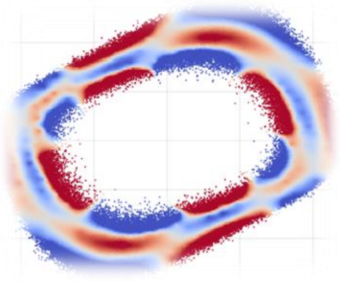
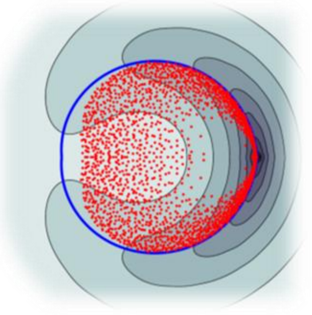


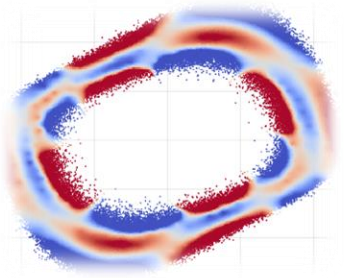


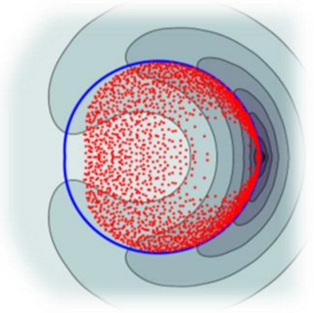
Can we compute spectral properties of (infinite-dimensional) Koopman operators?

Solves the problem of computing spectra of general Koopman operators on L^2 **spaces** and controlling projection error of inf dim → fin dim. (EDMD does **not** converge in general)

Matthew Colbrook
University of Cambridge
20/05/2024

- C., Townsend, *"Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems"* **Communications on Pure and Applied Mathematics**, 2024.
- C., *"The mpEDMD Algorithm for Data-Driven Computations of Measure-Preserving Dynamical Systems,"* **SIAM Journal on Numerical Analysis**, 2023.
- C., Drysdale, Horning, *"Rigged Dynamic Mode Decomposition: Data-Driven Generalized Eigenfunction Decompositions for Koopman Operators"*, arxiv preprint.
- C., *"The Multiverse of Dynamic Mode Decomposition Algorithms,"* **Handbook of Numerical Analysis**, 2024.





Can we compute spectral properties of (infinite-dimensional) Koopman operators?

Unitary version of EDMD which converges (spectra, measures, KMD) for any measure-preserving system (EDMD does **not** converge in general)

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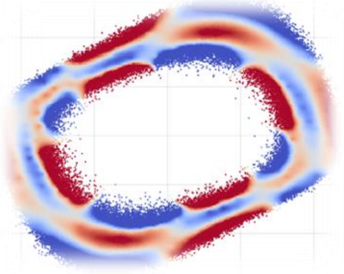

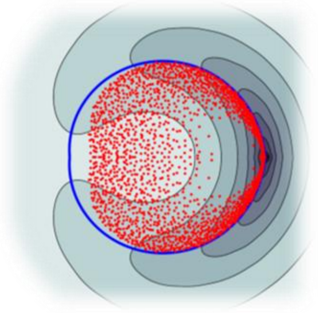

Can we compute spectral properties of (infinite-dimensional) Koopman operators?

Computes spectral measures and generalized eigenfunctions of measure-preserving systems.

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- C., Townsend, *"Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems"* **Communications on Pure and Applied Mathematics**, 2024.
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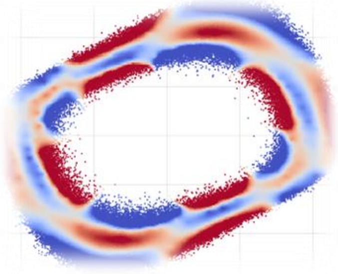

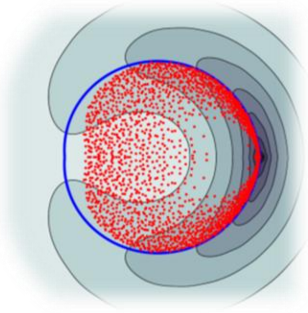

Can we compute spectral properties of (infinite-dimensional) Koopman operators?

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Recent review of DMD methods from a spectral computations point of view.

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- C., *"The mpEDMD Algorithm for Data-Driven Computations of Measure-Preserving Dynamical Systems,"* **SIAM Journal on Numerical Analysis**, 2023.
- C., Drysdale, Horning, *"Rigged Dynamic Mode Decomposition: Data-Driven Generalized Eigenfunction Decompositions for Koopman Operators"*, arxiv preprint.
- C., *"The Multiverse of Dynamic Mode Decomposition Algorithms,"* **Handbook of Numerical Analysis**, 2024.

The setup

State $x \in \Omega \subseteq \mathbb{R}^d$. ($\Omega$ does not need to be an attractor)

Unknown function $F: \Omega \to \Omega$ governs dynamics: $x_{n+1} = F(x_n)$.

Koopman operator: $\mathcal{K}$ acts on functions $g: \Omega \to \mathbb{C}$, $[\mathcal{K}g](x) = g(F(x))$.

Function space: $L^2(\Omega, \omega)$, positive measure ω, inner product $\langle \cdot, \cdot \rangle$.

Goal: Learn spectral properties from snapshot data $\{x^{(m)}, y^{(m)} = F(x^{(m)})\}_{m=1}^{M}$.

NB: No assumptions yet on snapshots or Ω or F or ω.
(Other than F non-singular w.r.t. ω, $\mathcal{K}$ a closed operator (can be unbounded).)

Warmup on $\ell^2(\mathbb{Z})$ - **** happens… even for measure-preserving systems

$$\begin{pmatrix} \ddots & \ddots & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \in \mathbb{C}^{N\times N}$$

Two-way infinite

- Spectrum is unit circle.
- Spectrum is stable.
- Continuous spectra.
- Unitary evolution.

- Spectrum is $\{0\}$.
- Spectrum is unstable.
- Discrete spectra.
- Nilpotent evolution.

Lots of Koopman operators are built up from operators like these!

Dangers when we truncate/discretize $\mathcal{K} \longrightarrow \mathbb{K} \in \mathbb{C}^{N\times N}$

$$\mathrm{Sp}(\mathcal{K}) = \{\lambda \in \mathbb{C} : \mathcal{K} - \lambda I \ \text{ is not invertible}\}$$

- **Too much:** Spurious eigenvalues $\lambda \in \mathrm{Sp}(\mathbb{K})$ far from $\mathrm{Sp}(\mathcal{K})$
- **Too little:** $\mathrm{Sp}(\mathbb{K})$ misses parts of $\mathrm{Sp}(\mathcal{K})$
- **Continuous spectra** ($\mathrm{Sp}(\mathcal{K})$ not just eigenvalues!)
- **Verification (e.g., subspace)**
- **Instability** (non-normal $\mathcal{K}$, non-normal discretizations of normal $\mathcal{K}$)

Methods like EDMD do not avoid these dangers as $N \to \infty$!

- C., Townsend, *"Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems,"* **Commun. Pure Appl. Math.**, 2024.

Outline

- General systems:
  - Residual Dynamic Mode Decomposition.
- Measure-preserving systems:
  - Rigged Dynamic Mode Decomposition
  - Measure-Preserving Extended Dynamic Mode Decomposition.
- The **Solvability Complexity Index** – *classification* of problems and *optimality* of algorithms.
- Where are we? Open questions and future research.

Outline

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Extended Dynamic Mode Decomposition (EDMD)

Functions $\psi_j : \Omega \to \mathbb{C}, j = 1, \dots, N$

$$\{x^{(m)}, y^{(m)} = F(x^{(m)})\}_{m=1}^{M}$$

- Schmid, "*Dynamic mode decomposition of numerical and experimental data,*" **J. Fluid Mech.**, 2010.
- Rowley, Mezić, Bagheri, Schlatter, Henningson, *"Spectral analysis of nonlinear flows,"* **J. Fluid Mech.**, 2009.
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Functions $\psi_j : \Omega \to \mathbb{C},\ j = 1, \dots, N$

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quadrature points

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j\left(x^{(m)}\right)} \psi_k\left(x^{(m)}\right) = \Bigg[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}^*}_{\Psi_X} \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_{W} \underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}}_{\Psi_X} \Bigg]_{jk}$$

quadrature weights

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j\left(x^{(m)}\right)} \underbrace{\psi_k\left(y^{(m)}\right)}_{[\mathcal{K}\psi_k](x^{(m)})} = \Bigg[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}^*}_{\Psi_X} \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_{W} \underbrace{\begin{pmatrix} \psi_1(y^{(1)}) & \cdots & \psi_N(y^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(y^{(M)}) & \cdots & \psi_N(y^{(M)}) \end{pmatrix}}_{\Psi_Y} \Bigg]_{jk}$$

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- Rowley, Mezić, Bagheri, Schlatter, Henningson, *"Spectral analysis of nonlinear flows,"* **J. Fluid Mech.**, 2009.
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Galerkin Approximation

$$\mathcal{K} \longrightarrow \mathbb{K} = (\Psi_X{}^* W \Psi_X)^{-1} \Psi_X{}^* W \Psi_Y = (\sqrt{W}\Psi_X)^{\dagger} \sqrt{W}\Psi_Y \in \mathbb{C}^{N \times N}$$

- Schmid, *"Dynamic mode decomposition of numerical and experimental data,"* **J. Fluid Mech.**, 2010.
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Residual DMD

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\, \psi_k\left(x^{(m)}\right) = \Big[\underbrace{\Psi_X{}^* W \Psi_X}_{G}\Big]_{jk}$$

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- C., Ayton, Szőke, "*Residual Dynamic Mode Decomposition,*" **J. Fluid Mech.**, 2023.
- Code*:* https://github.com/MColbrook/Residual-Dynamic-Mode-Decomposition

