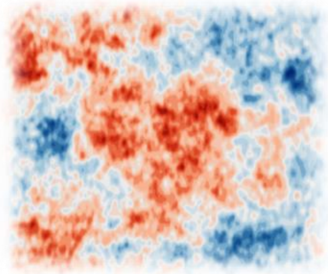
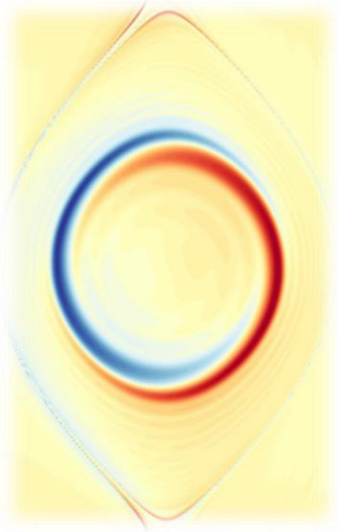




Data-Driven Spectral Measures and Generalized Eigenfunctions of Koopman Operators

Matthew Colbrook

University of Cambridge

18/06/2024

- C., Townsend, *"Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems"* **Communications on Pure and Applied Mathematics**, 2024.
- C., *"The mpEDMD Algorithm for Data-Driven Computations of Measure-Preserving Dynamical Systems,"* **SIAM Journal on Numerical Analysis**, 2023.
- C., Drysdale, Horning, *"Rigged Dynamic Mode Decomposition: Data-Driven Generalized Eigenfunction Decompositions for Koopman Operators"*, arxiv preprint.
- C., *"The Multiverse of Dynamic Mode Decomposition Algorithms,"* **Handbook of Numerical Analysis**, 2024.

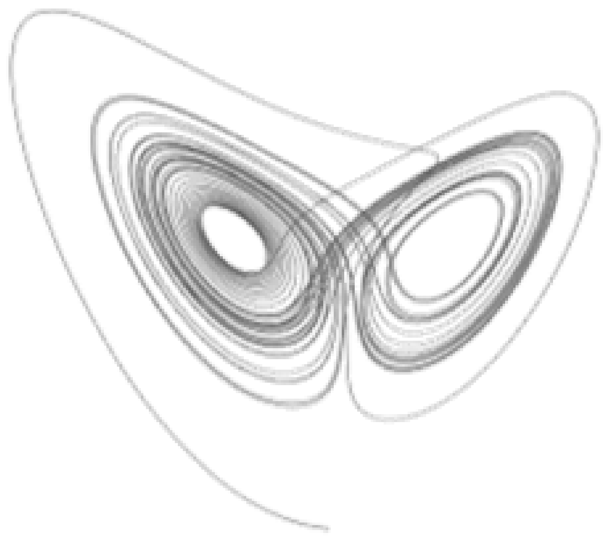
Data-driven dynamical systems

State $x \in \Omega \subseteq \mathbb{R}^d$.

Unknown function $F: \Omega \rightarrow \Omega$ governs dynamics: $x_{n+1} = F(x_n)$.

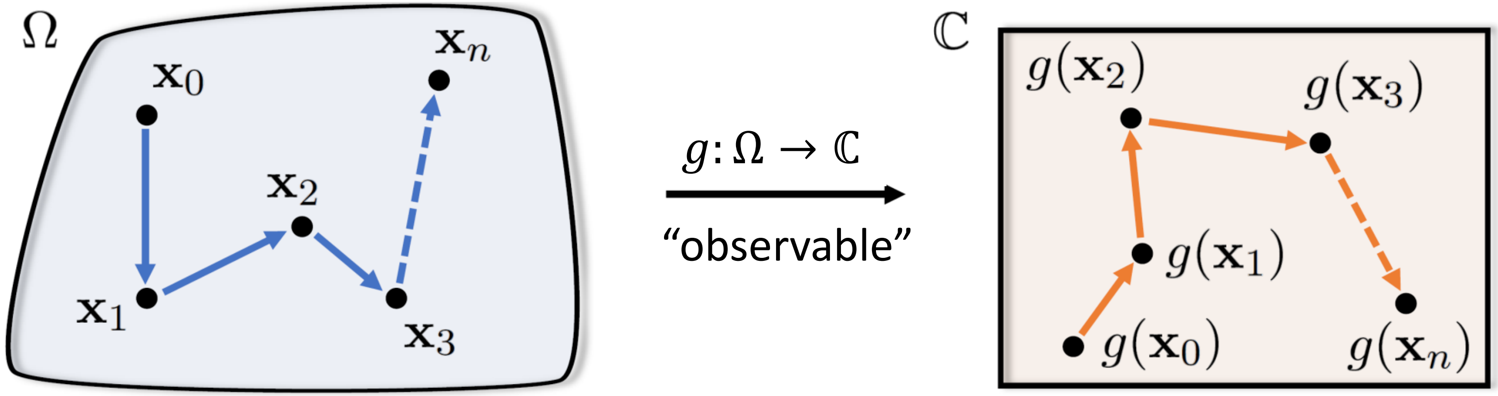
Goal: Learning from data $\{x^{(m)}, y^{(m)} = F(x^{(m)})\}_{m=1}^{M}$.

Applications: chemistry, climatology, control, electronics, epidemiology, finance, fluids, molecular dynamics, neuroscience, plasmas, robotics, video processing, etc.

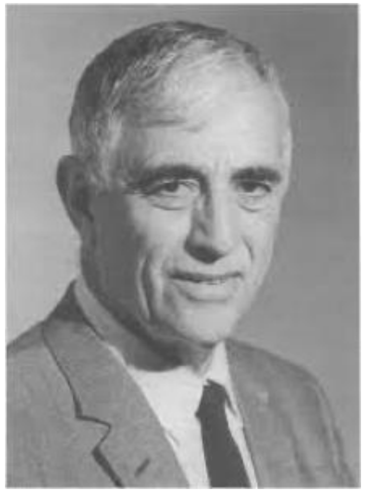


Surveys:
- Brunton, Budišić, Kaiser, Kutz, *"Modern Koopman theory for dynamical systems,"* SIAM Review, 2022.
- Budišić, Mohr, Mezić, *"Applied Koopmanism,"* Chaos, 2012.
- C., *"The Multiverse of Dynamic Mode Decomposition Algorithms,"* Handbook of Numerical Analysis, 2024.

Koopman Operator $\mathcal{K}$: A global linearization



Koopman



von Neumann

- Koopman, *"Hamiltonian systems and transformation in Hilbert space,"* **Proc. Natl. Acad. Sci. USA**, 1931.
- Koopman, v. Neumann, *"Dynamical systems of continuous spectra,"* **Proc. Natl. Acad. Sci. USA**, 1932.

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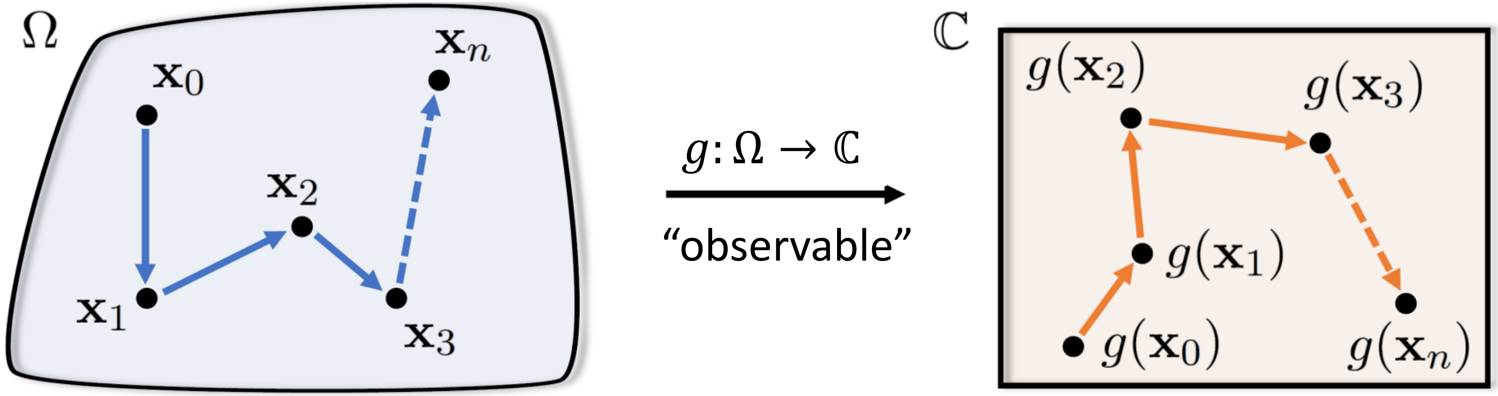

- $\mathcal{K}$ acts on functions $g: \Omega \to \mathbb{C}$, $[\mathcal{K}g](x) = g(F(x))$.
- **Function space:** $L^2(\Omega, \omega)$, positive measure ω, inner product $\langle \cdot, \cdot \rangle$.

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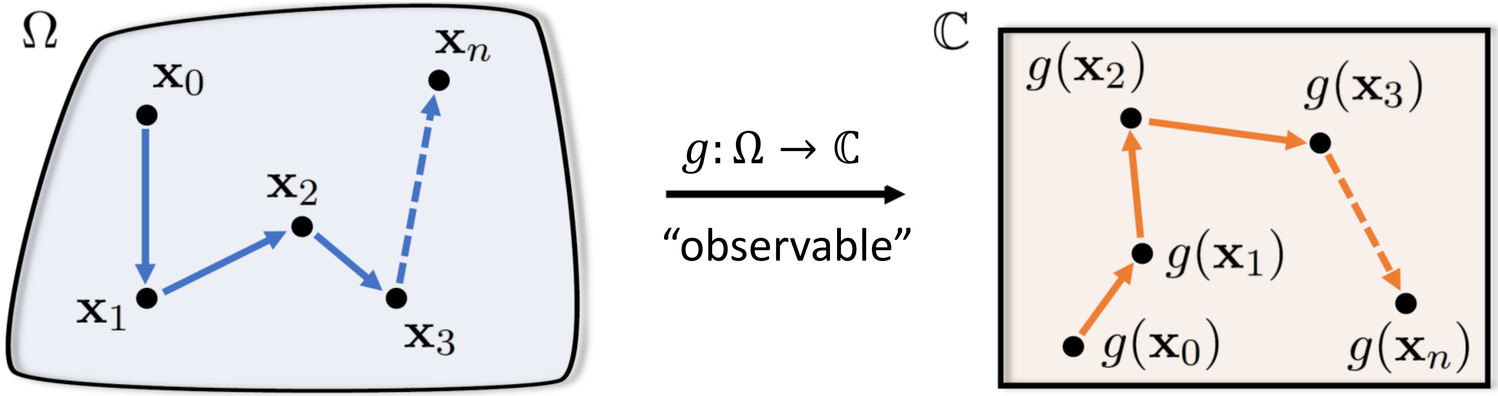

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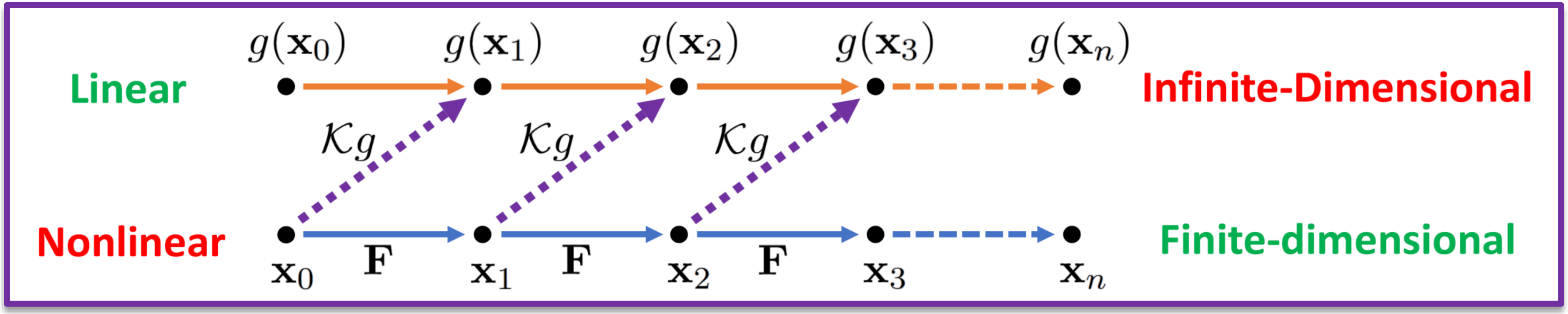

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- Koopman, v. Neumann, "*Dynamical systems of continuous spectra,*" **Proc. Natl. Acad. Sci. USA**, 1932.

