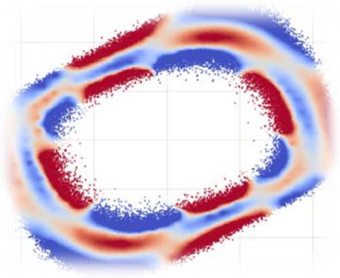
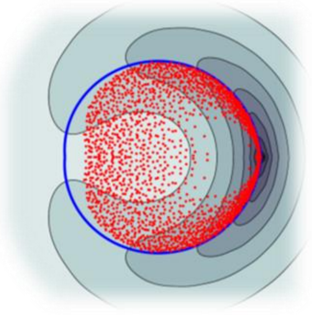
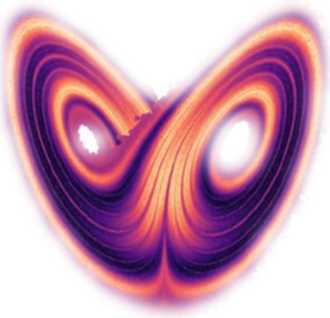






The Hitchhiker's Guide to the DMD Multiverse

Matthew Colbrook

University of Cambridge

23/05/2024

- C., Townsend, *“Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems”* **Communications on Pure and Applied Mathematics**, 2024.
- C., *"The mpEDMD Algorithm for Data-Driven Computations of Measure-Preserving Dynamical Systems,"* **SIAM Journal on Numerical Analysis**, 2023.
- C., Drysdale, Horning, *“Rigged Dynamic Mode Decomposition: Data-Driven Generalized Eigenfunction Decompositions for Koopman Operators”*, arxiv preprint.
- C., *“The Multiverse of Dynamic Mode Decomposition Algorithms,”* **Handbook of Numerical Analysis**, 2024.

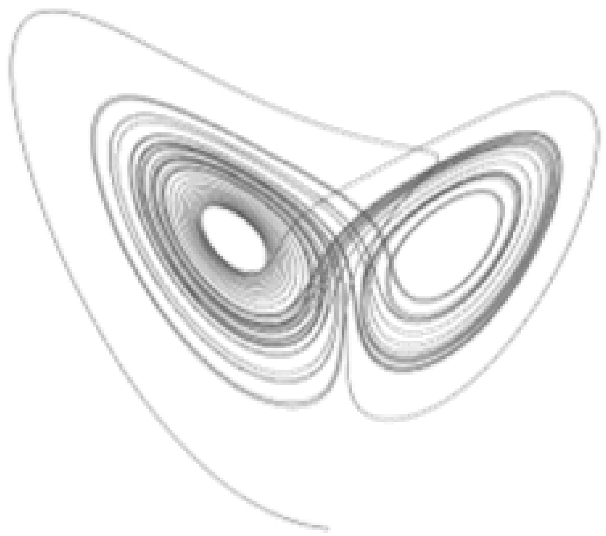
Data-driven dynamical systems

State $x \in \Omega \subseteq \mathbb{R}^d$.

Unknown function $F: \Omega \to \Omega$ governs dynamics: $x_{n+1} = F(x_n)$.

Goal: Learning from data $\{x^{(m)}, y^{(m)} = F(x^{(m)})\}_{m=1}^{M}$.

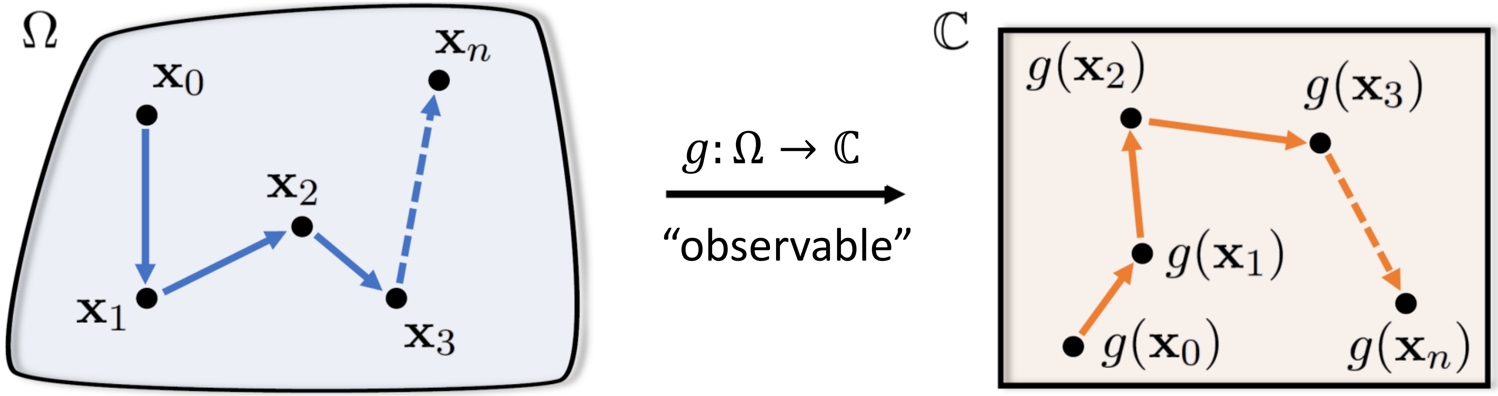
Applications: chemistry, climatology, control, electronics, epidemiology, finance, fluids, molecular dynamics, neuroscience, plasmas, robotics, video processing, etc.



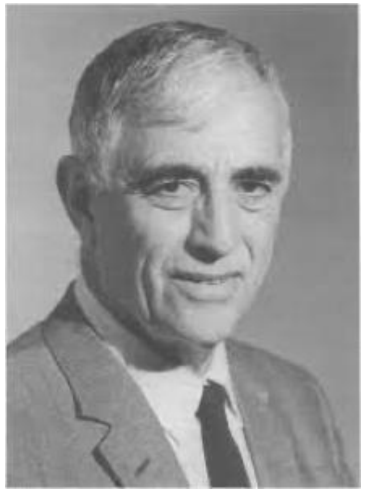
Surveys:

- Brunton, Budišić, Kaiser, Kutz, *"Modern Koopman theory for dynamical systems,"* SIAM Review, 2022.
- Budišić, Mohr, Mezić, *"Applied Koopmanism,"* Chaos, 2012.
- C., *"The Multiverse of Dynamic Mode Decomposition Algorithms,"* Handbook of Numerical Analysis, 2024.

Koopman Operator $\mathcal{K}$: A global linearization



Koopman



von Neumann

- Koopman, “*Hamiltonian systems and transformation in Hilbert space,*” **Proc. Natl. Acad. Sci. USA**, 1931.
- Koopman, v. Neumann, “*Dynamical systems of continuous spectra,*” **Proc. Natl. Acad. Sci. USA**, 1932.

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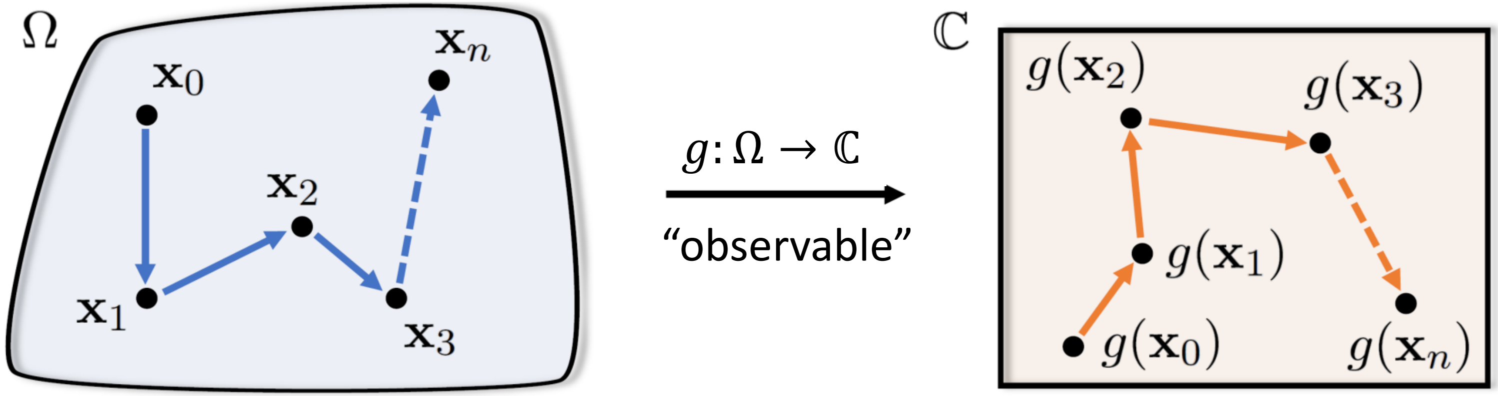

- $\mathcal{K}$ acts on functions $g: \Omega \to \mathbb{C}$, $[\mathcal{K}g](x) = g(F(x))$.
- **Function space:** $L^2(\Omega, \omega)$, positive measure ω, inner product $\langle \cdot, \cdot \rangle$.

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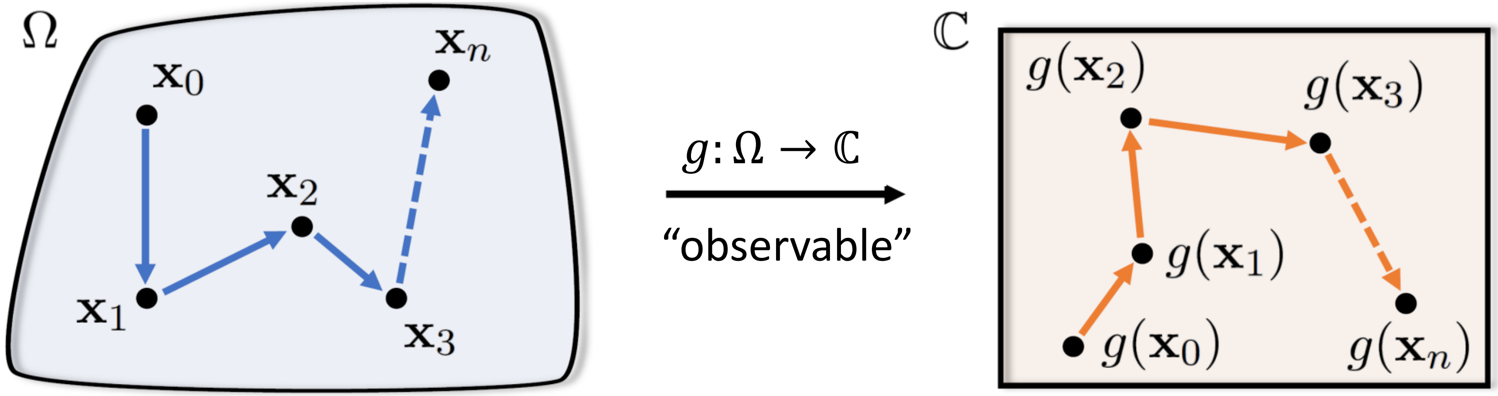

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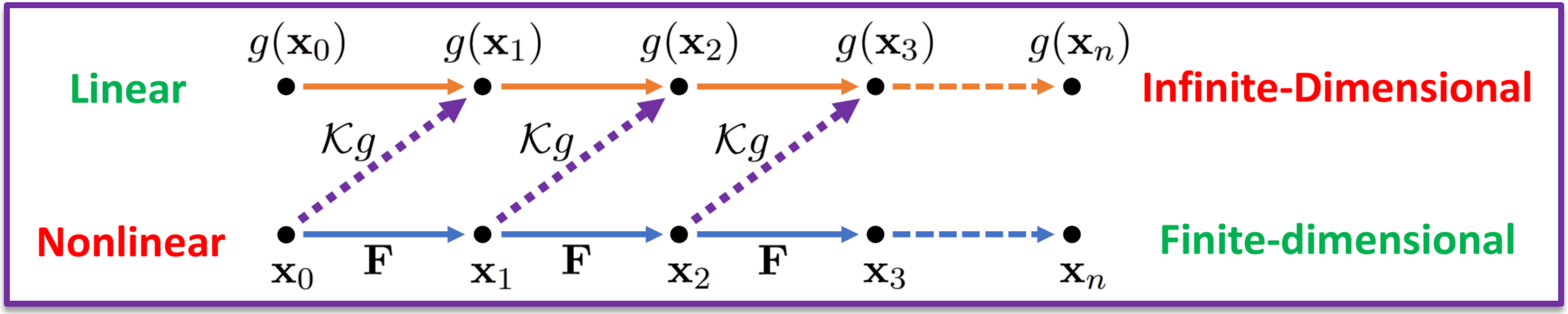

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- Koopman, v. Neumann, *"Dynamical systems of continuous spectra,"* **Proc. Natl. Acad. Sci. USA**, 1932.

Koopman mode decomposition

$x_{n+1} = F(x_n)$

$[\mathcal{K}g](x) = g(F(x))$

eigenfunction of $\mathcal{K}$

continuous spectrum

$$g(x) = \sum_{\text{eigenvalues } \lambda_j} c_{\lambda_j} \varphi_{\lambda_j}(x) + \int_{-\pi}^{\pi} \phi_{\theta,g}(x)\, \mathrm{d}\theta$$

$$g(x_n) = [\mathcal{K}^n g](x_0) = \sum_{\text{eigenvalues } \lambda_j} c_{\lambda_j} {\lambda_j}^n \varphi_{\lambda_j}(x_0) + \int_{-\pi}^{\pi} e^{in\theta} \phi_{\theta,g}(x_0)\, \mathrm{d}\theta$$

Encodes: geometric features, invariant measures, transient behavior, long-time behavior, coherent structures, quasiperiodicity, etc.

GOAL: Data-driven approximation of $\mathcal{K}$ and its spectral properties.

- Mezić, "*Spectral properties of dynamical systems, model reduction and decompositions,*" **Nonlinear Dynamics**, 2005.

