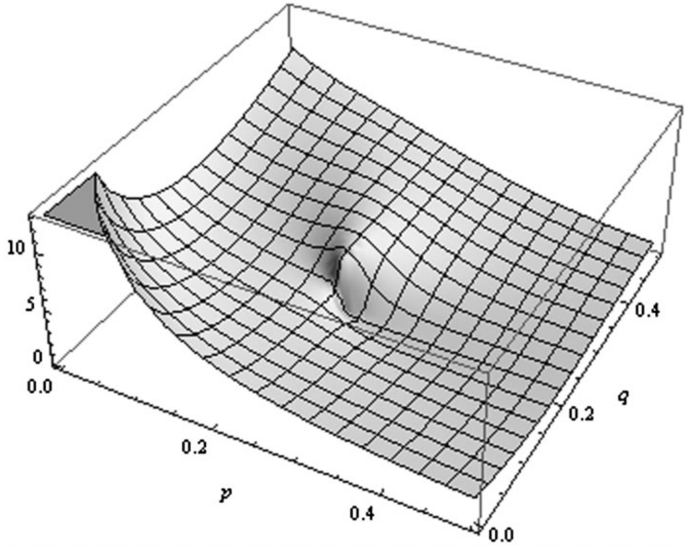

FIG. 5. Marshallian Demand Function $x(p, q)$ clearly indicating the Region of Giffenity in which x Increases as a Function of p, at Constant q

the other, and it is evident therefore that inferiority of good 1 for the IUF is *neither necessary nor sufficient* for inferiority of good 2 for the mirror DUF. The closed Giffen loop is also shown in this figure: it lies entirely within both $\mathcal{R}$ and $\mathcal{R}'$, consistent with the well-known fact that Giffenity implies inferiority but not vice versa.

4 EXTENSION TO THE n-GOOD SITUATION

The reflexion theorem is essentially a two-good theorem, and does not admit obvious extension to the n-good situation. That is, if an n-good IUF $V(p_1, \ldots, p_n)$ is such that one or more goods is Giffen, we cannot say whether the mirror DUF $U(x_1, \ldots, x_n) = -V(x_1, \ldots, x_n)$ will lead to Giffenity, and if it does, which goods will be Giffen. The problem posed here clearly leads to an interesting avenue for future research.

However, the above construction of a symmetric IUF exhibiting Giffenity in both goods (in non-overlapping regions) does admit natural extension. With demands $\mathbf{x} = \{x_i\}$, prices $\mathbf{p} = \{p_i\}$, and normalized budget constraint $\mathbf{p}.\mathbf{x} = \sum_{i=1}^{n} p_i x_i = 1$, we consider a family of surfaces in $\mathbf{p}$-space[3]

[3]We are concerned only with the region in which $p_i > a$ for all i, so the left-hand side of (26) is positive. The right-hand side must therefore also be positive, so when n is odd, we must have $a < \lambda^{-1}$. This is not a serious restriction, because Giffenity is in fact confined to the region in which $a < \lambda^{-1}$.

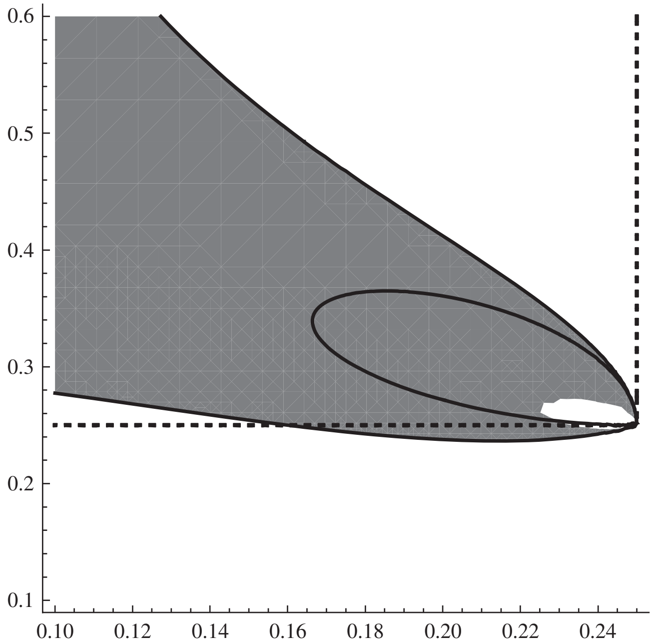

FIG. 6. The Region $\mathcal{R}$ (Shaded) in which Good 1 (for the IUF (22)) is Inferior, and the Region $\mathcal{R}'$ (Enclosed by the Dashed Lines) in which Good 2 (for the Mirror DUF) is Inferior. Part of $\mathcal{R}$ Lies Outside $\mathcal{R}'$ and Part of $\mathcal{R}'$ Lies Outside $\mathcal{R}$, indicating that Inferiority does not have the Reflexion Property. The Giffen Region (Inside the Closed Loop) Lies in the Interior of both $\mathcal{R}$ and $\mathcal{R}'$, as it should, since Giffenity Implies Inferiority but not vice versa. The Small Patch of White in the Shaded Region is an Indication of Resolution Problems near the Singular Point