Koopman mode decomposition

$x_{n+1} = F(x_n)$

$[\mathcal{K} g](x) = g(F(x))$

eigenfunction of $\mathcal{K}$

continuous spectrum

$$g(x) = \sum_{\text{eigenvalues } \lambda_j} c_{\lambda_j} \varphi_{\lambda_j}(x) + \int_{-\pi}^{\pi} \phi_{\theta,g}(x)\, \mathrm{d}\theta$$

$$g(x_n) = [\mathcal{K}^n g](x_0) = \sum_{\text{eigenvalues } \lambda_j} c_{\lambda_j} {\lambda_j}^n \varphi_{\lambda_j}(x_0) + \int_{-\pi}^{\pi} e^{in\theta} \phi_{\theta,g}(x_0)\, \mathrm{d}\theta$$

Encodes: geometric features, invariant measures, transient behavior, long-time behavior, coherent structures, quasiperiodicity, etc.

GOAL: Data-driven approximation of $\mathcal{K}$ and its spectral properties.

- Mezić, "*Spectral properties of dynamical systems, model reduction and decompositions,*" **Nonlinear Dynamics**, 2005.

Koopman mode decomposition

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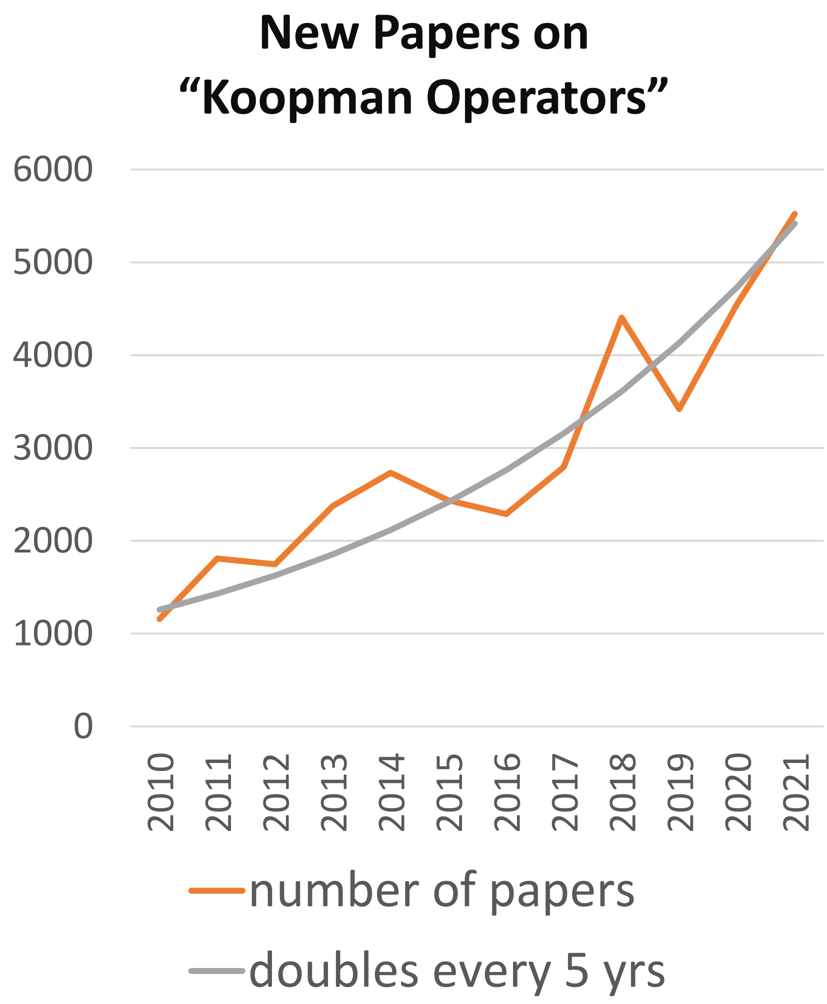

eigenfunction of $\mathcal{K}$

continuous spectrum

$$\sum_{\text{...values } \lambda_j} c_{\lambda_j}\varphi_{\lambda_j}(x) + \int_{-\pi}^{\pi} \phi_{\theta,g}(x)\, \mathrm{d}\theta$$

$$\sum_{\text{...values } \lambda_j} c_{\lambda_j}{\lambda_j}^n\varphi_{\lambda_j}(x_0) + \int_{-\pi}^{\pi} e^{in\theta}\phi_{\theta,g}(x_0)\, \mathrm{d}\theta$$

...s, invariant measures, transient behavior, ...t structures, quasiperiodicity, etc.

GOAL: Data-driven approximation of $\mathcal{K}$ and its **spectral properties.**

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Setting: Measure-preserving systems

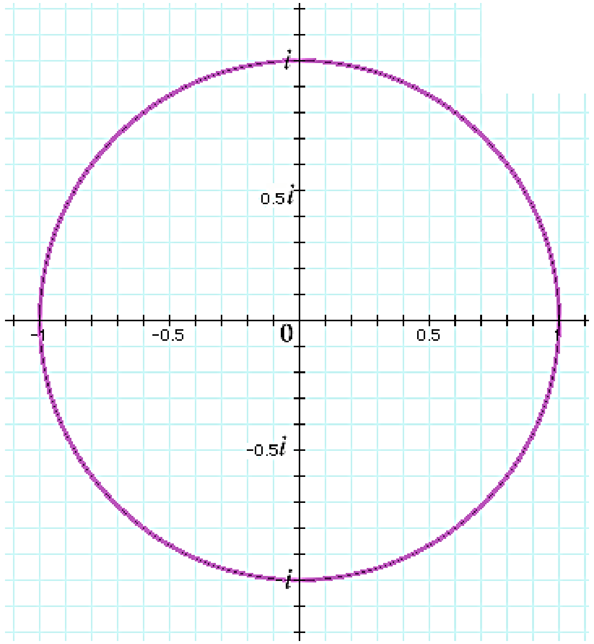
$$[\mathcal{K}g](x) = g\big(F(x)\big), \qquad g \in L^2(\Omega, \omega)$$

F preserves $\omega \Leftrightarrow \|\mathcal{K}g\| = \|g\|$ (isometry)

$$\Leftrightarrow \mathcal{K}^*\mathcal{K} = I$$

$$\Rightarrow \mathrm{Sp}(\mathcal{K}) \subseteq \{z : |z| \leq 1\}$$

(NB: unitary extensions of $\mathcal{K}$ via Wold decomposition.)



Shift example (on $\ell^2(\mathbb{Z})$)

$$\begin{pmatrix} \ddots & \ddots & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \in \mathbb{C}^{N\times N}$$

Two-way infinite

- Spectrum is unit circle.
- Spectrum is stable.
- Continuous spectra.
- Unitary evolution.

- Spectrum is $\{0\}$.
- Spectrum is unstable.
- Discrete spectra.
- Nilpotent evolution.

Lots of Koopman operators are built up from operators like these!

How to fix a Jordan block

$$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \in \mathbb{C}^{N\times N}$$

polar decomposition →

Circulant matrix

$$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ 1 & & & 0 \end{pmatrix} \in \mathbb{C}^{N\times N}$$

- Spectrum is $\{0\}$.
- Spectrum is unstable.
- Nilpotent evolution.

- Spectrum converges to unit circle as $N \to \infty$.
- Spectrum is stable.
- Unitary evolution.

Extended Dynamic Mode Decomposition (EDMD)

$$\Psi(x) = [\psi_1(x) \ \ \dots \ \ \psi_N(x)], \qquad g = \sum_{j=1}^{N} \mathbf{g}_j \psi_j = \Psi \mathbf{g} \in \operatorname{span}\{\psi_1, \dots, \psi_N\}$$

$$\min_{\mathbb{K} \in \mathbb{C}^{N \times N}} \left\{ \int_{\Omega} \max_{\|\mathbf{g}\|_2 = 1} |\Psi(x)\mathbb{K}\mathbf{g} - [\mathcal{K}g](x)|^2 \, \mathrm{d}\omega(x) = \int_{\Omega} \|\Psi(x)\mathbb{K} - \Psi(F(x))\|_2^2 \, \mathrm{d}\omega(x) \right\}$$

quadrature

$$\left\{x^{(m)}, y^{(m)} = F(x^{(m)})\right\}_{m=1}^{M}$$

$$\min_{\mathbb{K} \in \mathbb{C}^{N \times N}} \sum_{m=1}^{M} w_m \left\| \Psi(x^{(m)})\mathbb{K} - \Psi(y^{(m)}) \right\|_2^2$$

$\mathbb{K}$: **Galerkin method** on $V_N = \operatorname{span}\{\psi_1, \dots, \psi_N\}$

- Schmid, *"Dynamic mode decomposition of numerical and experimental data,"* **J. Fluid Mech.**, 2010.
- Rowley, Mezić, Bagheri, Schlatter, Henningson, *"Spectral analysis of nonlinear flows,"* **J. Fluid Mech.**, 2009.
- Kutz, Brunton, Brunton, Proctor, *"Dynamic mode decomposition: data-driven modeling of complex systems,"* **SIAM**, 2016.
- Williams, Kevrekidis, Rowley *"A data-driven approximation of the Koopman operator: Extending dynamic mode decomposition,"* **J. Nonlinear Sci.**, 2015.

A simple alteration

$$G_{jk} = \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})} \psi_k(x^{(m)}) \approx \langle \psi_k, \psi_j \rangle$$

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Measure-preserving: $\mathbf{g}^* G \mathbf{g} \approx \|g\|^2 = \|\mathcal{K} g\|^2 \approx \mathbf{g}^* \mathbb{K}^* G \mathbb{K} \mathbf{g}$

A simple alteration

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Enforce: $G = \mathbb{K}^* G \mathbb{K}$

A simple alteration

$$G_{jk} = \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})} \psi_k(x^{(m)}) \approx \langle \psi_k, \psi_j \rangle$$

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Enforce: $G = \mathbb{K}^* G \mathbb{K}$

quadrature

Orthogonal Procrustes problem

$$\min_{\substack{\mathbb{K} \in \mathbb{C}^{N \times N} \\ G = \mathbb{K}^* G \mathbb{K}}} \sum_{m=1}^{M} w_m \left\| \Psi(x^{(m)}) \mathbb{K} G^{-1/2} - \Psi(y^{(m)}) G^{-1/2} \right\|_2^2$$

The mpEDMD algorithm

Algorithm 4.1 The mpEDMD algorithm

Input: Snapshot data $\mathbf{X} \in \mathbb{C}^{d\times M}$ and $\mathbf{Y} \in \mathbb{C}^{d\times M}$, quadrature weights $\{w_m\}_{m=1}^{M}$, and a dictionary of functions $\{\psi_j\}_{j=1}^{N}$.

1: Compute the matrices $\mathbf{\Psi}_X$ and $\mathbf{\Psi}_Y$ and $\mathbf{W} = \mathrm{diag}(w_1, \ldots, w_M)$.
2: Compute an economy QR decomposition $\mathbf{W}^{1/2}\mathbf{\Psi}_X = \mathbf{QR}$, where $\mathbf{Q} \in \mathbb{C}^{M\times N}, \mathbf{R} \in \mathbb{C}^{N\times N}$.
3: Compute an SVD of $(\mathbf{R}^{-1})^*\mathbf{\Psi}_Y^*\mathbf{W}^{1/2}\mathbf{Q} = \mathbf{U}_1\mathbf{\Sigma}\mathbf{U}_2^*$.
4: Compute the eigendecomposition $\mathbf{U}_2\mathbf{U}_1^* = \hat{\mathbf{V}}\mathbf{\Lambda}\hat{\mathbf{V}}^*$ (via a Schur decomposition).
5: Compute $\mathbb{K} = \mathbf{R}^{-1}\mathbf{U}_2\mathbf{U}_1^*\mathbf{R}$ and $\mathbf{V} = \mathbf{R}^{-1}\hat{\mathbf{V}}$.

Output: Koopman matrix $\mathbb{K}$ with eigenvectors $\mathbf{V}$ and eigenvalues $\mathbf{\Lambda}$.

$V_N = \mathrm{span}\,\{\psi_1, \ldots, \psi_N\}$
$\mathcal{P}_{V_N}: L^2(\Omega, \omega) \to V_N$
orthogonal projection

As $M \to \infty$, **unitary part** of polar decomposition of $\mathcal{P}_{V_N}\mathcal{K}\mathcal{P}_{V_N}^*$.