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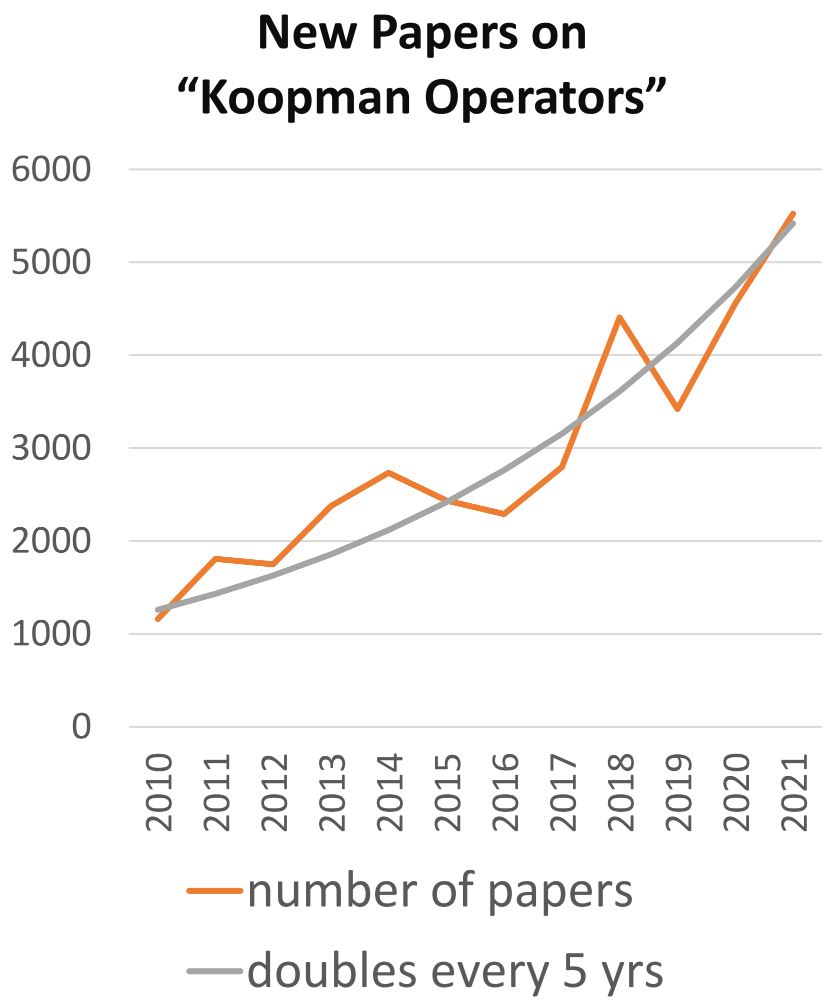
New Papers on
“Koopman Operators”
6000
5000
4000
3000
2000
1000
0
2010
2011
2012
2013
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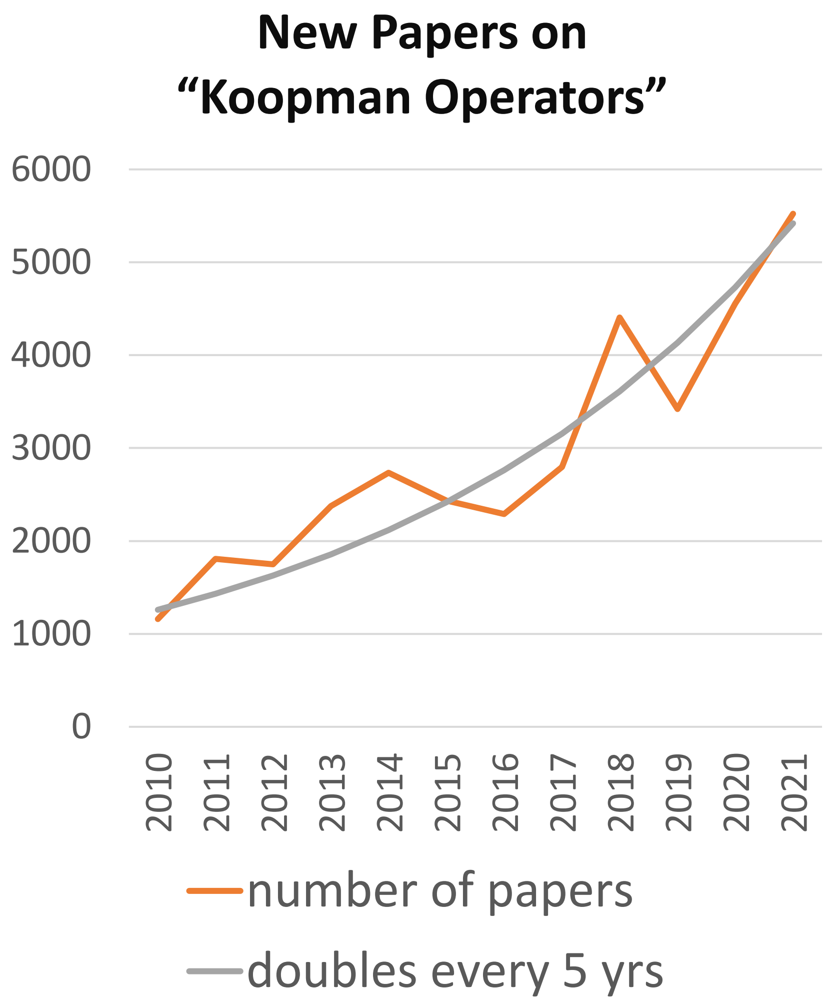
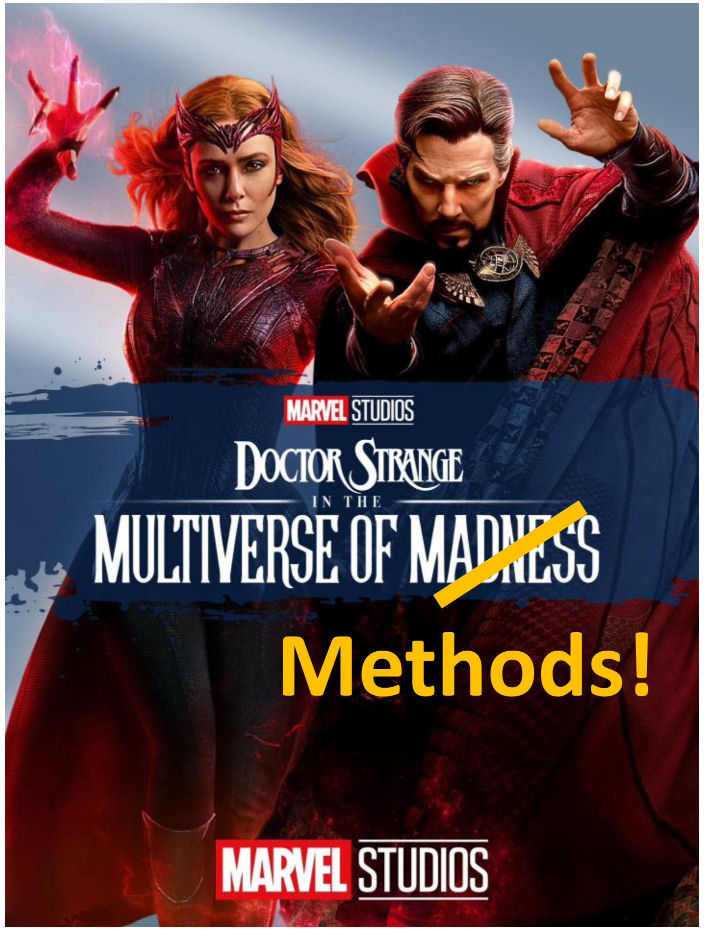
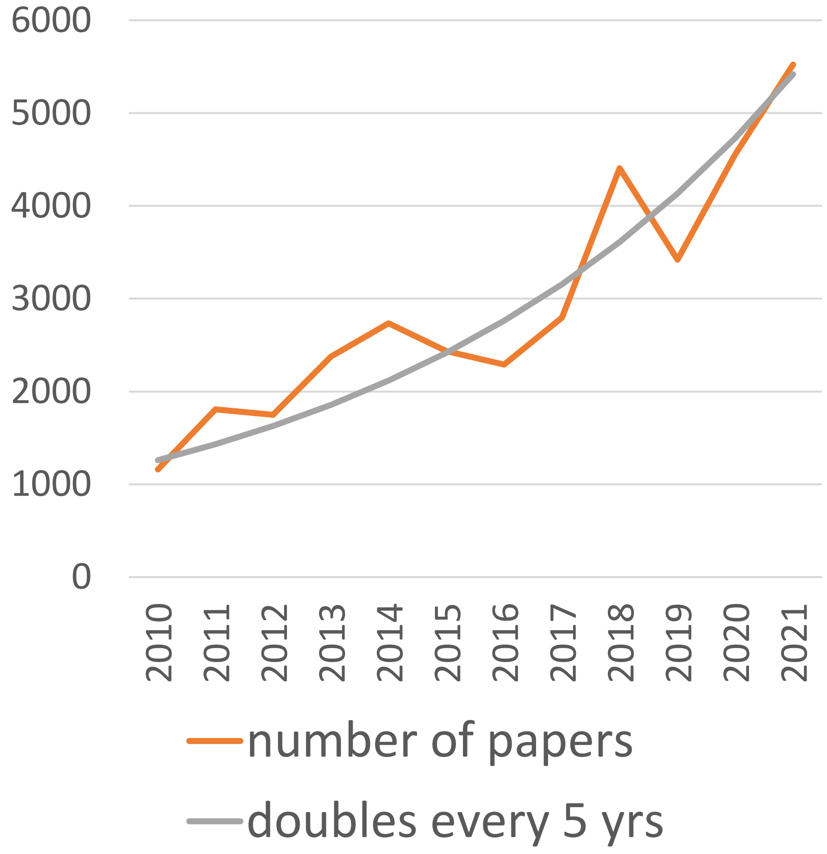
2021
number of papers
doubles every 5 yrs

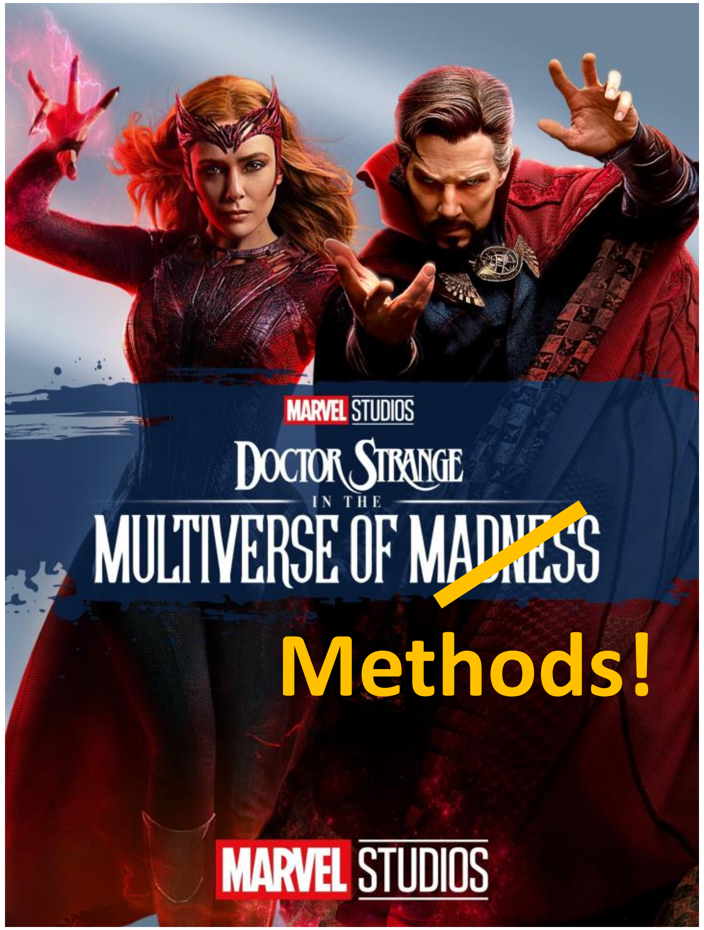




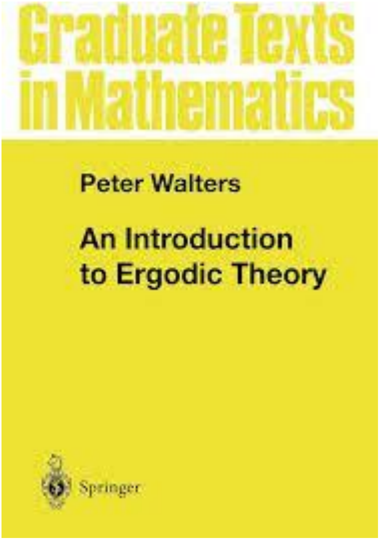
• C., *"The Multiverse of Dynamic Mode Decomposition Algorithms,"* **Handbook of Numerical Analysis**, 2024.

New Papers on “Koopman Operators”





Koopman operators are classical in ergodic theory.



Why all this sudden interest?

- C., *“The Multiverse of Dynamic Mode Decomposition Algorithms,”* **Handbook of Numerical Analysis**, 2024.

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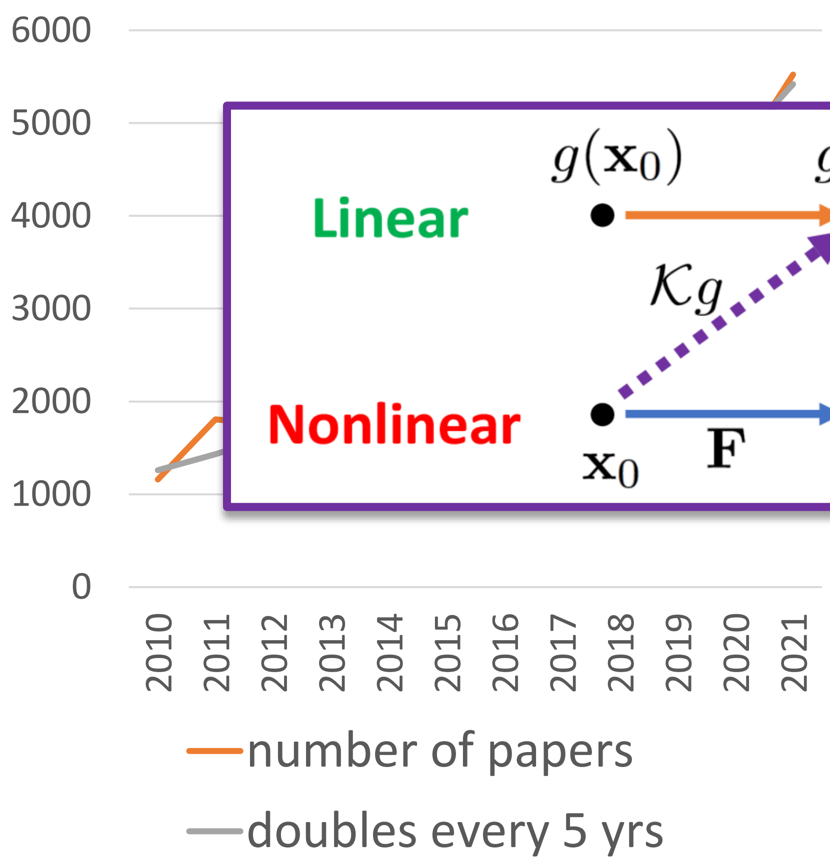