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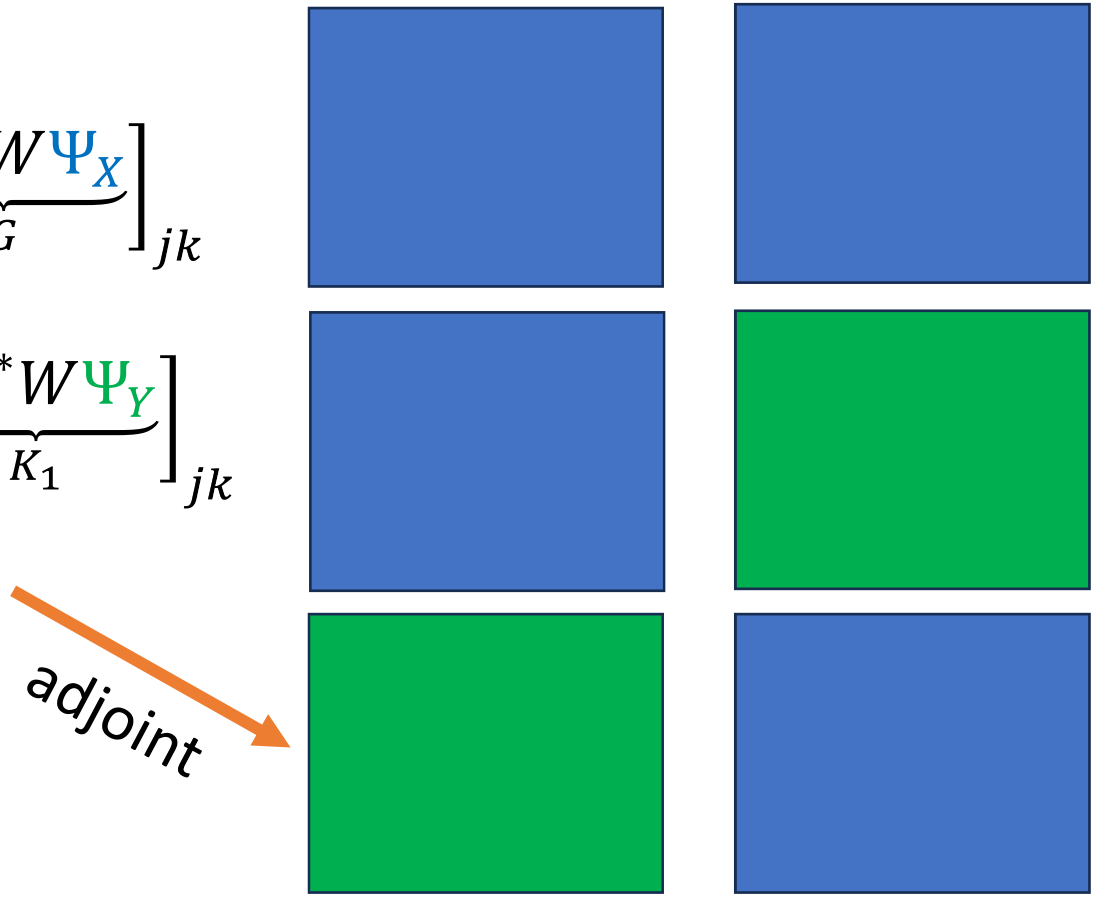

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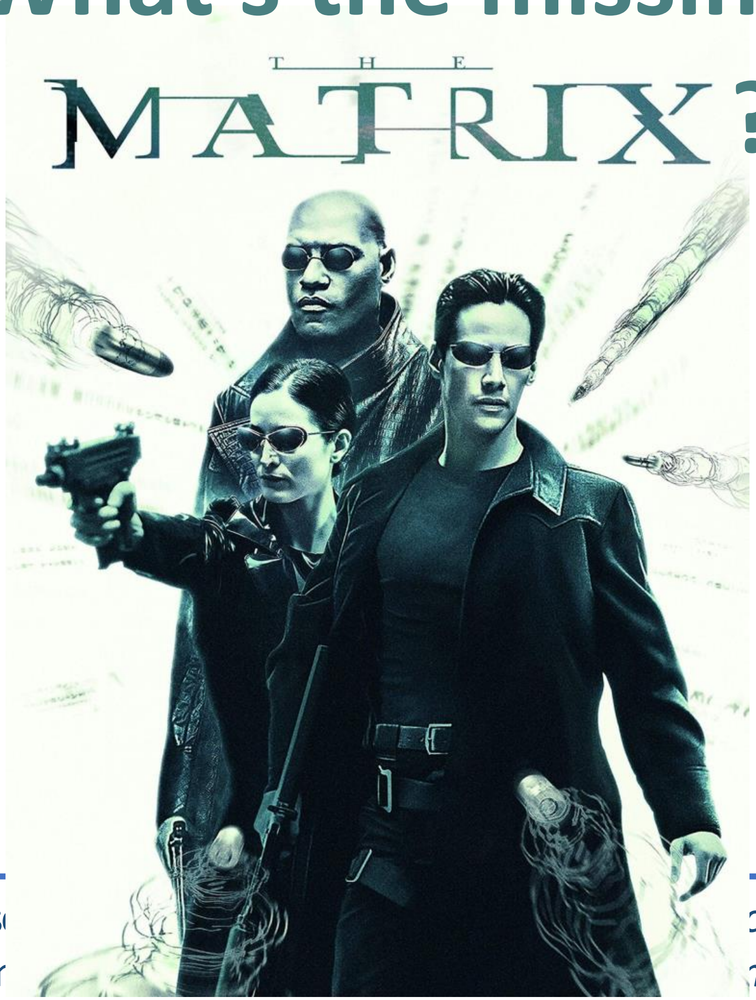
Residual DMD

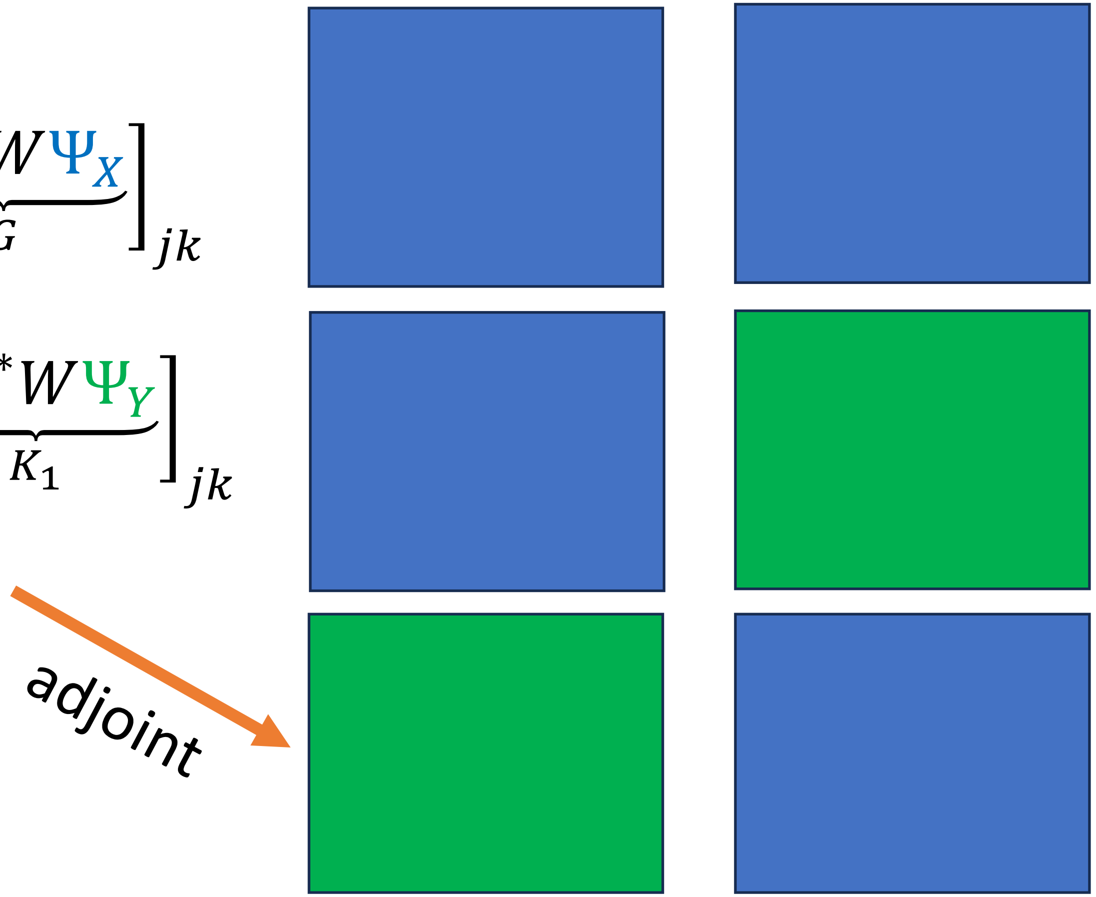
What’s the missing



?

$$= \Big[\underbrace{\Psi_X{}^{*} W \Psi_X}_{G}\Big]_{jk}$$

$$) \quad = \Big[\underbrace{\Psi_X{}^{*} W \Psi_Y}_{K_1}\Big]_{jk}$$



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Residuals: $g = \sum_{j=1}^{N} \mathbf{g}_j \psi_j,\ \|\mathcal{K}g - \lambda g\|^2 = \langle \mathcal{K}g - \lambda g, \mathcal{K}g - \lambda g \rangle$

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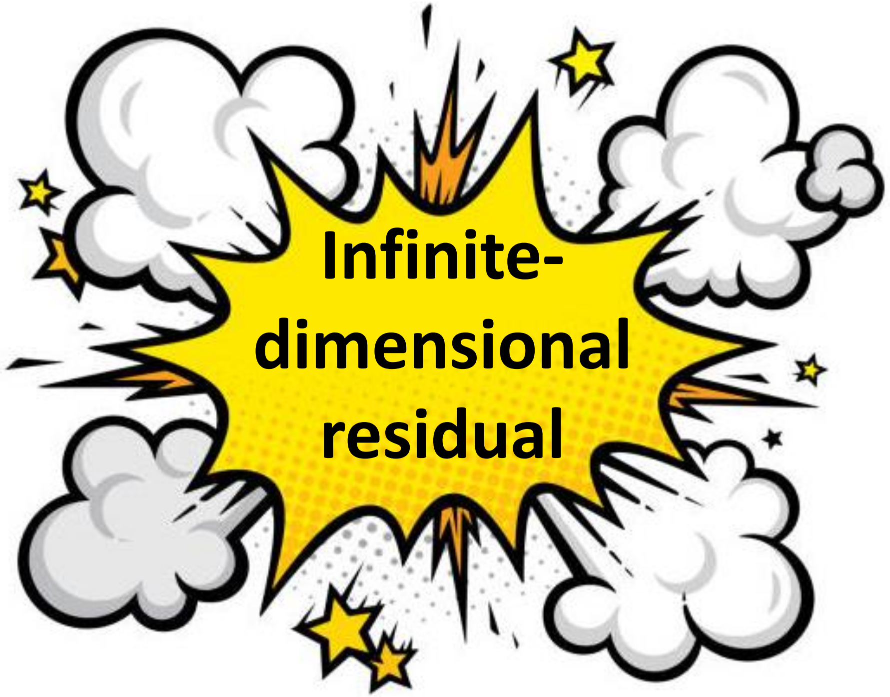

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ResDMD: Avoiding the dangers

If quadrature rule converges (ask me later)

Convergent methods for general $\mathcal{K}$

- **1 limit** $\lim_{M\to\infty}$: Avoid spurious eigenvalues.
- **2 limits** $\lim_{N\to\infty}\lim_{M\to\infty}$: Compute $\mathrm{Sp}_\varepsilon(\mathcal{K}) = \cup_{\|\mathcal{B}\|\leq\varepsilon}\mathrm{Sp}(\mathcal{K}+\mathcal{B})$.
- **3 limits** $\lim_{\varepsilon\downarrow 0}\lim_{N\to\infty}\lim_{M\to\infty}$: Compute $\mathrm{Sp}(\mathcal{K})$.
- Verification: dictionaries, approximate eigenfunctions, coherency,…
- Error bounds of forecasts.

