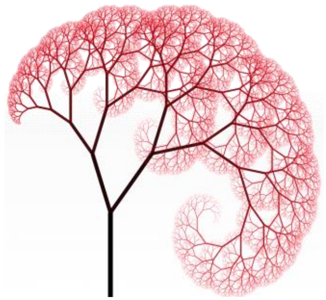
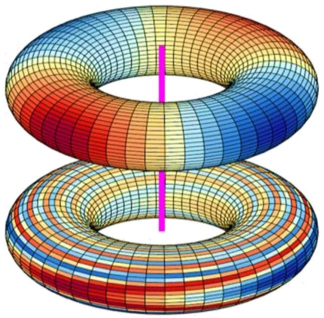




# Towards a classification theory for data-driven dynamical systems

Matthew Colbrook
University of Cambridge
28/10/2024

*"To <u>classify</u> is to bring order into chaos."* - **George Pólya**

C., Mezić, Stepanenko *"Limits and Powers of Koopman Learning,"* arxiv preprint, 2024.

For papers and talk slides/videos, visit:
http://www.damtp.cam.ac.uk/user/mjc249/home.html

# Data-driven dynamical systems

- Compact metric space $(\mathcal{X}, d)$ – the state space
- $x \in \mathcal{X}$ – the state

cts $F: \mathcal{X} \to \mathcal{X}$ – the dynamics: $x_{n+1} = F(x_n)$

Dynamics (geometry)
19th century

- Borel measure $\omega$ on $\mathcal{X}$
- Function space $L^2 = L^2(\mathcal{X}, \omega)$ (elements $g$ called "observables")
- Koopman operator $\mathcal{K}_F: L^2 \to L^2$; $[\mathcal{K}_F g](x) = g(F(x))$
- ***Available*** snapshot data: $\left\{\left(x^{(m)}, y^{(m)} = F\left(x^{(m)}\right)\right): m = 1, \dots, M\right\}$

**NB:** Pointwise definition of $\mathcal{K}_F$ needs $F\#\omega \ll \omega$ – this will hold throughout.
**NB:** $\mathcal{K}_F$ bounded equivalent to $\mathrm{d}F\#\omega/\mathrm{d}\omega \in L^\infty$ – this will hold throughout (can be dropped).

# Data-driven dynamical systems

- Compact metric space $(\mathcal{X}, d)$ – the state space
- $x \in \mathcal{X}$ – the state

  cts $F: \mathcal{X} \to \mathcal{X}$ – the dynamics: $x_{n+1} = F(x_n)$

**Dynamics (geometry)**
**19th century**

- Borel measure $\omega$ on $\mathcal{X}$
- Function space $L^2 = L^2(\mathcal{X}, \omega)$ (elements $g$ called "observables")
- Koopman operator $\mathcal{K}_F: L^2 \to L^2$; $[\mathcal{K}_F g](x) = g(F(x))$

**Analysis**
**20th century**

- ***<u>Available</u>*** snapshot data: $\left\{\left(x^{(m)}, y^{(m)} = F\left(x^{(m)}\right)\right) : m = 1, \ldots, M\right\}$

**NB:** Pointwise definition of $\mathcal{K}_F$ needs $F\#\omega \ll \omega$ – this will hold throughout.
**NB:** $\mathcal{K}_F$ bounded equivalent to $\mathrm{d}F\#\omega/\mathrm{d}\omega \in L^\infty$ – this will hold throughout (can be dropped).

# Data-driven dynamical systems

- Compact metric space $(\mathcal{X}, d)$ – the state space
- $x \in \mathcal{X}$ – the state
- ***Unknown*** cts $F: \mathcal{X} \to \mathcal{X}$ – the dynamics: $x_{n+1} = F(x_n)$

**Dynamics (geometry)**
**19th century**

- Borel measure $\omega$ on $\mathcal{X}$
- Function space $L^2 = L^2(\mathcal{X}, \omega)$ (elements $g$ called "observables")
- Koopman operator $\mathcal{K}_F: L^2 \to L^2$; $[\mathcal{K}_F g](x) = g(F(x))$

**Analysis**
**20th century**

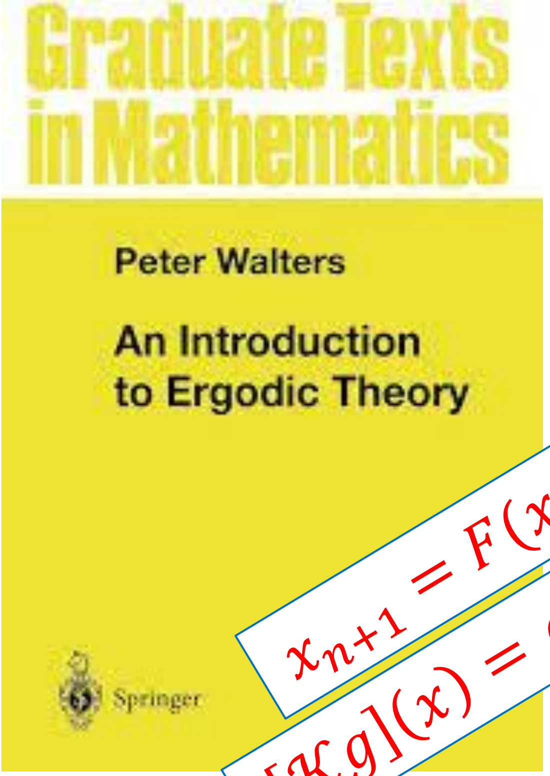
- ***Available*** snapshot data: $\left\{\left(x^{(m)}, y^{(m)} = F\left(x^{(m)}\right)\right): m = 1, \dots, M\right\}$

**Data**
**21st century**

**NB:** Pointwise definition of $\mathcal{K}_F$ needs $F\#\omega \ll \omega$ – this will hold throughout.
**NB:** $\mathcal{K}_F$ bounded equivalent to $\mathrm{d}F\#\omega/\mathrm{d}\omega \in L^\infty$ – this will hold throughout (can be dropped).

# Why you should care about Koopman

Fundamental in ergodic theory




$$x_{n+1} = F(x_n)$$

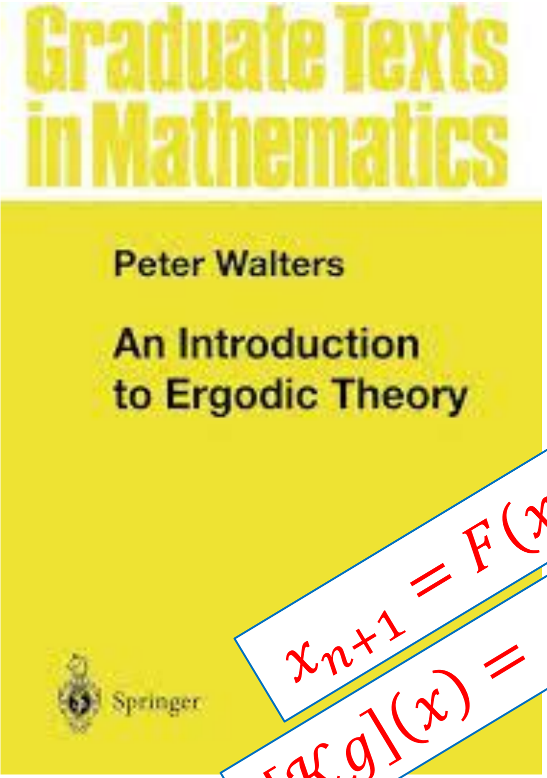
$$[\mathcal{K}g](x) = g(F(x))$$

E.g., key to ergodic theorems of
Birkhoff and von Neumann.

**Trades:** Nonlinear, finite-dimensional $\Longrightarrow$ Linear, infinite-dimensional.

# Why you should care about Koopman

Fundamental in ergodic theory




$x_{n+1} = F(x_n)$

$[\mathcal{K}g](x) = g(F(x))$

E.g., key to ergodic theorems of Birkhoff and von Neumann.

Can provide a *diagonalization* of a nonlinear system.

eigenfunction of $\mathcal{K}$

continuous spectrum

$$g(x) = \sum_{\text{eigenvalues } \lambda_j} c_{\lambda_j} \varphi_{\lambda_j}(x) + \int_{-\pi}^{\pi} \phi_{\theta,g}(x)\, \mathrm{d}\theta$$

$$g(x_n) = [\mathcal{K}^n g](x_0)$$

$$= \sum_{\text{eigenvalues } \lambda_j} c_{\lambda_j} \boxed{\lambda_j{}^n} \varphi_{\lambda_j}(x_0) + \int_{-\pi}^{\pi} \boxed{e^{in\theta}} \phi_{\theta,g}(x_0)\, \mathrm{d}\theta$$

**Spectral properties encode:** geometric features, invariant measures, transient behavior, long-time behavior, coherent structures, quasiperiodicity, etc.