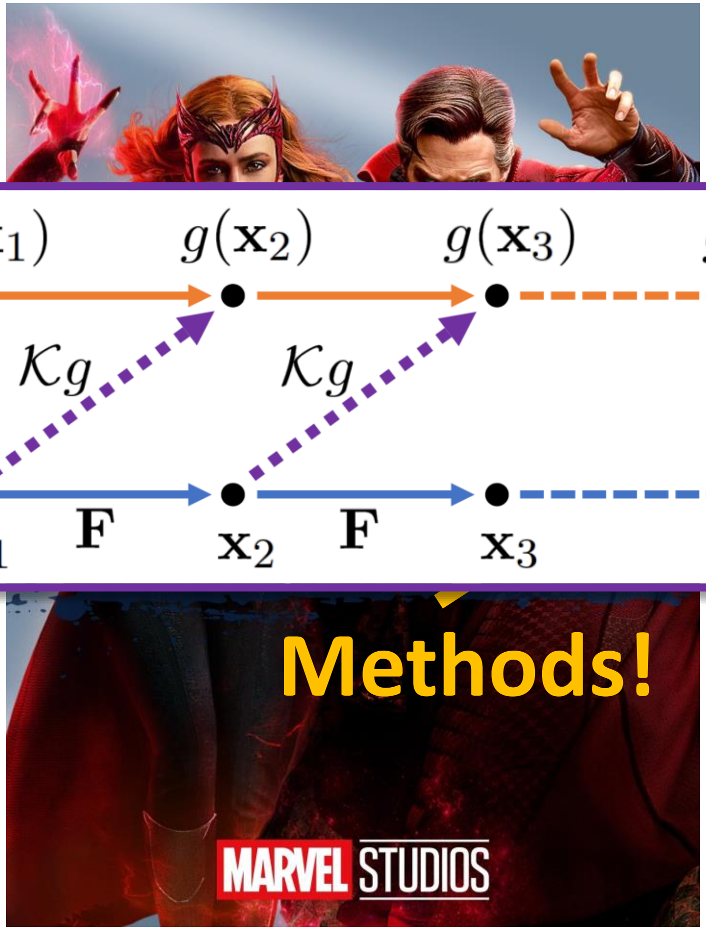

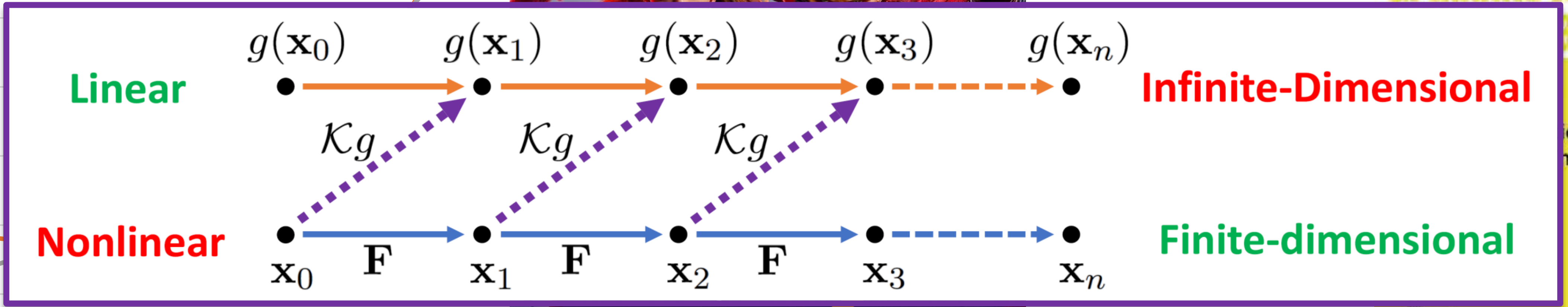

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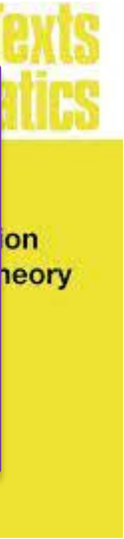

Why all this sudden interest?

- **Data-driven**
- **Deal with nonlinearity**
- **Easy-to-use methods**

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Warmup on $\ell^2(\mathbb{Z})$

$$\begin{pmatrix} \ddots & \ddots & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \in \mathbb{C}^{N\times N}$$

Two-way infinite

- Spectrum is unit circle.
- Spectrum is stable.
- Continuous spectra.
- Unitary evolution.

- Spectrum is $\{0\}$.
- Spectrum is unstable.
- Discrete spectra.
- Nilpotent evolution.

Lots of Koopman operators are built up from operators like these!

Dangers when we truncate/discretize $\mathcal{K} \longrightarrow \mathbb{K} \in \mathbb{C}^{N\times N}$

$$\mathrm{Sp}(\mathcal{K}) = \{\lambda \in \mathbb{C}: \mathcal{K} - \lambda I \text{ is not invertible}\}$$

- **Too much:** Spurious eigenvalues $\lambda \in \mathrm{Sp}(\mathbb{K})$ far from $\mathrm{Sp}(\mathcal{K})$
- **Too little:** $\mathrm{Sp}(\mathbb{K})$ misses parts of $\mathrm{Sp}(\mathcal{K})$
- **Continuous spectra** ($\mathrm{Sp}(\mathcal{K})$ not just eigenvalues!)
- **Verification (e.g., subspace)**
- **Instability** (non-normal $\mathcal{K}$, non-normal discretizations of normal $\mathcal{K}$)

Methods like EDMD do not avoid these dangers as $N \to \infty$!

- C., Townsend, *"Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems,"* **Commun. Pure Appl. Math.**, 2024.

Outline

- <u>General</u> systems:
  - Residual Dynamic Mode Decomposition.
- <u>Measure-preserving</u> systems:
  - Measure-Preserving Extended Dynamic Mode Decomposition.
  - Rigged Dynamic Mode Decomposition.
- The **Solvability Complexity Index** – *classification* of problems and *optimality* of algorithms.

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Extended Dynamic Mode Decomposition (EDMD)

Functions $\psi_j : \Omega \to \mathbb{C}, j = 1, \ldots, N$

$$\left\{x^{(m)}, y^{(m)} = F\left(x^{(m)}\right)\right\}_{m=1}^{M}$$

- Schmid, *"Dynamic mode decomposition of numerical and experimental data,"* **J. Fluid Mech.**, 2010.
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quadrature points

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j\left(x^{(m)}\right)} \psi_k\left(x^{(m)}\right) = \Bigg[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}^{*}}_{\Psi_X} \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_{W} \underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}}_{\Psi_X}\Bigg]_{jk}$$

quadrature weights

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j\left(x^{(m)}\right)} \underbrace{\psi_k\left(y^{(m)}\right)}_{[\mathcal{K}\psi_k](x^{(m)})} = \Bigg[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}^{*}}_{\Psi_X} \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_{W} \underbrace{\begin{pmatrix} \psi_1(y^{(1)}) & \cdots & \psi_N(y^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(y^{(M)}) & \cdots & \psi_N(y^{(M)}) \end{pmatrix}}_{\Psi_Y}\Bigg]_{jk}$$

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Galerkin Approximation

$$\mathcal{K} \longrightarrow \mathbb{K} = \left(\Psi_X{}^* W \Psi_X\right)^{-1} \Psi_X{}^* W \Psi_Y = \left(\sqrt{W}\Psi_X\right)^{\dagger} \sqrt{W}\Psi_Y \in \mathbb{C}^{N \times N}$$

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- Williams, Kevrekidis, Rowley “*A data-driven approximation of the Koopman operator: Extending dynamic mode decomposition,*” **J. Nonlinear Sci.**, 2015.

Residual DMD (ResDMD)

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\, \psi_k(x^{(m)}) = [\underbrace{\Psi_X{}^* W \Psi_X}_{G}]_{jk}$$

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\, \underbrace{\psi_k(y^{(m)})}_{[\mathcal{K}\psi_k](x^{(m)})} = [\underbrace{\Psi_X{}^* W \Psi_Y}_{K_1}]_{jk}$$

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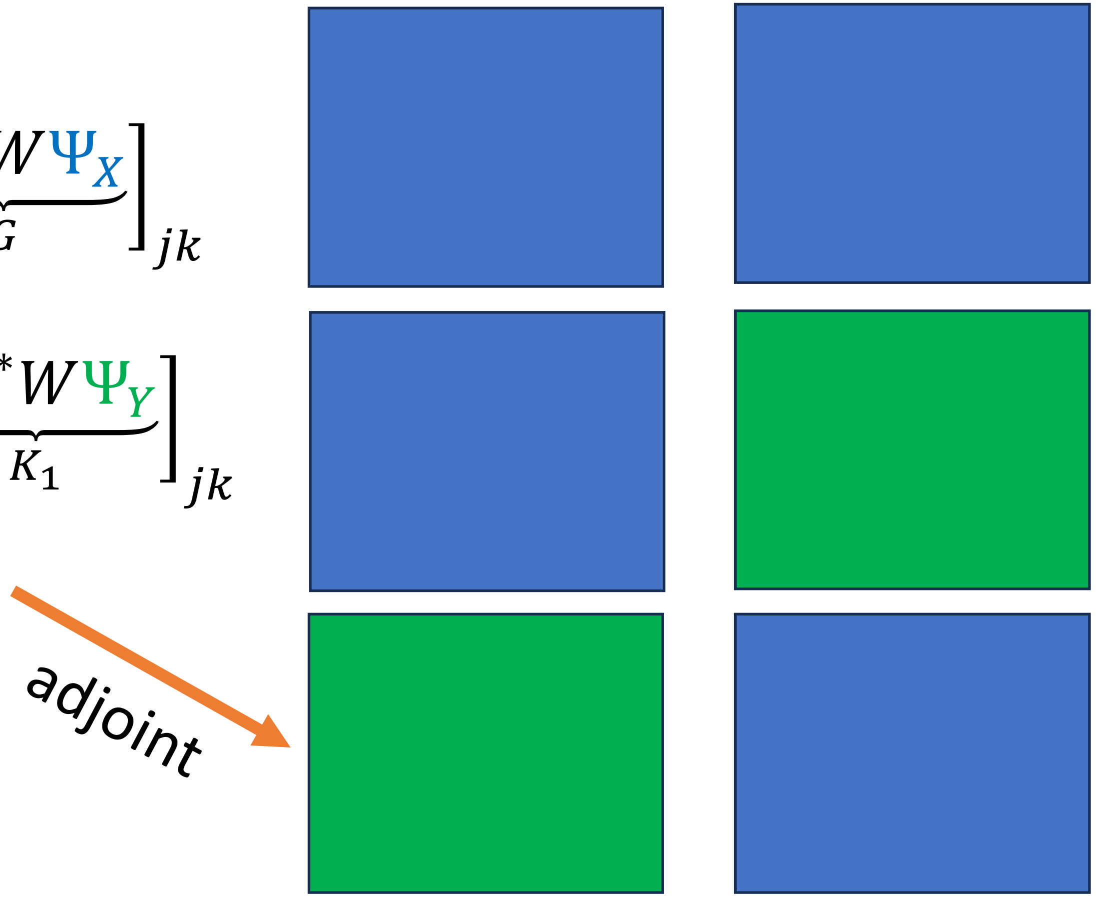
- C., Ayton, Szőke, "*Residual Dynamic Mode Decomposition*," **J. Fluid Mech.**, 2023.
- Code: https://github.com/MColbrook/Residual-Dynamic-Mode-Decomposition

Residual DMD (ResDMD)

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\, \psi_k\left(x^{(m)}\right) = \Big[\underbrace{\Psi_X{}^* W \Psi_X}_{G}\Big]_{jk}$$

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\, \underbrace{\psi_k\left(y^{(m)}\right)}_{[\mathcal{K}\psi_k](x^{(m)})} = \Big[\underbrace{\Psi_X{}^* W \Psi_Y}_{K_1}\Big]_{jk}$$



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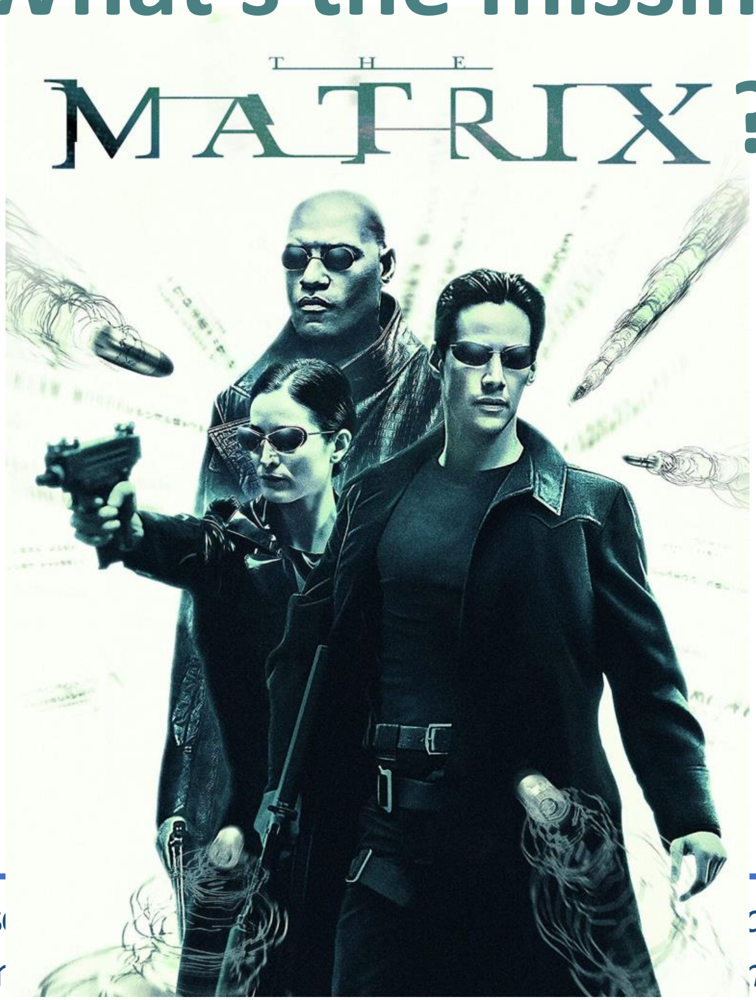
Residual DMD (ResDMD)