M = number of snapshots
N = number of basis functions

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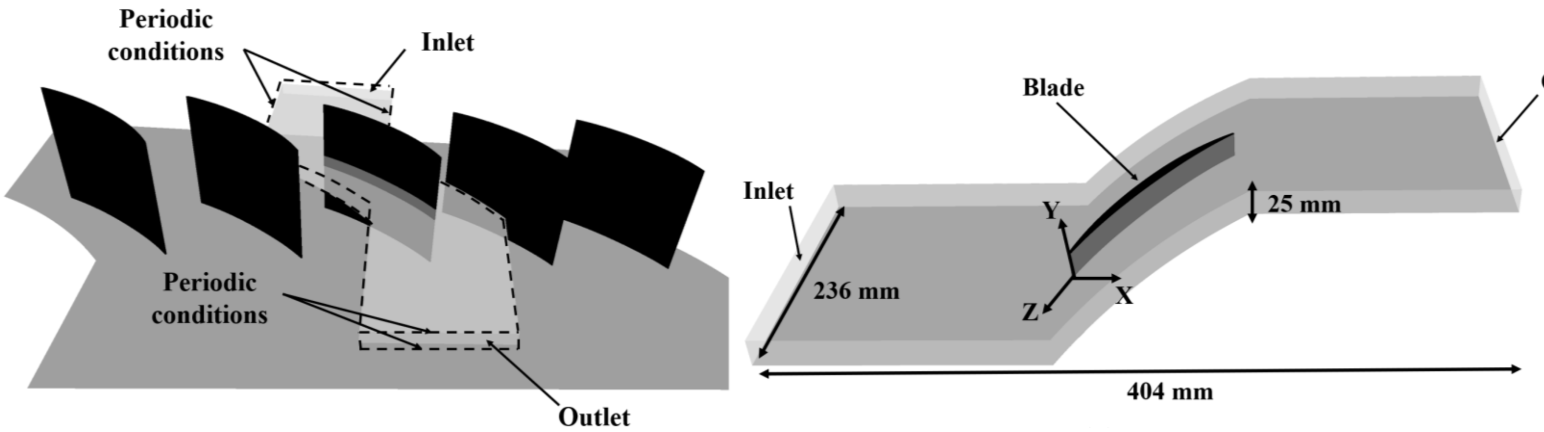
- Verification: dictionaries, approximate eigenfunctions, coherency,…
- Error bounds of forecasts.
- Extends to stochastic systems (+ variance through Koopman).

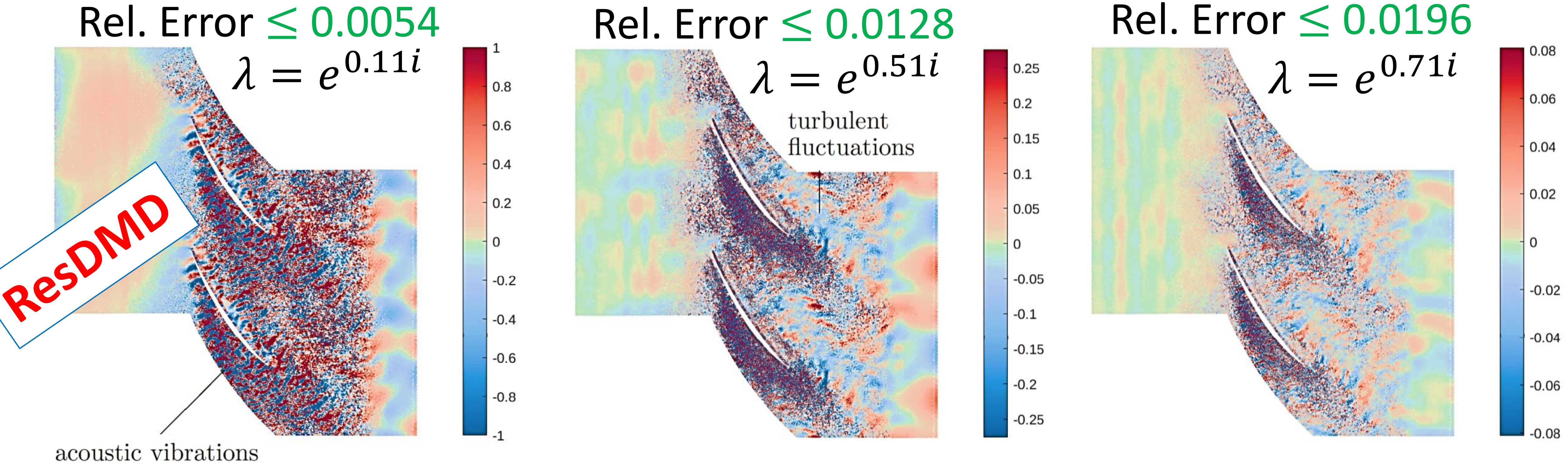
C., Li, Raut, Townsend, *"Beyond expectations: Residual Dynamic Mode Decomposition and Variance for Stochastic Dynamical Systems,"* **Nonlinear Dynamics**, 2024.

Example: Verified spectra and modes

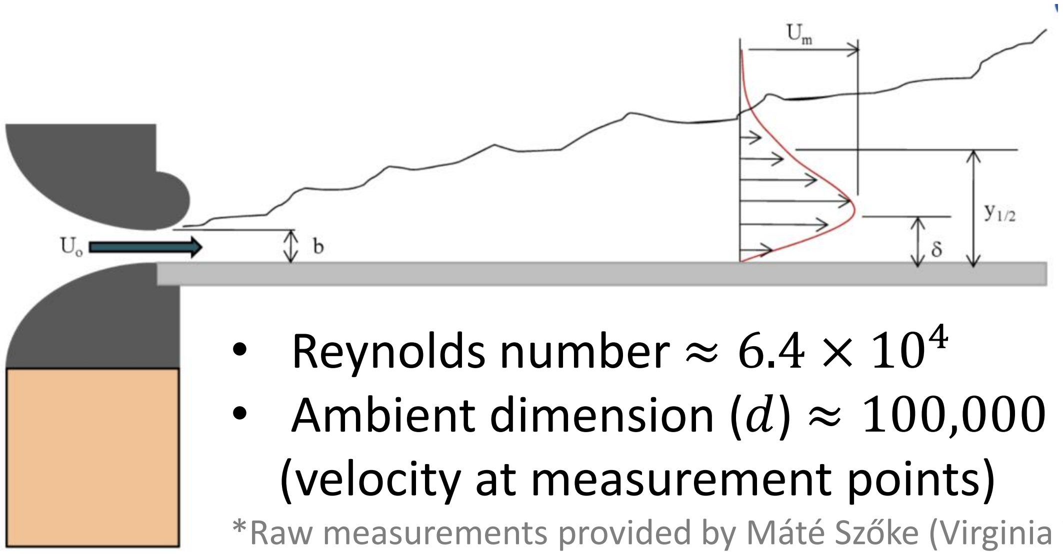


- Reynolds number $\approx 3.9\times 10^5$
- Ambient dimension (d) $\approx$ 300,000 (number of measurement points)

*Raw measurements provided by Stephane Moreau (Sherbrooke)

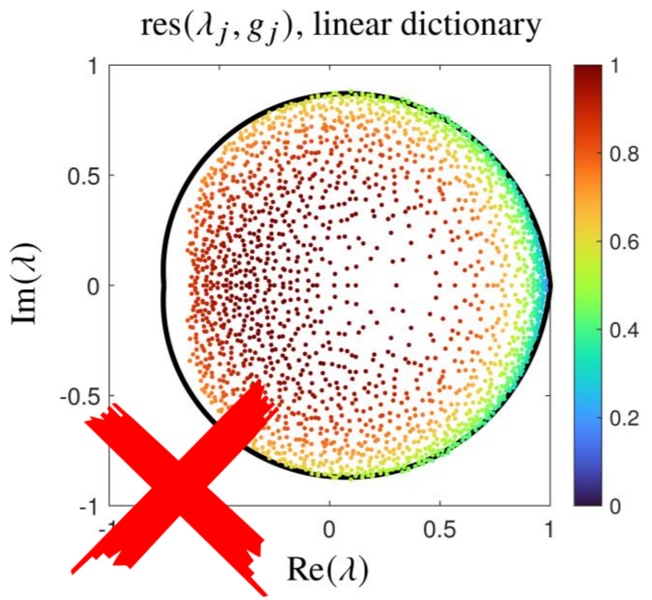


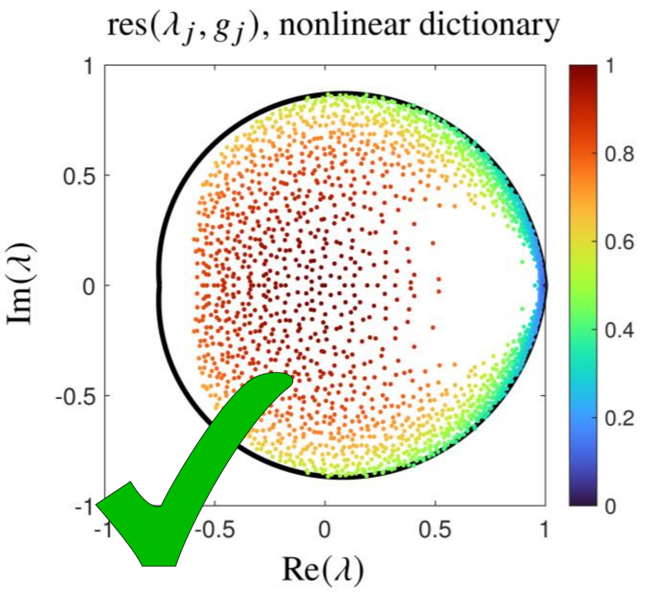
Example: Verified dictionary



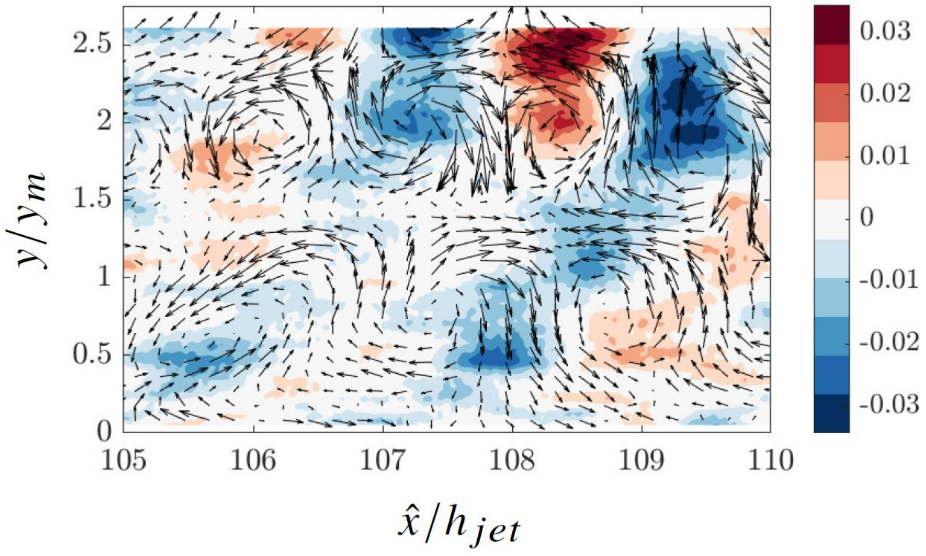
- Reynolds number $\approx 6.4 \times 10^4$
- Ambient dimension (d) $\approx$ 100,000 (velocity at measurement points)

