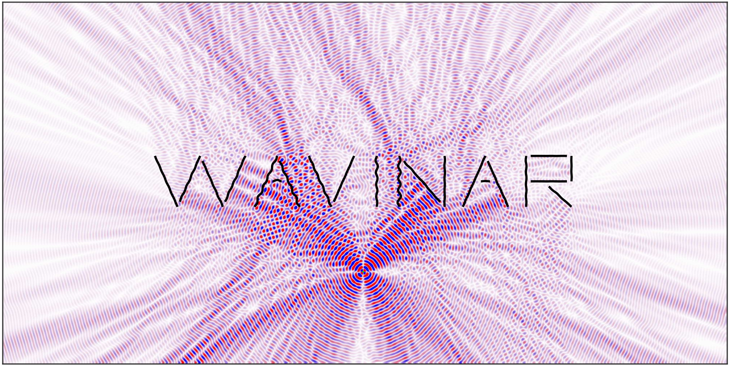
Scattering, Acoustic Black Holes and Mathieu Functions

A boundary spectral method for diffraction by multiple variable poro-elastic plates

Matthew Colbrook
University of Cambridge



Collaborators for papers referenced in this talk

Anastasia Kisil (Manchester)	Lorna Ayton (Cambridge)	Matthew Priddin (Cambridge)
		

With special thanks also to Justin Jaworski at Lehigh who's discussed numerous aspects with me, and who I hope to be working with on future projects soon!

Sketch of talk

Goal: Numerically solve scattering problems with complex boundary conditions. Want: **accurate**, **fast** and **flexible** (and **easy-to-use**?).

Outline:

- Motivation
- Building a numerical method
- Acoustic black holes
- Conclusions and future work
- Extra slides: Comparison with BEM - feel free to ask about this in discussion

Take home message:

- Classical separation of variables can be made into an effective spectral method for solving 2D scattering problems (multiple plates), satisfying these requirements.
- It's particularly flexible with respect to boundary conditions.
- We can use it to study problems such as acoustic black holes.

Motivation

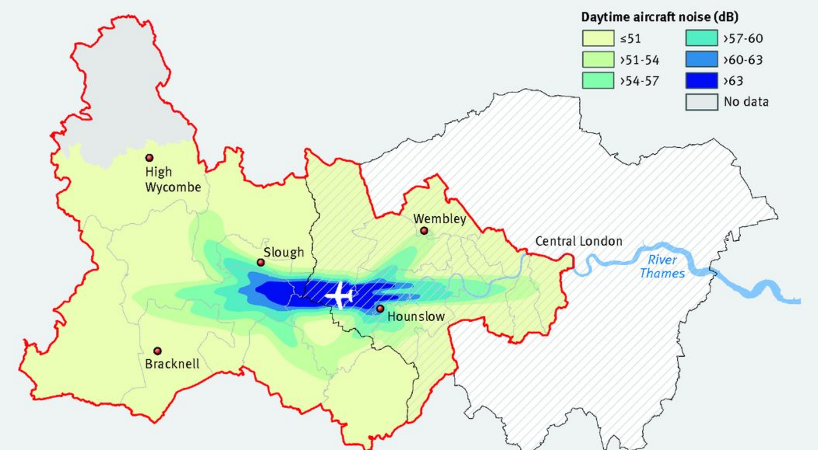
- **Application:** A big problem in aero-acoustics is noise reduction.
- **Current challenge:** developing **fast** and **accurate** numerical tools for scattering problems.
  $\rightarrow$ predict effect of physical parameters and external forces.
- Can we model complicated boundary conditions such as **elasticity**? (this is difficult via traditional methods)

Elastic $\rightarrow$ absorbs energy $\rightarrow$ reduced noise

Wind turbines

See, e.g., C. & Ayton, JSV, 2019 for modelling elastic tips of turbines.

Airport noise (at least before the virus!)