What's the missing



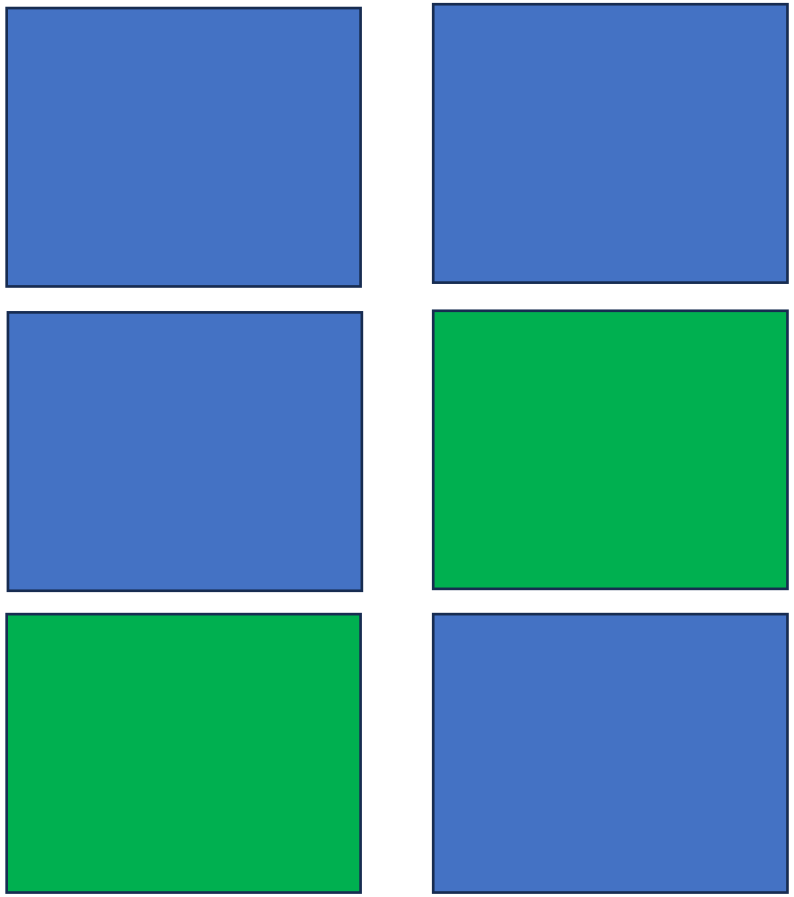
?

$$= \Big[\underbrace{\Psi_X{}^* W \Psi_X}_{G}\Big]_{jk}$$

)

$$= \Big[\underbrace{\Psi_X{}^* W \Psi_Y}_{K_1}\Big]_{jk}$$

adjoint



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$$\langle \mathcal{K}\psi_k, \mathcal{K}\psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(y^{(m)})}\, \psi_k\left(y^{(m)}\right) = \Big[\underbrace{\Psi_Y{}^* W \Psi_Y}_{K_2}\Big]_{jk}$$

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Residual DMD (ResDMD)

$$\langle\psi_k,\psi_j\rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\,\psi_k\big(x^{(m)}\big) = \big[\underbrace{\Psi_X{}^{*}W\Psi_X}_{G}\big]_{jk}$$

$$\langle\mathcal{K}\psi_k,\psi_j\rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\,\underbrace{\psi_k\big(y^{(m)}\big)}_{[\mathcal{K}\psi_k](x^{(m)})} = \big[\underbrace{\Psi_X{}^{*}W\Psi_Y}_{K_1}\big]_{jk}$$

$$\langle\mathcal{K}\psi_k,\mathcal{K}\psi_j\rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(y^{(m)})}\,\psi_k\big(y^{(m)}\big) = \big[\underbrace{\Psi_Y{}^{*}W\Psi_Y}_{K_2}\big]_{jk}$$

Residuals: $g = \sum_{j=1}^{N} \mathbf{g}_j\psi_j,\ \|\mathcal{K}g-\lambda g\|^2 = \langle\mathcal{K}g-\lambda g, \mathcal{K}g-\lambda g\rangle$

- C., Townsend, "*Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems,*" **Commun. Pure Appl. Math.**, 2023.
- C., Ayton, Szőke, "*Residual Dynamic Mode Decomposition,*" **J. Fluid Mech.**, 2023.
- Code*:* https://github.com/MColbrook/Residual-Dynamic-Mode-Decomposition

Residual DMD (ResDMD)

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\, \psi_k\left(x^{(m)}\right) = \Big[\underbrace{\Psi_X{}^* W \Psi_X}_{G}\Big]_{jk}$$

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\, \underbrace{\psi_k\left(y^{(m)}\right)}_{[\mathcal{K}\psi_k](x^{(m)})} = \Big[\underbrace{\Psi_X{}^* W \Psi_Y}_{K_1}\Big]_{jk}$$

$$\langle \mathcal{K}\psi_k, \mathcal{K}\psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(y^{(m)})}\, \psi_k\left(y^{(m)}\right) = \Big[\underbrace{\Psi_Y{}^* W \Psi_Y}_{K_2}\Big]_{jk}$$

Residuals: $g = \sum_{j=1}^{N} \mathbf{g}_j \psi_j$, $\|\mathcal{K}g - \lambda g\|^2 = \sum_{k,j=1}^{N} \mathbf{g}_k \overline{\mathbf{g}_j} \langle \mathcal{K}\psi_k - \lambda\psi_k, \mathcal{K}\psi_j - \lambda\psi_j \rangle$

- C., Townsend, "*Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems*," **Commun. Pure Appl. Math.**, 2023.
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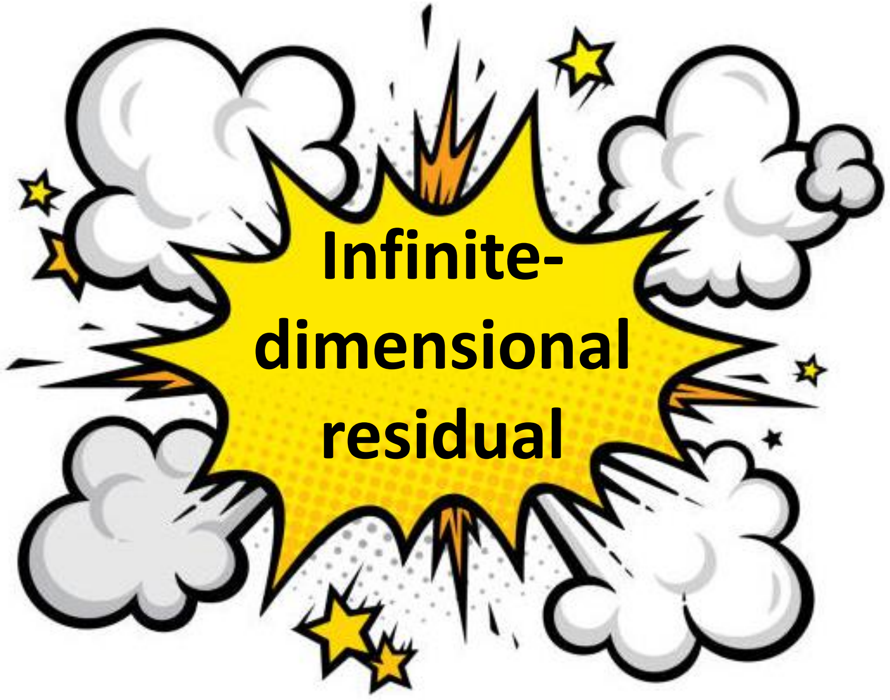
Residual DMD (ResDMD)

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\, \psi_k(x^{(m)}) = [\underbrace{\Psi_X{}^* W \Psi_X}_{G}]_{jk}$$

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\, \underbrace{\psi_k(y^{(m)})}_{[\mathcal{K}\psi_k](x^{(m)})} = [\underbrace{\Psi_X{}^* W \Psi_Y}_{K_1}]_{jk}$$

$$\langle \mathcal{K}\psi_k, \mathcal{K}\psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(y^{(m)})}\, \psi_k(y^{(m)}) = [\underbrace{\Psi_Y{}^* W \Psi_Y}_{K_2}]_{jk}$$



Residuals: $g = \sum_{j=1}^{N} \mathbf{g}_j \psi_j$, $\|\mathcal{K}g - \lambda g\|^2 = \lim_{M\to\infty} \mathbf{g}^* [K_2 - \lambda K_1{}^* - \bar{\lambda} K_1 + |\lambda|^2 G]\mathbf{g}$

- C., Townsend, "*Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems,*" **Commun. Pure Appl. Math.**, 2023.
- C., Ayton, Szőke, "*Residual Dynamic Mode Decomposition,*" **J. Fluid Mech.**, 2023.
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ResDMD: Avoiding the dangers

If quadrature rule converges

Convergent methods for general $\mathcal{K}$

- **1 limit** $\lim_{M\to\infty}$: Avoid spurious eigenvalues.
- **2 limits** $\lim_{N\to\infty}\lim_{M\to\infty}$: Compute $\mathrm{Sp}_\varepsilon(\mathcal{K}) = \cup_{\|\mathcal{B}\|\leq\varepsilon}\mathrm{Sp}(\mathcal{K}+\mathcal{B})$.
- **3 limits** $\lim_{\varepsilon\downarrow 0}\lim_{N\to\infty}\lim_{M\to\infty}$: Compute $\mathrm{Sp}(\mathcal{K})$.
- Verification: dictionaries, approximate eigenfunctions, coherency,…
- Error bounds of forecasts.

M = number of snapshots
N = number of basis functions

ResDMD: Avoiding the dangers

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- Error bounds of forecasts.
- Extends to kernel methods and $M < N$ (dual residual).

• C., *"Another look at Residual Dynamic Mode Decomposition in the regime of fewer Snapshots than Dictionary Size,"* arXiv preprint (2024).

ResDMD: Avoiding the dangers

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M = number of snapshots
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- Verification: dictionaries, approximate eigenfunctions, coherency,…
- Error bounds of forecasts.
- Extends to kernel methods and $M < N$ (dual residual).
- Extends to stochastic systems (+ variance through Koopman).

- C., *"Another look at Residual Dynamic Mode Decomposition in the regime of fewer Snapshots than Dictionary Size,"* arXiv preprint (2024).
- C., Li, Raut, Townsend, *"Beyond expectations: Residual Dynamic Mode Decomposition and Variance for Stochastic Dynamical Systems,"* **Nonlinear Dynamics**, 2024.

Quadrature with trajectory data

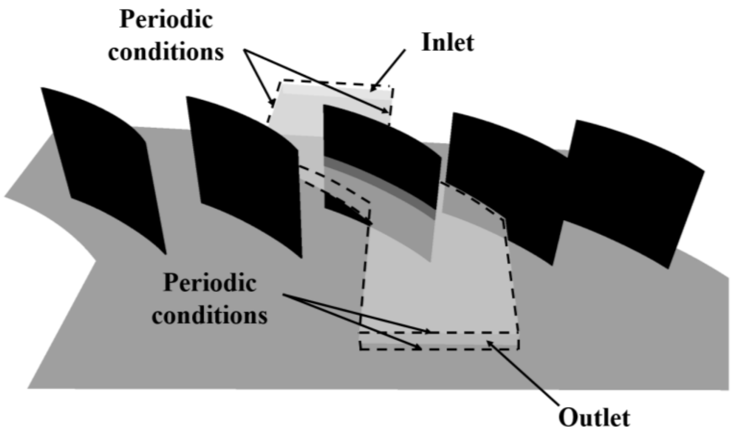
E.g., $$\langle \mathcal{K}\psi_k, \psi_j \rangle = \lim_{M\to\infty} \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\, \underbrace{\psi_k\left(y^{(m)}\right)}_{[\mathcal{K}\psi_k](x^{(m)})}$$

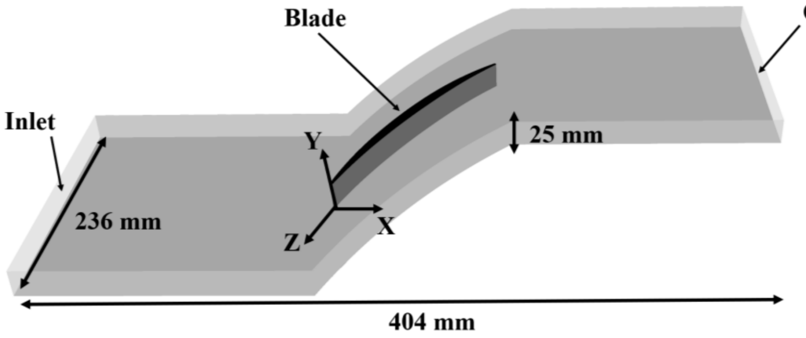
Three examples:

- **High-order quadrature:** $\{x^{(m)}, w_m\}_{m=1}^{M}$ M-point quadrature rule.
  Rapid convergence. Requires free choice of $\{x^{(m)}\}_{m=1}^{M}$ and small d.
- **Random sampling:** $\{x^{(m)}\}_{m=1}^{M}$ selected at random.
  Large d. Slow Monte Carlo $O(M^{-1/2})$ rate of convergence.
- **Ergodic sampling:** $x^{(m+1)} = F(x^{(m)})$.
  Single trajectory, large d. Requires ergodicity, convergence can be slow.

Most common

Example: Verified spectra and modes

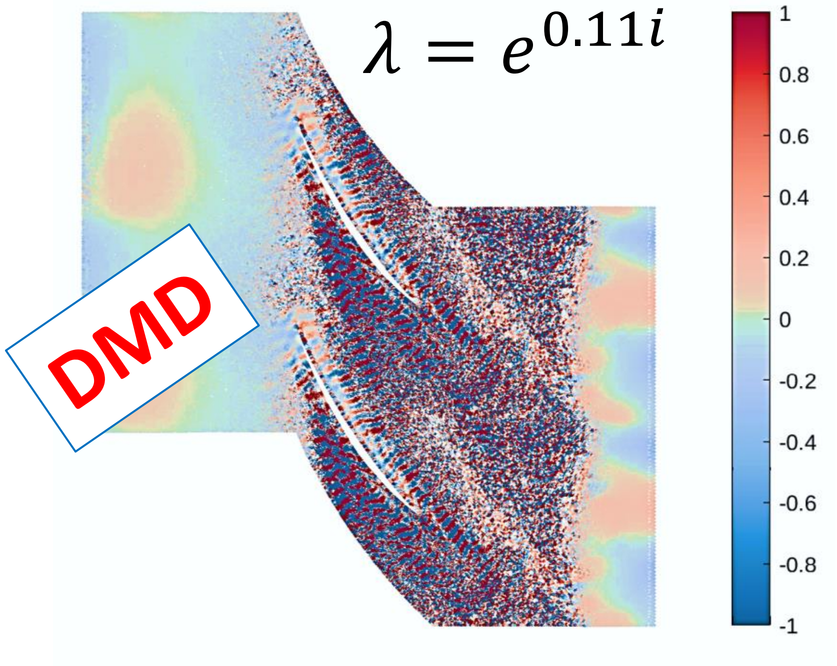




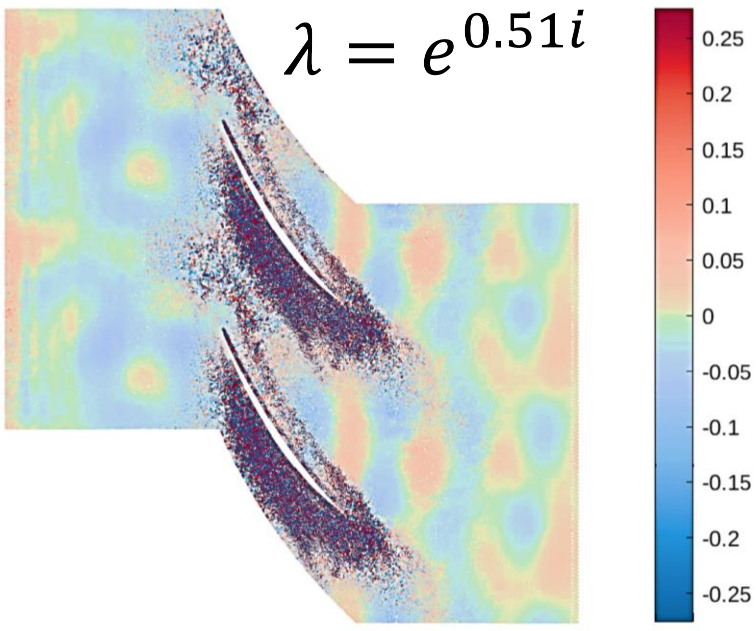
- Reynolds number $\approx 3.9 \times 10^5$
- Ambient dimension $(d) \approx 300{,}000$ (number of measurement points)