- C., *"The mpEDMD Algorithm for Data-Driven Computations of Measure-Preserving Dynamical Systems,"* **SINUM**, 2023.
- Code: https://github.com/MColbrook/Measure-preserving-Extended-Dynamic-Mode-Decomposition

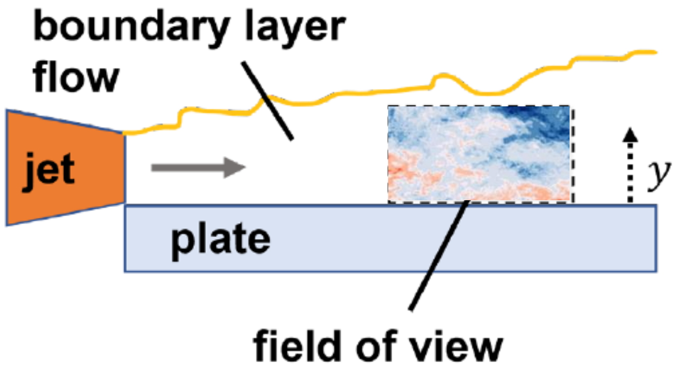
Convergence properties (theorems in paper)

- Spectral measures.
- Functional calculus, L^2 forecasting etc.
- Koopman mode decomposition.
- Spectrum.
- Resolvent (see later!)

Key ingredient: **unitary** discretization.

Turbulence (real data)

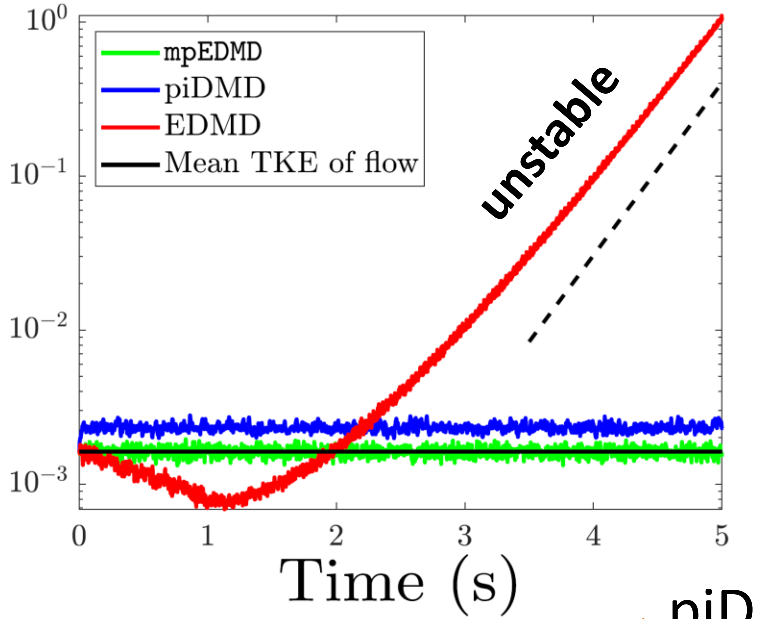
Experimental setup



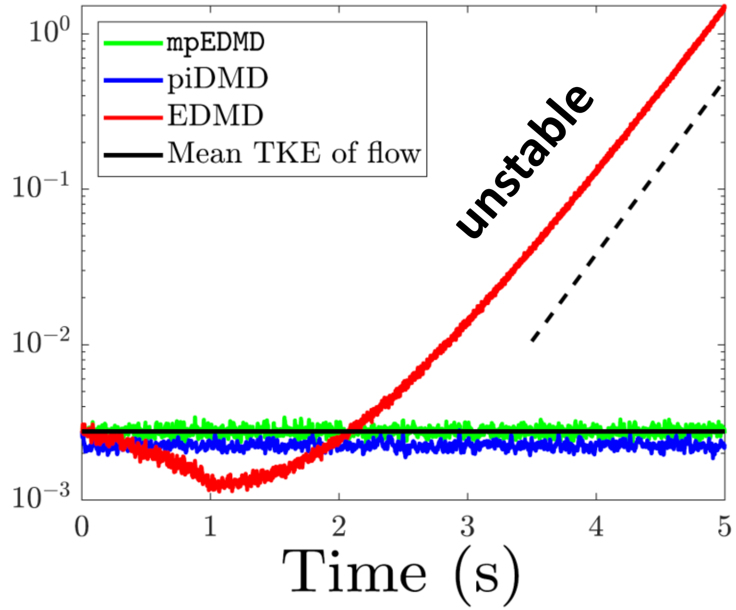
- Reynolds number $\approx 6.4 \times 10^4$
- Ambient dimension (d) $\approx 100{,}000$ (velocity at measurement points)

*PIV data provided by Máté Szőke (Virginia Tech)

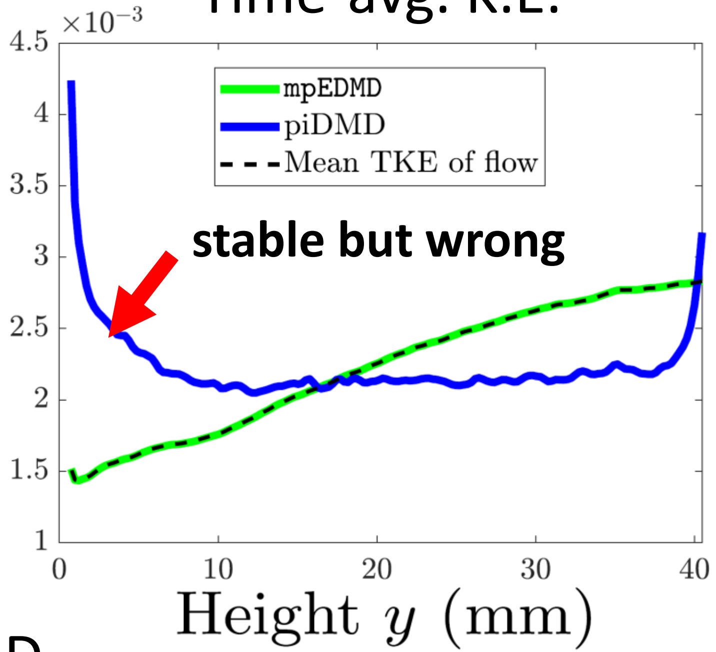
Turbulent K.E. y=5mm



Turbulent K.E. y=35mm



Time-avg. K.E.

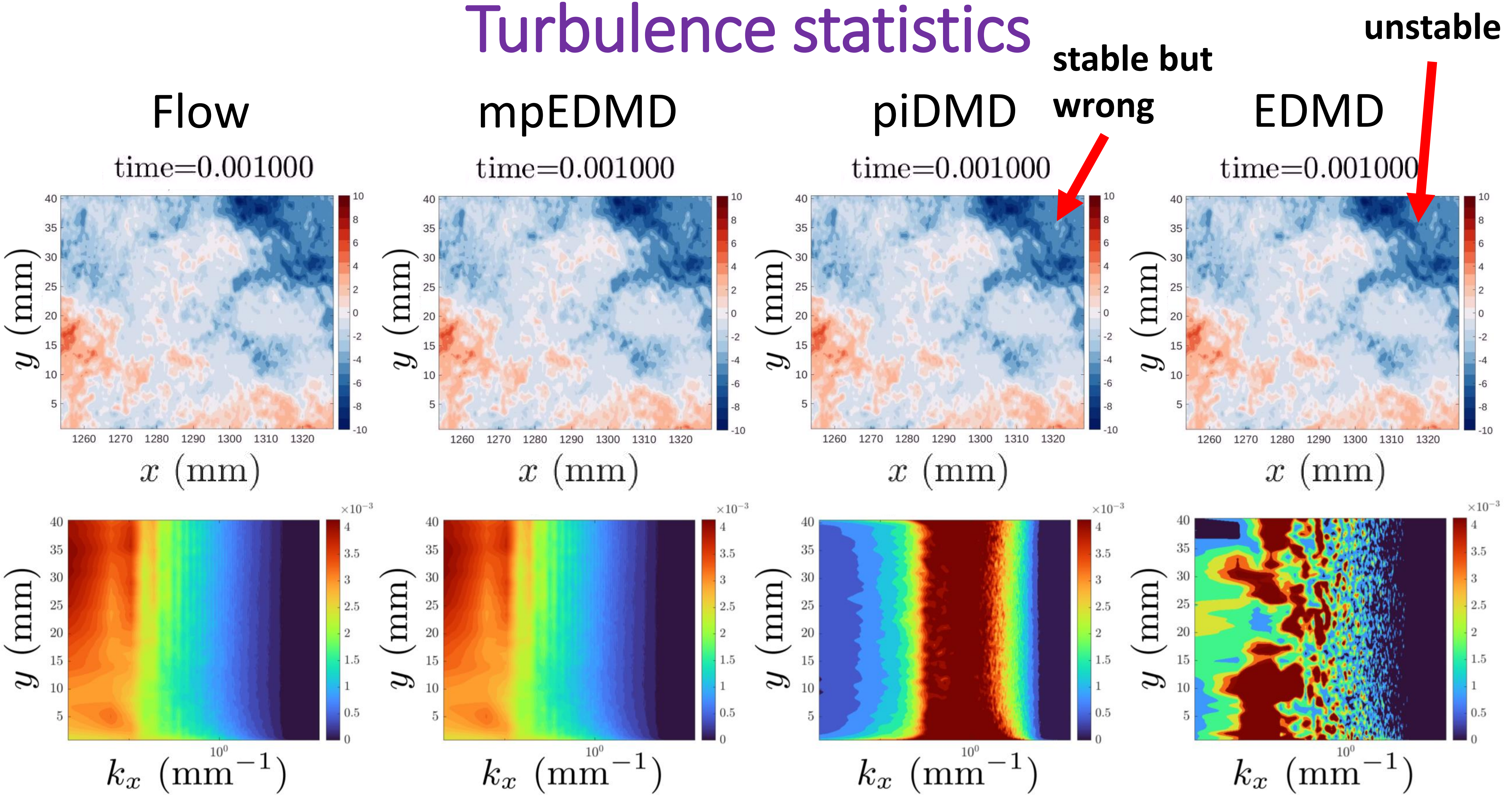


piDMD

EDMD

- Baddoo, Herrmann, McKeon, Kutz, Brunton, *"Physics-informed dynamic mode decomposition (piDMD),"* preprint.
- Williams, Kevrekidis, Rowley *"A data-driven approximation of the Koopman operator: Extending dynamic mode decomposition,"* **J. Nonlinear Sci.**, 2015.

Turbulence statistics



Back to the shift!

$$e_j \to e_{j-1}$$

"Solve" $(U - zI)u_z = 0$

$$u_z = \sum_{j=-\infty}^{\infty} z^j e_j$$

$$U = \begin{pmatrix} \ddots & \ddots & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix}$$

Two-way infinite

Back to the shift!

Doesn't live in $\ell^2(\mathbb{Z})$!!!

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Two-way infinite

Let $|z| = 1$, $\phi = \sum_{j=-\infty}^{\infty} \phi_j e_j$ where ϕ_j decay faster than any inverse polynomial.

Test functions

Back to the shift!