**Trades:** Nonlinear, finite-dimensional $\Longrightarrow$ Linear, infinite-dimensional.

# Why you should care about Koopman

Fundamental in ergodic theory




Can provide a *diagonalization* of a nonlinear system.

eigenfunction of $\mathcal{K}$

continuous spectrum

$$g(x) = \sum_{\text{eigenvalues } \lambda_j} c_{\lambda_j}\varphi_{\lambda_j}(x) + \int_{-\pi}^{\pi} \phi_{\theta,g}(x)\, d\theta$$

**+ HUGE recent interest in their spectral properties…**




$x_{n+1} = F(x_n)$

$[\mathcal{K}g](x) = g(F(x))$

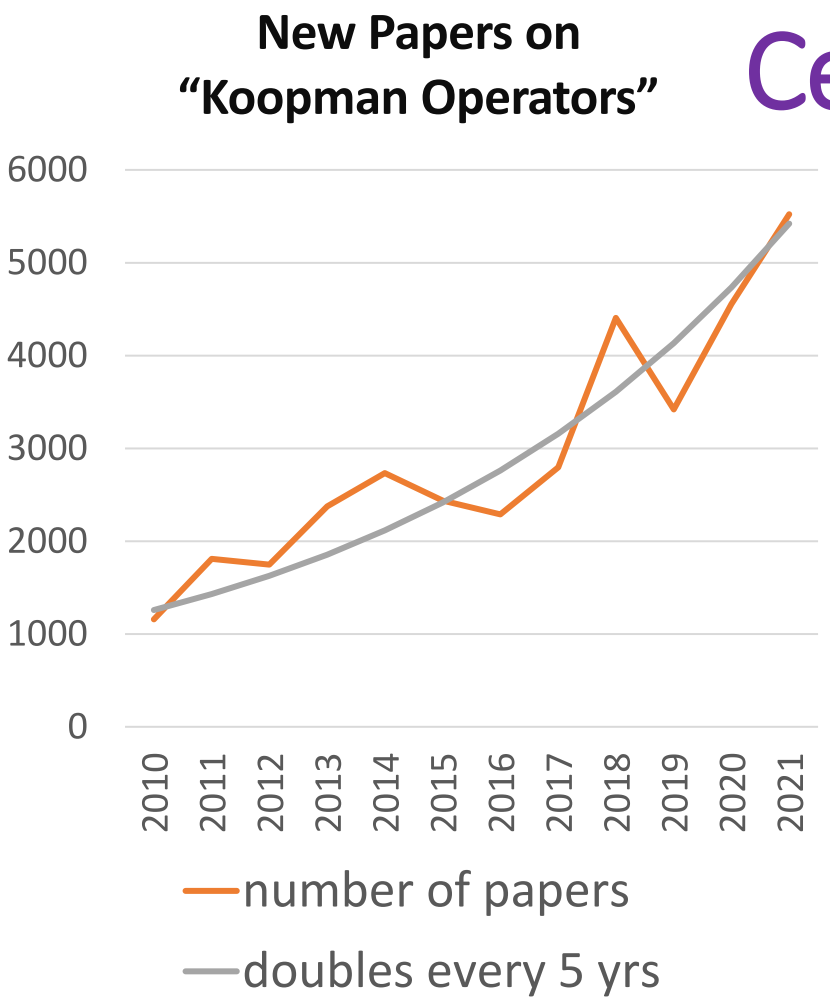
$$= \sum_{\text{eigenvalues } \lambda_j} c_{\lambda_j}\lambda_j^{\,n}\varphi_{\lambda_j}(x_0) + \int_{-\pi}^{\pi} e^{in\theta}\phi_{\theta,g}(x_0)\, d\theta$$

E.g., key to ergodic theorems of Birkhoff and von Neumann.

**Spectral properties encode:** geometric features, invariant measures, transient behavior, long-time behavior, coherent structures, quasiperiodicity, etc.

**Trades:** Nonlinear, finite-dimensional $\Longrightarrow$ Linear, infinite-dimensional.

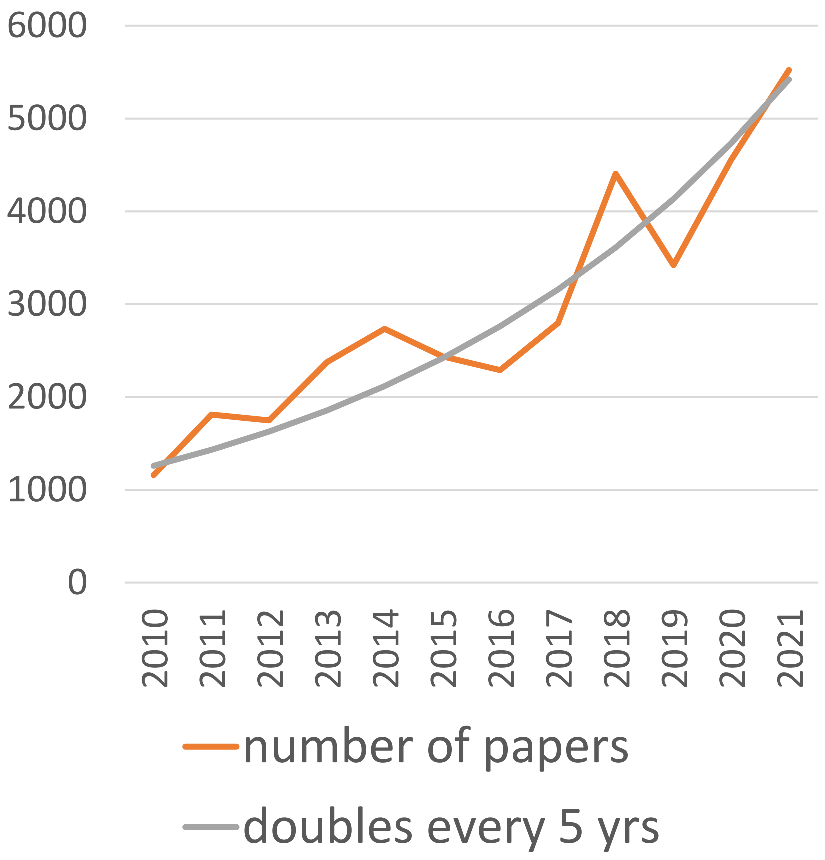
# Central in data-driven era




**Loads of applications!!!**

# Central in data-driven era

**New Papers on "Koopman Operators"**




**Loads of applications!!!**

**2012**


CHAOS 22, 047510 (2012)

Applied Koopmanism[a)]

Marko Budišić, Ryan Mohr, and Igor Mezić

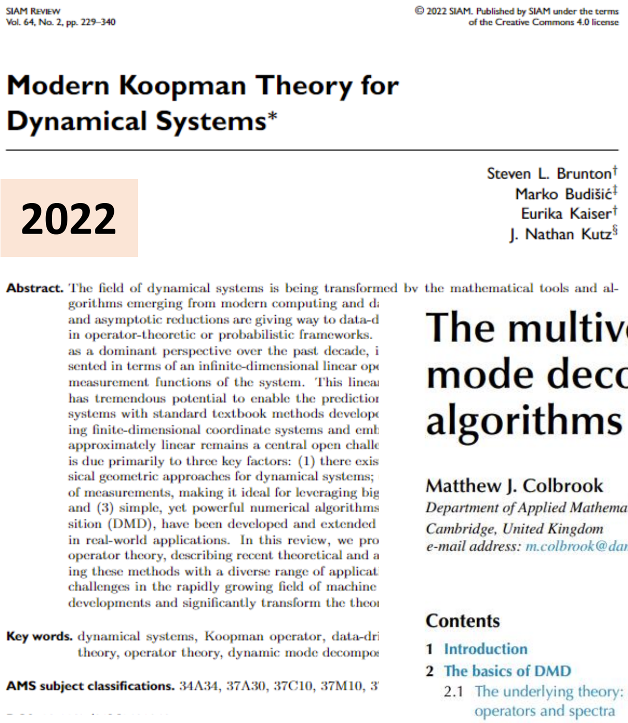
Department of Mechanical Engineering, University of California, Santa Barbara, California 93106-5070, USA

(Received 11 June 2012; accepted 30 November 2012; publishe

A majority of methods from dynamical system analysis, espec on Poincaré's geometric picture that focuses on "dynamics of st our field for a century, it has shown difficulties in handling h uncertain systems, which are more and more common in engine "big data" measurements. This overview article presents an alt systems, based on the "dynamics of observables" picture. Tl operator: an infinite-dimensional, linear operator that is noneth nonlinear dynamics. The first goal of this paper is to make it c different papers and contexts all relate to each other through s operator. The second goal is to present these methods in a conc framework accessible to researchers who would like to apply tl them. Finally, we aim to provide a road map through the litera described in detail. We describe three main concepts: Ko eigenquotients, and continuous indicators of ergodicity. For ea of theoretical concepts required to define and study them, n developed for their analysis, and, when possible, applicatio Koopman framework is showing potential for crossing over fro industrial practice. Therefore, the paper highlights its strengths Additionally, we point out areas where an additional research pu adopted as an off-the-shelf framework for analysis and desi *Physics*. [http://dx.doi.org/10.1063/1.4772195]

**A majority of methods from dynamical systems analysis, especially those in applied settings, rely on Poincaré's geometric picture that focuses on "dynamics of states." While this picture has fueled our field for a century, it has shown difficulties in handling high-dimensional, ill-described, and** approa of ergo theory Throug erature

**2022**


SIAM Review Vol. 64, No. 2, pp. 229–340

© 2022 SIAM. Published by SIAM under the terms of the Creative Commons 4.0 license

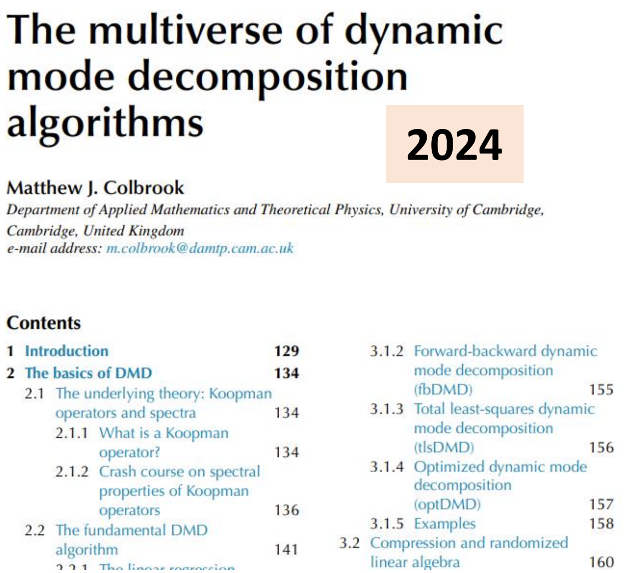
**Modern Koopman Theory for Dynamical Systems***