*Raw measurements provided by Máté Szőke (Virginia Te

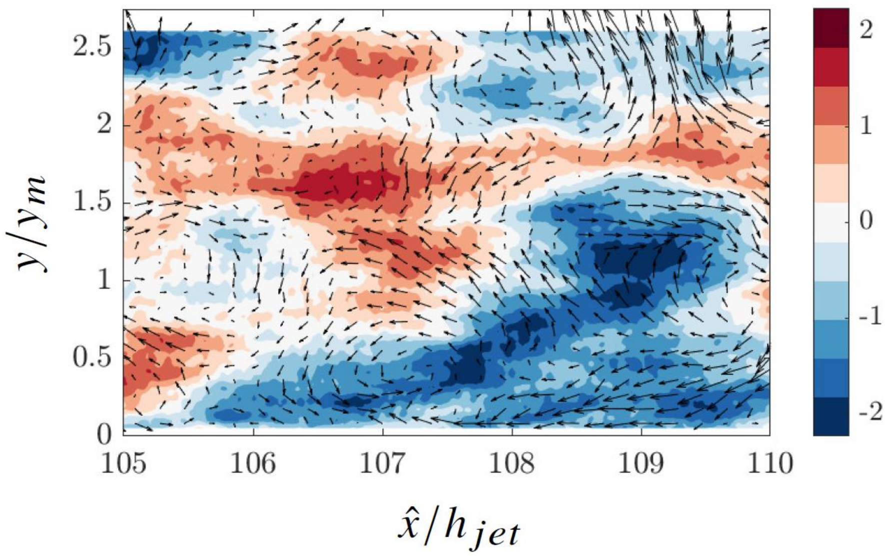




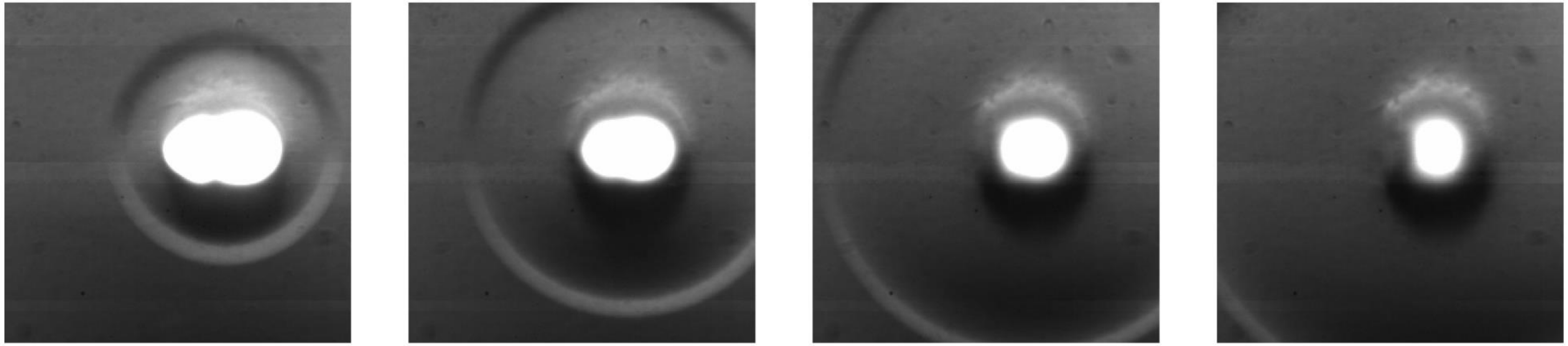
$\lambda = 0.9439 + 0.2458i$, error ≤ 0.0765



$\lambda = 0.8948 + 0.1065i$, error ≤ 0.1105

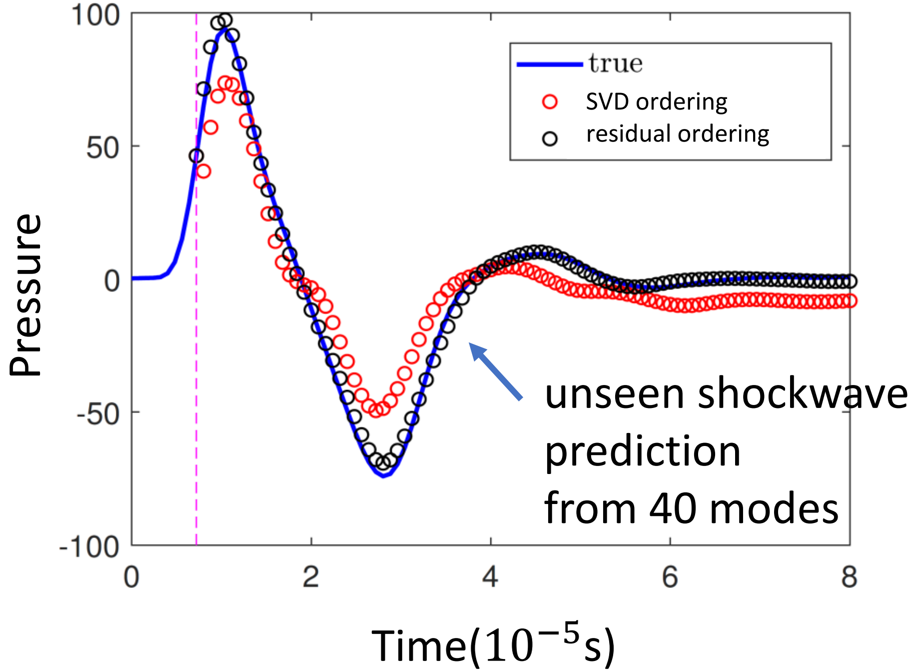


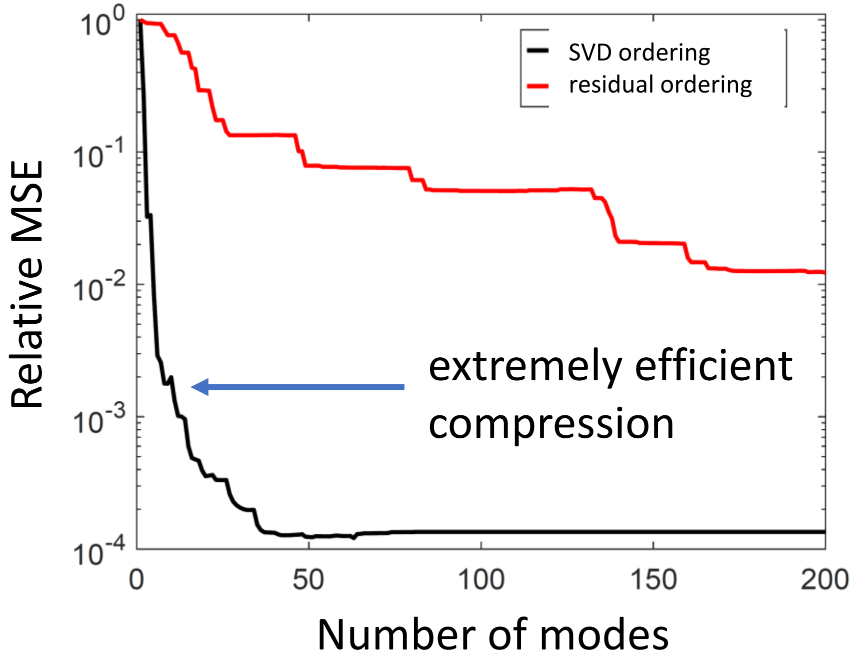
Example: Verified KMD and compression



a) $t = 5\ \mu s$ b) $t = 10\ \mu s$ c) $t = 15\ \mu s$ d) $t = 20\ \mu s$

Matt Szőke's laser cannon!





Outline

- General systems:
  - Residual Dynamic Mode Decomposition.
- Measure-preserving systems:
  - **Rigged Dynamic Mode Decomposition**
  - **Measure-Preserving Extended Dynamic Mode Decomposition.**
- The **Solvability Complexity Index** – *classification* of problems and *optimality* of algorithms.
- Where are we? Open questions and future research.

Measure-preserving systems

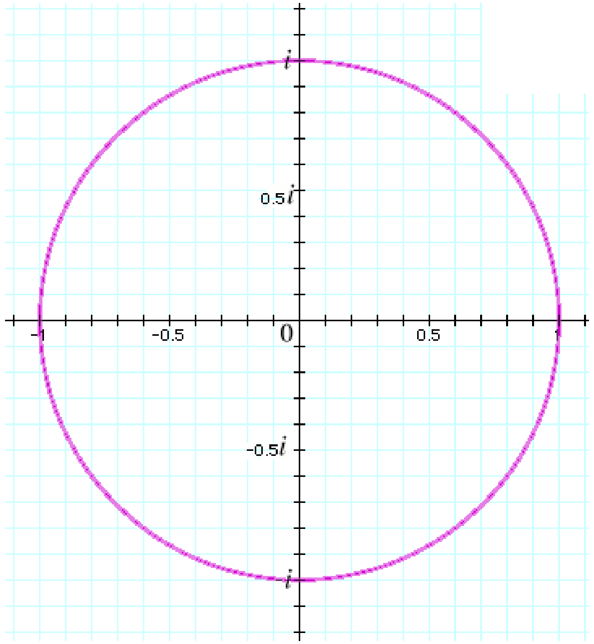
$$[\mathcal{K}g](x) = g(F(x)), \qquad g \in L^2(\Omega, \omega)$$

F preserves $\omega \Leftrightarrow \|\mathcal{K}g\| = \|g\|$ (isometry)

$\Leftrightarrow \mathcal{K}^*\mathcal{K} = I$

$\Rightarrow \mathrm{Sp}(\mathcal{K}) \subseteq \{z : |z| \leq 1\}$

(NB: unitary extensions of $\mathcal{K}$ via Wold decomposition.)