Source: The BMJ 2013;347:f5432

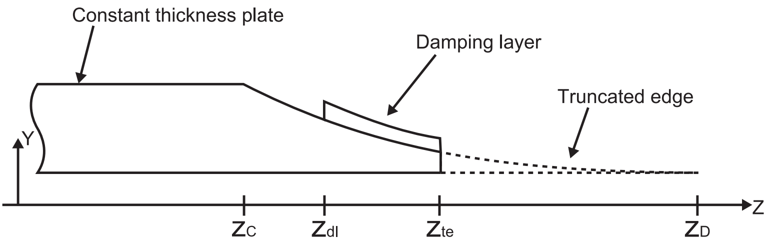
Average noise levels near Heathrow - a major health concern.

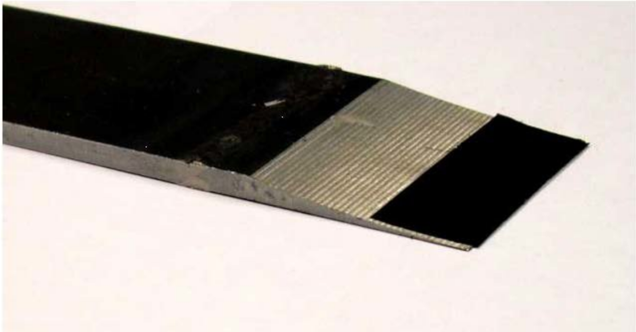
Owls are silent predators - can we copy them?

See this talk (in particular the awesome video demonstration):
http://www.newton.ac.uk/seminar/20190815133014001

For details on paper that combines numerical method with data from owls wings, see Lorna Ayton's wavinar on 7th July.

Acoustic black holes and metamaterials





Scattering problem

Acoustic 2D scattering governed by the Helmholtz equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + k_0^2\phi = 0, \quad (x, y) \in \mathcal{D}.$$

Typical boundary conditions on $\partial\mathcal{D}$:

- Zero normal velocity (Neumann: prescribed $\partial\phi/\partial n = \phi_n$)
- Continuity of pressure (Dirichlet: prescribed ϕ)
- Impedance/porosity
  (Robin: prescribed linear combination of ϕ_n and ϕ)
- Elastic deformation (more on this later)

Sommerfeld radiation condition at infinity (radiates <u>to</u> infinity):

$$\lim_{r\to\infty} r^{\frac{1}{2}} \left(\frac{\partial}{\partial r} - ik_0\right) \phi(r,\theta) = 0$$

Crucial for well-posed problem (and important physically)!

Elastic boundary conditions for a single plate

Porous plate $-d \leq x \leq d$, $y = 0$ with evenly-spaced circular apertures of radius R and fractional open area α_H. Plate deformation $\eta(x)$ satisfies:

$$B_0(x)\eta(x) + \sum_{l=1}^{4} B_l(x) \frac{\partial^l \eta}{\partial x^l}(x) = -\rho_f c_0^2 \left(1 + \frac{4\alpha_H}{\pi}\right) [\phi](x).$$

Kinematic condition for incident field ϕ_{I}:

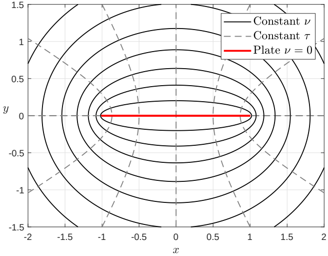
$$\left.\frac{\partial \phi}{\partial y}\right|_{y=0} + \left.\frac{\partial \phi_{\mathrm{I}}}{\partial y}\right|_{y=0} = k_0^2 \Big[(1 - \alpha_H)\eta + \alpha_H \eta_a\Big].$$

$\eta_a = 2[\phi]/(\pi k_0^2 R) =$ average fluid displacement in apertures.

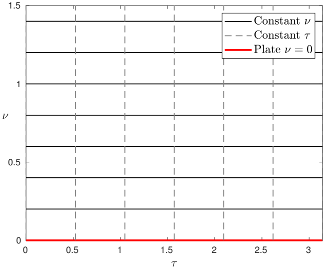
Endpoint x_0 either free $\eta''(x_0) = \eta'''(x_0) = 0$ or clamped $\eta(x_0) = \eta'(x_0) = 0$.