Steven L. Brunton[†]
Marko Budišić[‡]
Eurika Kaiser[†]
J. Nathan Kutz[§]

**Abstract.** The field of dynamical systems is being transformed by the mathematical tools and al- gorithms emerging from modern computing and d and asymptotic reductions are giving way to data-d in operator-theoretic or probabilistic frameworks. as a dominant perspective over the past decade, i sented in terms of an infinite-dimensional linear op measurement functions of the system. This linea has tremendous potential to enable the predictior systems with standard textbook methods develope ing finite-dimensional coordinate systems and emb approximately linear remains a central open challe is due primarily to three key factors: (1) there exis sical geometric approaches for dynamical systems; of measurements, making it ideal for leveraging big and (3) simple, yet powerful numerical algorithms sition (DMD), have been developed and extended in real-world applications. In this review, we pro operator theory, describing recent theoretical and a ing these methods with a diverse range of applicat challenges in the rapidly growing field of machine developments and significantly transform the theo

**Key words.** dynamical systems, Koopman operator, data-dri theory, operator theory, dynamic mode decompo

**AMS subject classifications.** 34A34, 37A30, 37C10, 37M10, 3


# The multiverse of dynamic mode decomposition algorithms

**2024**

Matthew J. Colbrook

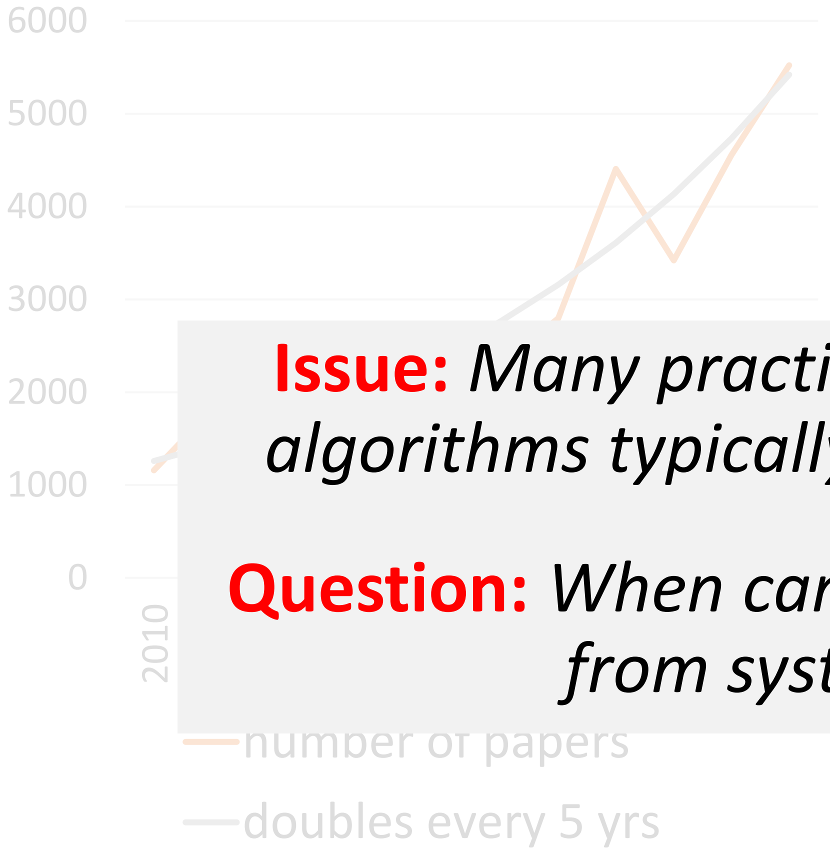
*Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, United Kingdom*

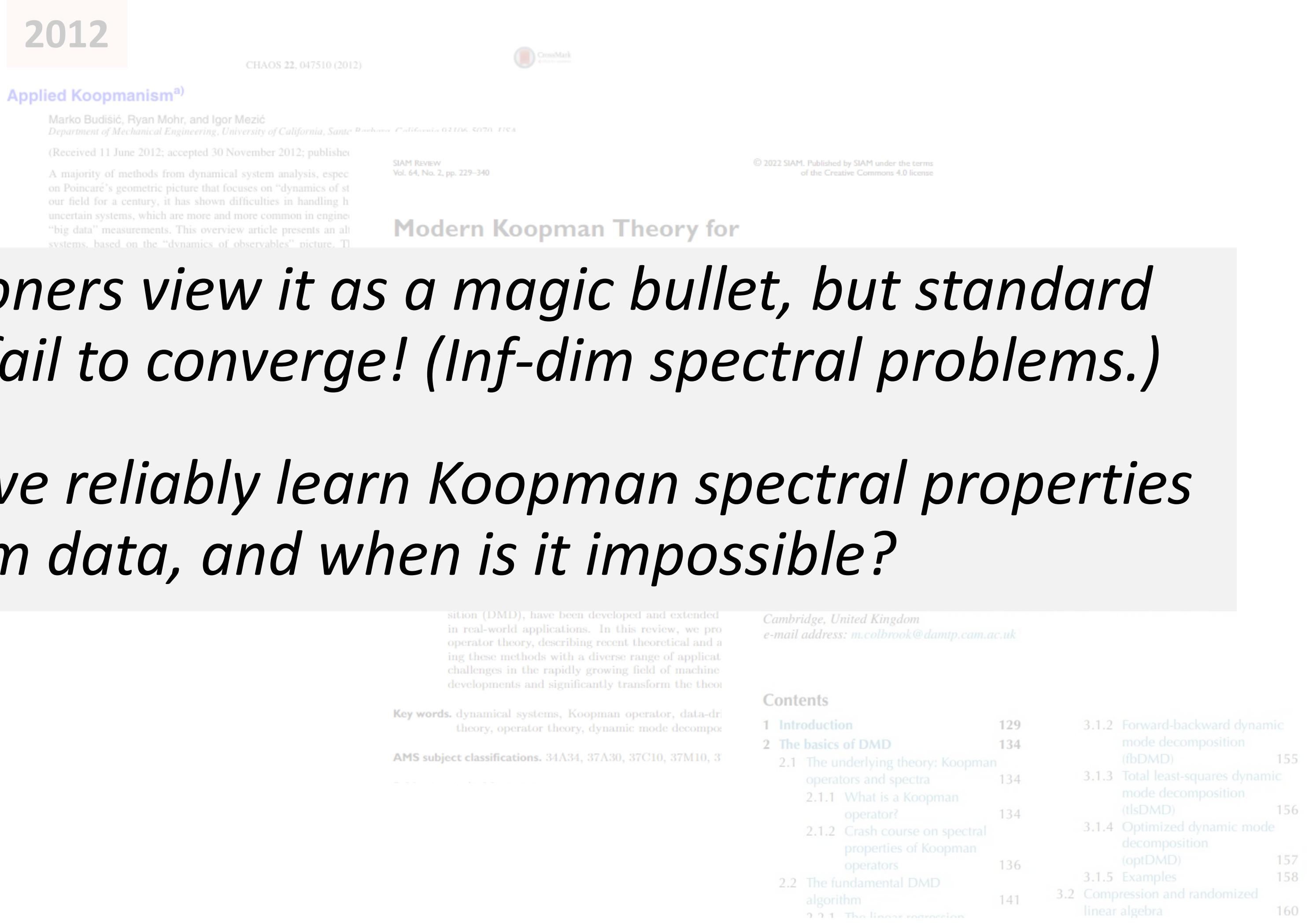
*e-mail address:* m.colbrook@damtp.cam.ac.uk

**Contents**

# Central in data-driven era

New Papers on "Koopman Operators"




**Issue:** *Many practitioners view it as a magic bullet, but standard algorithms typically fail to converge! (Inf-dim spectral problems.)*

**Question:** *When can we reliably learn Koopman spectral properties from system data, and when is it impossible?*

Loads of applications!!!

# Outline

**Question:** *When can we reliably learn Koopman spectral properties from system data, and when is it impossible?*

- Constructing adversaries – *impossibility theorem* **(a general strategy)**
- Towers of algorithms – *possibility theorem* **(problem-specific algorithm)**
- The Solvability Complexity Index Hierarchy – *classifications* **(general tool)**
- Where does this leave us?

# Theorem A (impossibility)

Implies $\mathcal{K}$ is unitary

*Class of systems:* $\Omega_{\mathbb{D}} = \{F: \overline{\mathbb{D}} \to \overline{\mathbb{D}} \mid F \text{ cts, measure preserving, invertible}\}.$

*Data an algorithm can use:* $\mathcal{T}_F = \{(x, y_m) \mid x \in \overline{\mathbb{D}}, \|F(x) - y_m\| \leq 2^{-m}\}.$

**Theorem A:** There **does not exist** any sequence of deterministic algorithms $\{\Gamma_n\}$ using $\mathcal{T}_F$ such that $\lim_{n\to\infty} \Gamma_n(F) = \mathrm{Sp}(\mathcal{K}_F)\ \ \forall F \in \Omega_{\mathbb{D}}$.

***NB:*** Similarly, no random algorithms converging with probability $> 1/2$.

# Proof idea: Constructing an adversary

$F_0$: rotation by $\pi$, $\mathrm{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$

**Phase transition lemma:** Let $X = \{x_1, \ldots, x_N\}$,$Y = \{y_1, \ldots, y_N\}$ be distinct points in annulus $\mathcal{A} = \{x \in \mathbb{D} | 0 < R < \|x\| < r < 1\}$ with $X \cap Y = \emptyset$. There exists a measure-preserving homeomorphism $H$ such that $H$ acts as the identity on $\mathbb{D} \backslash \mathcal{A}$ and $H(y_j) = F_0(H(x_j)), j = 1, \ldots, N$.