Problem: Often $\mathcal{K}$ doesn't have basis of eigenfunctions (i.e., **continuous spectra**)

Gelfand's theorem → diagonalisation

- **Finite matrix:** $B \in \mathbb{C}^{n\times n}$, $B^*B = BB^*$, orthonormal basis of e-vectors $\{v_j\}_{j=1}^{n}$

$$v = \sum_{j=1}^{n} (v_j{}^* v) v_j, \qquad Bv = \sum_{j=1}^{n} \lambda_j (v_j{}^* v) v_j \qquad \forall v \in \mathbb{C}^n$$

- **Infinite dimensions:** Unitary $\mathcal{K}$. Typically, **no basis of eigenfunctions**!
Some technical assumptions (can always be realized):

$$g = \int_{[-\pi,\pi]_{\mathrm{per}}} \langle g_\theta^* | g \rangle \, g_\theta \mathrm{d}\nu(\theta), \qquad \mathcal{K}g = \int_{[-\pi,\pi]_{\mathrm{per}}} e^{i\theta} \langle g_\theta^* | g \rangle \, g_\theta \mathrm{d}\nu(\theta)$$

$g \in S \subset L^2(\Omega, \omega)$

Koopman modes

generalized eigenfunctions
distributions $\in \mathcal{S}^*$

$e^{i\theta} = \lambda$

Koopman Mode Decomposition

Rigged DMD: Smoothing

Carathéodory function:

$$F_g(z) = (\mathcal{K} + zI)(\mathcal{K} - zI)^{-1}g = \int_{[-\pi,\pi]_{\mathrm{per}}} \frac{e^{i\theta} + z}{e^{i\theta} - z} \langle g_\theta^* | g \rangle \, g_\theta \, \mathrm{d}\nu(\theta)$$

Rigged DMD: Smoothing

Carathéodory function:

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Let $r = 1 + \varepsilon > 1, \theta_0 \in [-\pi, \pi]_{\mathrm{per}}$,

$$\frac{1}{4\pi}\left[F_g\left(r^{-1} e^{i\theta_0}\right) - F_g\left(r e^{i\theta_0}\right)\right]$$

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$$= \frac{1}{2\pi} \int_{[-\pi,\pi]_{\mathrm{per}}} \frac{r^2 - 1}{1 + r^2 - 2r\cos(\theta_0 - \theta)} \langle g_\theta^* | g \rangle \, g_\theta \, \mathrm{d}\nu(\theta)$$

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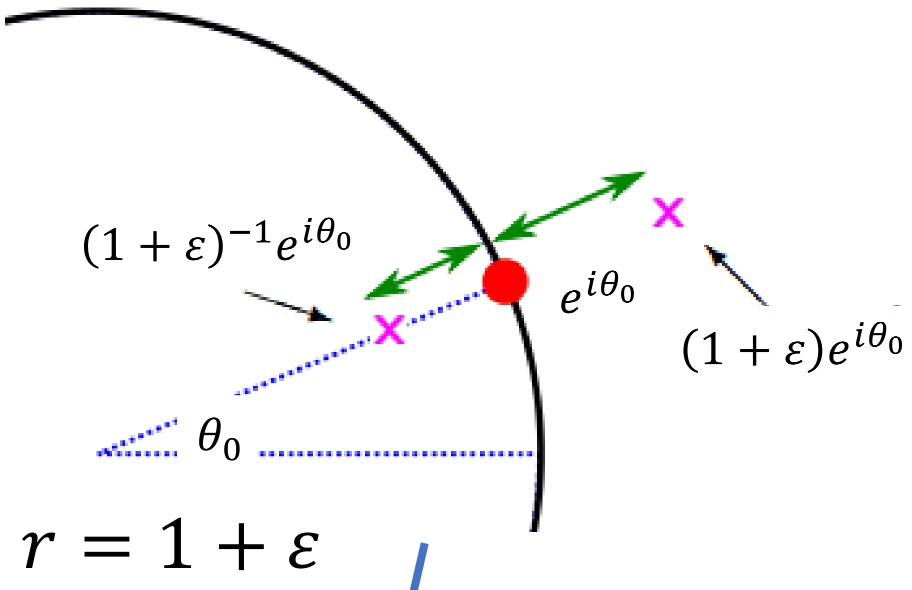
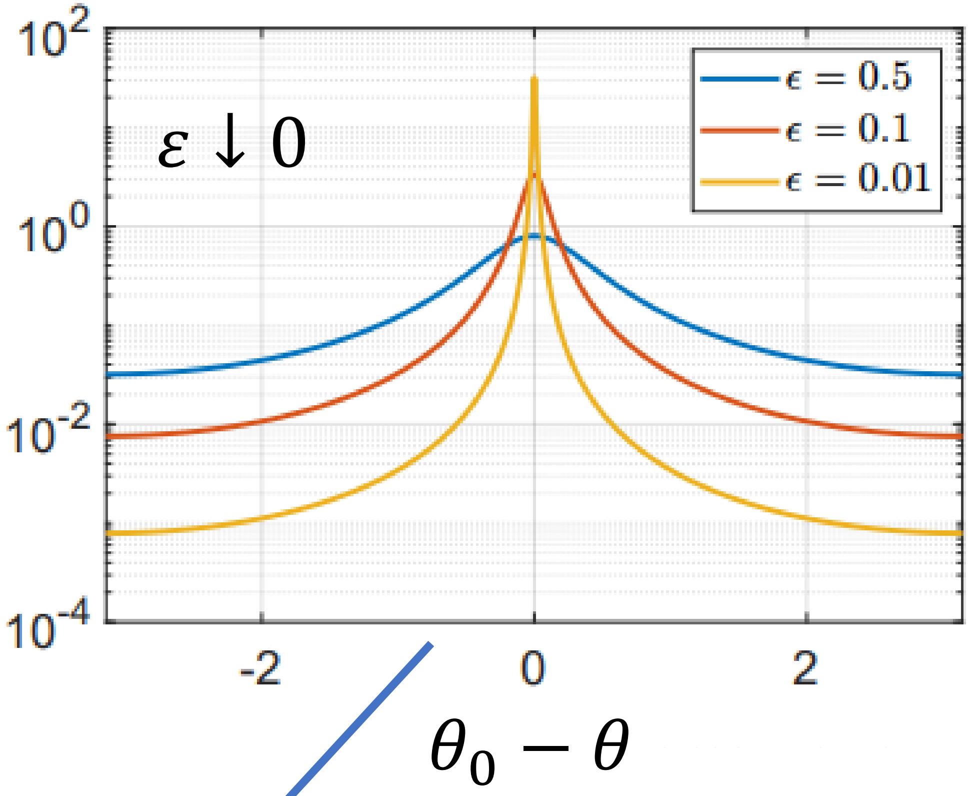
Poisson kernel

$$\frac{1}{4\pi}\left[F_g\left(r^{-1} e^{i\theta_0}\right) - F_g\left(r e^{i\theta_0}\right)\right] = \frac{1}{2\pi} \int_{[-\pi,\pi]_{\mathrm{per}}} \frac{r^2 - 1}{1 + r^2 - 2r\cos(\theta_0 - \theta)} \langle g_\theta^* | g \rangle \, g_\theta \, \mathrm{d}\nu(\theta)$$

Smoothed generalized eigenfunction



Poisson kernel



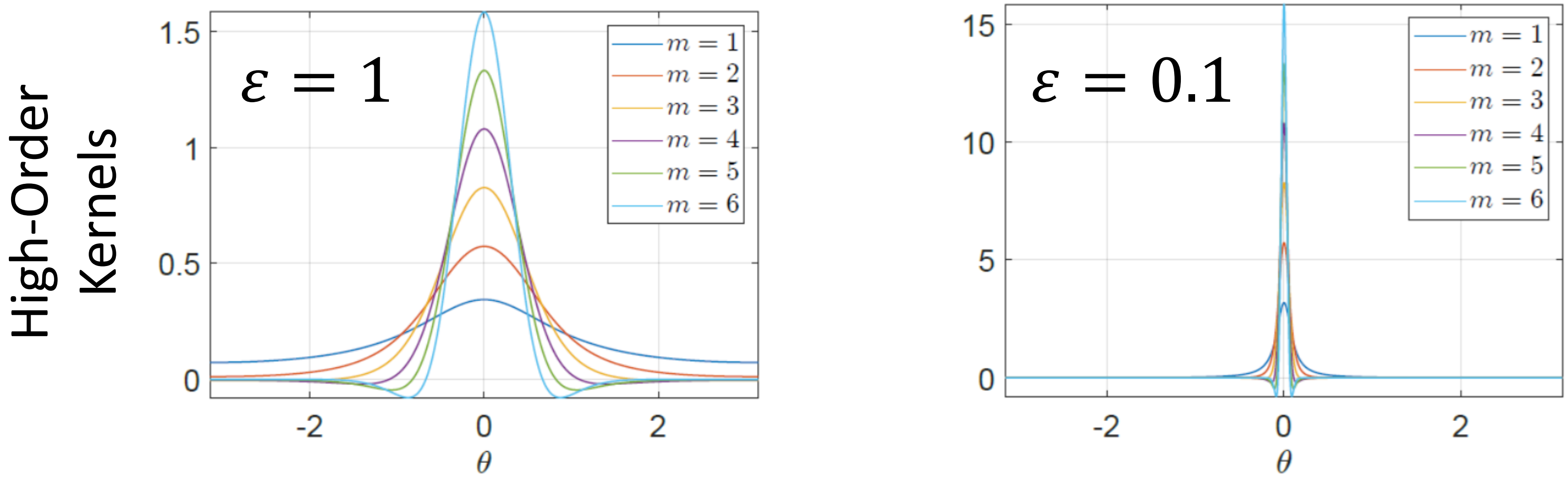
$$\frac{1}{4\pi}\left[F_g\left(r^{-1}e^{i\theta_0}\right) - F_g\left(re^{i\theta_0}\right)\right] = \frac{1}{2\pi}\int_{[-\pi,\pi]_{\mathrm{per}}} \frac{r^2-1}{1+r^2-2r\cos(\theta_0-\theta)} \langle g_\theta^*|g\rangle\, g_\theta \,\mathrm{d}\nu(\theta)$$

Smoothed generalized eigenfunction

Better smoothing kernels as $\varepsilon \downarrow 0$

- Poisson kernel: **slow** convergence $\mathcal{O}(\varepsilon \log(1/\varepsilon))$.
- Construct high-order kernels using $F_g(z)$.