$$\prod_{i=1}^{n}(p_i - a) = (1 - \lambda a)^n \tag{26}$$

where $0 < \lambda < 1$ and each surface of the family is labelled by the parameter a. One member of this family passes through the origin $\mathbf{p} = \mathbf{0}$; for this member, $(-a)^n = (1 - \lambda a)^n$, and the appropriate (real) root is given by $-a = 1 - \lambda a$, i.e. $a = -(1 - \lambda)^{-1}$. Also, the surface for which $a = \lambda^{-1}$ has a singularity (a right-angled corner) at the point $\mathbf{p} = \lambda^{-1}(1, 1, \ldots, 1)$ on the 'diagonal' $p_1 = p_2 = \ldots = p_n$. It makes sense therefore to restrict a to the range $-(1 - \lambda)^{-1} \le a \le \lambda^{-1}$. The corresponding family of surfaces fills the region

$$\mathcal{V}: \quad 0 \le p_i \le \lambda^{-1} \qquad (i = 1, 2, \ldots, n) \tag{27}$$

Equation (26) may be regarded as a polynomial equation for a with n roots, some of which (for given p_i) may occur in complex conjugate pairs. One of the roots $a(p_i)$ is however always real when $p_i \in \mathcal{V}$, namely that for which

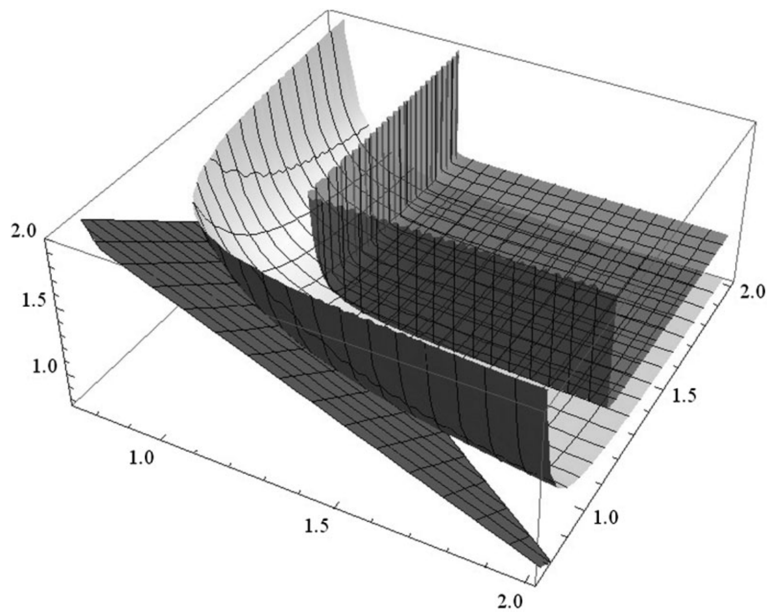

FIG. 7. Two of the Invariance Surfaces (26) with $n = 3$, and the Budget Plane Tangent to One of them

$a(\mathbf{0}) = -(1 - \lambda)^{-1}$, and we may adopt this real root as the 'label' which is constant on any member of the family (26). This root is an increasing function of each p_i in the range $0 < p_i < \lambda^{-1}$, so we may define an appropriate IUF $V(p_i)$ as any decreasing function of a, most simply $V(p_i) \equiv -a(p_i)$. The indifference surfaces $V = \text{const.}$ are then evidently just the family of (generalized) rectangular hyperboloids (26).

The polynomial equation (26) can be solved explicitly for the required root $a(p_i)$ if $n = 2$, 3 or 4, but not for $n \geq 5$ (Abel's impossibility theorem; in this case, the solution may only be found numerically). Here however, we may illustrate what happens in the three-good situation ($n = 3$), which provides sufficient indication of the more general situation $n > 3$. To simplify the notation, let $\mathbf{x} = (x, y, z)$, $\mathbf{p} = (p, q, r)$; the budget constraint is then represented by the plane $px + qy + rz = 1$ in $\mathbf{p}$-space, with unit normal $\mathbf{n} = (x, y, z)/|\mathbf{x}|$. Figure 7 shows two of the surfaces (26) for different values of a and the budget-plane tangent to one of them. In the (x, y, z) space, this plane intersects the x-axis at the point $A(p^{-1}, 0, 0)$, the y-axis at $B(0, q^{-1}, 0)$ and the z-axis at $C(0, 0, r^{-1})$. As p increases keeping q and r fixed, this plane rotates about the line BC, and the point of tangency (x, y, z) on the surface to which it is tangent changes accordingly.