***Conjugacy of <u>data</u> ($x_j \rightarrow y_j$) with $F_0$***

**Idea:** Use lemma to trick any algorithm into oscillating between spectra.

- Brown and Halperin. "*On certain area-preserving maps.*" **Annals of Mathematics**, 1935.

# Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{T}_F$, $\lim_{n\to\infty} \Gamma_n(F) = \mathrm{Sp}(\mathcal{K}_F)\ \ \forall F \in \Omega_{\mathbb{D}}$.

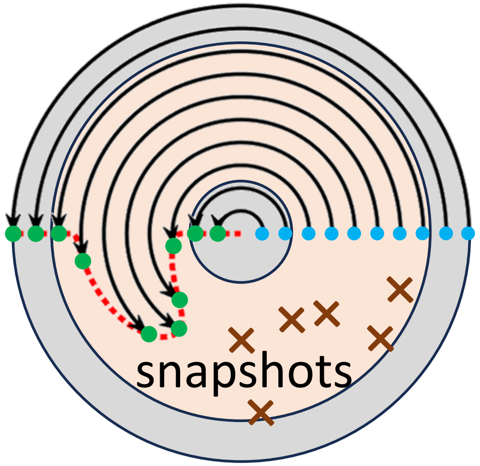
Build an adversarial $F$…

$\mathcal{T}_F = \{(x, y_m) \mid \|F(x) - y_m\| \leq 2^{-m}\}$

# Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{T}_F$, $\lim_{n\to\infty}\Gamma_n(F) = \mathrm{Sp}(\mathcal{K}_F)\ \ \forall F \in \Omega_{\mathbb{D}}$.

Build an adversarial $F$…




$\widetilde{F_1}(r,\theta) = \big(r, \theta + \pi + \phi(r)\big)$, $\mathrm{supp}(\phi) \subset [1/4, 3/4]$

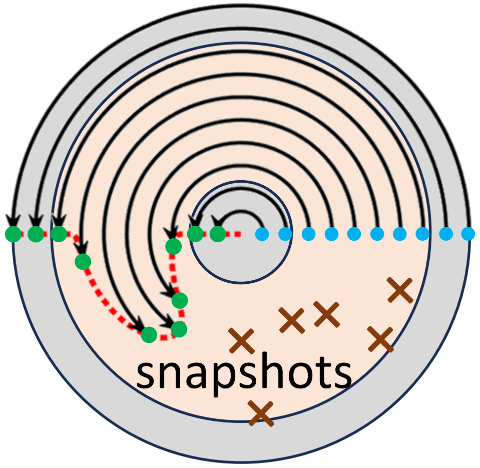
$\mathrm{Sp}(\mathcal{K}_{\widetilde{F_1}}) = \mathbb{T}$ (unit circle).

$\mathcal{T}_F = \{(x, y_m) \mid \|F(x) - y_m\| \leq 2^{-m}\}$

# Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{T}_F$, $\lim_{n\to\infty} \Gamma_n(F) = \mathrm{Sp}(\mathcal{K}_F)\ \ \forall F \in \Omega_{\mathbb{D}}$.

Build an adversarial $F$...




$\widetilde{F_1}(r,\theta) = \big(r, \theta + \pi + \phi(r)\big)$, $\mathrm{supp}(\phi) \subset [1/4, 3/4]$

$\mathrm{Sp}(\mathcal{K}_{\widetilde{F_1}}) = \mathbb{T}$ (unit circle).

$\lim_{n\to\infty} \Gamma_n(\widetilde{F_1}) = \mathrm{Sp}(\mathcal{K}_{\widetilde{F_1}}) \Rightarrow \exists n_1$ s.t. $\mathrm{dist}(i, \Gamma_{n_1}(\widetilde{F_1})) \leq 1.$

**BUT** $\Gamma_{n_1}$ uses finite amount of info to output $\Gamma_{n_1}(\widetilde{F_1})$.

Let $X$, $Y$ correspond to these snapshots.

$\mathcal{T}_F = \{(x, y_m) \mid \|F(x) - y_m\| \leq 2^{-m}\}$

# Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{T}_F$, $\lim_{n\to\infty}\Gamma_n(F) = \mathrm{Sp}(\mathcal{K}_F)\ \ \forall F \in \Omega_{\mathbb{D}}$.

Build an adversarial $F$…

$\widetilde{F_1}(r,\theta) = \big(r, \theta + \pi + \phi(r)\big)$, $\mathrm{supp}(\phi) \subset [1/4, 3/4]$

$\mathrm{Sp}(\mathcal{K}_{\widetilde{F_1}}) = \mathbb{T}$ (unit circle).

$\lim_{n\to\infty}\Gamma_n\big(\widetilde{F_1}\big) = \mathrm{Sp}(\mathcal{K}_{\widetilde{F_1}}) \Rightarrow \exists n_1$ s.t. $\mathrm{dist}(i, \Gamma_{n_1}\big(\widetilde{F_1}\big)) \leq 1.$

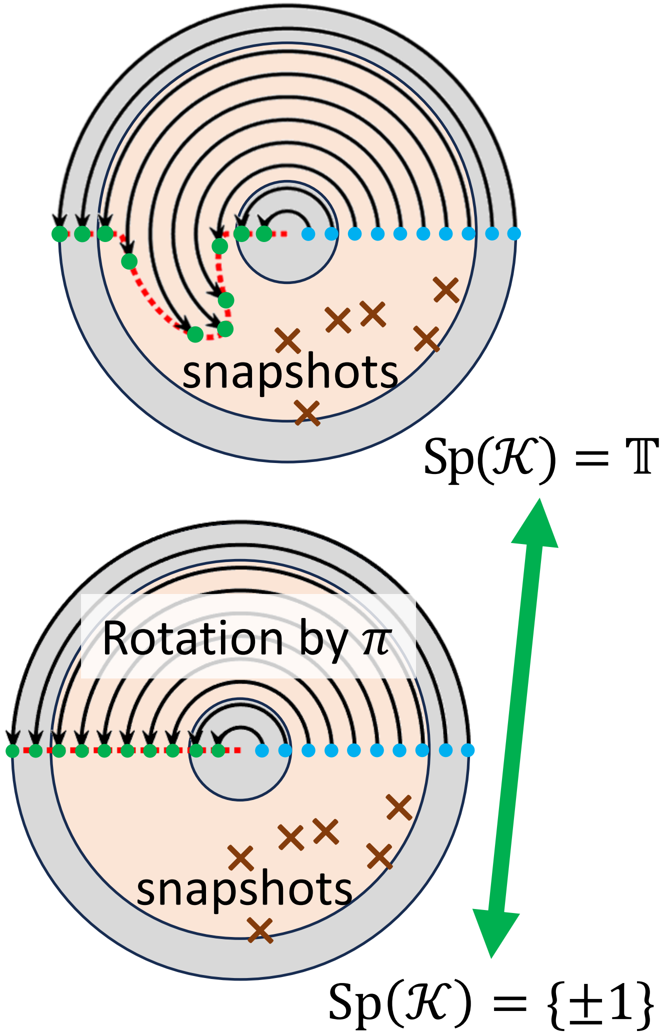
**BUT** $\Gamma_{n_1}$ uses finite amount of info to output $\Gamma_{n_1}\big(\widetilde{F_1}\big)$.

Let $X$, $Y$ correspond to these snapshots.

Lemma: $F_1 = H_1^{-1} \circ F_0 \circ H_1$ on annulus $\mathcal{A}_1$.

Consistent data $\Rightarrow \Gamma_{n_1}(F_1) = \Gamma_{n_1}\big(\widetilde{F_1}\big)$, $\mathrm{dist}(i, \Gamma_{n_1}(F_1)) \leq 1$

**BUT** $\mathrm{Sp}(\mathcal{K}_{F_1}) = \mathrm{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$



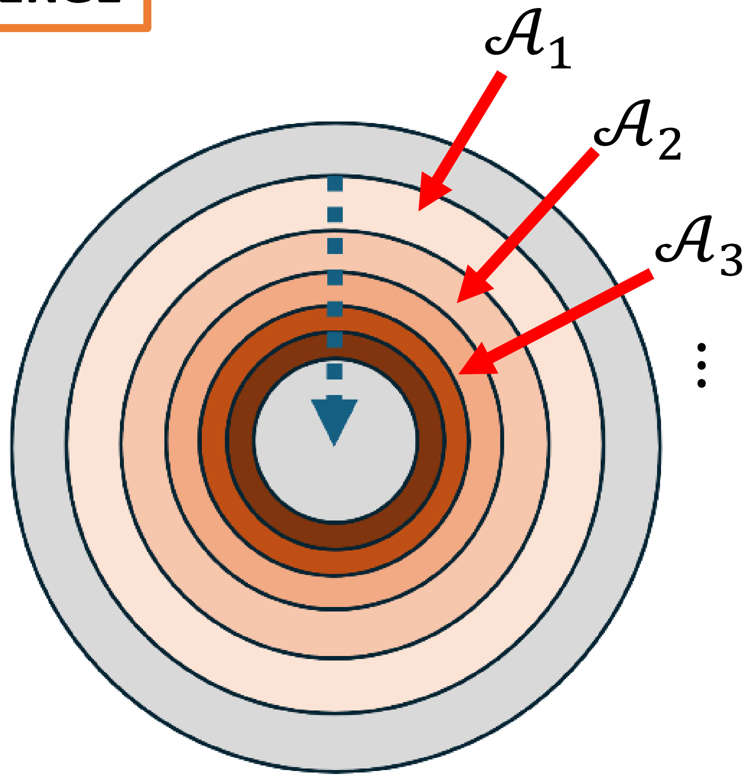
# Proof idea: Constructing an adversary

**Inductive step:** Repeat on annuli, $F_k = H_k^{-1} \circ F_0 \circ H_k$ on $\mathcal{A}_k$. $F = \lim_{k\to\infty} F_k$