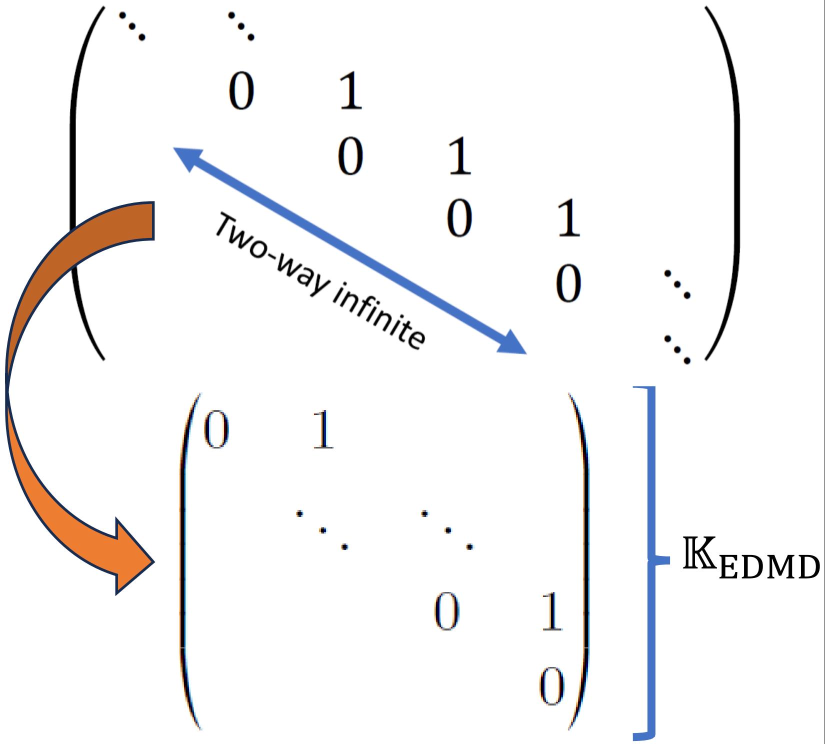
Smaller ε requires more data



- **Theorems:** **fast** $\mathcal{O}(\varepsilon^m \log(1/\varepsilon))$ convergence for
  - Generalized eigenfunctions (topology of $\mathcal{S}^*$).
  - Spectral measures (traces of generalized eigenfunctions): pointwise, L^p, weak,…
  - Forecasting (i.e., iterating Koopman mode decomposition), coherency etc.

Final ingredient: F_g requires $(\mathcal{K} - zI)^{-1}$

EDMD diverges:



Acts on $\mathrm{span}\{e_{-N}, \dots, e_N\}$

E.g., if $|z| < 1$,

$$(\mathbb{K}_{\mathrm{EDMD}} - zI)^{-1} e_0 = \sum_{j=1}^{N} \frac{-1}{z^j} e_{-j}$$

Exponential blowup as $N \to \infty$.

Final ingredient: F_g requires $(\mathcal{K} - zI)^{-1}$

EDMD diverges:

$$\begin{pmatrix} \ddots & \ddots & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix}$$

Two-way infinite

$$\mathbb{K}_{\mathrm{EDMD}} = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

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mpEDMD converges:

$$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

$$\mathbb{K}_{\mathrm{mpEDMD}} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ 1 & & & 0 \end{pmatrix}$$

General method: unitary part of a **polar decomposition** of EDMD!

C., *"The mpEDMD Algorithm for Data-Driven Computations of Measure-Preserving Dynamical Systems,"* **SINUM**, 2023.

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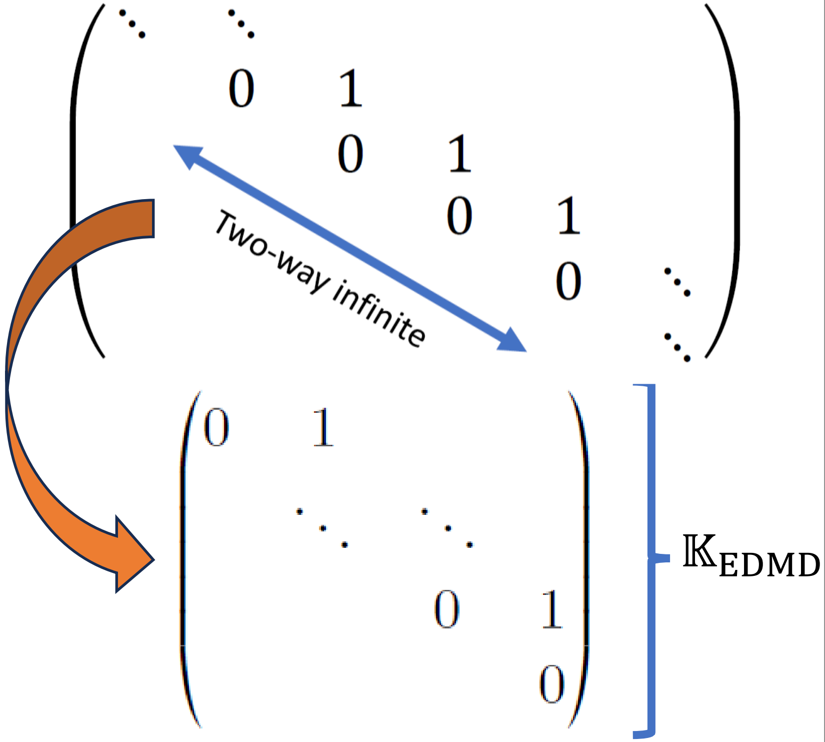

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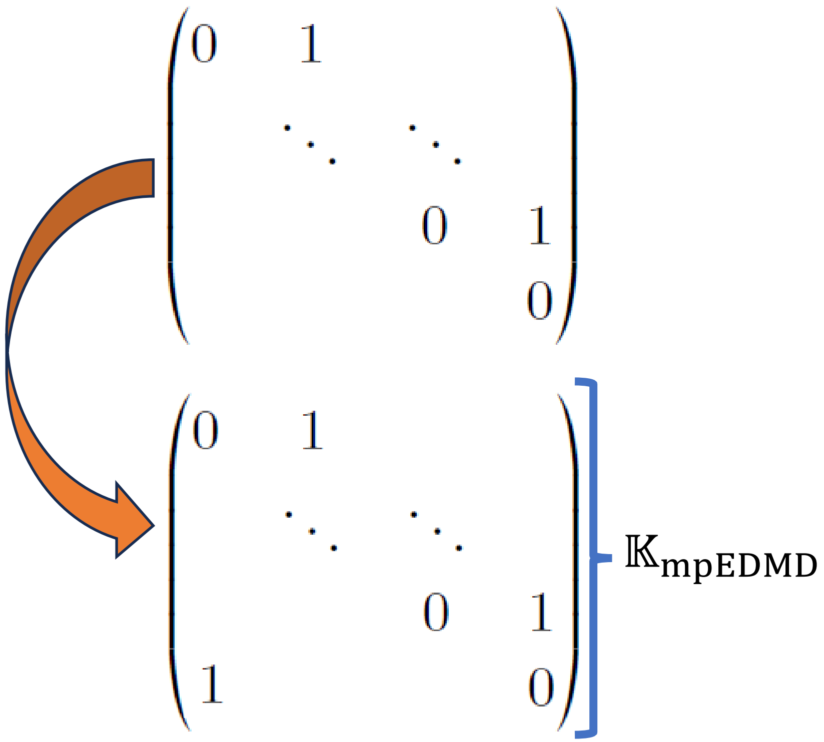

General method: unitary part of a **polar decomposition** of EDMD!

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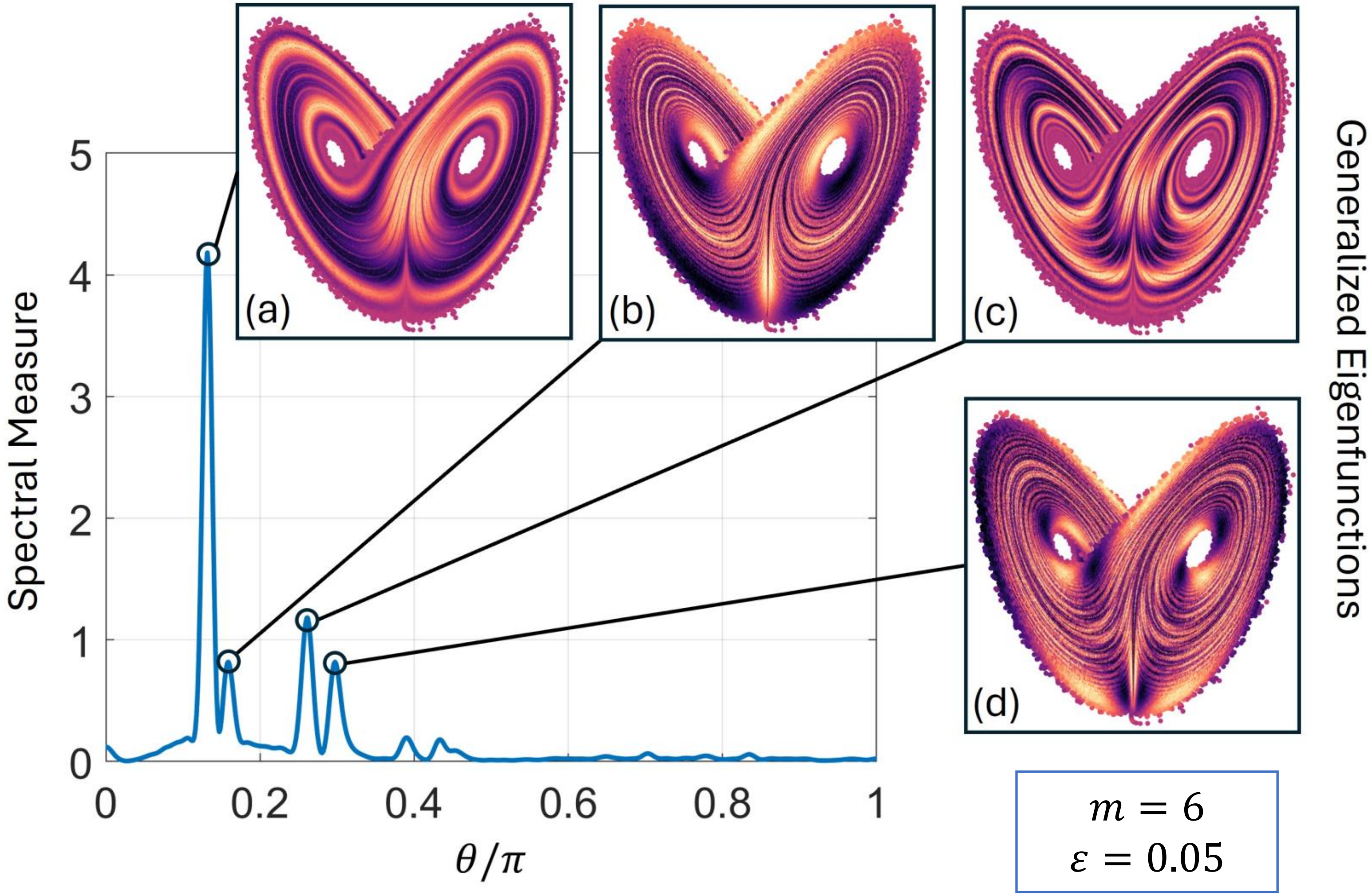
Rigged DMD converges:

- For general $\mathcal{K}$: $(\mathbb{K}_{\text{mpEDMD}} - zI)^{-1}\mathbf{g}$ converges to $(\mathcal{K} - zI)^{-1} g$ as $\lim_{N\to\infty} \lim_{M\to\infty}$
- Hence, Rigged DMD converges as $\lim_{\varepsilon\downarrow 0} \lim_{N\to\infty} \lim_{M\to\infty}$
- ResDMD allows us to select $\varepsilon = \varepsilon(N)$ adaptively (convergence in **2 limits**)

Example: Lorenz system

$\dot{x}_1 = 10(x_2 - x_1),\ \dot{x}_2 = x_1(28 - x_3) - x_2,\ \dot{x}_3 = x_1 x_2 - 8/3\, x_3,\quad \Delta_t = 0.05,\ \ \Omega = \text{attractor},\ \ \omega = \text{SRB measure}$



$m = 6$
$\varepsilon = 0.05$

No formula for generalized eigenfunctions!!