*Raw measurements provided by Stephane Moreau (Sherbrooke)

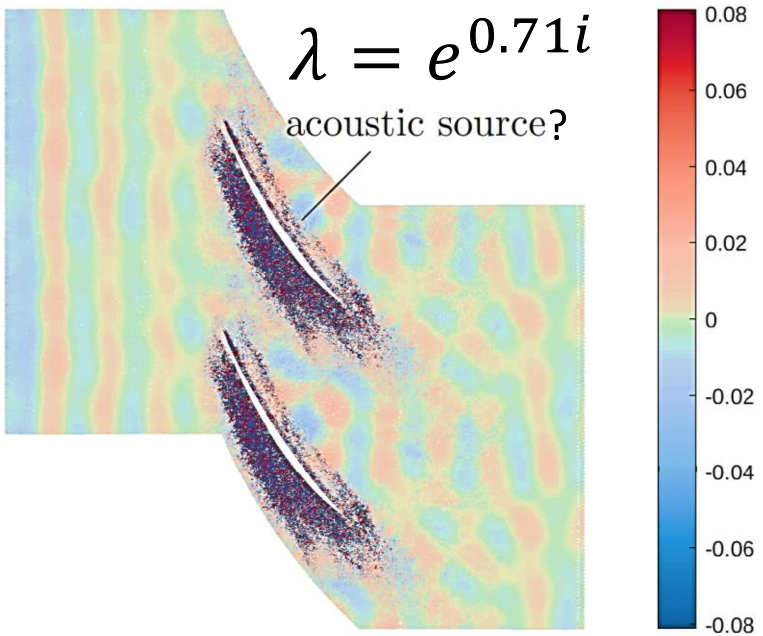
Rel. Error = ?



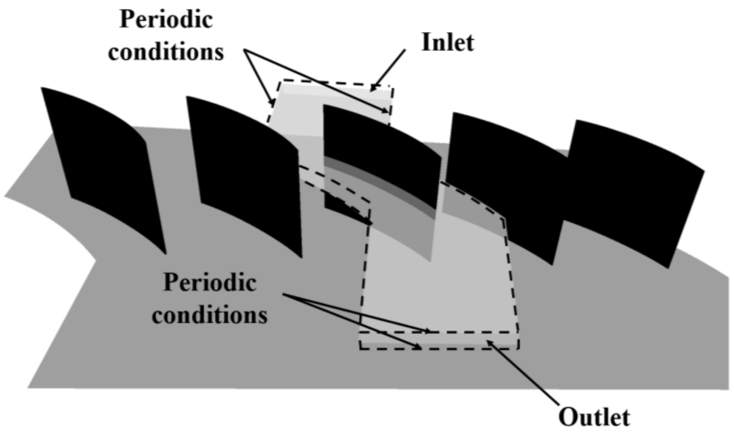
Rel. Error = ?

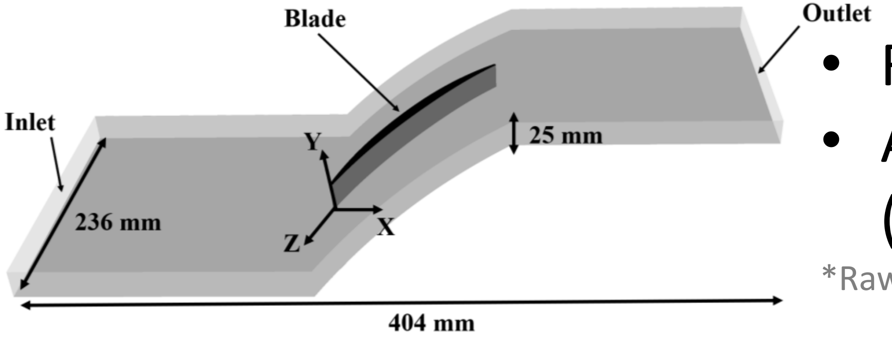


Rel. Error = ?



Example: Verified spectra and modes

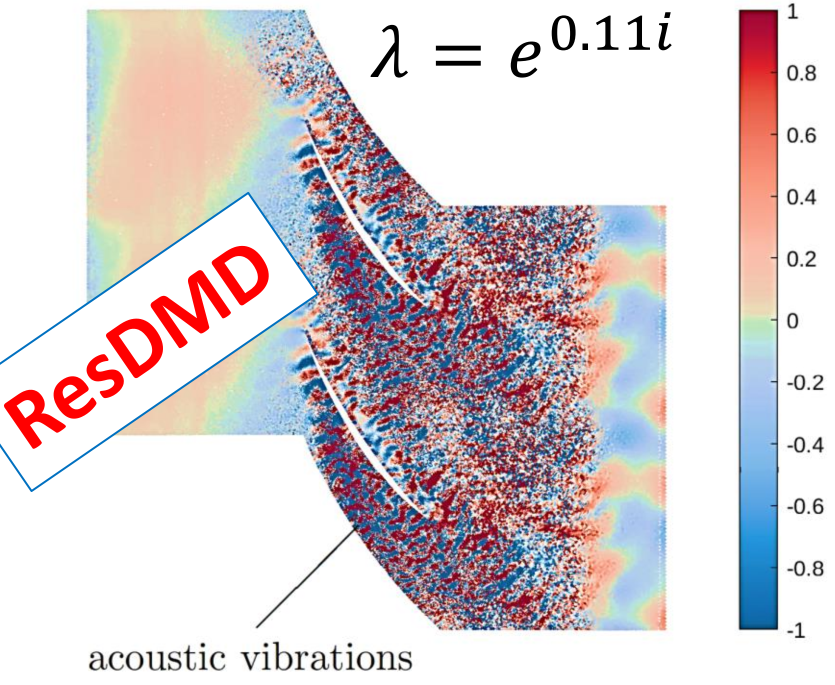




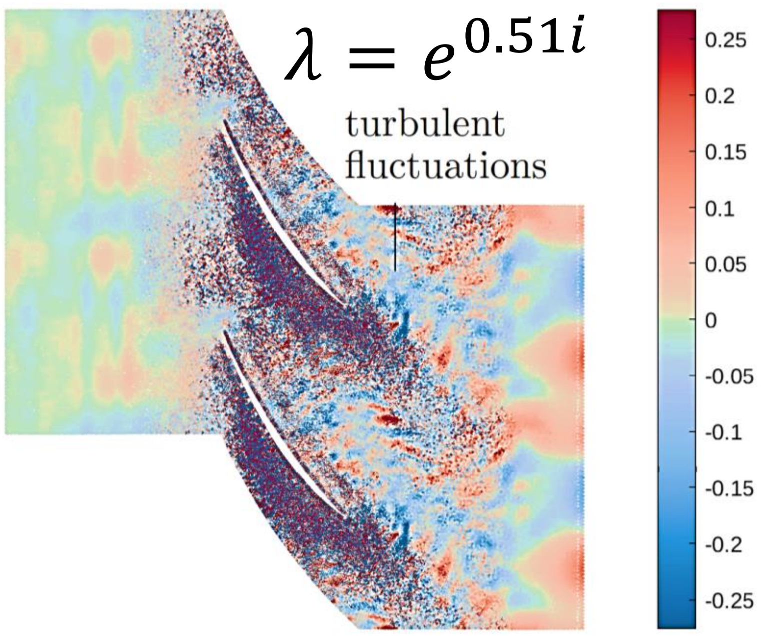
- Reynolds number $\approx 3.9 \times 10^5$
- Ambient dimension $(d) \approx 300{,}000$ (number of measurement points)

*Raw measurements provided by Stephane Moreau (Sherbrooke)

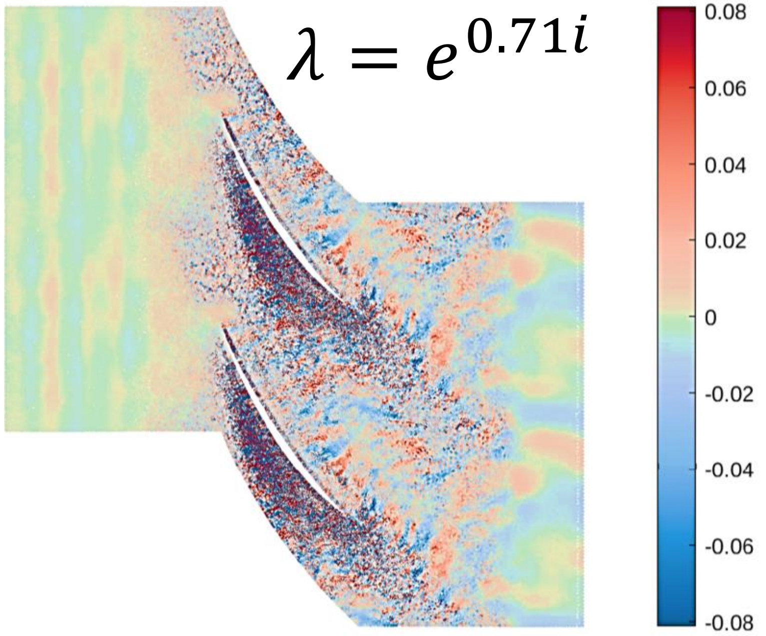
Rel. Error ≤ 0.0054



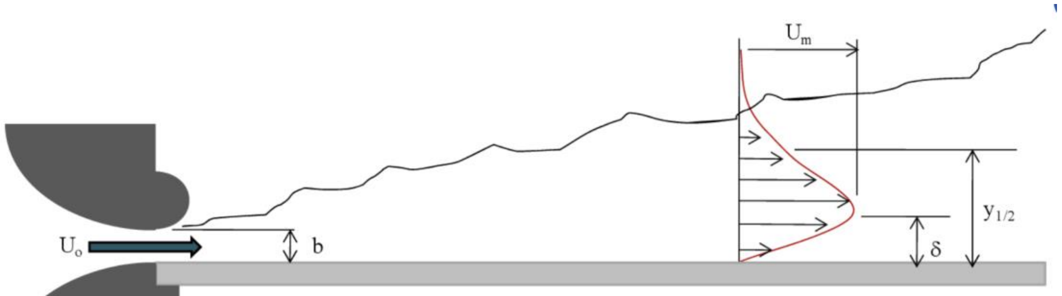
Rel. Error ≤ 0.0128

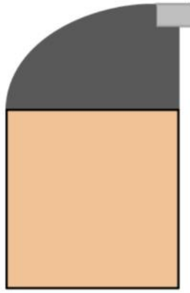


Rel. Error ≤ 0.0196



Example: Verified dictionary

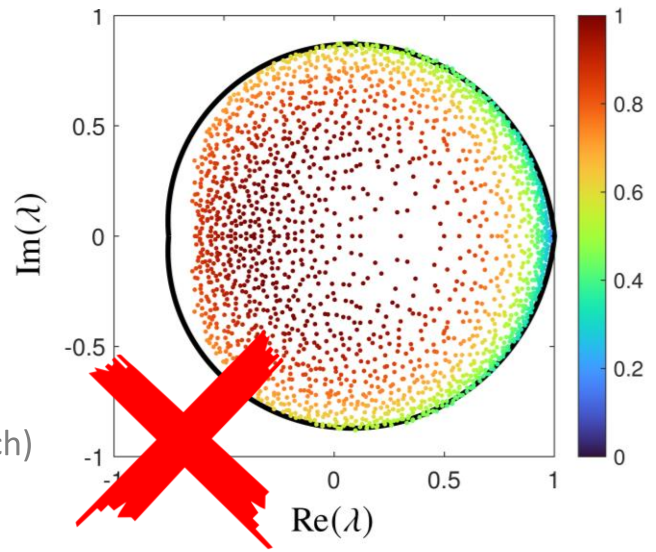




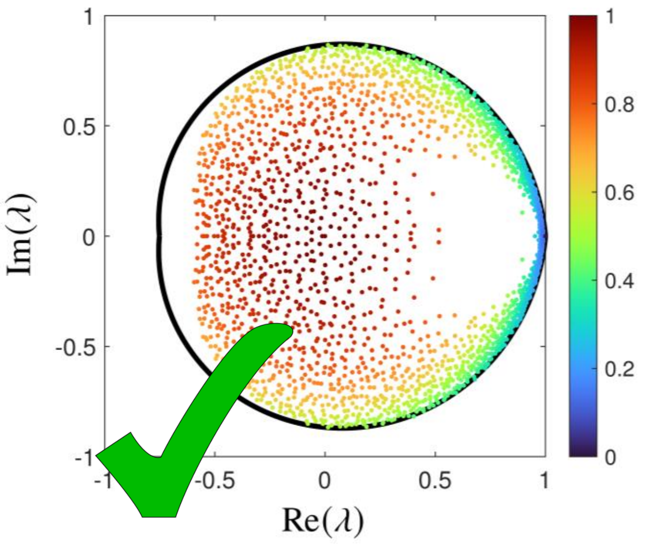
- Reynolds number $\approx 6.4 \times 10^4$
- Ambient dimension (d) $\approx$ 100,000 (velocity at measurement points)

*Raw measurements provided by Máté Szőke (Virginia Tech)