Consistent data $\Rightarrow \Gamma_{n_k}(F) = \Gamma_{n_k}(\widetilde{F_k})$, $\mathrm{dist}(i, \Gamma_{n_k}(F)) \leq 1$, $n_k \to \infty$

**BUT** $\mathrm{Sp}(\mathcal{K}_F) = \mathrm{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$

**CANNOT CONVERGE**




Cascade of disks

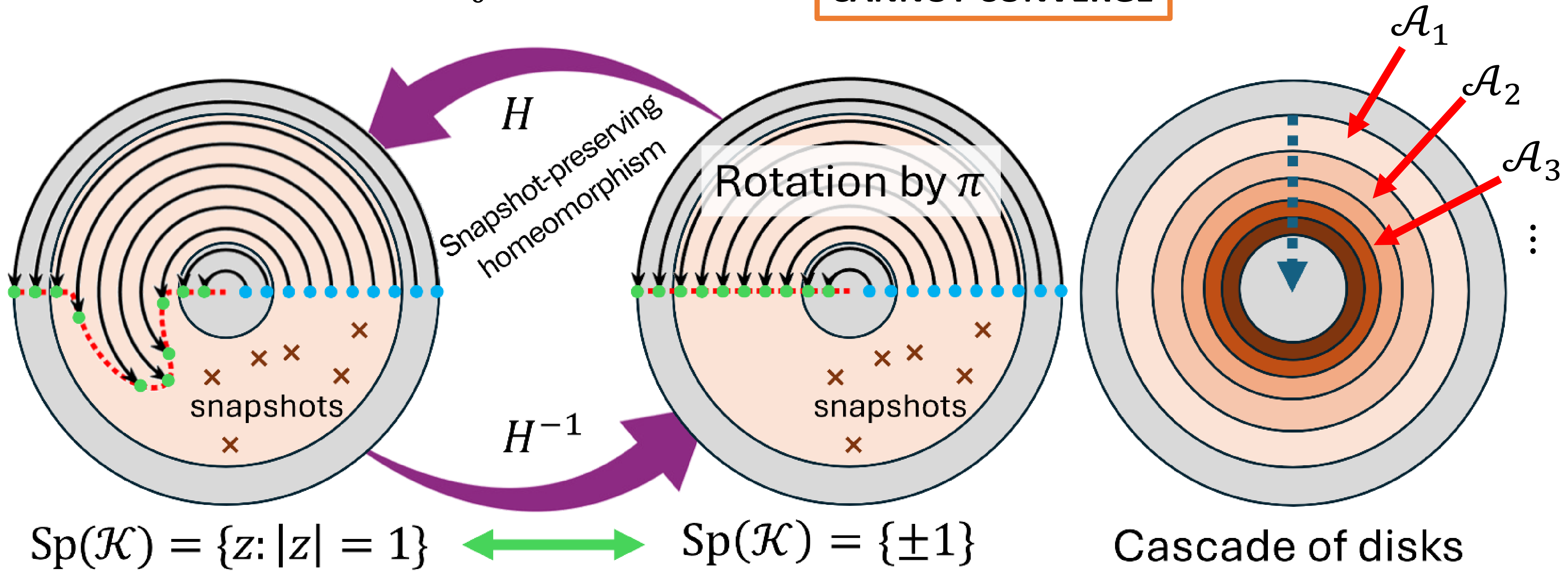
# Proof idea: Constructing an adversary

**Inductive step:** Repeat on annuli, $F_k = H_k^{-1} \circ F_0 \circ H_k$ on $\mathcal{A}_k$. $F = \lim_{k\to\infty} F_k$

Consistent data $\Rightarrow \Gamma_{n_k}(F) = \Gamma_{n_k}(\widetilde{F_k})$, $\mathrm{dist}(i, \Gamma_{n_k}(F)) \leq 1$, $n_k \to \infty$

**BUT** $\mathrm{Sp}(\mathcal{K}_F) = \mathrm{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$

**CANNOT CONVERGE**



# Theorem B (possibility)

$\Omega_{\mathcal{X}}^{m} = \{F: \mathcal{X} \to \mathcal{X} \mid F \text{ cts, measure preserving}\}.$
$\mathcal{T}_F = \{(x, y_m) | x \in \mathcal{X}, \|F(x) - y_m\| \leq 2^{-m}\}.$

**Theorem B:** There **exists** *deterministic* algorithms $\{\Gamma_{n_2,n_1}\}$ using input data $\mathcal{T}_F$ such that $\lim_{n_2\to\infty}\lim_{n_1\to\infty}\Gamma_{n_2,n_1}(F) = \mathrm{Sp}(\mathcal{K}_F)\ \ \forall F \in \Omega_{\mathcal{X}}^{m}$.

**Double limit** $\lim_{n_2\to\infty}\lim_{n_1\to\infty}$

# Proof sketch

- $\lim_{N\to\infty}\lim_{M\to\infty}\gamma_{N,M}(F,z) = \|(\mathcal{K}_F - zI)^{-1}\|^{-1}.$

**$\boldsymbol{N}$ = size of basis, $\boldsymbol{M}$ = amount of data (quadrature).**

# Proof sketch

- $\lim_{N\to\infty}\lim_{M\to\infty}\gamma_{N,M}(F,z) = \|(\mathcal{K}_F - zI)^{-1}\|^{-1}$.

  **$N$ = size of basis, $M$ = amount of data (quadrature).**

- Measure-preserving $\Rightarrow \|(\mathcal{K}_F - zI)^{-1}\|^{-1} = \mathrm{dist}(z, \mathrm{Sp}(\mathcal{K}_F))$.

# Proof sketch

- $\lim_{N\to\infty}\lim_{M\to\infty}\gamma_{N,M}(F,z) = \|(\mathcal{K}_F - zI)^{-1}\|^{-1}$.

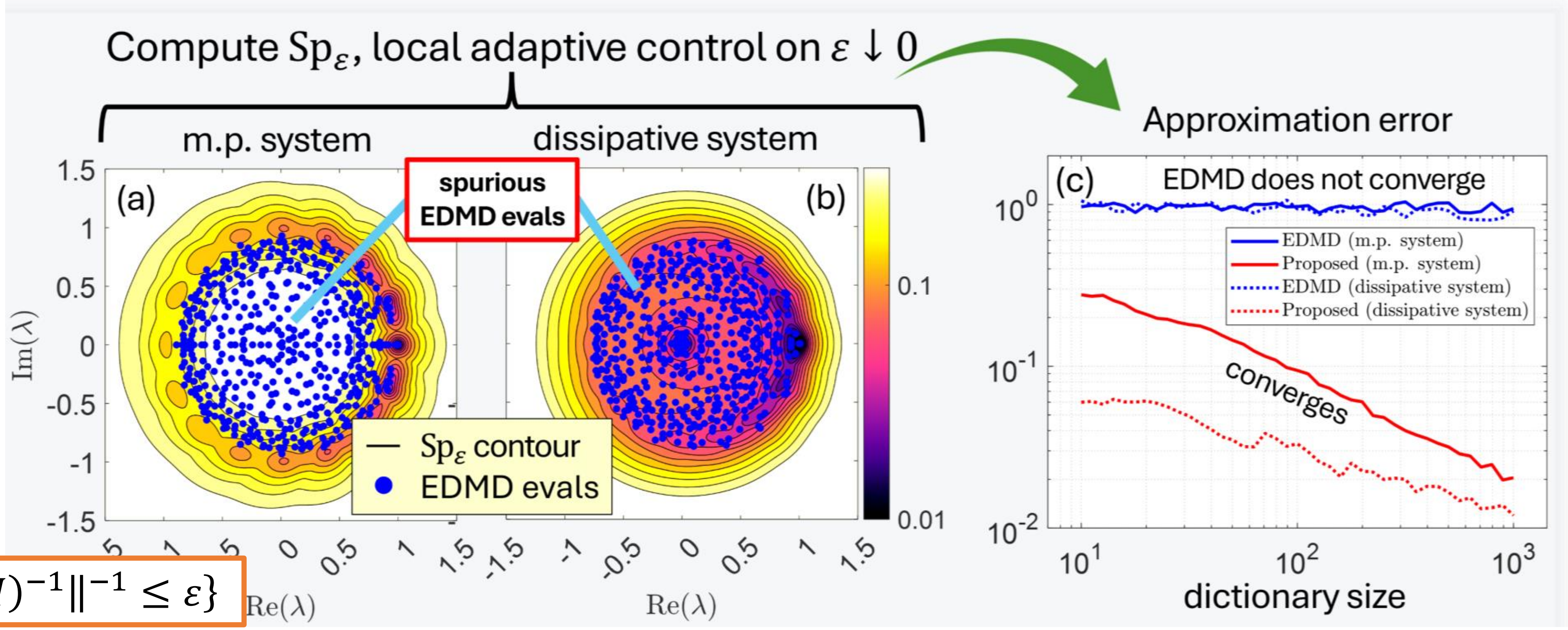
  **$\boldsymbol{N}$ = size of basis, $\boldsymbol{M}$ = amount of data (quadrature).**
- Measure-preserving $\Rightarrow \|(\mathcal{K}_F - zI)^{-1}\|^{-1} = \mathrm{dist}(z, \mathrm{Sp}(\mathcal{K}_F))$.
- Local $N$-adaptive minimisation of $\gamma_{N,M}(F,z)$ to approximate $\mathrm{Sp}(\mathcal{K}_F)$

# Proof sketch

- $\lim_{N\to\infty}\lim_{M\to\infty}\gamma_{N,M}(F,z) = \|(\mathcal{K}_F - zI)^{-1}\|^{-1}$.