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Generalised eigenfunctions u_z and generalised eigenvalues $\{z : |z| = 1\}$

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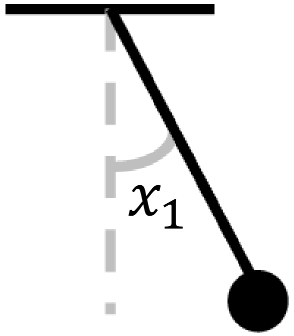
Generalised eigenfunctions u_z and generalised eigenvalues $\{z : |z| = 1\}$

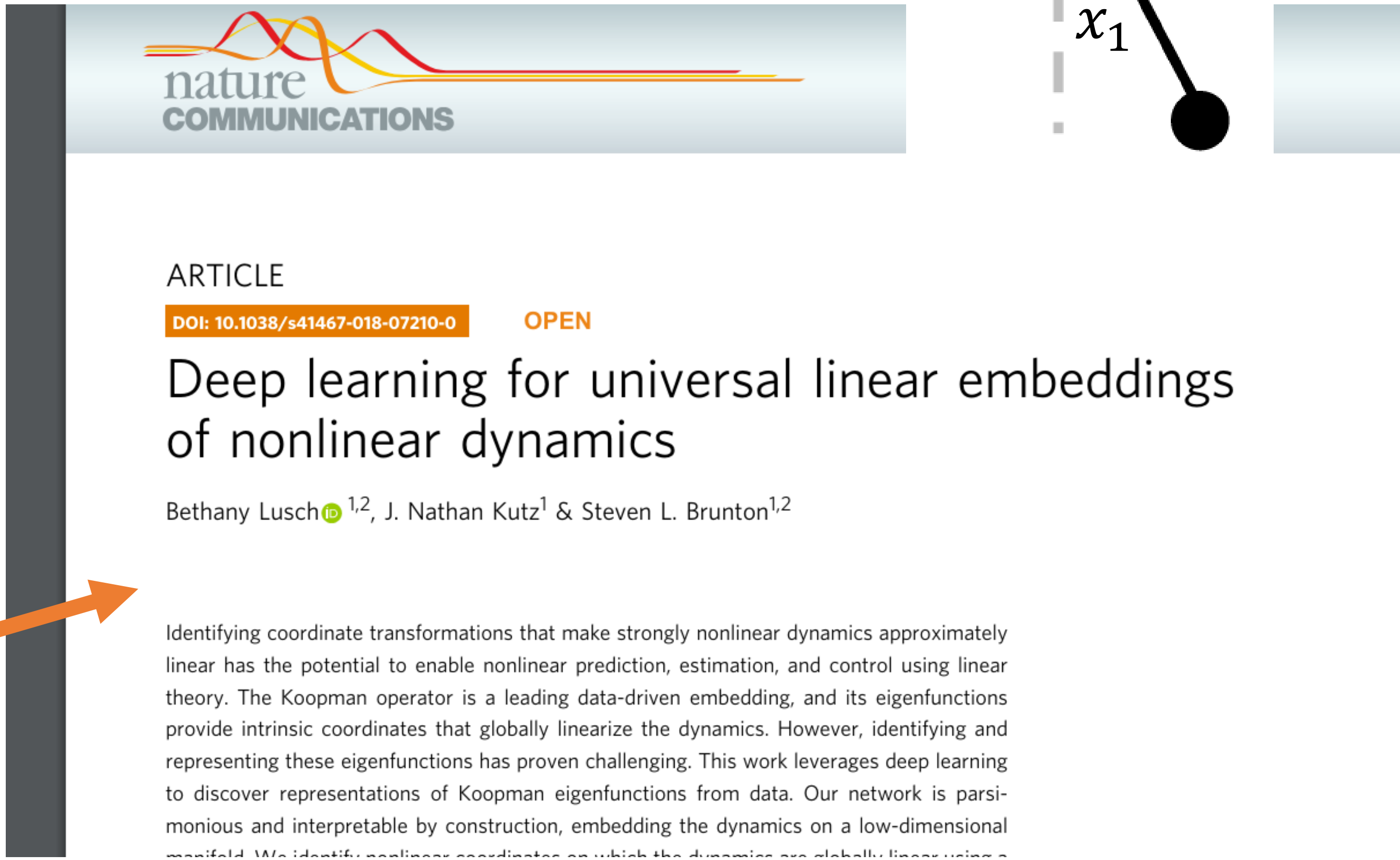
RIGGED HILBERT SPACE

Example: Nonlinear pendulum



$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin(x_1)$$
$$\Omega = [-\pi, \pi]_{\text{per}} \times \mathbb{R},$$
$$\Delta_t = 1,$$
$\omega =$ Lebesgue measure

Considered a challenge in Koopman theory!


nature COMMUNICATIONS

ARTICLE

DOI: 10.1038/s41467-018-07210-0 OPEN

Deep learning for universal linear embeddings of nonlinear dynamics

Bethany Lusch[1,2], J. Nathan Kutz[1] & Steven L. Brunton[1,2]

Identifying coordinate transformations that make strongly nonlinear dynamics approximately linear has the potential to enable nonlinear prediction, estimation, and control using linear theory. The Koopman operator is a leading data-driven embedding, and its eigenfunctions provide intrinsic coordinates that globally linearize the dynamics. However, identifying and representing these eigenfunctions has proven challenging. This work leverages deep learning to discover representations of Koopman eigenfunctions from data. Our network is parsimonious and interpretable by construction, embedding the dynamics on a low-dimensional

Explicit diagonalization using Radon transform!

- Action-angle coordinates (n degrees of freedom):

$$\dot{\mathbf{I}} = \mathbf{0}, \qquad \dot{\boldsymbol{\theta}} = \mathbf{I}, \qquad \Omega = \mathbb{R}^n \times [-\pi, \pi]^n_{\mathrm{per}}$$

Explicit diagonalization using Radon transform!

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$$\dot{\mathbf{I}} = \mathbf{0}, \qquad \dot{\boldsymbol{\theta}} = \mathbf{I}, \qquad \Omega = \mathbb{R}^n \times [-\pi, \pi]^n_{\mathrm{per}}$$

- g in Schwartz space,

$$g(\mathbf{I}, \boldsymbol{\theta}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} \hat{g}_{\mathbf{k}}(\mathbf{I}) e^{i\mathbf{k}\cdot\boldsymbol{\theta}}, \qquad \hat{g}_{\mathbf{k}}(\mathbf{I}) = \frac{1}{(2\pi)^n} \int_{[-\pi,\pi]^n_{\mathrm{per}}} g(\mathbf{I}, \boldsymbol{\theta})\, e^{-i\mathbf{k}\cdot\boldsymbol{\theta}} \mathrm{d}\boldsymbol{\theta}$$

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$$\hat{g}_{\mathbf{k}}(\mathbf{I}) = \sum_{j=1}^{\infty}\sum_{m=-\infty}^{\infty}\int_{[-\pi,\pi]_{\mathrm{per}}} \left\langle g_\theta^{(\mathbf{k},m,j)^*} | g \right\rangle g_\theta^{(\mathbf{k},m,j)} \mathrm{d}\theta$$

$$g_\theta^{(\mathbf{k},m,j)} = \frac{1}{(2\pi)^n}\delta(\theta + 2\pi m - \Delta t\mathbf{k}\cdot\mathbf{I})\psi_j^{(k)} e^{i\mathbf{k}\cdot\boldsymbol{\theta}}$$

Plane wave

Generalised eigenfunctions

Supported on hyperplane

Orthonormal basis of hyperplane

Gelfand's theorem → diagonalisation

- **Finite matrix:** $B \in \mathbb{C}^{n\times n}$, $B^*B = BB^*$, orthonormal basis of e-vectors $\{v_j\}_{j=1}^{n}$

$$v = \sum_{j=1}^{n} (v_j{}^* v) v_j, \qquad Bv = \sum_{j=1}^{n} \lambda_j (v_j{}^* v) v_j \qquad \forall v \in \mathbb{C}^n$$

- **Infinite dimensions:** Unitary $\mathcal{K}$. Typically, **no basis of eigenfunctions**! Some technical assumptions (can always be realized):

$$g = \int_{[-\pi,\pi]_{\mathrm{per}}} \langle g_\theta^* | g \rangle \, g_\theta \mathrm{d}\nu(\theta), \qquad \mathcal{K}g = \int_{[-\pi,\pi]_{\mathrm{per}}} e^{i\theta} \langle g_\theta^* | g \rangle \, g_\theta \mathrm{d}\nu(\theta)$$

$g \in \mathcal{S} \subset L^2(\Omega, \omega)$

Koopman modes

generalized eigenfunctions
distributions $\in \mathcal{S}^*$

$e^{i\theta} = \lambda$

Koopman Mode Decomposition

Rigged DMD: Smoothing

Carathéodory function:

$$F_g(z) = (\mathcal{K} + zI)(\mathcal{K} - zI)^{-1}g = \int_{[-\pi,\pi]_{\text{per}}} \boxed{\frac{e^{i\theta} + z}{e^{i\theta} - z}} \langle g_\theta^* | g \rangle \, g_\theta \, \mathrm{d}\nu(\theta)$$

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Let $r = 1 + \varepsilon > 1, \theta_0 \in [-\pi, \pi]_{\text{per}},$

$$\frac{1}{4\pi}\left[F_g\left(r^{-1}e^{i\theta_0}\right) - F_g\left(re^{i\theta_0}\right)\right]$$

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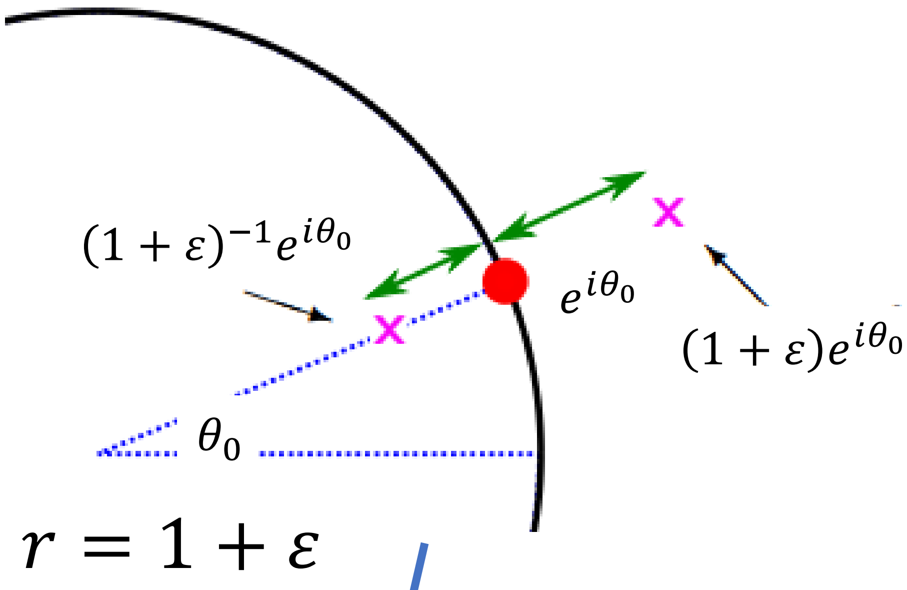
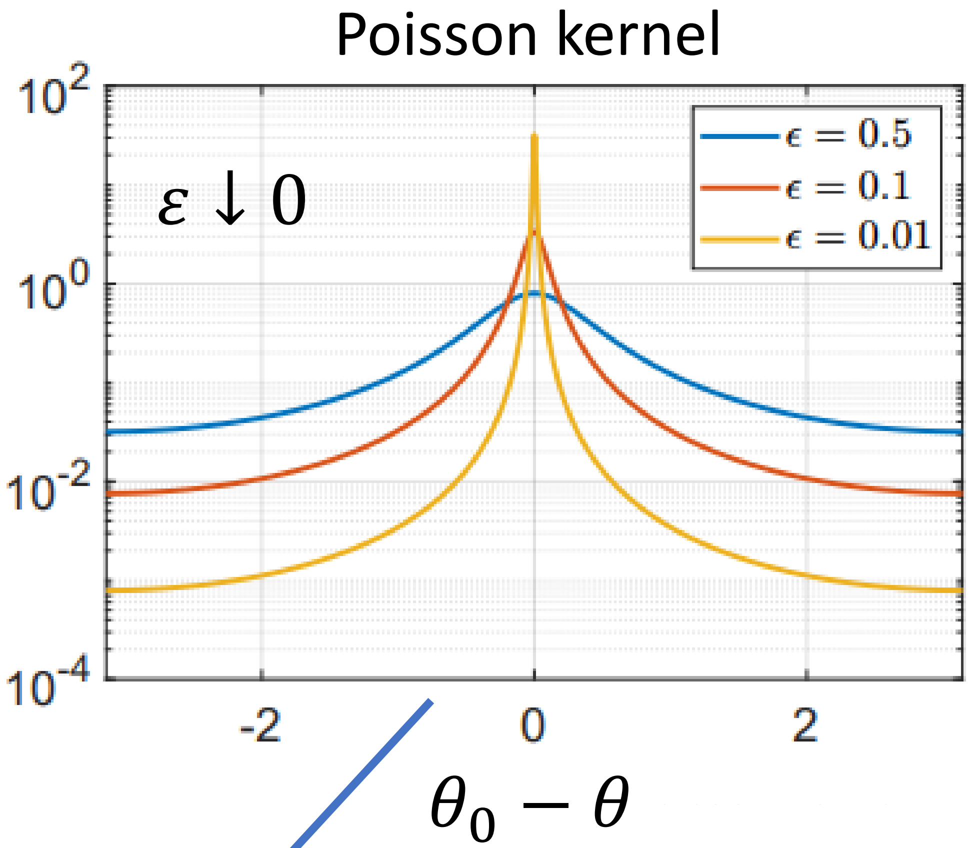
Poisson kernel