Separation of variables

Elliptic coordinates $x = d\cosh(\nu)\cos(\tau), y = d\sinh(\nu)\sin(\tau)$



$\Rightarrow$



$$\phi(\nu,\tau) = \sum_{m=1}^{\infty} a_m \mathrm{se}_m(\tau)\mathrm{Hse}_m(\nu).$$

Will determine the unknown coefficients using collocation.

Angular Mathieu functions

Expand in a rapidly convergent sine series:

$$\mathrm{se}_m(Q;\tau) = \mathrm{se}_m(\tau) = \sum_{l=1}^{\infty} B_l^{(m)} \sin(l\tau), \quad Q = d^2 k_0^2/4.$$

For the even order solutions, eigenvalue problem becomes

$$\begin{pmatrix} 2^2 - \lambda_{2m} & Q & & & \\ Q & 4^2 - \lambda_{2m} & Q & & \\ & Q & 6^2 - \lambda_{2m} & Q & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} B_2^{(2m)} \\ B_4^{(2m)} \\ B_6^{(2m)} \\ \vdots \end{pmatrix} = 0.$$

A similar system holds for the odd order solutions.

Radial Mathieu functions

Expand in a rapidly convergent Bessel function series:

$$\mathrm{Hse}_m(\nu) = \sum_{l=1}^{\infty} \frac{(-1)^{l+m} B_l^{(m)}}{C_m} \Big[J_{l-1}(\mathrm{e}^{-\nu}\sqrt{Q}) H^{(1)}_{l+p_m}(\mathrm{e}^{\nu}\sqrt{Q}) - J_{l+p_m}(\mathrm{e}^{-\nu}\sqrt{Q}) H^{(1)}_{l-1}(\mathrm{e}^{\nu}\sqrt{Q}) \Big],$$

where $p_m = 1$ if m is even and $p_m = 0$ if m is odd.

Normalisation constants C_m such that $\mathrm{Hse}'_m(0) = 1$.

WARNING: Care needed in some regimes to avoid underflow and overflow associated with cancellations between the Bessel and Hankel functions. Solve this using asymptotics (details in paper).

Bottom line: With a bit of care, both types of Mathieu functions can be accurately and efficiently evaluated $\Rightarrow$ can be used with collocation.

Employing the boundary conditions

Expansion of η in Chebyshev polynomials of the first kind

$$\eta(x) = \sum_{j=0}^{N-1} b_j T_j\left(\frac{x}{d}\right).$$

Collocate thin plate equation at $N-4$ Chebyshev points

$$\sum_{j=0}^{N-1} \frac{b_j \pi}{2\rho_f c_0^2} \sum_{l=0}^{4} \frac{B_l(x)}{d^l} T_j^{(l)}\left(\frac{x}{d}\right) + (\pi + 4\alpha_H) \sum_{m=1}^{M} a_m \mathrm{se}_m\Big(\cos^{-1}\left(\frac{x}{d}\right)\Big)\mathrm{Hse}_m(0) = 0.$$

Collocate kinematic relation at M Chebyshev points

$$\sqrt{d^2 - x^2} \cdot \frac{\partial \phi_{\mathrm{I}}}{\partial y}(x) + \sum_{m=1}^{M} a_m \mathrm{se}_m\left(\cos^{-1}\left(\frac{x}{d}\right)\right)\left[1 - \frac{4\alpha_H \mathrm{Hse}_m(0)}{\pi R}\sqrt{d^2 - x^2}\right]$$

$$= k_0^2(1-\alpha_H)\sqrt{d^2 - x^2} \sum_{j=0}^{N-1} b_j T_j\left(\frac{x}{d}\right).$$

$+4$ relations for η BCs $\Rightarrow (M+N) \times (M+N)$ system for coefficients.

Bottom line: Easy to employ complicated BCs with collocation and (standard) spectral methods.