  **$N$ = size of basis, $M$ = amount of data (quadrature).**
- Measure-preserving $\Rightarrow \|(\mathcal{K}_F - zI)^{-1}\|^{-1} = \mathrm{dist}(z, \mathrm{Sp}(\mathcal{K}_F))$.
- Local $N$-adaptive minimisation of $\gamma_{N,M}(F,z)$ to approximate $\mathrm{Sp}(\mathcal{K}_F)$

"Extended Dynamic Mode Decomposition" is the gold-standard Galerkin method.




$\mathrm{Sp}_\varepsilon(\mathcal{K}_F) = \{z \in \mathbb{C} \mid \|(\mathcal{K}_F - zI)^{-1}\|^{-1} \le \varepsilon\}$

# Limits of limits: Towers of algorithms

**Def:** $\{\Gamma_{n_k,\ldots,n_1}\}$ with $\lim_{n_k\to\infty} \ldots \lim_{n_1\to\infty} \Gamma_{n_k,\ldots,n_1}$ convergent a ***tower of algorithms.***

algorithms

First appeared in dynamical systems theory:

Steve Smale

"Is there any purely iterative convergent rational map for polynomial zero finding?"

Curtis McMullen

"Yes for cubic, no for higher degree. Quartic and quintic can be solved using towers of algorithms. Sextic cannot be solved in any number of limits."


- Smale, *"On the efficiency of algorithms of analysis."* **Bull. Am. Math. Soc.**, 1985.
- McMullen, *"Families of rational maps and iterative root-finding algorithms."* **Annals Math.**, 1987.
- McMullen, *"Braiding of the attractor and the failure of iterative algorithms."* **Invent. Math.** 1988.
- Doyle, McMullen, *"Solving the quintic by iteration."* **Acta Math.**, 1989.

# Classifications: *Solvability Complexity Index (SCI)*

**SCI:** Fewest number of limits needed to solve a computational problem.


- Hansen, "On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators." **J. Am. Math. Soc.**, 2011.
- C., "*The foundations of infinite-dimensional spectral computations,*" **PhD diss.**, University of Cambridge, 2020.
- C., Hansen, "*The foundations of spectral computations via the solvability complexity index hierarchy,*" **J. Eur. Math. Soc.**, 2022.
- C., Antun, Hansen, "*The difficulty of computing stable and accurate neural networks,*" **Proc. Natl. Acad. Sci. USA**, 2022.
- Ben-Artzi, C., Hansen, Nevanlinna, Seidel, "*On the solvability complexity index hierarchy and towers of algorithms,*" arXiv, 2020.

# Classifications: *Solvability Complexity Index (SCI)*

**SCI:** Fewest number of limits needed to solve a computational problem.

**Theorem A:** $\text{SCI} > 1$

**Theorem B:** $\text{SCI} \leq 2$

combine

$\text{SCI} = 2$


- Hansen, "On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators." **J. Am. Math. Soc.**, 2011.
- C., "*The foundations of infinite-dimensional spectral computations,*" **PhD diss.**, University of Cambridge, 2020.
- C., Hansen, "*The foundations of spectral computations via the solvability complexity index hierarchy,*" **J. Eur. Math. Soc.**, 2022.
- C., Antun, Hansen, "*The difficulty of computing stable and accurate neural networks,*" **Proc. Natl. Acad. Sci. USA**, 2022.
- Ben-Artzi, C., Hansen, Nevanlinna, Seidel, "*On the solvability complexity index hierarchy and towers of algorithms,*" arXiv, 2020.

# Classifications: *Solvability Complexity Index (SCI)*

**SCI:** Fewest number of limits needed to solve a computational problem.

**Theorem A:** SCI $> 1$

**Theorem B:** SCI $\leq 2$

combine

SCI $= 2$

So far literature has only proven upper bounds, that need not be sharp…


- Hansen, "On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators." **J. Am. Math. Soc.**, 2011.
- C., "*The foundations of infinite-dimensional spectral computations*," **PhD diss.**, University of Cambridge, 2020.
- C., Hansen, "*The foundations of spectral computations via the solvability complexity index hierarchy*," **J. Eur. Math. Soc.**, 2022.
- C., Antun, Hansen, "*The difficulty of computing stable and accurate neural networks*," **Proc. Natl. Acad. Sci. USA**, 2022.
- Ben-Artzi, C., Hansen, Nevanlinna, Seidel, "*On the solvability complexity index hierarchy and towers of algorithms*," arXiv, 2020.

# Results from Koopman literature

**SCI:** Fewest number of limits needed to solve a computational problem.

| Algorithm | Comments/Assumptions | Spectral Problem's Corresponding SCI Upper Bound | | | |
|---|---|---|---|---|---|
| | | *KMD* | *Spectrum* | *Spectral Measure (if m.p.)* | *Spectral Type (if m.p.)* |
| Extended DMD [47] | general $L^2$ spaces | $\mathrm{SCI} \leq 2^*$ | N/C | N/C | n/a |
| Residual DMD [44] | general $L^2$ spaces | $\mathrm{SCI} \leq 2^*$ | $\mathrm{SCI} \leq 3^*$ | $\mathrm{SCI} \leq 2^*$ | varies, see [84]<br>e.g., a.c. density: $\mathrm{SCI} \leq 2^*$ |
| Measure-preserving EDMD [45] | m.p. systems | $\mathrm{SCI} \leq 1$ | N/C | $\mathrm{SCI} \leq 2^*$ (general)<br>$\mathrm{SCI} \leq 1$ (delay-embedding) | n/a |
| Hankel DMD [85] | m.p. ergodic systems | $\mathrm{SCI} \leq 2^*$ | N/C | N/C | n/a |
| Periodic approximations [86] | m.p. + $\omega$ a.c. | $\mathrm{SCI} \leq 2$ | N/C | $\mathrm{SCI} \leq 2$ (see [87]) | a.c. density: $\mathrm{SCI} \leq 3$ |
| Christoffel–Darboux kernel [40] | m.p. ergodic systems | $\mathrm{SCI} \leq 3$ | n/a | $\mathrm{SCI} \leq 2$ | e.g., a.c. density: $\mathrm{SCI} \leq 2$ |
| Generator EDMD [88] | cts.-time, samples $\nabla F$<br>(otherwise additional limit) | $\mathrm{SCI} \leq 2$ | N/C | $\mathrm{SCI} \leq 2$ (see [89]) | n/a |
| Compactification [42] | cts.-time, m.p. ergodic systems | $\mathrm{SCI} \leq 4$ | N/C | $\mathrm{SCI} \leq 4$ | n/a |
| Resolvent compactification [43] | cts.-time, m.p. ergodic systems | $\mathrm{SCI} \leq 5$ | N/C | $\mathrm{SCI} \leq 5$ | n/a |
| Diffusion maps [90] (see also [10]) | cts.-time, m.p. ergodic systems | $\mathrm{SCI} \leq 3$ | n/a | n/a | n/a |

**Are these sharp?**

**Previous techniques prove upper bounds on SCI.**
"N/C": method need not converge. "n/a": algorithm not applicable to problem.

Also in Ulam's method for Markov processes, SRB measure computation, control,…

# Classifications: *Solvability Complexity Index (SCI)*

**SCI:** Fewest number of limits needed to solve a computational problem.

- $\Delta_1$: One limit, full error control. E.g., $d(\Gamma_n(F), \mathrm{Sp}(\mathcal{K}_F)) \leq 2^{-n}$.
- $\Delta_{m+1}$: SCI $\leq m$.
- $\Sigma_m$: SCI $\leq m$, final limit from below.
  E.g., $\Sigma_1$: $\Gamma_n(F) \subset \mathrm{Sp}(\mathcal{K}_F) + B_{2^{-n}}(0)$.
- $\Pi_m$: SCI $\leq m$, final limit from above.
  E.g., $\Pi_1$: $\mathrm{Sp}(\mathcal{K}_F) \subset \Gamma_n(F) + B_{2^{-n}}(0)$.


- Hansen, “On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators.” **J. Am. Math. Soc.**, 2011.
- C., “*The foundations of infinite-dimensional spectral computations,*” **PhD diss.**, University of Cambridge, 2020.
- C., Hansen, “*The foundations of spectral computations via the solvability complexity index hierarchy,*” **J. Eur. Math. Soc.**, 2022.
- C., Antun, Hansen, "*The difficulty of computing stable and accurate neural networks,*" **Proc. Natl. Acad. Sci. USA**, 2022.
- Ben-Artzi, C., Hansen, Nevanlinna, Seidel, “*On the solvability complexity index hierarchy and towers of algorithms,*” arXiv, 2020.

# Classifications: *Solvability Complexity Index (SCI)*

**SCI:** Fewest number of limits needed to solve a computational problem.

- $\Delta_1$: One limit, full error control. E.g., $d(\Gamma_n(F), \mathrm{Sp}(\mathcal{K}_F)) \leq 2^{-n}$.
- $\Delta_{m+1}$: SCI $\leq m$.
- $\Sigma_m$: SCI $\leq m$, final limit from below.
  E.g., $\Sigma_1$: $\Gamma_n(F) \subset \mathrm{Sp}(\mathcal{K}_F) + B_{2^{-n}}(0)$.
- $\Pi_m$: SCI $\leq m$, final limit from above.
  E.g., $\Pi_1$: $\mathrm{Sp}(\mathcal{K}_F) \subset \Gamma_n(F) + B_{2^{-n}}(0)$.