Experimental Details

Single trajectory (ergodic system)

$M = 10000, N = 1000$

$$g(x_1, x_2, x_3) = \tanh\left(\frac{x_1 x_2 - 5x_3}{10}\right) - c$$

Krylov subspace: $V_N = \{g, \mathcal{K}g, \ldots, \mathcal{K}^{N-1}g\}$

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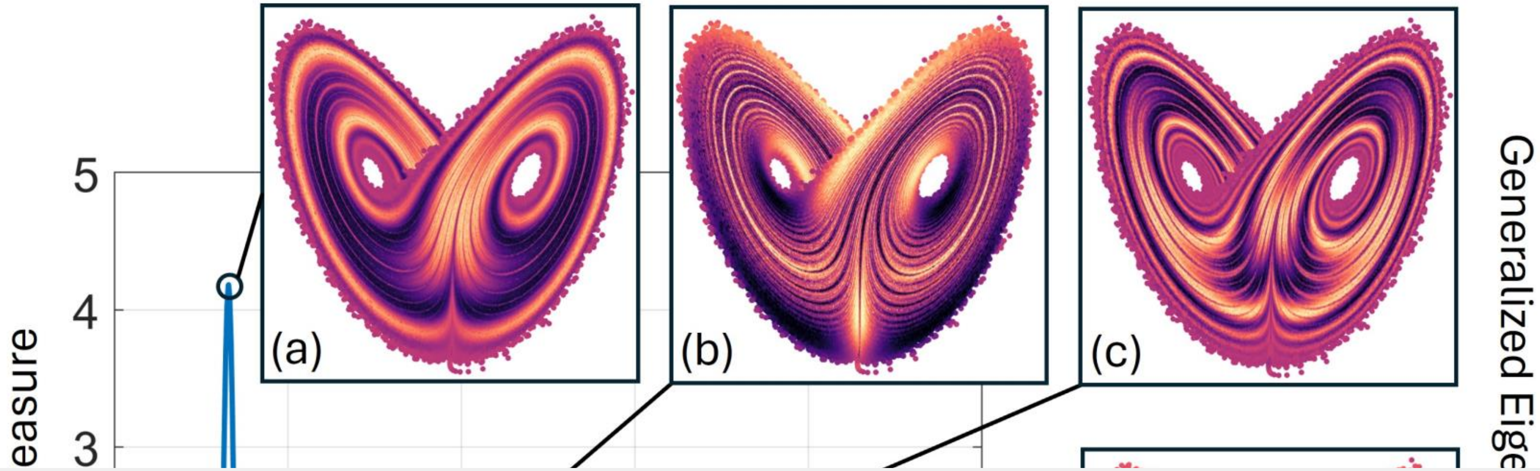
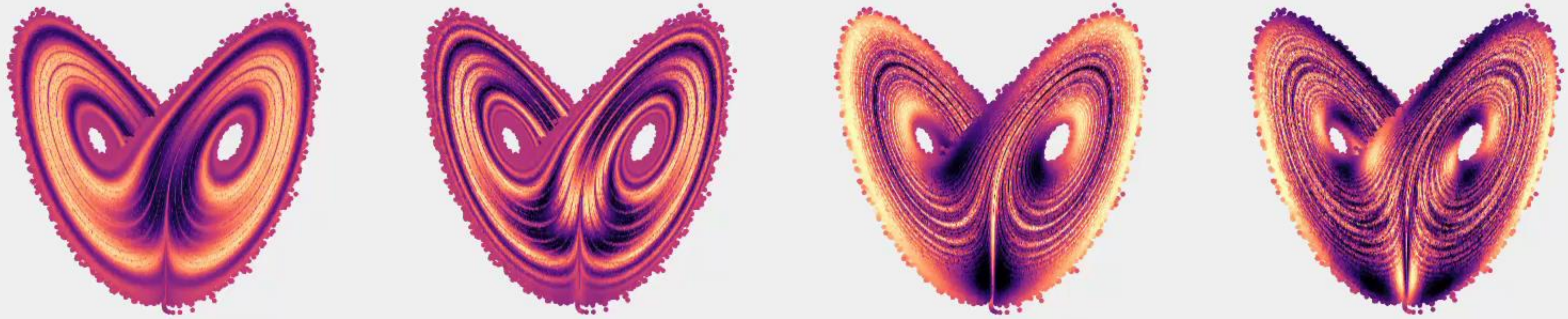


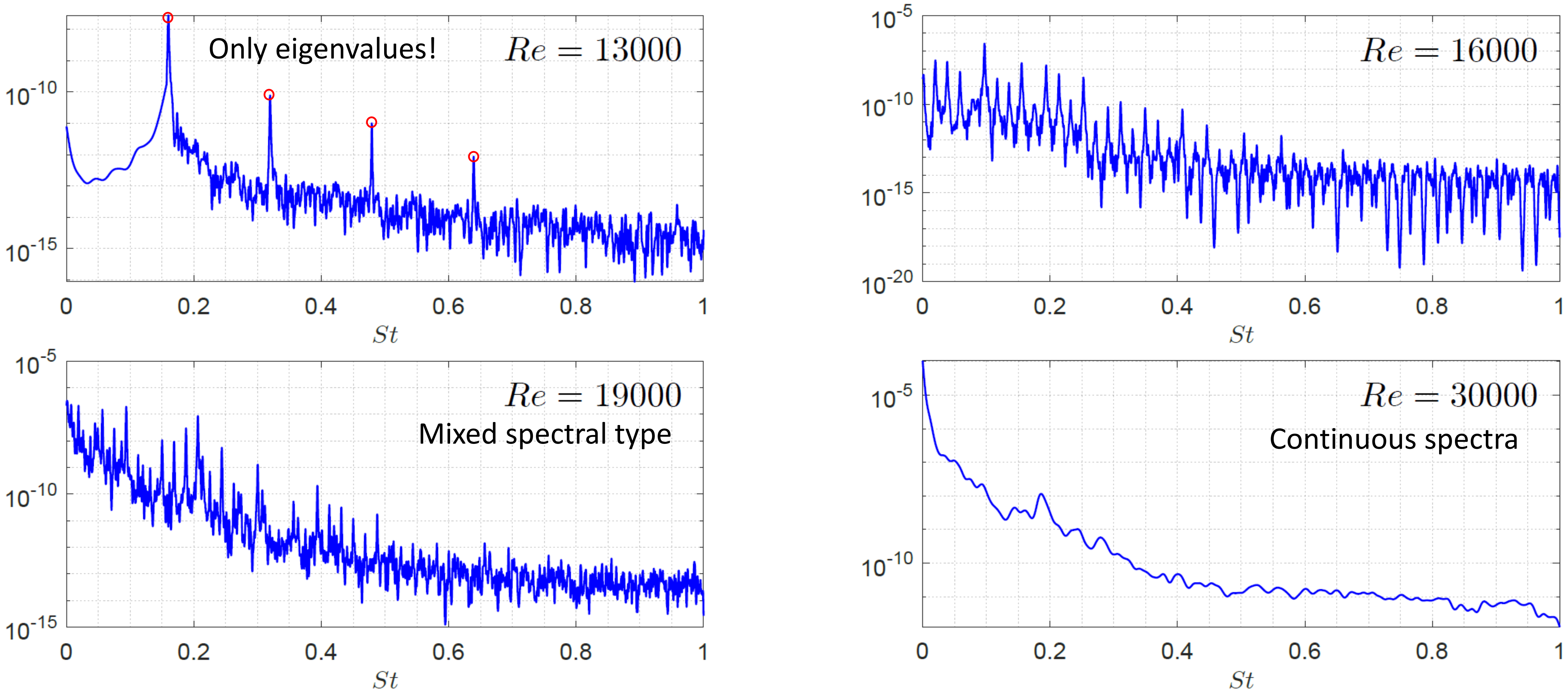
No formula for
generalized eigenfunctions!!

Experimental Details



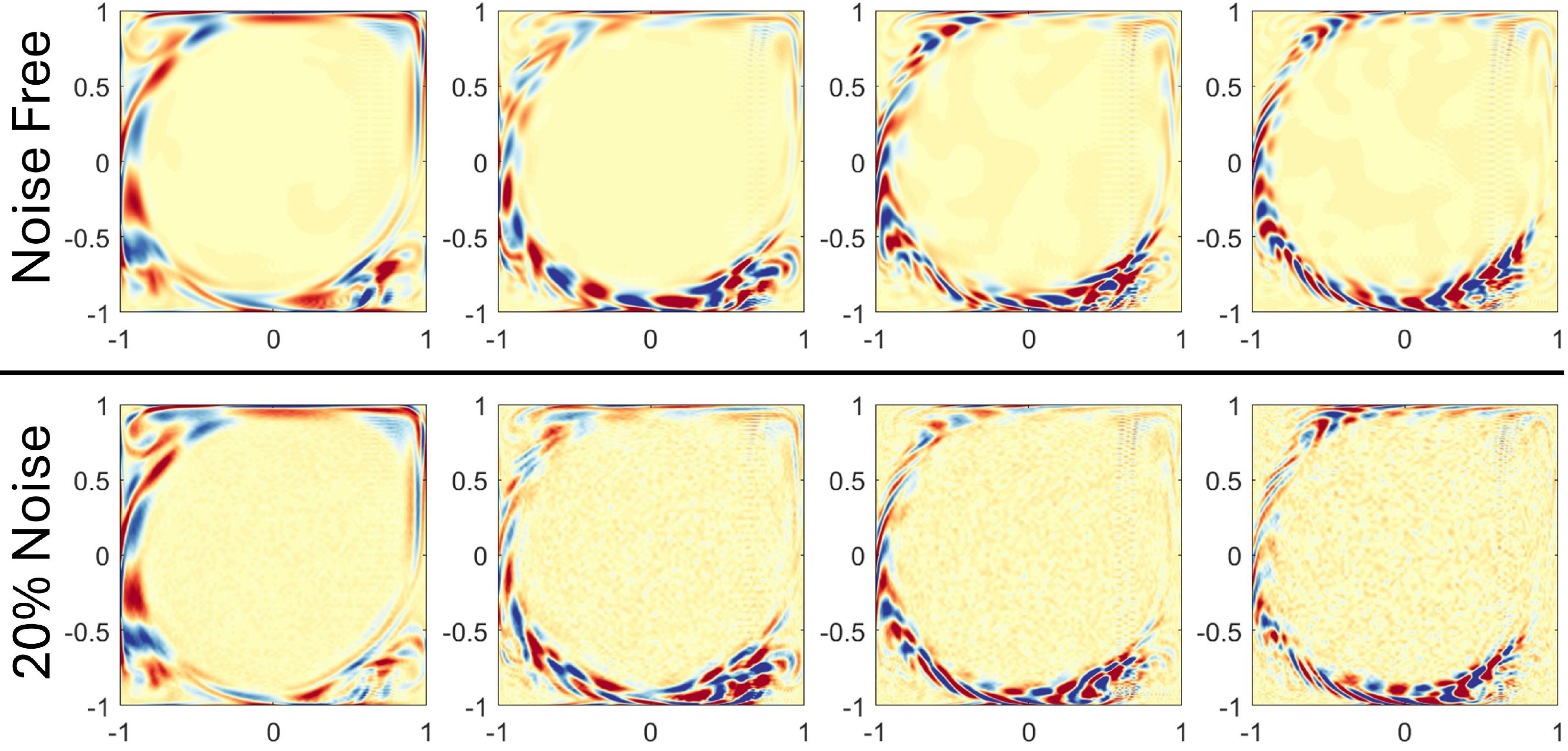
Example: Noisy cavity flow (spectral measures)

Experimental Details
Single trajectory
$M = 10000, N$ varies
Basis: POD modes
20% Gaussian noise



Example: Noisy cavity flow (generalized Koopman modes)

Re=30000
Deep in the continuous spectrum!!!