Smoothed generalized eigenfunction





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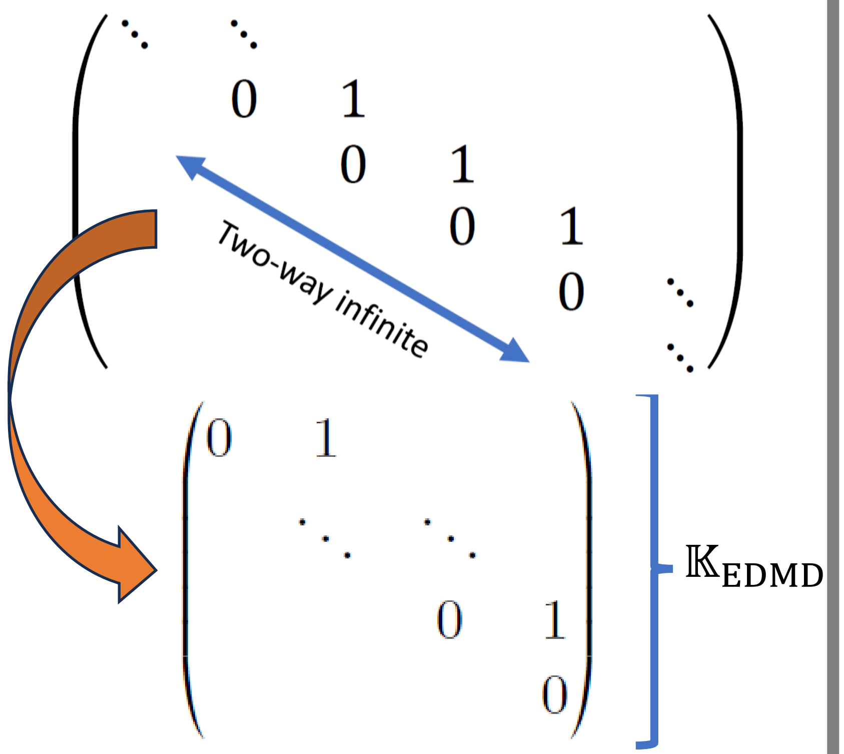
Smoothed generalized eigenfunction

F_g requires $(\mathcal{K} - zI)^{-1}$

EDMD diverges:



Acts on $\mathrm{span}\{e_{-N}, \ldots, e_N\}$

E.g., if $|z| < 1$,

$$(\mathbb{K}_{\mathrm{EDMD}} - zI)^{-1} e_0 = \sum_{j=1}^{N} \frac{-1}{z^j} e_{-j}$$

Exponential blowup as $N \to \infty$.

F_g requires $(\mathcal{K} - zI)^{-1}$

EDMD diverges:

$$\begin{pmatrix} \ddots & \ddots & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix}$$

Two-way infinite

$$\mathbb{K}_{\text{EDMD}} = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

Acts on $\text{span}\{e_{-N}, \dots, e_N\}$

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mpEDMD converges:

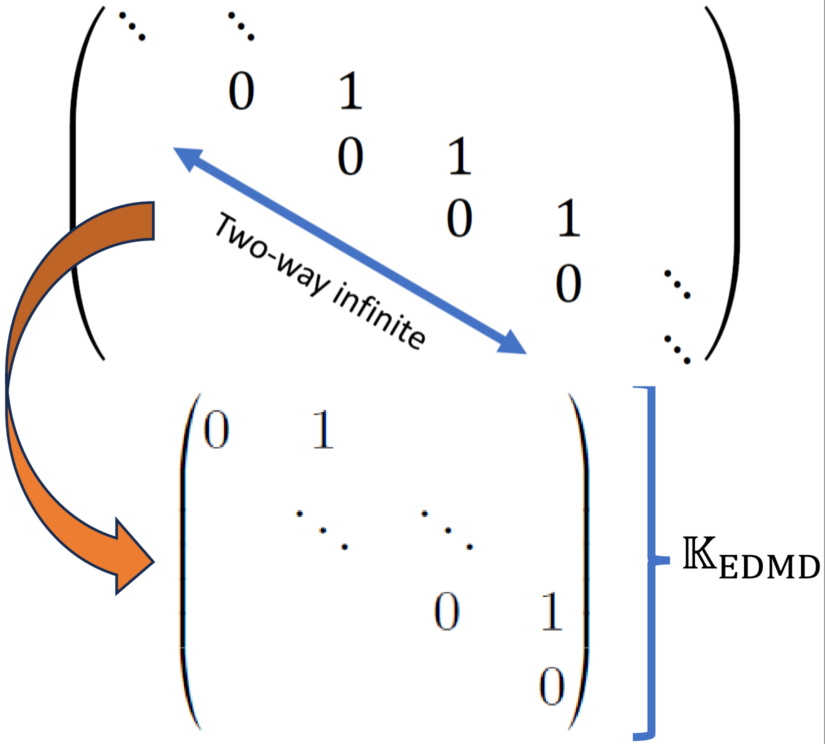
$$\begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

$$\mathbb{K}_{\text{mpEDMD}} = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 \\ 1 & & & 0 \end{pmatrix}$$

General method: unitary part of a **polar decomposition** of EDMD!

F_g requires $(\mathcal{K} - zI)^{-1}$

EDMD diverges:



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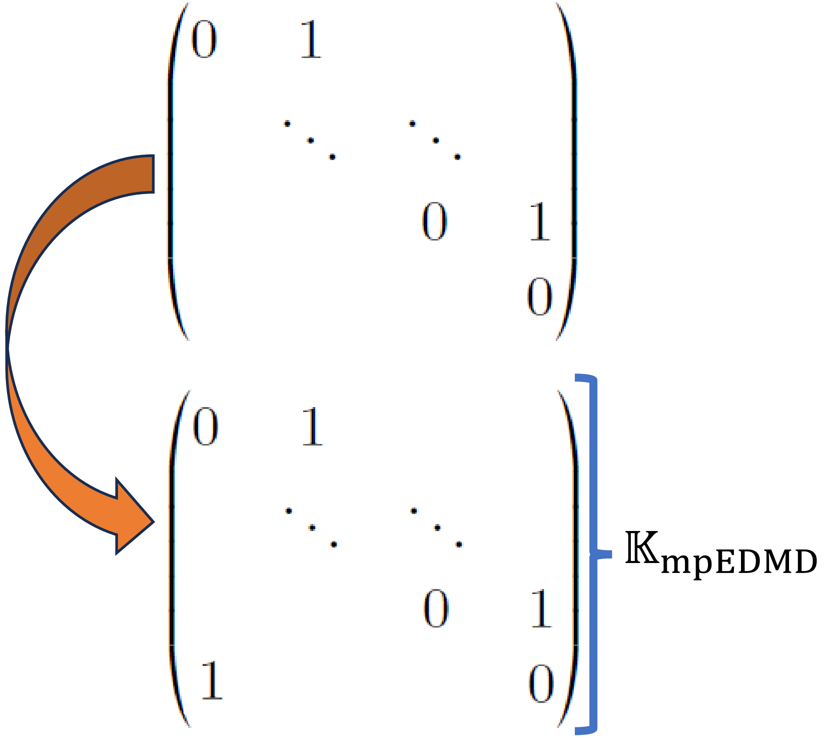


General method: unitary part of a **polar decomposition** of EDMD!

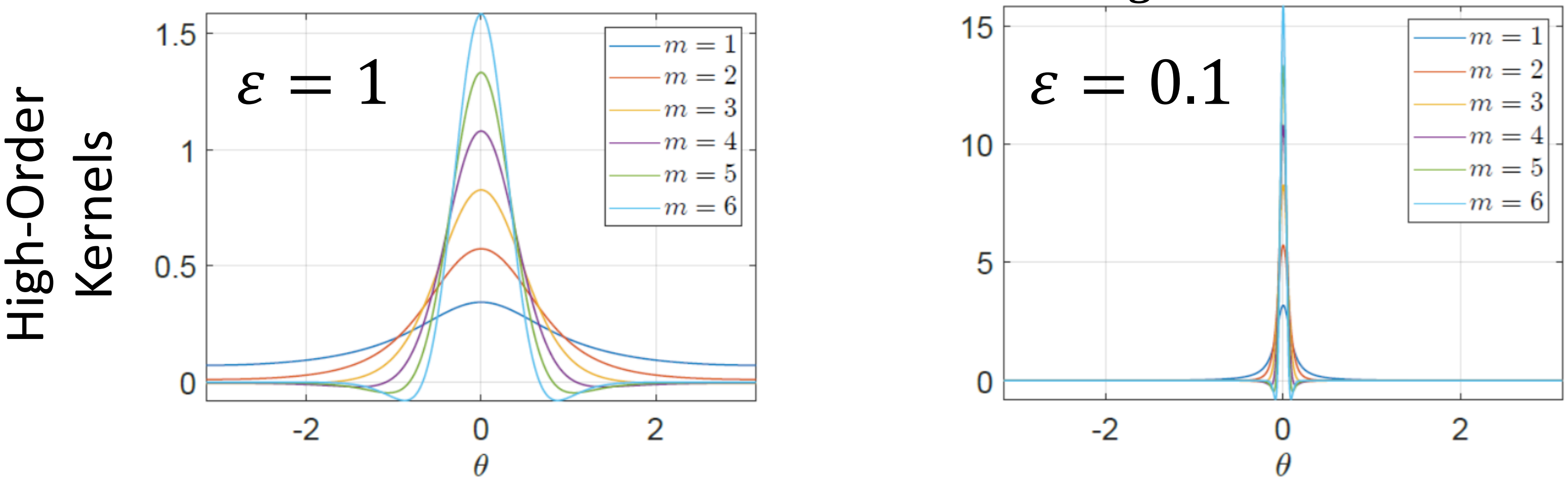
Rigged DMD converges:

- For general $\mathcal{K}$: $(\mathbb{K}_{\mathrm{mpEDMD}} - zI)^{-1}\mathbf{g}$ converges to $(\mathcal{K} - zI)^{-1} g$ as $\lim_{N\to\infty} \lim_{M\to\infty}$
- Hence, Rigged DMD converges as $\lim_{\varepsilon \downarrow 0} \lim_{N\to\infty} \lim_{M\to\infty}$
- ResDMD allows us to select $\varepsilon = \varepsilon(N)$ adaptively (convergence in **2 limits**)

Better smoothing kernels as $\varepsilon \downarrow 0$

- Poisson kernel: **slow** convergence $\mathcal{O}(\varepsilon \log(1/\varepsilon))$.
- Construct high-order *rational* kernels using $F_g(z)$.