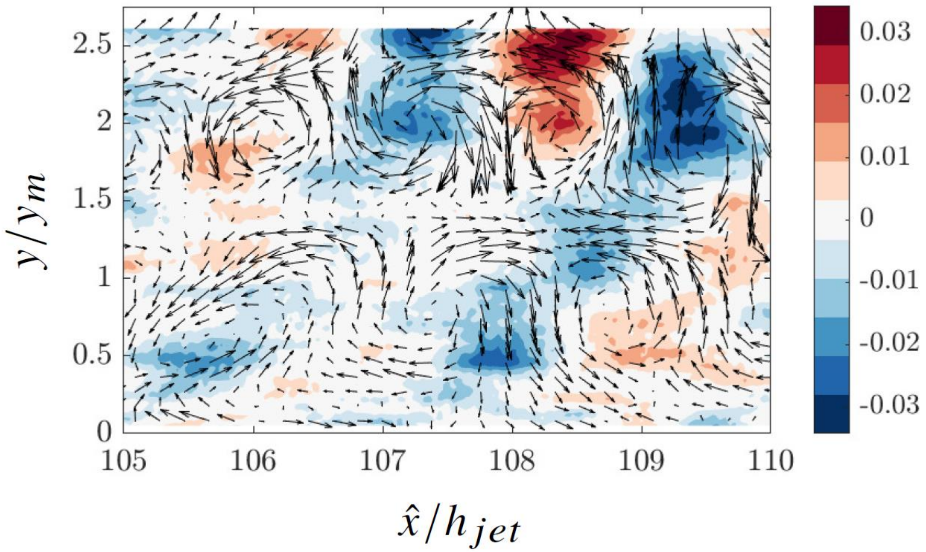
Residuals using Truncated SVD (linear)



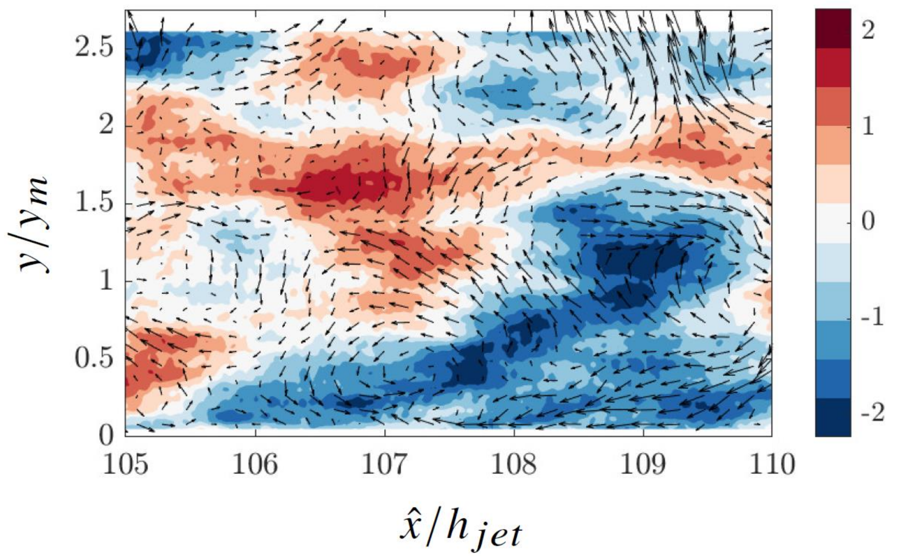
Residuals using Gaussian kernel method



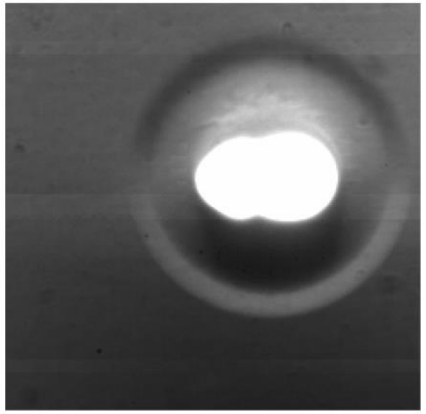
$\lambda = 0.9439 + 0.2458i$, error ≤ 0.0765



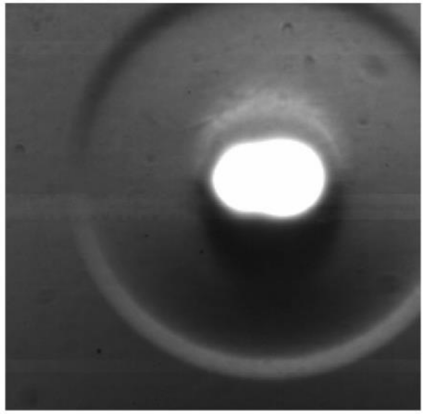
$\lambda = 0.8948 + 0.1065i$, error ≤ 0.1105



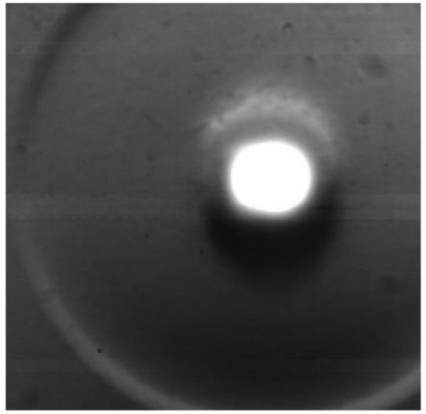
Example: Verified KMD and compression



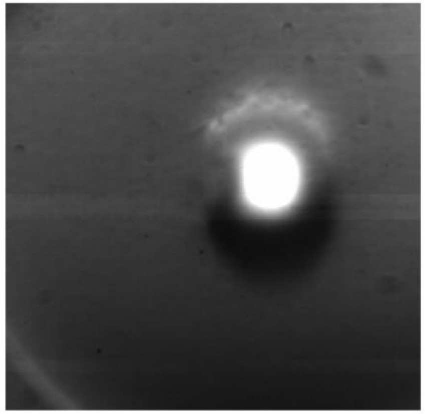
a) $t = 5\ \mu s$



b) $t = 10\ \mu s$

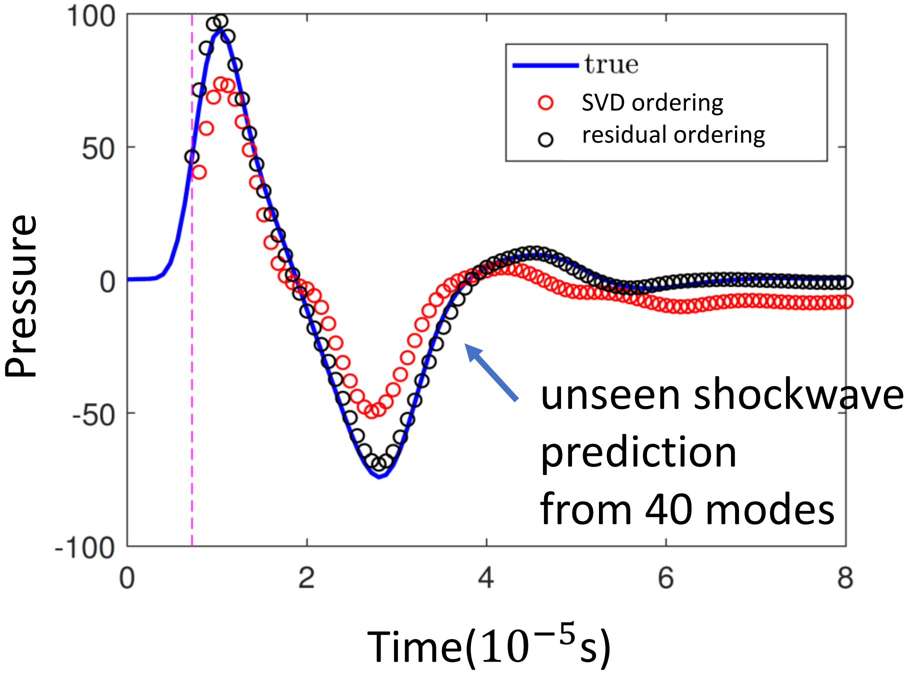


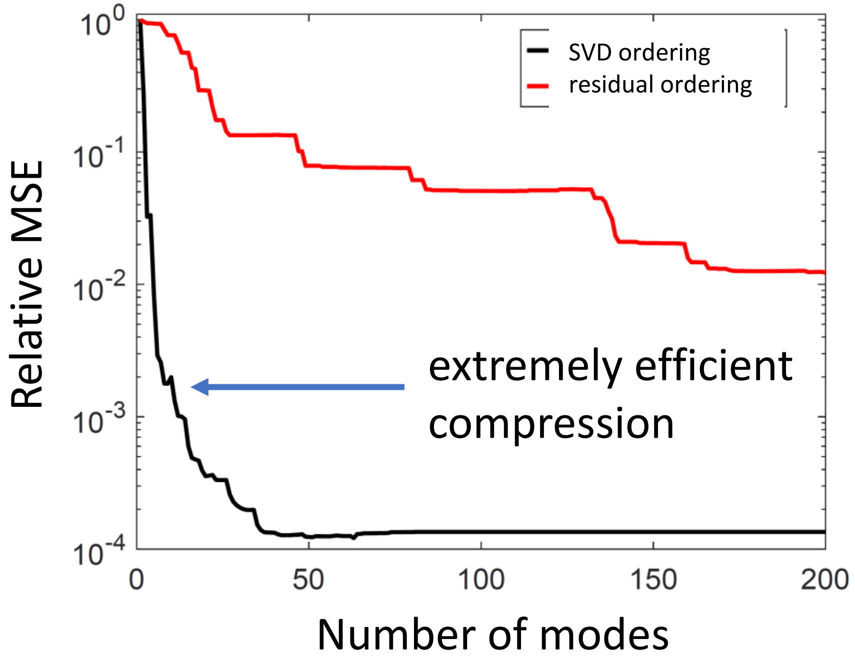
c) $t = 15\ \mu s$



d) $t = 20\ \mu s$

Matt Szőke's laser cannon!





Outline

- General systems:
  - Residual Dynamic Mode Decomposition.
- Measure-preserving systems:
  - **Measure-Preserving Extended Dynamic Mode Decomposition.**
  - Rigged Dynamic Mode Decomposition
- The **Solvability Complexity Index** – *classification* of problems and *optimality* of algorithms.

Measure-preserving systems

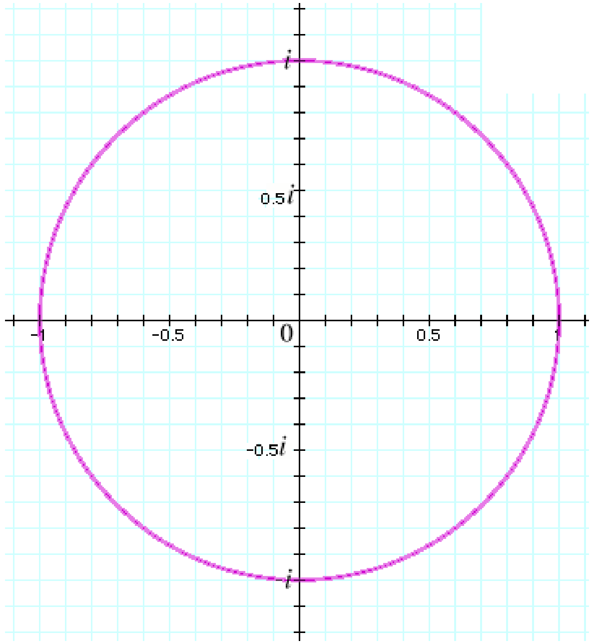
$$[\mathcal{K}g](x) = g\big(F(x)\big), \qquad g \in L^2(\Omega, \omega)$$

F preserves $\omega \Leftrightarrow \|\mathcal{K}g\| = \|g\|$ (isometry)

$\Leftrightarrow \mathcal{K}^*\mathcal{K} = I$

$\Rightarrow \mathrm{Sp}(\mathcal{K}) \subseteq \{z: |z| \leq 1\}$

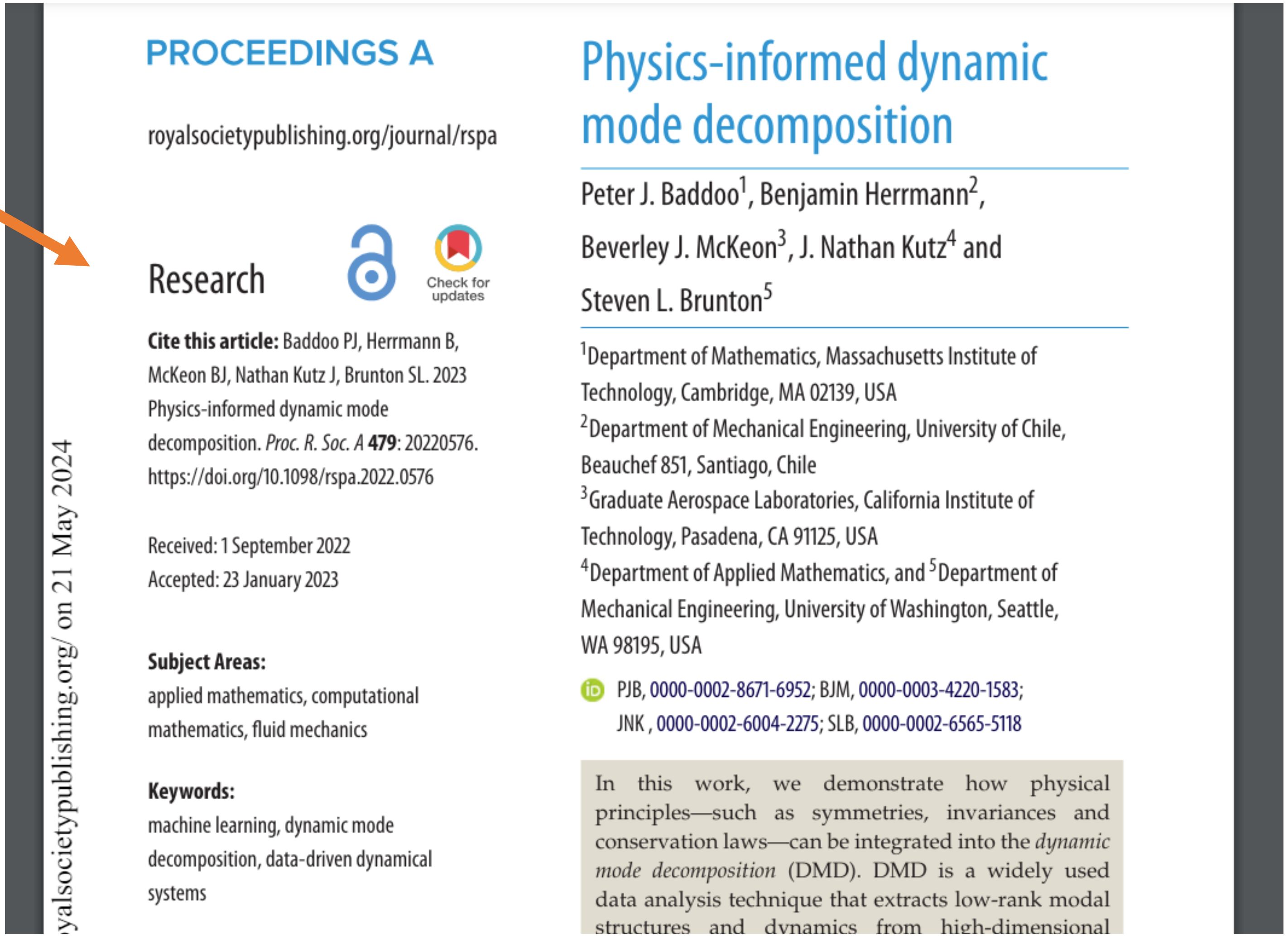
(NB: unitary extensions of $\mathcal{K}$ via Wold decomposition.)