Acoustic black hole

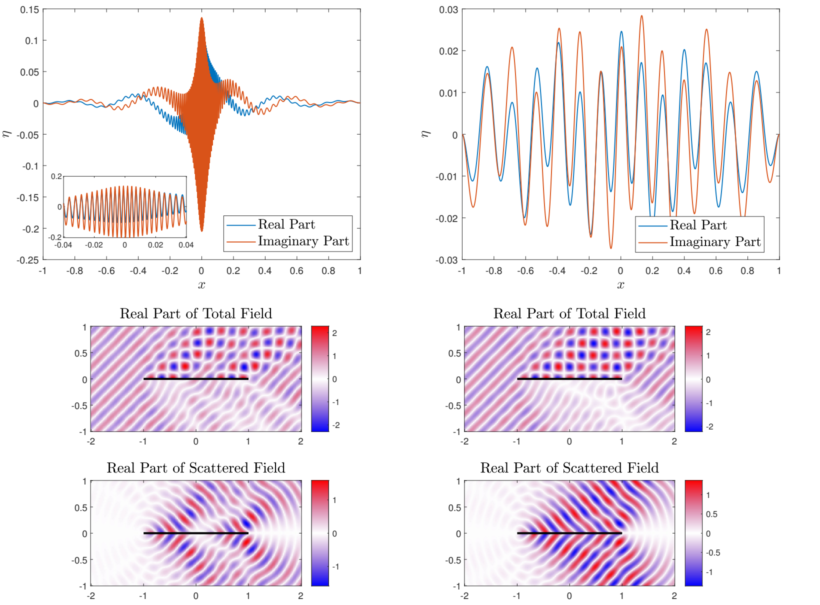
Aluminium plate of thickness $h(x)$ with

$$B(x) = \frac{Eh(x)^3}{12(1-\nu^2)}, \quad E = 69 \times 10^9 \mathrm{Pa}, \quad \nu = 0.35$$

$$\frac{d^2}{dx^2}\left(B(x)\eta''(x)\right) - m_0 h(x)\eta(x) = -\rho_f c_0^2 \left(1 + \frac{4\alpha_H}{\pi}\right)[\phi](x)$$

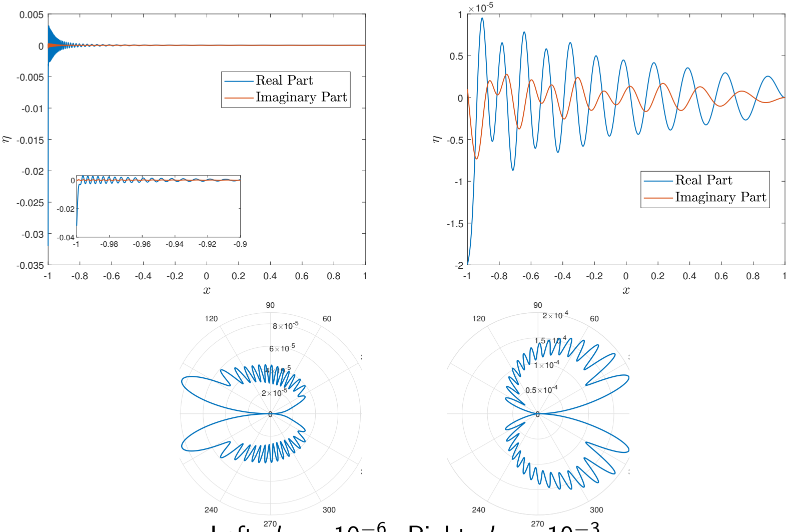
NB: in this talk, physical parameters chosen for aluminium plate in air.

Incident plane wave, $k_0 = 20$, $h(x) = 0.001x^2 + h_0$



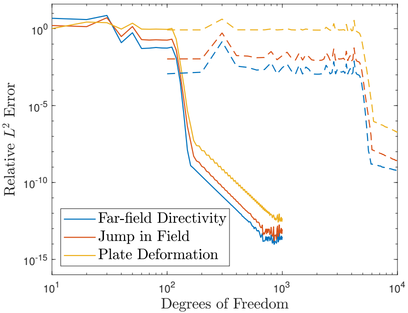
Left: $h_0 = 10^{-6}$. Right: $h_0 = 10^{-3}$.