Smaller ε requires more data



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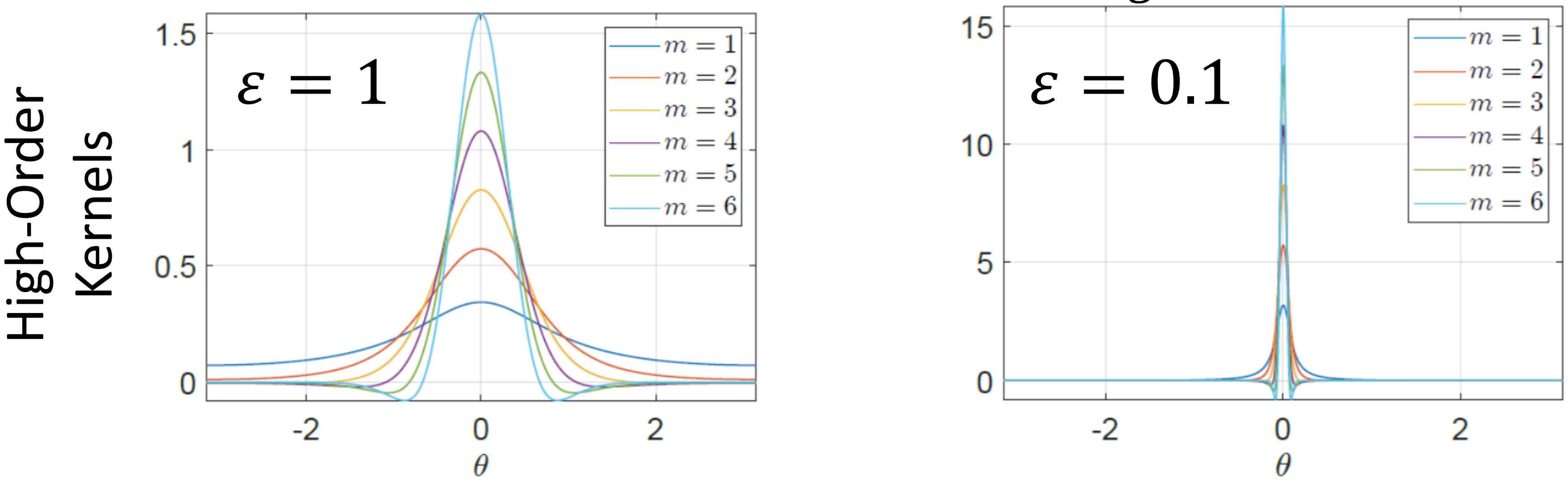
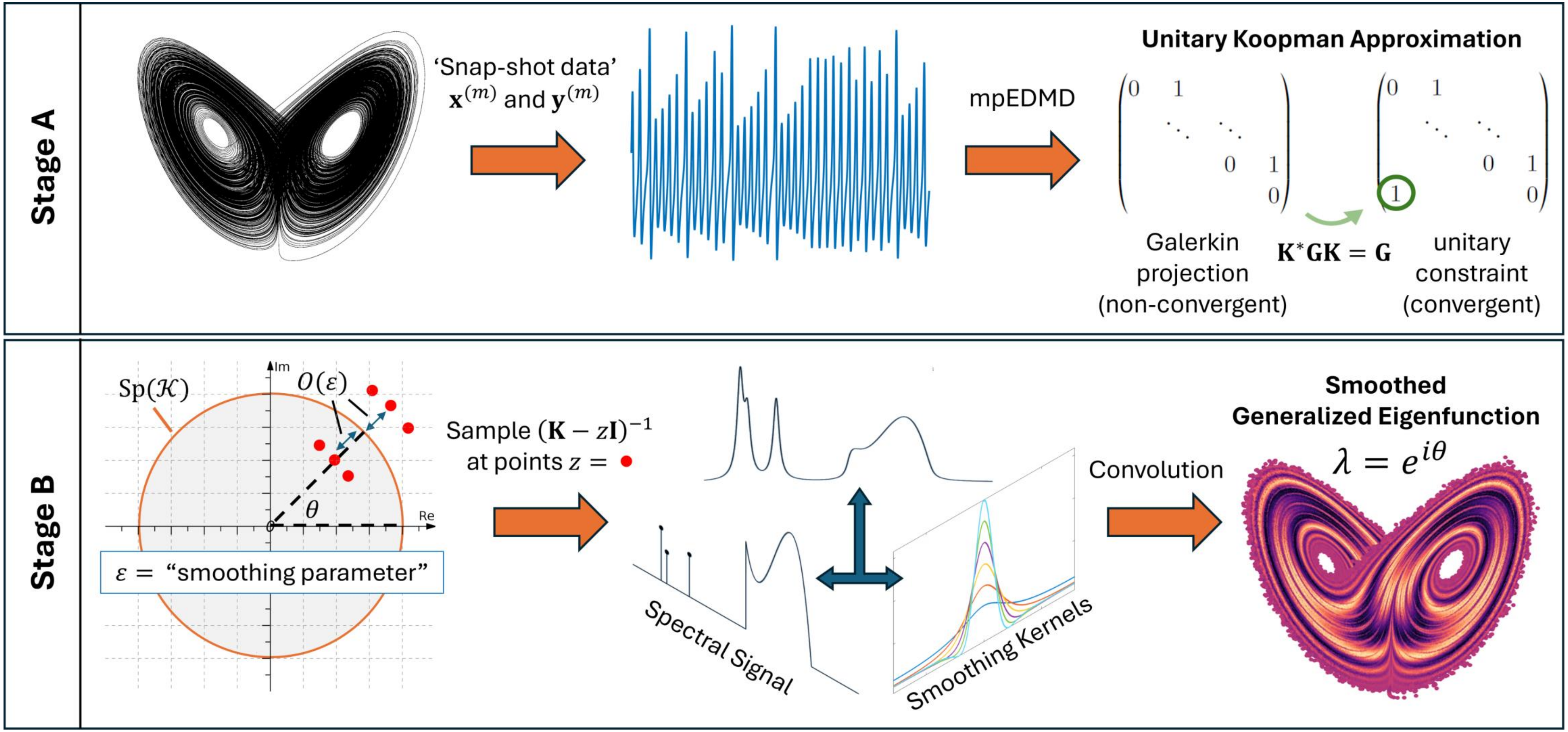


Theorem: Suppose quadrature rule converges & $\lim_{N\to\infty} \mathrm{dist}(h, V_N) = 0$ for any $h \in L^2(\Omega, \omega)$. Choosing $N = N(\varepsilon)$, **fast** $\mathcal{O}(\varepsilon^m \log(1/\varepsilon))$ convergence for:

- Generalized eigenfunctions (topology of $\mathcal{S}^*$).
- Spectral measures (gen. efun. projections): pointwise, L^p, weak,…
- Forecasting (i.e., iterating Koopman mode decomposition), coherency etc.

Rigged DMD

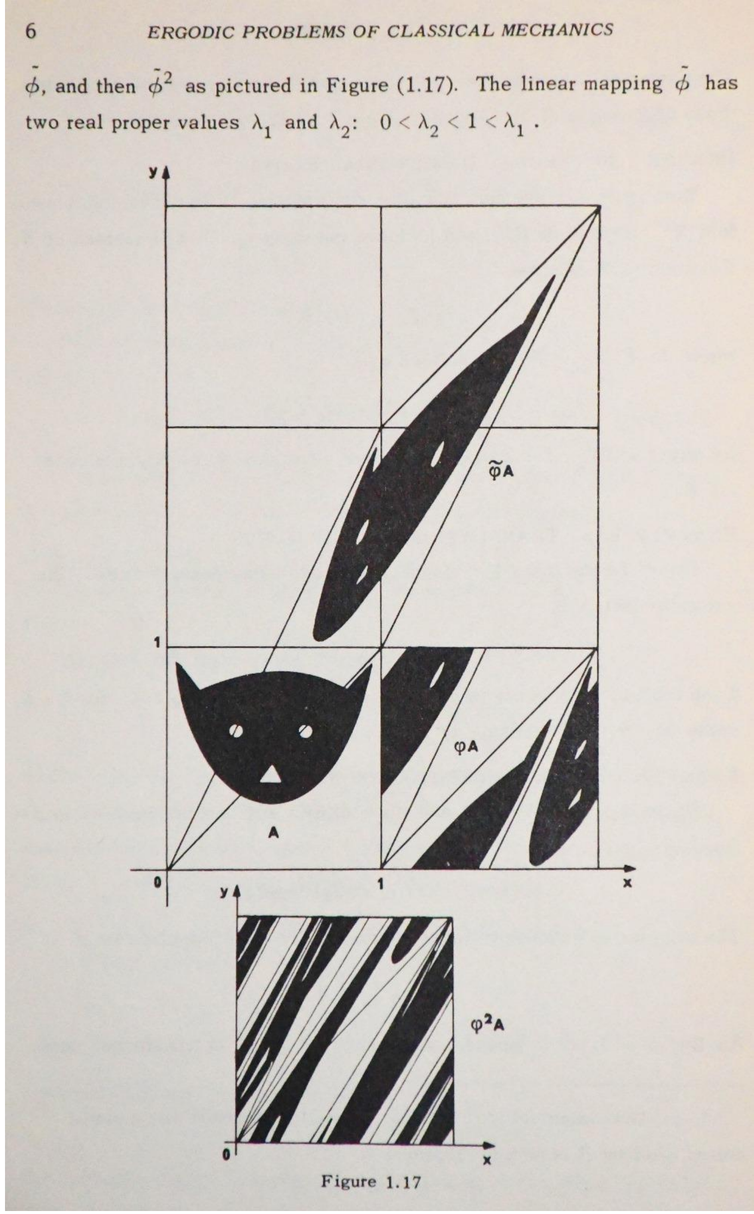


- C., Drysdale, Horning, *"Rigged Dynamic Mode Decomposition: Data-Driven Generalized Eigenfunction Decompositions for Koopman Operators"*, arxiv preprint.
- Code*:* https://github.com/MColbrook/Rigged-Dynamic-Mode-Decomposition

Example: Arnold's cat map

$F(x, y) = (2x + y, x + y) \mod 2\pi$

$\Omega = [-\pi, \pi]^2_{\text{per}}, \quad \omega =$ Lebesgue measure


6 ERGODIC PROBLEMS OF CLASSICAL MECHANICS

$\bar{\phi}$, and then $\bar{\phi}^2$ as pictured in Figure (1.17). The linear mapping $\bar{\phi}$ has two real proper values λ_1 and λ_2: $0 < \lambda_2 < 1 < \lambda_1$.

Figure 1.17

Arnold's "Ergodic Problems of Classical Mechanics"

Example: Arnold's cat map

Experimental details

Length-one trajectories, $M = 50 \times 50, N = 500$

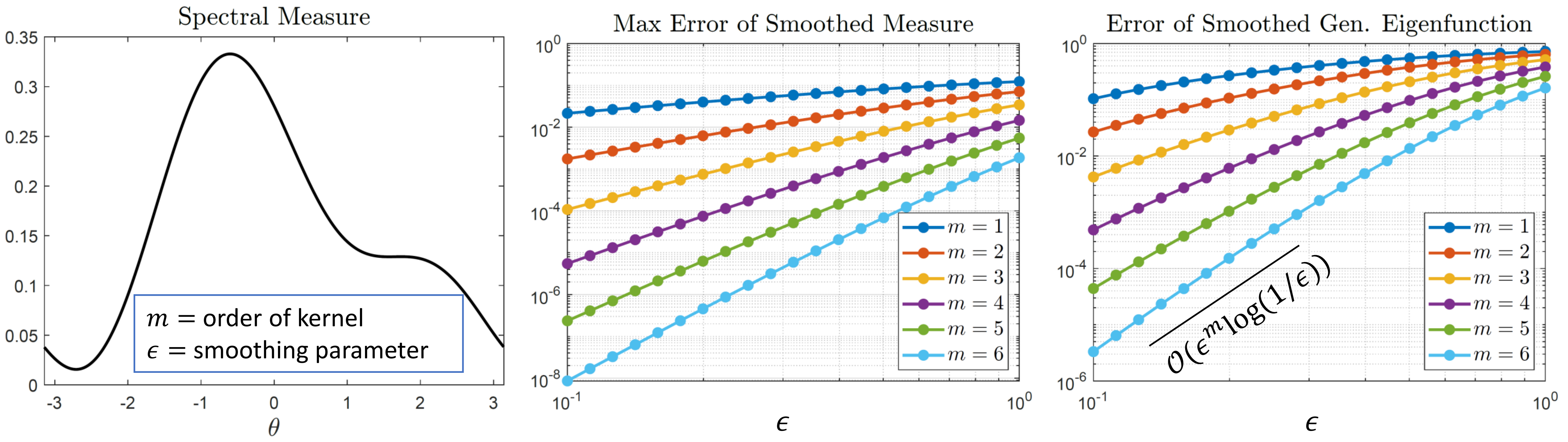
$g(x,y) = \sin(x) + \frac{1}{2}\sin(2x+y) + \frac{i}{4}\sin(5x+3y)$