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Wider program: Solvability Complexity Index

- ResDMD: convergence as $\lim_{N\to\infty}\lim_{M\to\infty}$
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Convergence in
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Convergence in **<u>multiple successive limits</u>**

- **FACT:** Unless you have very strong assumptions (e.g., uniform ergodicity, finite-dimensional invariant subspace etc.), ***<u>EVERY</u>*** convergent Koopman algorithm to date needs multiple limits. These limits can be different things.

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- C., Hansen, *"The foundations of spectral computations via the solvability complexity index hierarchy,"* **J. Eur. Math. Soc.**, 2022.
- C., Antun, Hansen, "*The difficulty of computing stable and accurate neural networks,*" **Proc. Natl. Acad. Sci. USA**, 2022.
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$\Longrightarrow$ Classification of problems, optimality of algorithms.

David Hilbert

Steve Smale

- C., Hansen, “*The foundations of spectral computations via the solvability complexity index hierar*… **h.**
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Sample: some results for bounded op. on $l^2(\mathbb{N})$

increasing difficulty →

Error control | **1 limit** | **2 limits** | **3 limits**

$$\Pi_0 = \Delta_0 = \Sigma_0$$

$$\Delta_0 \subsetneq \Delta_1 \subsetneq \Sigma_1 \cup \Pi_1 \subsetneq \Delta_2 \subsetneq \Sigma_2 \cup \Pi_2 \subsetneq \Delta_3 \subsetneq \Sigma_3 \cup \Pi_3 \cdots$$

$$\Delta_n \subsetneq \Pi_n \subsetneq \Delta_{n+1}, \quad \Delta_n \subsetneq \Sigma_n \subsetneq \Delta_{n+1}$$

Certain self-adjoint 1D quasiperiodic operators → Δ_1

Approx. sparse normal op → Σ_1

Compact operators → Δ_2

Normal operators → Σ_2

"Sparse" operators → Π_2

General operators → Π_3

Zoo of problems: spectral type (pure point, absolutely continuous, singularly continuous), Lebesgue measure and fractal dimensions of spectra, discrete spectra, essential spectra, eigenspaces + multiplicity, spectral radii, essential numerical ranges, geometric features of spectrum (e.g., capacity), spectral gap problem, resonances ...

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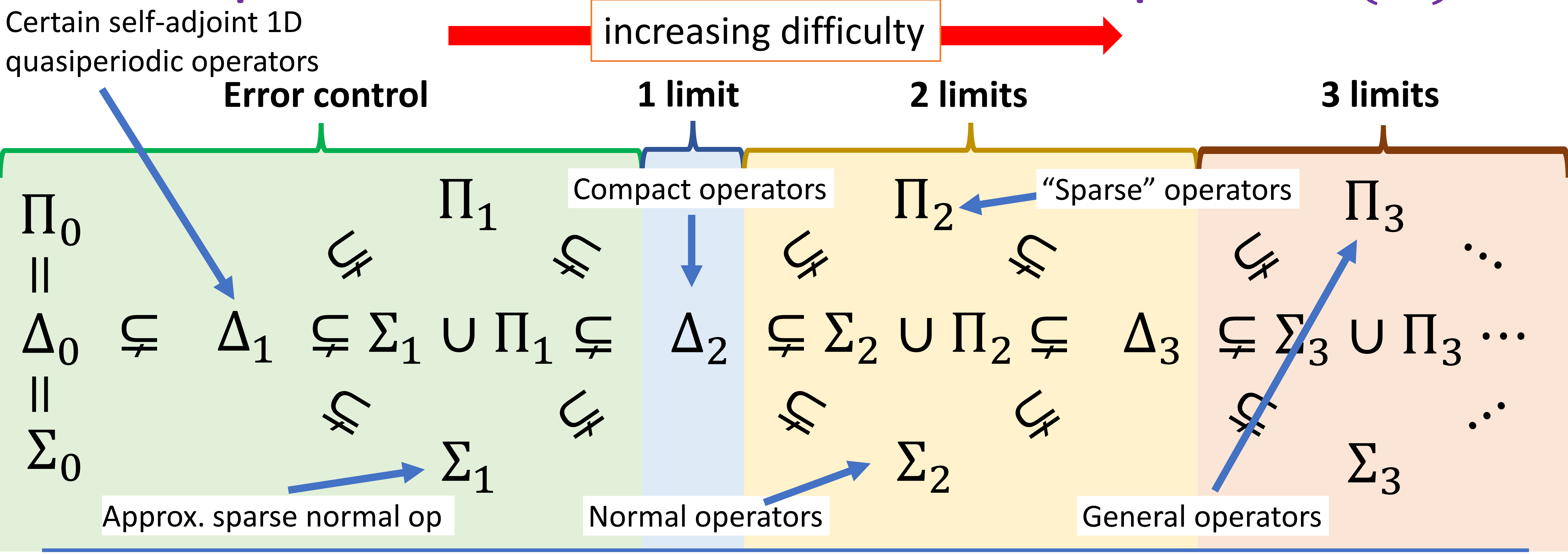

- C., *"The foundations of infinite-dimensional spectral computations,"* **PhD diss.**, University of Cambridge, 2020.
- Ben-Artzi, C., Hansen, Nevanlinna, Seidel, *"On the solvability complexity index hierarchy and towers of algorithms,"* preprint.
- C., Horning, Townsend, *"Computing spectral measures of self-adjoint operators,"* **SIAM Rev.**, 2021.
- Ben-Artzi, Marletta, Rösler, *"Computing the sound of the sea in a seashell,"* **Found. Comput. Math.**, 2022.
- Ben-Artzi, Marletta, Rösler, *"Computing scattering resonances,"* **J. Eur. Math. Soc.**, 2022.
- C., *"On the computation of geometric features of spectra of linear operators on Hilbert spaces,"* **Found. Comput. Math.**, 2022.
- Webb, Olver, *"Spectra of Jacobi operators via connection coefficient matrices,"* **Commun. Math. Phys.**, 2021.
- Rösler, Stepanenko, *"Computing eigenvalues of the Laplacian on rough domains,"* preprint.
- Rösler, Tretter, *"Computing Klein-Gordon Spectra,"* prepint.

Coming soon… SCI for Koopman (with Mezić)

- General systems (computing spectrum).
- Measure-preserving systems (spectrum, spectral type etc.)

Bottom line:

- Many problems are impossible in one limit, even with perfect and unlimited snapshots, probabilistic algorithms, nice smooth F on compact manifolds.

  E.g., computing spectrum (as a set) of smooth measure-preserving systems on unit disc.

- Problems can be tackled in multiple limits under very general conditions.

$\Longrightarrow$ New program on foundations and classification for Koopman.

Summary: Infinities matter

Practical + theoretical guarantees

- A complete picture has emerged on $L^2(\Omega, \omega)$
  - *General systems:* Compute spectral properties with error control.
    CONTROL INFINITE-DIMENSIONAL RESIDUALS
  - *Measure-preserving systems:* Continuous spectra (and generalized eigenfunctions)
    SMOOTHING KERNELS and the **RESOLVENT**.

Brief Summaries

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  SCI hierarchy: classify problems, provably optimal algorithms.

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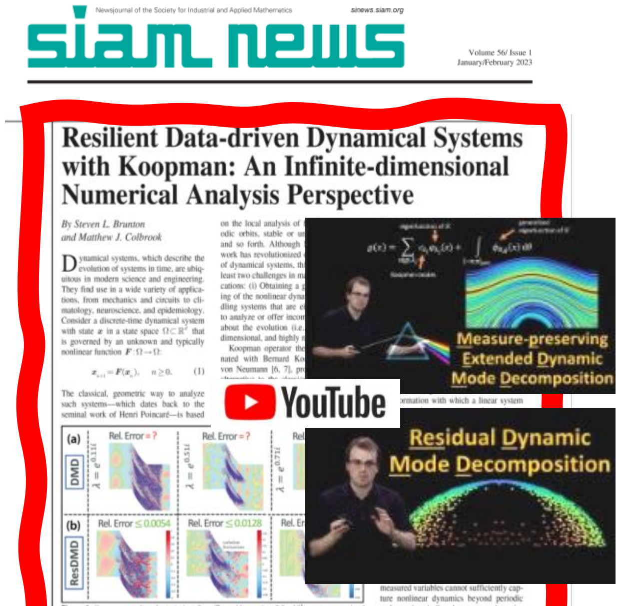

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- Other uses of residuals and ResDMD (control, analytic DMD,…)
- What about other function spaces?
- What further classifications can we prove?
  Only starting to scratch the surface!
- Beyond spectra: Applications such as control.

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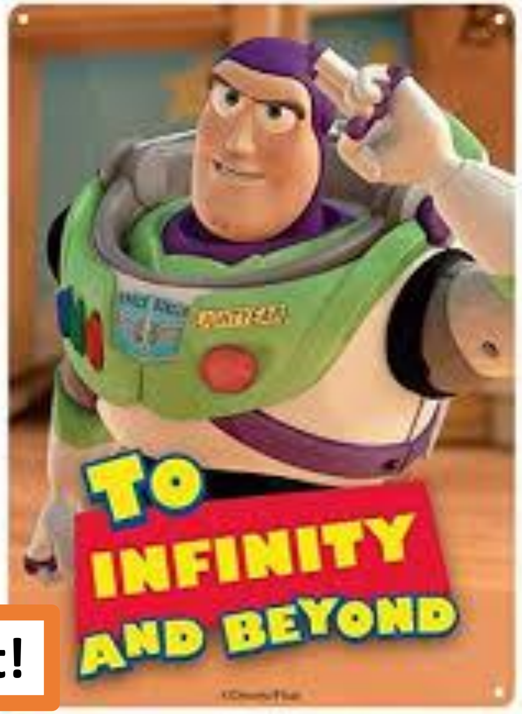
- What about other function spaces?
- What further classifications can we prove?
  Only starting to scratch the surface!
- Beyond spectra: Applications such as control.



Buzz was right!

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