**verification**

**trust output**

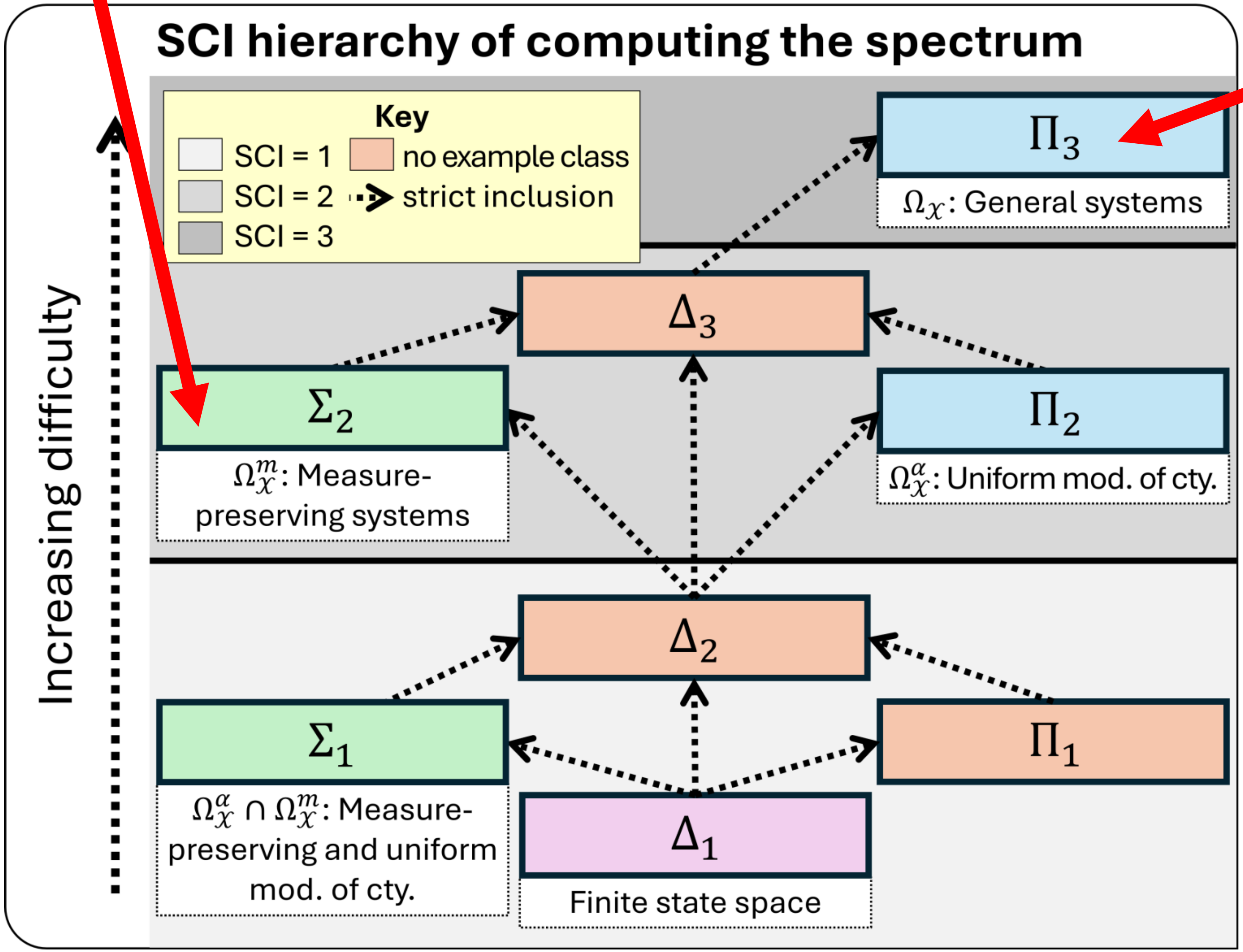
**covers spectrum**


- Hansen, "On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators." **J. Am. Math. Soc.**, 2011.
- C., "*The foundations of infinite-dimensional spectral computations,*" **PhD diss.**, University of Cambridge, 2020.
- C., Hansen, "*The foundations of spectral computations via the solvability complexity index hierarchy,*" **J. Eur. Math. Soc.**, 2022.
- C., Antun, Hansen, "*The difficulty of computing stable and accurate neural networks,*" **Proc. Natl. Acad. Sci. USA**, 2022.
- Ben-Artzi, C., Hansen, Nevanlinna, Seidel, "*On the solvability complexity index hierarchy and towers of algorithms,*" arXiv, 2020.

# Classification for Koopman I

**Theorems A + B**

**3 limits needed in general!**




**Different classes:**

$$\Omega_\mathcal{X} = \{F: \mathcal{X} \to \mathcal{X} \mid F \text{ cts}\}$$

$$\Omega_\mathcal{X}^m = \{F: \mathcal{X} \to \mathcal{X} \mid F \text{ cts, m.p.}\}$$

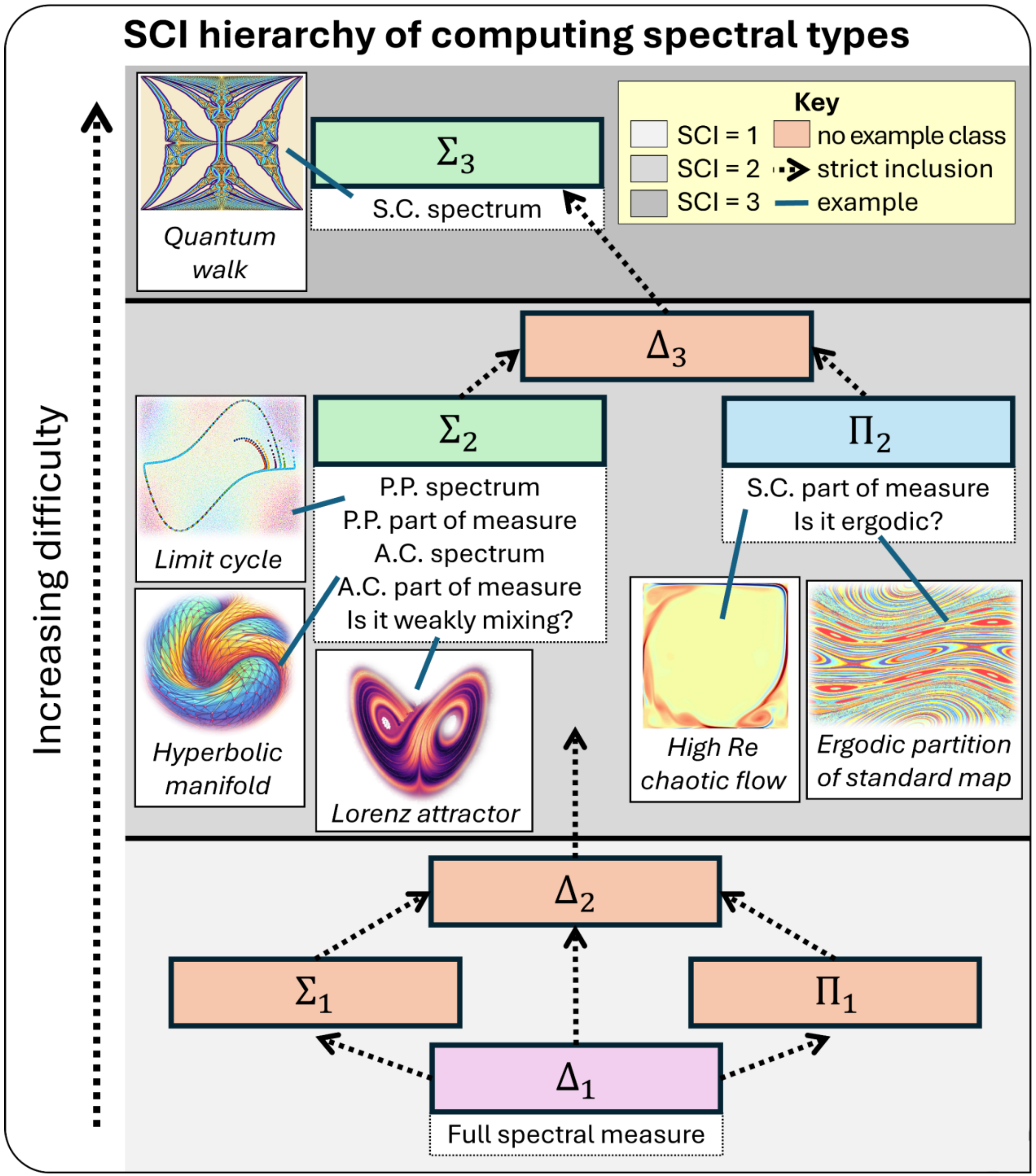
$$\Omega_\mathcal{X}^\alpha = \{F: \mathcal{X} \to \mathcal{X} \mid F \text{ mod. cty. } \alpha\}$$