Problem: We want our discretization to respect this property!

Structure-preserving DMD methods

- Enforce DMD matrix to lie on a manifold.
- **NB:** This is **much** easier for DMD which uses a linear choice of basis functions (which acts in state-space) than EDMD (which acts in coefficient space).
- We need something different....



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Research

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Subject Areas:
applied mathematics, computational mathematics, fluid mechanics

Keywords:
machine learning, dynamic mode decomposition, data-driven dynamical systems

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Physics-informed dynamic mode decomposition

Peter J. Baddoo[1], Benjamin Herrmann[2], Beverley J. McKeon[3], J. Nathan Kutz[4] and Steven L. Brunton[5]

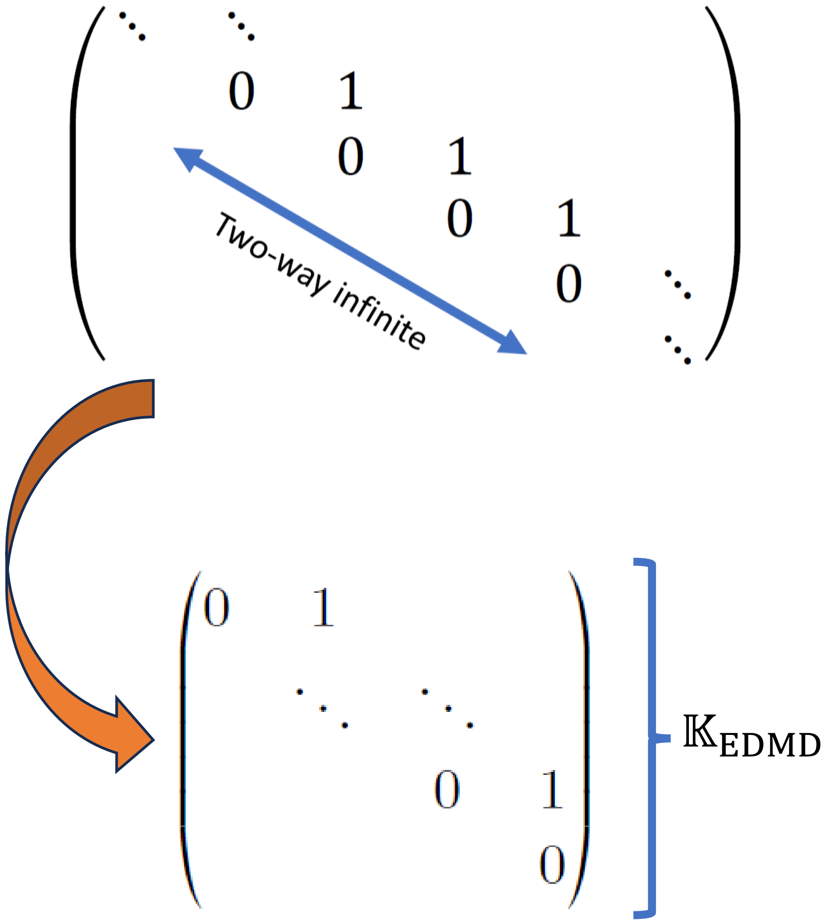
[1]Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
[2]Department of Mechanical Engineering, University of Chile, Beauchef 851, Santiago, Chile
[3]Graduate Aerospace Laboratories, California Institute of Technology, Pasadena, CA 91125, USA
[4]Department of Applied Mathematics, and [5]Department of Mechanical Engineering, University of Washington, Seattle, WA 98195, USA

PJB, 0000-0002-8671-6952; BJM, 0000-0003-4220-1583; JNK, 0000-0002-6004-2275; SLB, 0000-0002-6565-5118

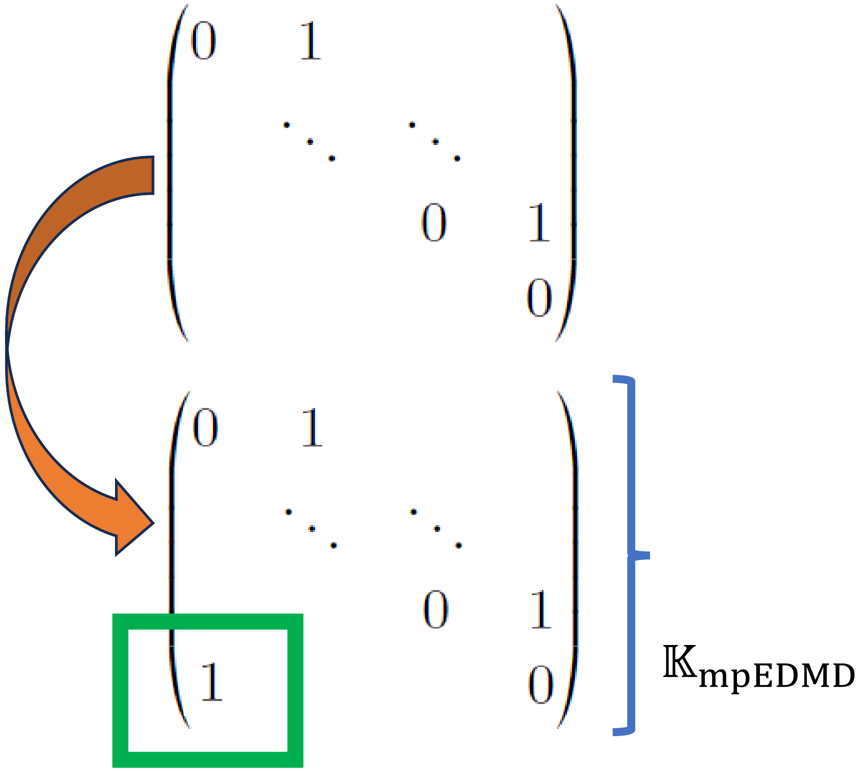
In this work, we demonstrate how physical principles—such as symmetries, invariances and conservation laws—can be integrated into the *dynamic mode decomposition* (DMD). DMD is a widely used data analysis technique that extracts low-rank modal structures and dynamics from high-dimensional

Back to the shift!

EDMD diverges:



mpEDMD converges:



- C., *"The mpEDMD Algorithm for Data-Driven Computations of Measure-Preserving Dynamical Systems,"* **SINUM**, 2023.
- Code*:* https://github.com/MColbrook/Measure-preserving-Extended-Dynamic-Mode-Decomposition

Back to the shift!

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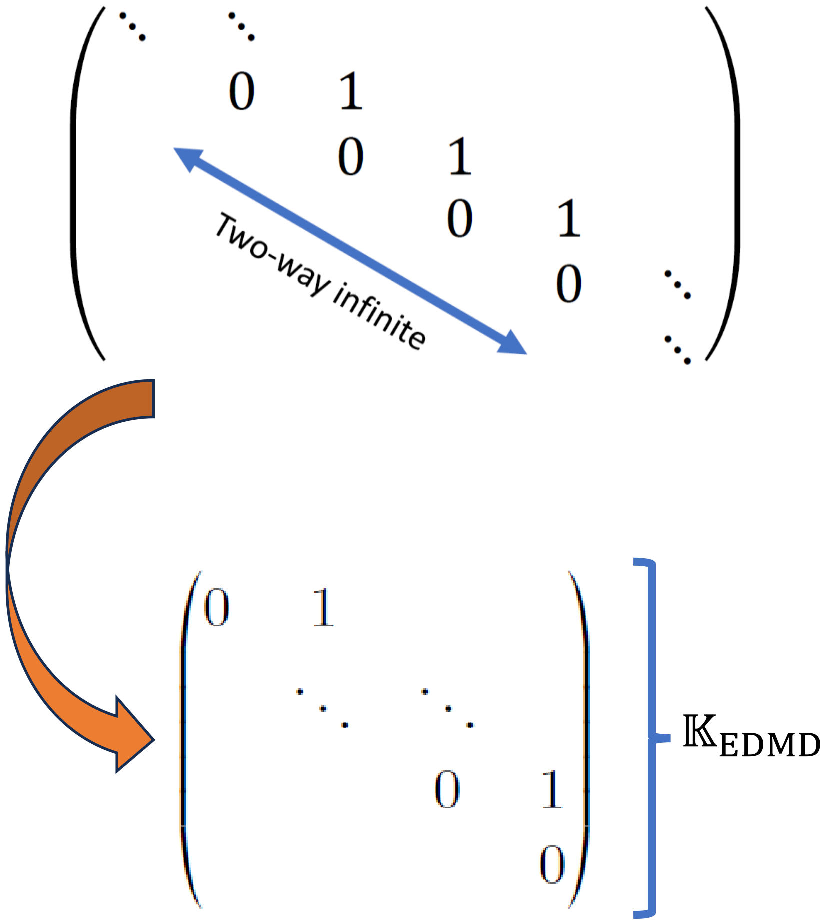

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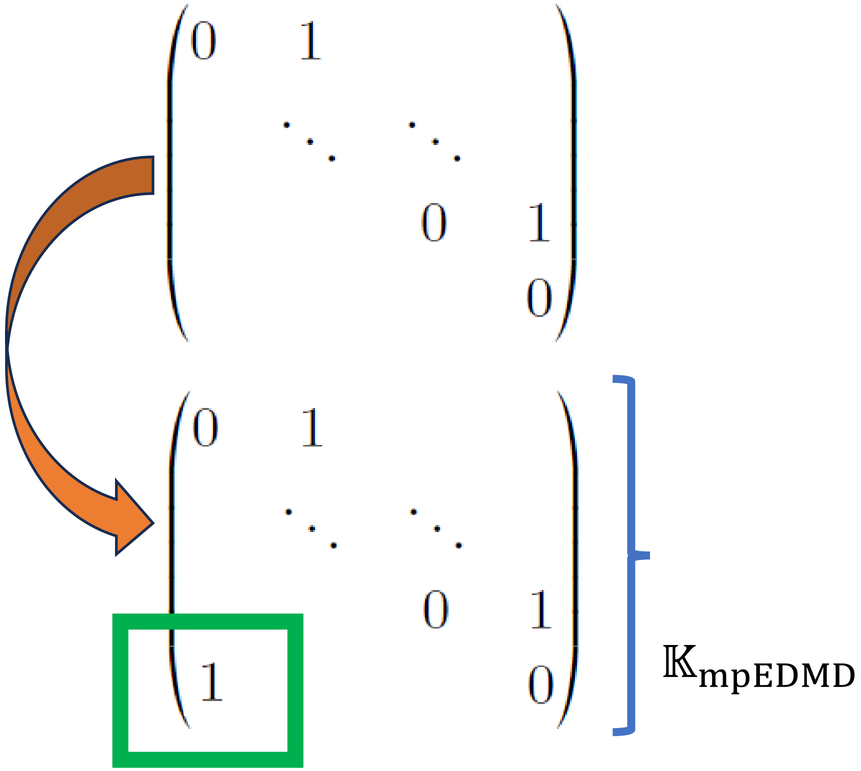

Let's make this into a general method…

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- Code: https://github.com/MColbrook/Measure-preserving-Extended-Dynamic-Mode-Decomposition

Back to EDMD!

$$\Psi(x) = [\psi_1(x) \;\; \dots \;\; \psi_N(x)], \qquad g = \sum_{j=1}^{N} \mathbf{g}_j \psi_j = \Psi \mathbf{g} \in \operatorname{span}\{\psi_1, \dots, \psi_N\}$$

$$\min_{\mathbb{K} \in \mathbb{C}^{N\times N}} \left\{ \int_\Omega \max_{\|\mathbf{g}\|_2 = 1} |\Psi(x)\mathbb{K}\mathbf{g} - [\mathcal{K}g](x)|^2 \, \mathrm{d}\omega(x) = \int_\Omega \|\Psi(x)\mathbb{K} - \Psi(F(x))\|_2^2 \, \mathrm{d}\omega(x) \right\}$$

quadrature

$$\left\{x^{(m)}, y^{(m)} = F(x^{(m)})\right\}_{m=1}^{M}$$

$$\min_{\mathbb{K} \in \mathbb{C}^{N\times N}} \sum_{m=1}^{M} w_m \left\| \Psi\left(x^{(m)}\right)\mathbb{K} - \Psi\left(y^{(m)}\right) \right\|_2^2$$

Least-squares problem

A simple alteration

$$G_{jk} = \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})} \psi_k(x^{(m)}) \approx \langle \psi_k, \psi_j \rangle$$

A simple alteration

$$G_{jk} = \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\psi_k(x^{(m)}) \approx \langle \psi_k, \psi_j \rangle$$

Measure-preserving: $\mathbf{g}^* G \mathbf{g} \approx \|g\|^2 = \|\mathcal{K} g\|^2 \approx \mathbf{g}^* \mathbb{K}^* G \mathbb{K} \mathbf{g}$

A simple alteration

$$G_{jk} = \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})} \psi_k(x^{(m)}) \approx \langle \psi_k, \psi_j \rangle$$

Measure-preserving: $\mathbf{g}^* G \mathbf{g} \approx \|g\|^2 = \|\mathcal{K} g\|^2 \approx \mathbf{g}^* \mathbb{K}^* G \mathbb{K} \mathbf{g}$

Enforce: $G = \mathbb{K}^* G \mathbb{K}$

A simple alteration

$$G_{jk} = \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\psi_k(x^{(m)}) \approx \langle \psi_k, \psi_j \rangle$$

Measure-preserving: $\mathbf{g}^* G \mathbf{g} \approx \|g\|^2 = \|\mathcal{K} g\|^2 \approx \mathbf{g}^* \mathbb{K}^* G \mathbb{K} \mathbf{g}$

Enforce: $G = \mathbb{K}^* G \mathbb{K}$

quadrature

Orthogonal Procrustes problem

$$\min_{\substack{\mathbb{K}\in\mathbb{C}^{N\times N} \\ G=\mathbb{K}^* G \mathbb{K}}} \sum_{m=1}^{M} w_m \left\| \Psi(x^{(m)}) \mathbb{K} G^{-1/2} - \Psi(y^{(m)}) G^{-1/2} \right\|_2^2$$

The mpEDMD algorithm

Algorithm 4.1 The mpEDMD algorithm

Input: Snapshot data $\mathbf{X} \in \mathbb{C}^{d\times M}$ and $\mathbf{Y} \in \mathbb{C}^{d\times M}$, quadrature weights $\{w_m\}_{m=1}^{M}$, and a dictionary of functions $\{\psi_j\}_{j=1}^{N}$.