Equation (26) is now a cubic for $a(p, q, r)$ which can be solved explicitly. In view of the symmetry in (p, q, r), the solution depends only on the symmetric

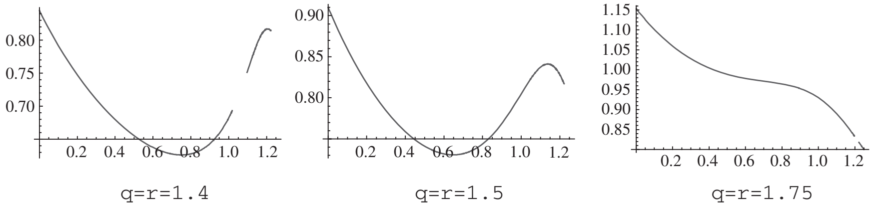

FIG. 8. Marshallian Demand Curves for Good 1 in the Three-good Problem; the Plots Show $x(p, q, r)$ against p for Three Different Choices of (q, r). When q and r are Sufficiently near the Singular Point (5/4, 5/4), There is a Segment of Positive Slope, Indicating Giffenity of Good 1 (the Gaps in these Curves Reflect Numerical Problems Encountered by Mathematica in this Evaluation). Far from this Point, there is no Giffenity

forms $p + q + r$, $qr + rp + pq$ and pqr. We have used Mathematica to obtain, as a function of these variables, the real root for which $a(0, 0, 0) = -(1 - \lambda)^{-1}$. For the choice $\lambda = 4/5$ on which we again focus, $a(0, 0, 0) = -5$. We set $V(p, q, r) = -a(p, q, r)$, and construct the Marshallian demand functions $x(p, q, r)$, $y(p, q, r)$, $z(p, q, r)$, using Roy's identity in the form

$$x = \frac{V_p}{pV_p + qV_q + rV_r} \qquad y = \frac{V_q}{pV_p + qV_q + rV_r} \qquad z = \frac{V_r}{pV_p + qV_q + rV_r} \tag{28}$$

We can then plot $x(p, q, r)$ as a function of p for various fixed values of q and r, thus exploring the space for a region $\mathcal{G}_1$ of Giffenity of good 1 where $\partial x/\partial p > 0$. These plots for three different choices of (q, r) are shown in Fig. 8, and the regions of rising slope in the first two figures indicates Giffenity of good 1. As the distance from the 'critical point' (5/4, 5/4) increases, just as for the two-good case, the Giffenity disappears. The region $\mathcal{G}_1$ of Giffenity of good 1 is apparently a lozenge-shaped region in the three-dimensional space of the variables (p, q, r) confined to the region $p < 5/4$, $q > 5/4$, $r > 5/4$, with a conical point at (5/4, 5/4, 5/4). By the symmetry of the geometry with respect to (p, q, r) or (x, y, z), it is apparent that corresponding lozenges $\mathcal{G}_2$, $\mathcal{G}_3$ of Giffenity of good 2 and good 3 are obtained by rotation of $\mathcal{G}_1$ about the diagonal $p = q = r$ through angles $2\pi/3$ and $4\pi/3$ respectively.

It is evident that similar considerations do generalize to the n-good problem. Thus, we may assert that the IUF that has indifference surfaces given by (26) exhibits Giffenity of good 1 in an 'n-dimensional lozenge' in the subdomain in which $p_1 < \lambda^{-1}$ and $p_i > \lambda^{-1}$ $(i = 2, 3, \ldots, n)$, and that every good has a similar region of Giffenity.

5 CONCLUSIONS

This paper was motivated by a 'challenge' that has stimulated much recent research: that of specifying a utility function that predicts Giffen behaviour while at the same time meeting a set of reasonable conditions. We claim to have met the challenge because we have specified a two-good IUF (equation (22)) that is defined in closed form over the entire positive (p, q) quadrant, is continuous, twice differentiable, monotonically decreasing in both arguments and globally quasi-convex, and from which it is possible, by virtue of Roy's identity, to derive Marshallian demands that are expressible in closed form. Moreover, by virtue of the symmetry of the specified IUF, we have produced a function that predicts that *both* goods are Giffen goods (in non-overlapping regions of price space). In this sense we have more than met the challenge.

The original challenge might be interpreted as requiring commencement from a DUF. We have instead started from an IUF. However, the reflexion theorem proved in Section 3.2 provides the vital link between an IUF and its 'mirror' DUF, that enables us to transform our IUF into a DUF that is also guaranteed to meet the challenge, by a straightforward reversal of sign. The force of this reflexion theorem is enhanced by our demonstration that, while the result invariably applies to Giffen behaviour, it does not in general apply to another well-known demand feature, namely inferiority.

Another sense in which we have more than met the Giffen challenge is that we have demonstrated that it is possible to generalize the IUF to more than two goods, and in this way, we have found an explicit IUF for which every good has the Giffen property.

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