Krylov subspace: $V_N = \{g, \mathcal{K}g, \ldots, \mathcal{K}^{N-1}g\}$

$F(x,y) = (2x+y, x+y) \mod 2\pi$

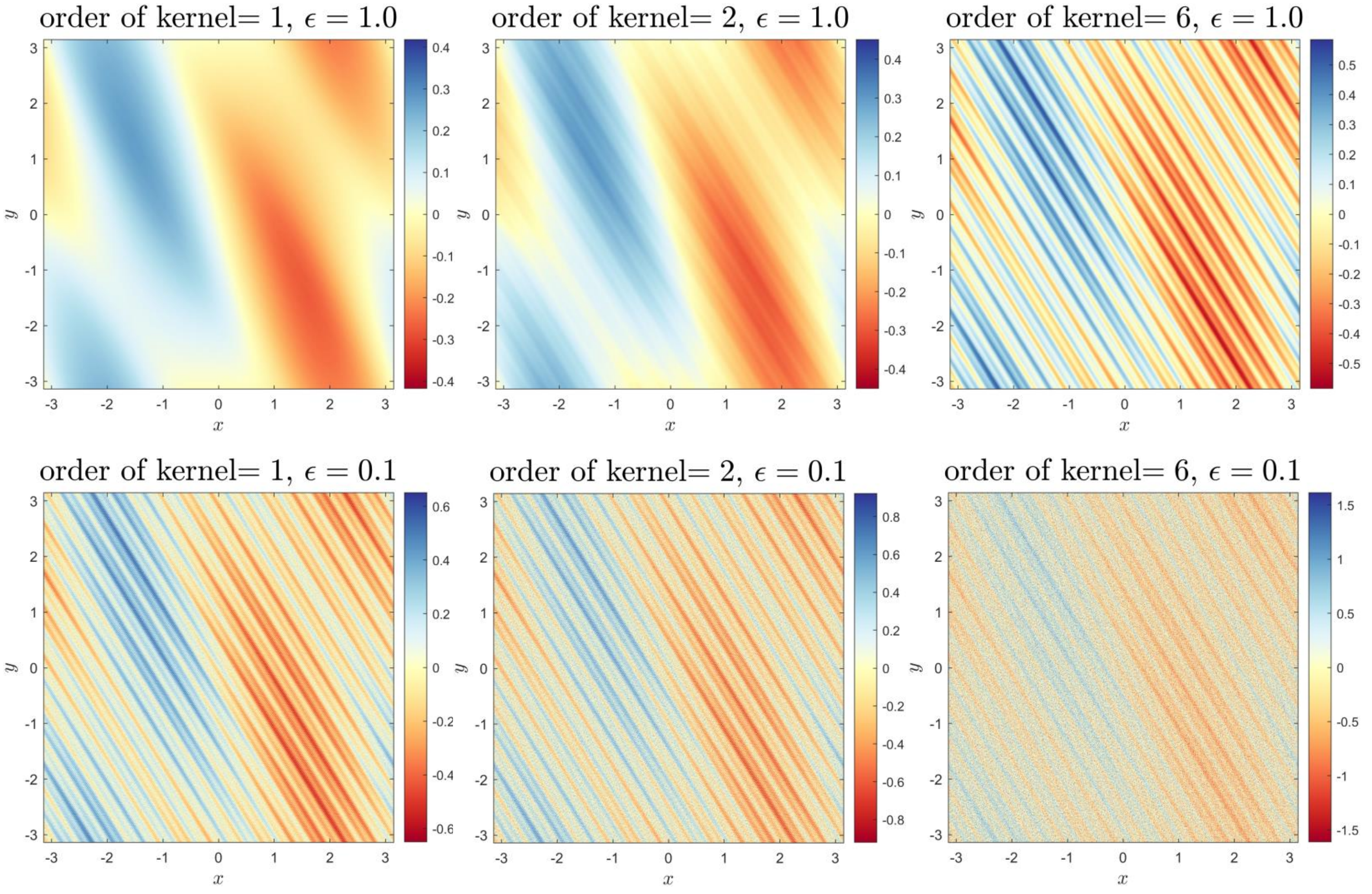
$\Omega = [-\pi,\pi]^2_{\text{per}}, \quad \omega =$ Lebesgue measure

Explicit formula: g_θ become more oscillatory as $\epsilon \downarrow 0$ (non-decaying Fourier series)



Higher kernel order (accuracy)

Higher resolution ($\varepsilon \downarrow 0$)



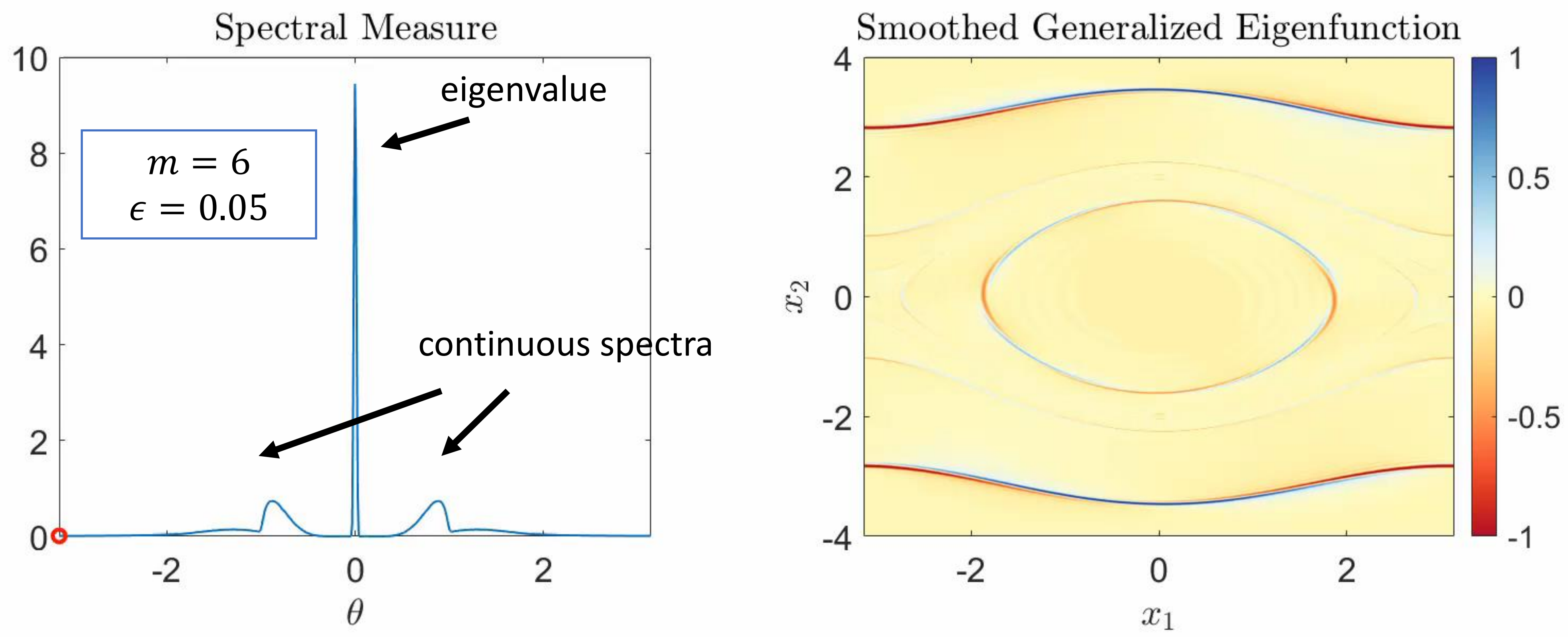
Increased oscillations of generalized eigenfunction

Example: Nonlinear pendulum

Experimental Details
Length-one trajectories over grid
$M = 500 \times 500, N = 300$
$g(x_1, x_2) = \exp(ix_1)/\cosh(x_2)$
Krylov subspace: $V_N = \{g, \mathcal{K}g, \dots, \mathcal{K}^{N-1}g\}$

$\dot{x}_1 = x_2,\ \dot{x}_2 = -\sin(x_1)$, $\Omega = [-\pi, \pi]_{\text{per}} \times \mathbb{R}$, $\Delta_t = 1$, $\omega =$ Lebesgue measure

Explicit formula: g_θ become plane waves concentrated on unions of lines of constant energy as $\epsilon \downarrow 0$.



$\mathcal{S}$ constructed from Krylov subspace

- If $\mathcal{K}$ is represented by an infinite matrix with finitely many non-zero entries in each column, can build $\mathcal{S}$ using weighted sequence spaces.
- Always possible using time-delay embedding:

$$\{\text{Unions (different } g\text{) of spaces } \operatorname{span}\{g, \mathcal{K}g, \ldots, \mathcal{K}^{N-1}g, \ldots\}\} \subset \mathcal{S}$$

- Generalises shift example: in coefficient space w.r.t. Krylov subspaces.

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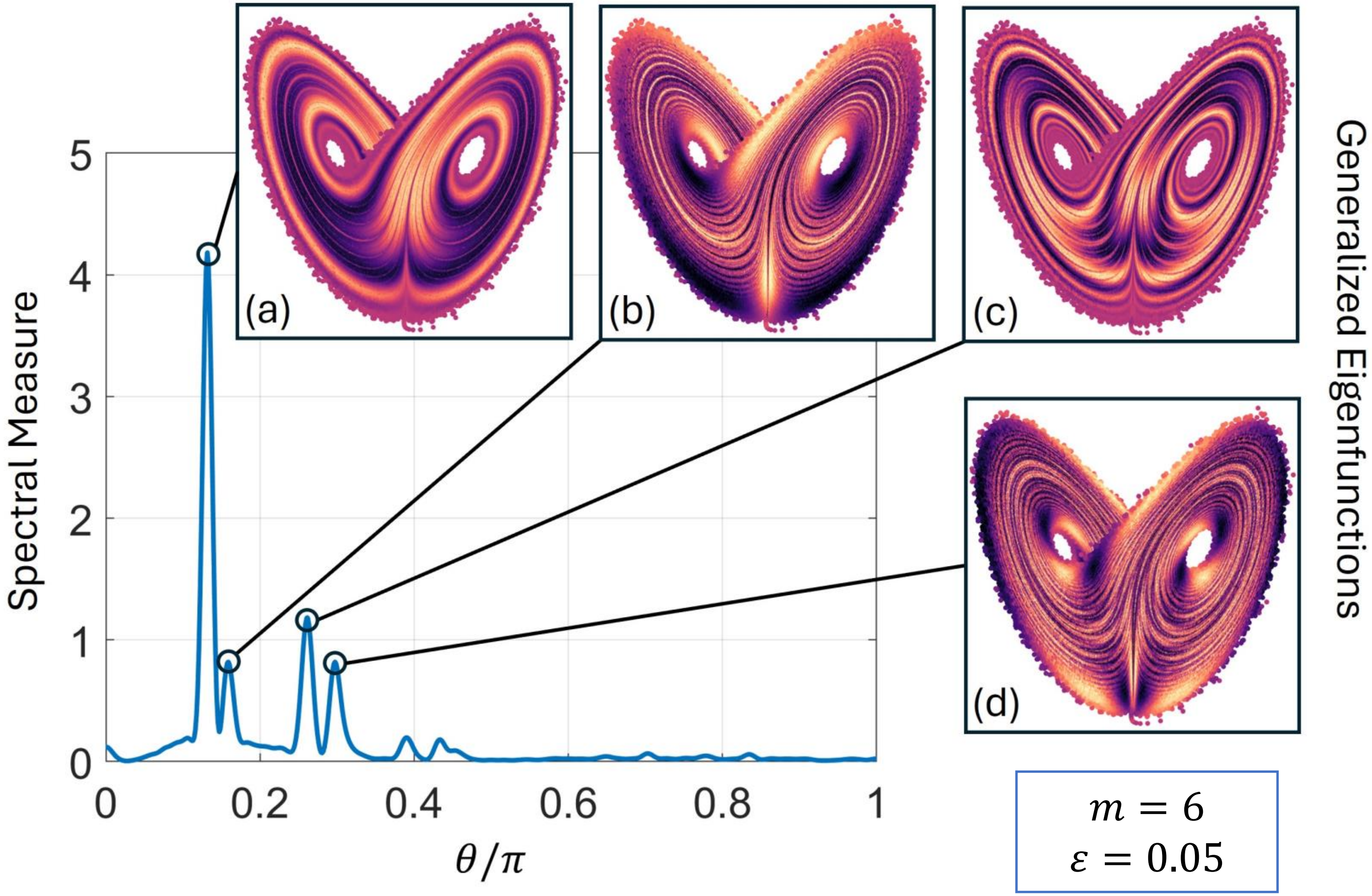
- Generalises shift example: in coefficient space w.r.t. Krylov subspaces.

Let's do this for Lorenz…

Example: Lorenz system

$\dot{x}_1 = 10(x_2 - x_1),\ \dot{x}_2 = x_1(28 - x_3) - x_2,\ \dot{x}_3 = x_1 x_2 - 8/3\, x_3, \quad \Delta_t = 0.05, \ \Omega = \text{attractor}, \ \omega = \text{SRB measure}$



$m = 6$
$\varepsilon = 0.05$

No formula for generalized eigenfunctions!!