$$[d_\mathcal{X}(F(x), F(y)) \leq \alpha(d_\mathcal{X}(x,y))]$$

**Optimal algorithms and classifications of dynamical systems.**

# Classification for Koopman II

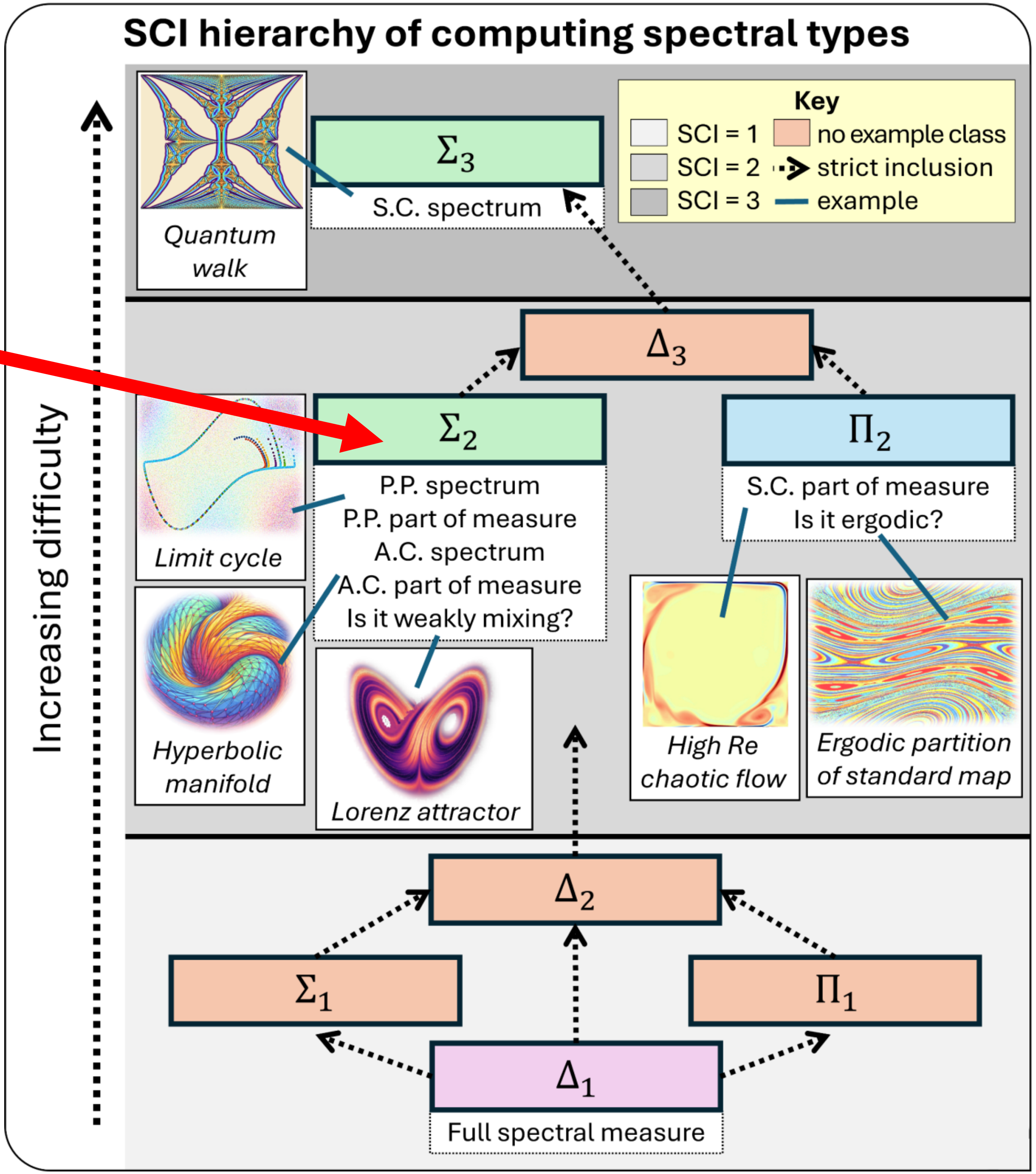




# Classification for Koopman II

**Example: Theorem C**
For smooth, measure-preserving systems on a torus, learning eigenfunctions or even determining if there are any has SCI $= 2$ (even if we can sample derivatives).

Finding finite-dimensional embeddings in which the dynamics are linear (e.g., autoencoders, latent space representation) is very hard!



# General tool in data-driven dynamical systems/PDEs

Adversarial arguments generalize to:

- "Learning the $F$". E.g., SINDy $(x_{n+1} = F(x_n))$
- Solving PDEs with neural networks (PINNs)
- Learning PDEs from forcing-solution pairs (e.g., hyperbolic)


- Brunton, Proctor, Kutz, "*Discovering governing equations from data by sparse identification of nonlinear dynamical systems*," **Proc. Natl. Acad. Sci. USA**, 2016.
- Karniadakis, Kevrekidis, Lu, Perdikaris, Wang, Yang, "*Physics-informed machine learning*," **Nature Reviews Physics**, 2021.
- Boulle, Halikias, Townsend, "*Elliptic PDE learning is provably data-efficient*," **Proc. Natl. Acad. Sci. USA**, 2023.

# Where does this leave us?

- Many problems **NECESSARILY** require multiple limits.
- New tools for **lower bounds** (impossibility results) for Koopman learning.
- Combine with **upper bounds** (algorithms)
  $\Rightarrow$ **classify difficulty** of problems + **prove optimality** of algorithms.
- Ergodic theory + approximation theory + computational analysis
  $\Rightarrow$ started to map out this terrain.

# Where does this leave us?

- Many problems **NECESSARILY** require multiple limits.
- New tools for **lower bounds** (impossibility results) for Koopman learning.
- Combine with **upper bounds** (algorithms)
  $\Rightarrow$ **classify difficulty** of problems + **prove optimality** of algorithms.
- Ergodic theory + approximation theory + computational analysis
  $\Rightarrow$ started to map out this terrain.
- Future work:
  - Other function spaces.
  - Partial observations, continuous-time.
  - Control and uses of Koopman.
  - Other data-driven dynamical system methods.

# Where does this leave us?

- Many problems **NECESSARILY** require multiple limits.
- New tools for **lower bounds** (impossibility results) for Koopman learning.
- Combine with **upper bounds** (algorithms)
  $\Rightarrow$ **classify difficulty** of problems + **prove optimality** of algorithms.
- Ergodic theory + approximation theory + computational analysis
  $\Rightarrow$ started to map out this terrain.
- Future work:
  - Other function spaces.
  - Partial observations, continuous-time.
  - Control and uses of Koopman.
  - Other data-driven dynamical system methods.

**Where does your problem/method fit into the SCI hierarchy? Is it optimal?**

# References

[1] Colbrook, Matthew J., and Alex Townsend. "Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems." Communications on Pure and Applied Mathematics 77.1 (2024): 221-283.

[2] Colbrook, Matthew J., Lorna J. Ayton, and Máté Szőke. "Residual dynamic mode decomposition: robust and verified Koopmanism." Journal of Fluid Mechanics 955 (2023): A21.

[3] Colbrook, M. J., Li, Q., Raut, R. V., & Townsend, A. "Beyond expectations: residual dynamic mode decomposition and variance for stochastic dynamical systems." Nonlinear Dynamics 112.3 (2024): 2037-2061.

[4] Colbrook, Matthew J. "The Multiverse of Dynamic Mode Decomposition Algorithms." arXiv preprint arXiv:2312.00137 (2023).

[5] Colbrook, Matthew J. "The mpEDMD algorithm for data-driven computations of measure-preserving dynamical systems." SIAM Journal on Numerical Analysis 61.3 (2023): 1585-1608.

[6] Colbrook, Matthew J., Catherine Drysdale, and Andrew Horning. "Rigged Dynamic Mode Decomposition: Data-Driven Generalized Eigenfunction Decompositions for Koopman Operators." arXiv preprint arXiv:2405.00782 (2024).

[7] Boullé, Nicolas, and Matthew J. Colbrook. "Multiplicative Dynamic Mode Decomposition." arXiv preprint arXiv:2405.05334 (2024).

[8] Colbrook, Matthew J. "Another look at Residual Dynamic Mode Decomposition in the regime of fewer Snapshots than Dictionary Size." Physica D: Nonlinear Phenomena 469 (2024).

[9] Colbrook, Matthew. “The foundations of infinite-dimensional spectral computations.” Diss. University of Cambridge, 2020.

[10] Ben-Artzi, J., Colbrook, M. J., Hansen, A. C., Nevanlinna, O., & Seidel, M. (2020). “Computing Spectra--On the Solvability Complexity Index Hierarchy and Towers of Algorithms.” arXiv preprint arXiv:1508.03280.

[11] Colbrook, Matthew J., Vegard Antun, and Anders C. Hansen. "The difficulty of computing stable and accurate neural networks: On the barriers of deep learning and Smale’s 18th problem." Proceedings of the National Academy of Sciences 119.12 (2022): e2107151119.

[12] Colbrook, Matthew, Andrew Horning, and Alex Townsend. "Computing spectral measures of self-adjoint operators." SIAM review 63.3 (2021): 489-524.

[13] Colbrook, Matthew J., Bogdan Roman, and Anders C. Hansen. "How to compute spectra with error control." Physical Review Letters 122.25 (2019): 250201.

[14] Colbrook, Matthew J., and Anders C. Hansen. "The foundations of spectral computations via the solvability complexity index hierarchy." Journal of the European Mathematical Society (2022).

[15] Colbrook, Matthew J. "Computing spectral measures and spectral types." Communications in Mathematical Physics 384 (2021): 433-501.

[16] Colbrook, Matthew J., and Anders C. Hansen. "On the infinite-dimensional QR algorithm." Numerische Mathematik 143 (2019): 17-83.

[17] Colbrook, Matthew J. "On the computation of geometric features of spectra of linear operators on Hilbert spaces." Foundations of Computational Mathematics (2022): 1-82.

[18] Brunton, Steven L., and Matthew J. Colbrook. "Resilient Data-driven Dynamical Systems with Koopman: An Infinite-dimensional Numerical Analysis Perspective.“

[19] Colbrook, Matthew J., Igor Mezić, and Alexei Stepanenko. "Limits and Powers of Koopman Learning.“ arXiv preprint arxiv:2407.06312 (2024).