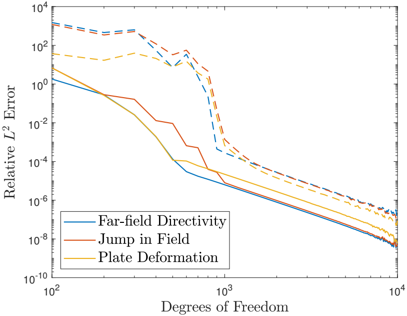
Quadrupole at $(x, y) = (-1, 0.001)$, $k_0 = 25$, $h(x) = 0.001(x+1)^2 + h_0$



Left: $h_0 = 10^{-6}$. Right: $h_0 = 10^{-3}$.

In case you were worried about convergence...





Left: Incident plane wave for $h_0 = 10^{-6}$ (dashed) and $h_0 = 10^{-3}$ (full). Right: Quadrupole for $h_0 = 10^{-6}$ (dashed) and $h_0 = 10^{-3}$ (full).

Bottom line: Several digits of relative accuracy, even for these singular elastic BCs.

Pros and cons vs other boundary type methods

Pros	Cons
No singular integrals or quadrature	No proof of convergence
Very flexible w.r.t. BCs	
Implicit sine series for far field	
Can stably evaluate near field	
Much easier to use than state of art BEM	
Deals with multiple plates	No curved boundaries yet
More accurate than basic BEM	No analysis of singularities?
Faster than basic BEM	Dense system - no numerical analysis of structure of linear system yet, e.g. low rank, FMM,...

Bottom line: Proposed method is more suited to the kinds of problems and applications we are looking at and low-mid frequency scattering off plates. More work needed for other regimes such as very large k_0.

Conclusion

Numerical:

- Can cope with complex boundary conditions.
- Achieved goal of **accurate**, **fast** and **flexible**.
- Bonus: (very) easy to use and modify.

Future work will take advantage of these in applications but **physical:**

- Acoustic BHs can lead to “transparent” plates.
- Acoustic BHs can produce counter-intuitive scattering and sound absorption.
- (Not shown) Acoustic BCs can lead to reduced scattered sound.

Can we also employ dampeners to absorb sound?

Future work will also look at other geometries.

References for method in this talk

(Porous/Robin BCs) M.J. Colbrook, M.J. Priddin. "Fast and spectrally accurate numerical methods for perforated screens." *Submitted, should appear soon!*

(Elastic BCs) M.J. Colbrook, A.V. Kisil. "Scattering, Acoustic Black Holes and Mathieu Functions: A boundary spectral method for diffraction by multiple variable poro-elastic plates." *Submitted*.

(Application with owls) L.J. Ayton, M.J. Colbrook, T.F. Geyer, P. Chaitanya, E. Sarradj. "Reducing aerofoil-turbulence interaction noise through chordwise-varying porosity." *Submitted*.

For further papers in this program, slides of this talk and numerical code:
http://www.damtp.cam.ac.uk/user/mjc249/home.html
https://github.com/MColbrook/MathieuFunctionCollocation

See also related papers:

(Basic M function method + UTM) M.J. Colbrook, L.J. Ayton, A.S. Fokas. "The unified transform for mixed boundary condition problems in unbounded domains." *Proceedings of the Royal Society A*, 2019.

(Elastic UTM) M.J. Colbrook, L.J. Ayton. "A spectral collocation method for acoustic scattering by multiple elastic plates." *JSV*, 2019.

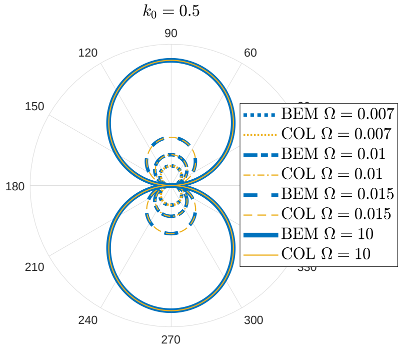
Comparison with BEM

Compare with *Cavalieri, Wolf, & Jaworski, "Numerical solution of acoustic scattering by finite perforated elastic plates", Proceedings A 2016*.