Experimental Details

Single trajectory (ergodic system)

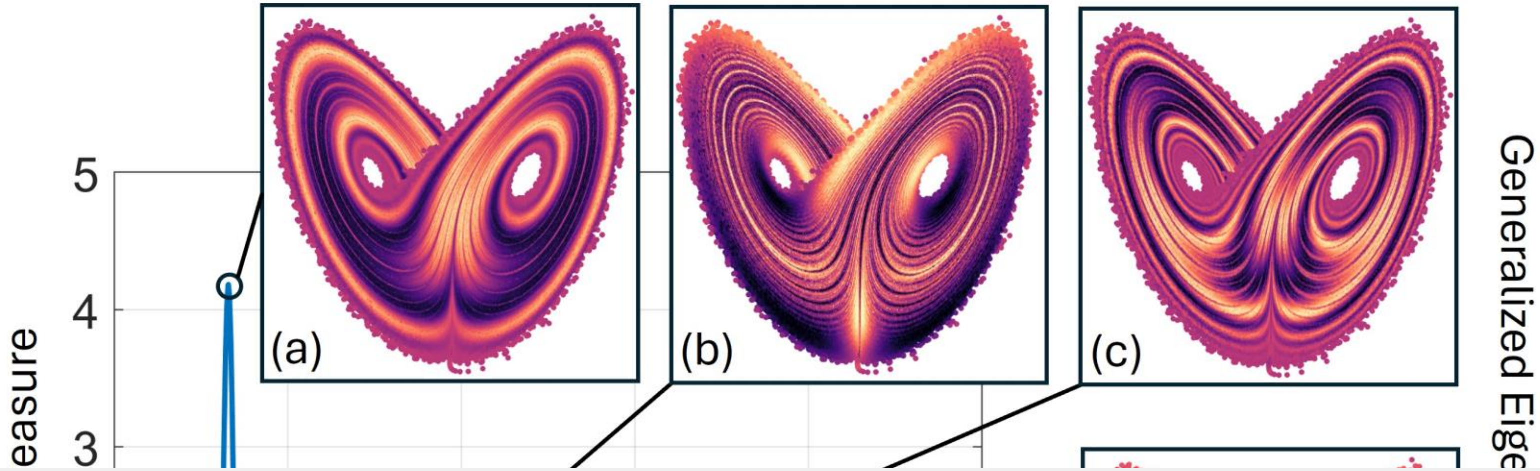
$M = 10000, N = 1000$

$$g(x_1, x_2, x_3) = \tanh\left(\frac{x_1 x_2 - 5x_3}{10}\right) - c$$

Krylov subspace: $V_N = \{g, \mathcal{K}g, \dots, \mathcal{K}^{N-1}g\}$

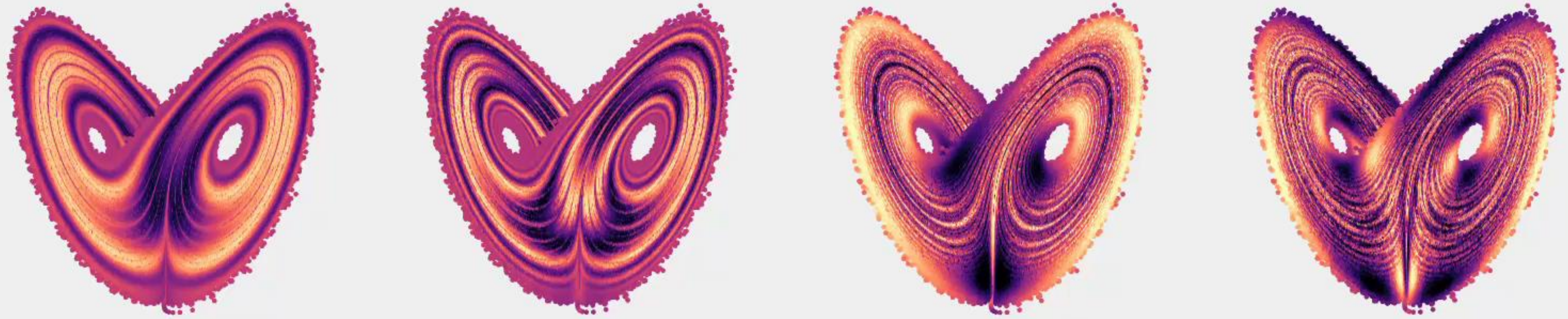
Example: Lorenz system

$\dot{x}_1 = 10(x_2 - x_1),\ \dot{x}_2 = x_1(28 - x_3) - x_2,\ \dot{x}_3 = x_1 x_2 - 8/3\, x_3, \quad \Delta_t = 0.05,\ \ \Omega = \text{attractor},\ \omega =$ SRB measure



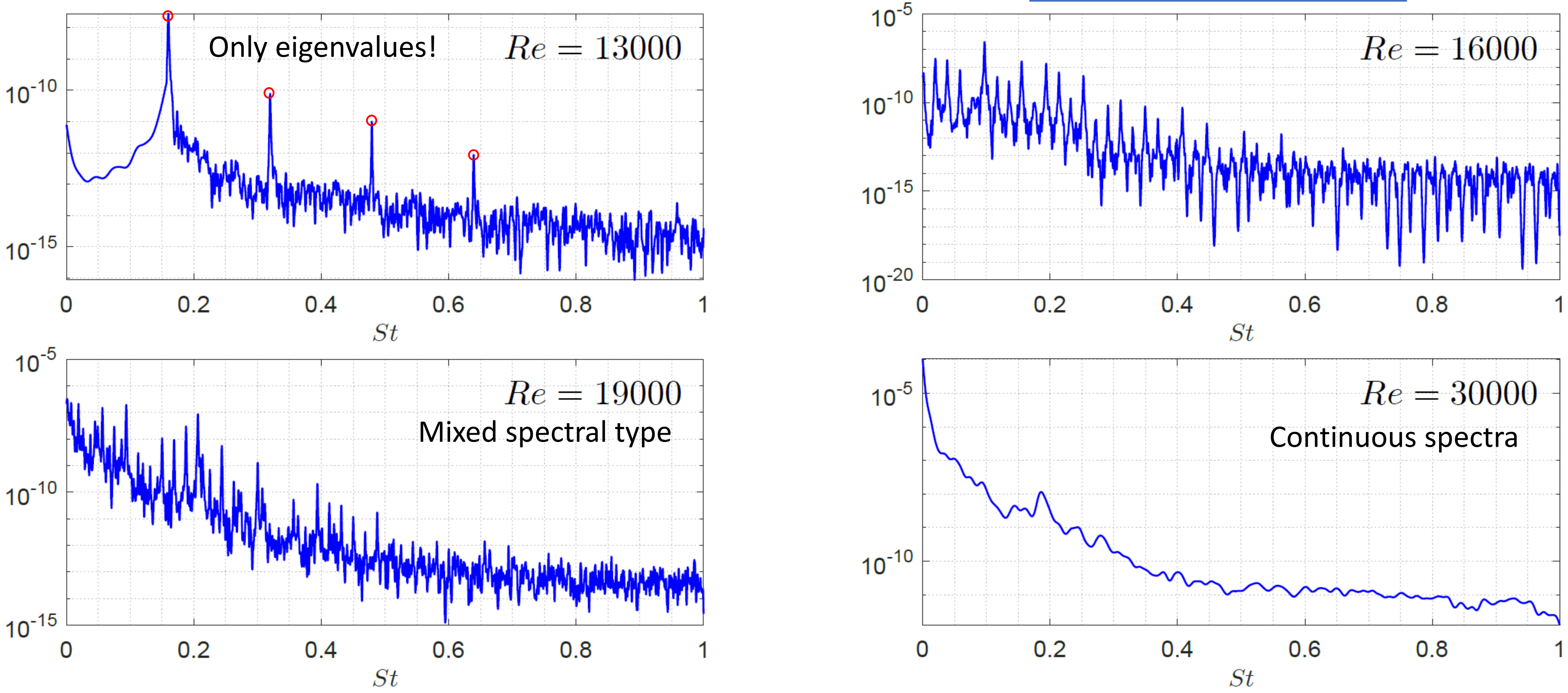
No formula for
generalized eigenfunctions!!

Experimental Details



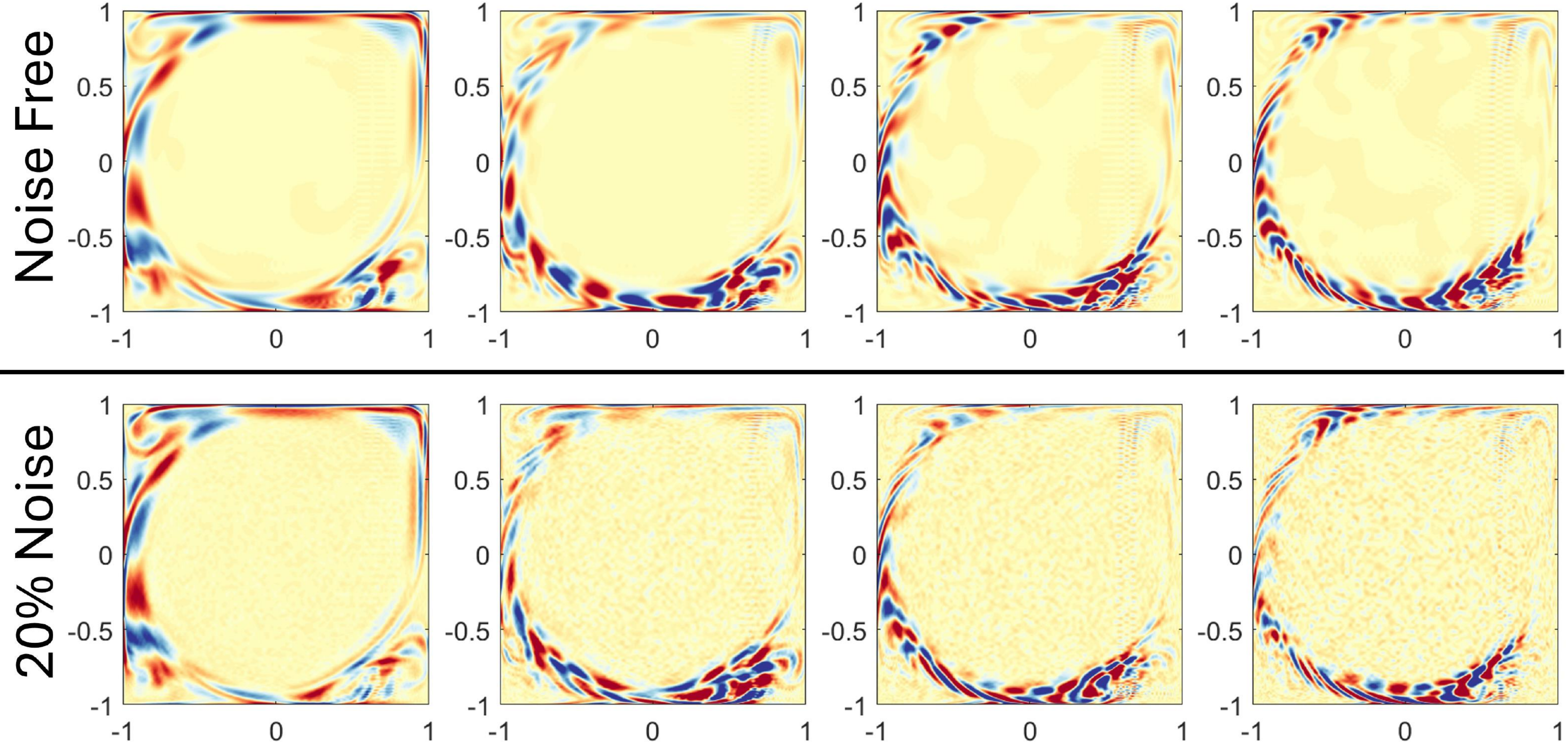
Example: Noisy cavity flow (spectral measures)

Single trajectory
$M = 10000, N$ varies
Basis: POD modes
20% Gaussian noise
*Raw measurements provided
Arbabi and Mezić (PRF 2017)



Example: Noisy cavity flow (generalized Koopman modes)

Re=30000
Deep in the continuous spectrum!!!



Practical + dictionary agnostic + theoretical guarantees

Summary

Interest in Koopman boils down to a data-driven inf-dim spectral problem.

- **mpEDMD**
  - EDMD + enforcing measure-preserving (polar decomposition of Galerkin)
  - Convergence of spectral measures, spectra, Koopman mode decomposition.
  - Long-time stability, improved qualitative behavior, increased stability to noise.
- **Rigged DMD**
  - Continuous spectra and generalized eigenfunctions.
  - Smoothing kernels + resolvent (using mpEDMD).
  - High-order convergence.

Future work
- Use in control.
- Other function spaces? E.g., RKHS

[General (non-measure-preserving) systems: ResDMD]

Brief Summaries

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