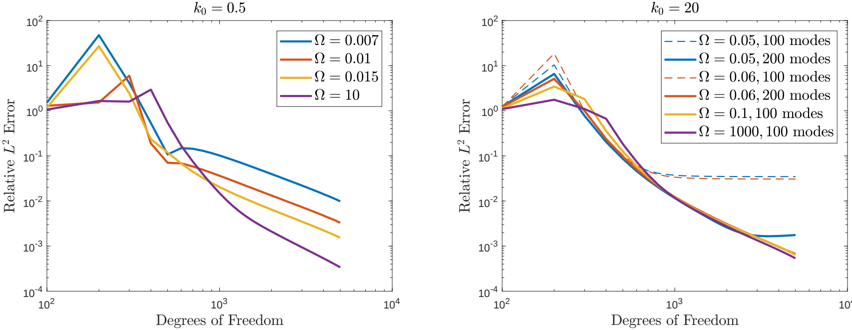
Uses BEM method with basis functions constructed using vibration modes of the plate (computed using standard spectral methods).

$$(1-\alpha_H)\frac{\partial^4 \eta}{\partial x^4} - \frac{k_0^4}{\Omega^4}\eta = -\left(1+\frac{4\alpha_H}{\pi}\right)\frac{\epsilon}{\Omega^6}k_0^3[\phi],$$
$$\frac{\partial \phi}{\partial y}\Big|_{y=0} + \frac{\partial \phi_{\mathrm{I}}}{\partial y}\Big|_{y=0} = (1-\alpha_H)k_0^2\eta + \frac{2\alpha_H}{\pi R}[\phi].$$

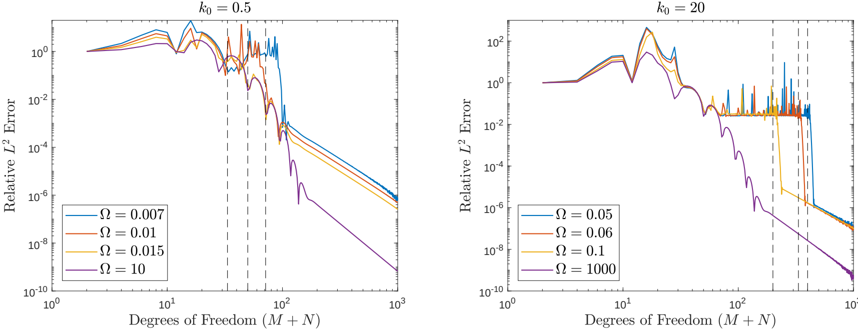
<u>Constant</u> parameters:
$\Omega =$ vacuum bending wave Mach number
$\epsilon = 0.0021 =$ fluid-loading



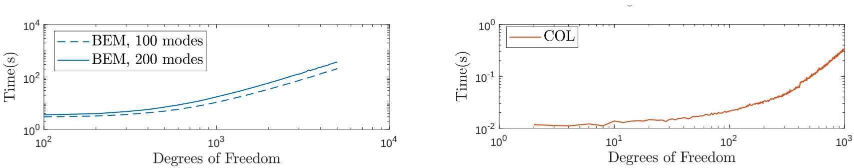
$k_0 = 0.5$
BEM $\Omega = 0.007$
COL $\Omega = 0.007$
BEM $\Omega = 0.01$
COL $\Omega = 0.01$
BEM $\Omega = 0.015$
COL $\Omega = 0.015$
BEM $\Omega = 10$
COL $\Omega = 10$
90
60
120
150
180
210
240
270
300
$k_0 = 20$
BEM $\Omega = 0.05$
COL $\Omega = 0.05$
BEM $\Omega = 0.06$
COL $\Omega = 0.06$
BEM $\Omega = 0.1$
COL $\Omega = 0.1$
BEM $\Omega = 1000$
COL $\Omega = 1000$



Left: Convergence of elastic BEM for $k_0 = 0.5$ (100 modes). Right: Same but for $k_0 = 20$ (number of modes shown).



Left: Convergence of Mathieu function collocation for $k_0 = 0.5$. The vertical dashed lines are positioned at the bending wavenumbers $k_B = k_0/\Omega$ (too small to plot for $\Omega = 10$). Right: Same but for $k_0 = 20$.



Left: Times taken for elastic BEM. Right: Same but for Mathieu function collocation. Note the difference in orders of magnitude on the horizontal and vertical axes - the Mathieu function collocation approach is much faster.