1: Compute the matrices $\mathbf{\Psi}_X$ and $\mathbf{\Psi}_Y$ and $\mathbf{W} = \mathrm{diag}(w_1, \ldots, w_M)$.
2: Compute an economy QR decomposition $\mathbf{W}^{1/2}\mathbf{\Psi}_X = \mathbf{QR}$, where $\mathbf{Q} \in \mathbb{C}^{M\times N}$, $\mathbf{R} \in \mathbb{C}^{N\times N}$.
3: Compute an SVD of $(\mathbf{R}^{-1})^*\mathbf{\Psi}_Y^*\mathbf{W}^{1/2}\mathbf{Q} = \mathbf{U}_1\mathbf{\Sigma}\mathbf{U}_2^*$.
4: Compute the eigendecomposition $\mathbf{U}_2\mathbf{U}_1^* = \hat{\mathbf{V}}\mathbf{\Lambda}\hat{\mathbf{V}}^*$ (via a Schur decomposition).
5: Compute $\mathbb{K} = \mathbf{R}^{-1}\mathbf{U}_2\mathbf{U}_1^*\mathbf{R}$ and $\mathbf{V} = \mathbf{R}^{-1}\hat{\mathbf{V}}$.

Output: Koopman matrix $\mathbb{K}$ with eigenvectors $\mathbf{V}$ and eigenvalues $\mathbf{\Lambda}$.

$V_N = \mathrm{span}\{\psi_1, \ldots, \psi_N\}$
$\mathcal{P}_{V_N}: L^2(\Omega, \omega) \to V_N$
orthogonal projection

As $M \to \infty$, **unitary part** of polar decomposition of $\mathcal{P}_{V_N}\mathcal{K}\mathcal{P}_{V_N}^*$.

Convergence: spectral measures (see later), Koopman mode decomposition,…

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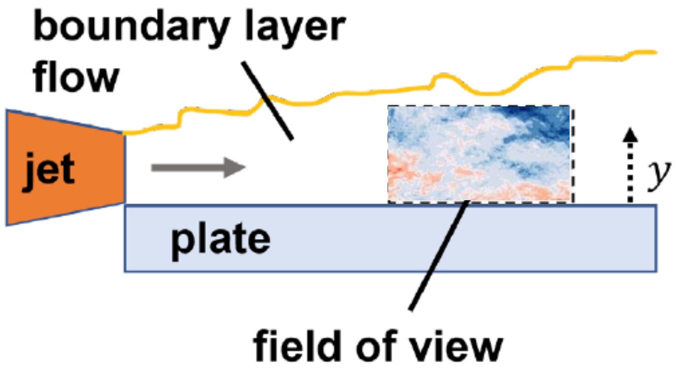
- Code: https://github.com/MColbrook/Measure-preserving-Extended-Dynamic-Mode-Decomposition

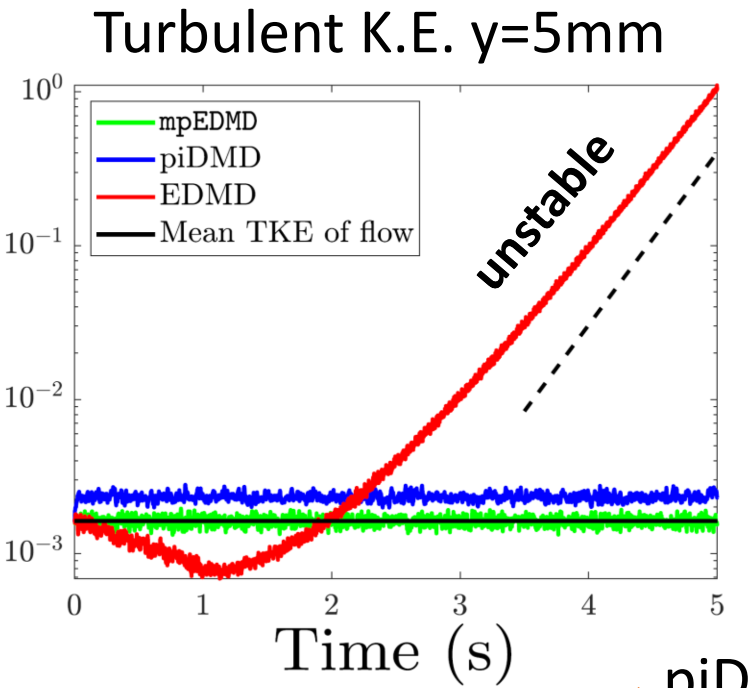
Turbulence (real data)

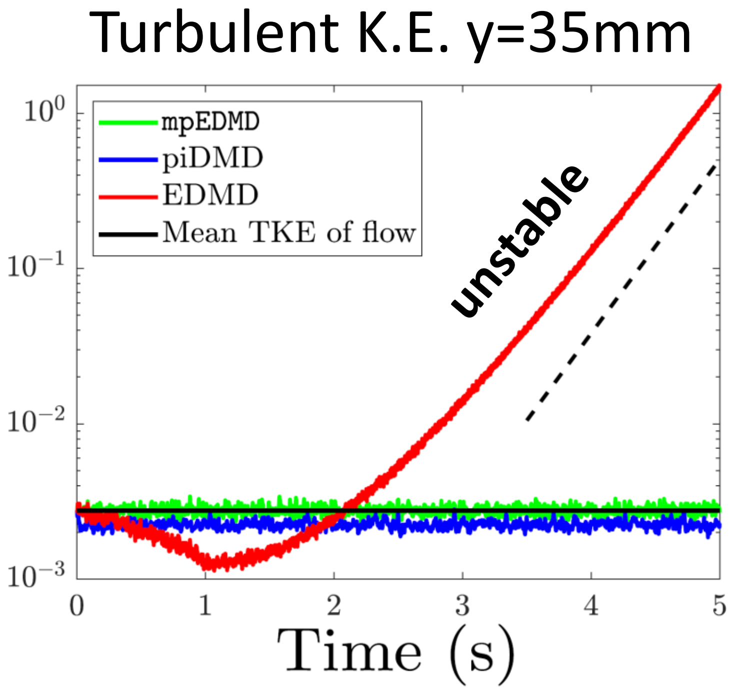
Experimental setup

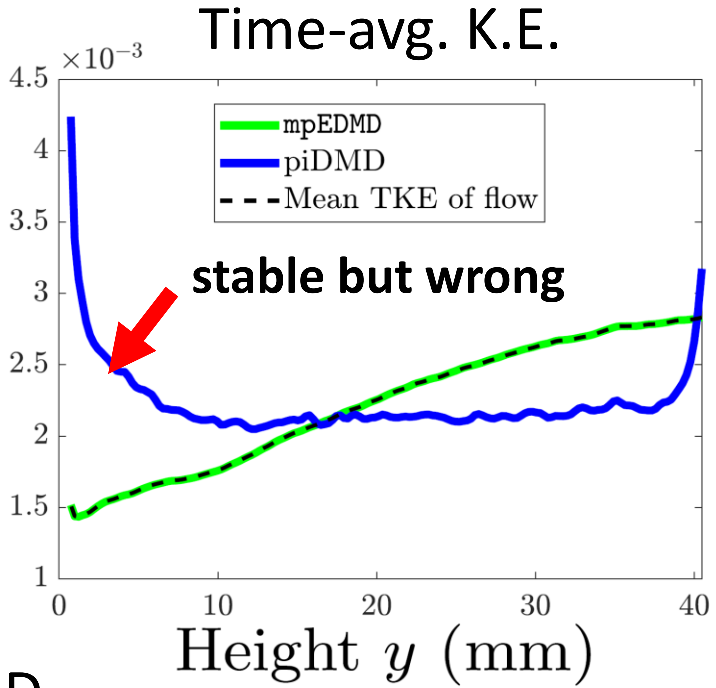


- Reynolds number $\approx 6.4 \times 10^4$
- Ambient dimension (d) $\approx$ 100,000 (velocity at measurement points)

*PIV data provided by Máté Szőke (Virginia Tech)





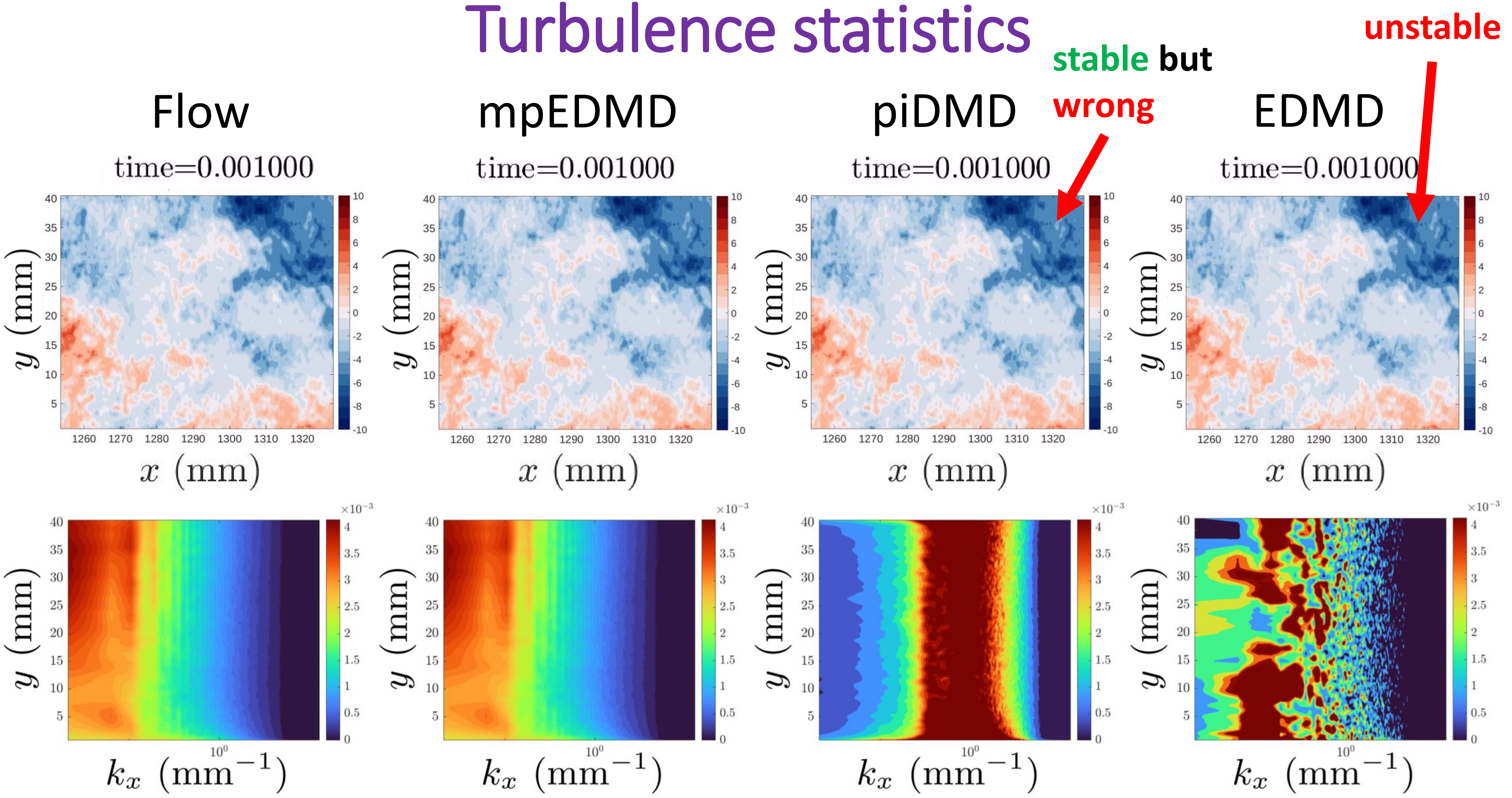


piDMD

EDMD

- Baddoo, Herrmann, McKeon, Kutz, Brunton, *"Physics-informed dynamic mode decomposition (piDMD),"* preprint.
- Williams, Kevrekidis, Rowley *"A data-driven approximation of the Koopman operator: Extending dynamic mode decomposition,"* **J. Nonlinear Sci.**, 2015.

Turbulence statistics



Outline

- <u>General</u> systems:
  - Residual Dynamic Mode Decomposition.
- <u>Measure-preserving</u> systems:
  - Measure-Preserving Extended Dynamic Mode Decomposition.
  - **Rigged Dynamic Mode Decomposition.**
- The **Solvability Complexity Index** – *classification* of problems and *optimality* of algorithms.

Problem: Often $\mathcal{K}$ doesn’t have basis of eigenfunctions (i.e., **continuous spectra**)

Back to the shift!

$$e_j \to e_{j-1}$$

"Solve" $(U - zI)u_z = 0$

$$u_z = \sum_{j=-\infty}^{\infty} z^j e_j$$

$$U = \begin{pmatrix} \ddots & \ddots & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix}$$

Two-way infinite

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Doesn't live in $\ell^2(\mathbb{Z})$!!!

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Two-way infinite

Let $|z| = 1$, $\phi = \sum_{j=-\infty}^{\infty} \phi_j e_j$ where ϕ_j decay faster than any inverse polynomial.

Test functions

Back to the shift!

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Test functions

$$\langle u_z, \phi \rangle = \sum_{j=-\infty}^{\infty} z^j \overline{\phi_j}, \qquad \langle U u_z, \phi \rangle = \langle u_z, U^* \phi \rangle = \sum_{j=-\infty}^{\infty} z^j \overline{\phi_{j-1}} = z \langle u_z, \phi \rangle$$

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Generalised eigenfunctions u_z and generalised eigenvalues $\{z: |z| = 1\}$

Another example: Nonlinear pendulum

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin(x_1)$$
$$\Omega = [-\pi, \pi]_{\text{per}} \times \mathbb{R},$$
$$\Delta_t = 1,$$
$$\omega = \text{ Lebesgue measure}$$

Considered a challenge in Koopman theory!

nature COMMUNICATIONS

ARTICLE

DOI: 10.1038/s41467-018-07210-0 OPEN

Deep learning for universal linear embeddings of nonlinear dynamics

Bethany Lusch[1,2], J. Nathan Kutz[1] & Steven L. Brunton[1,2]

Identifying coordinate transformations that make strongly nonlinear dynamics approximately linear has the potential to enable nonlinear prediction, estimation, and control using linear theory. The Koopman operator is a leading data-driven embedding, and its eigenfunctions provide intrinsic coordinates that globally linearize the dynamics. However, identifying and representing these eigenfunctions has proven challenging. This work leverages deep learning to discover representations of Koopman eigenfunctions from data. Our network is parsimonious and interpretable by construction, embedding the dynamics on a low-dimensional

Explicit diagonalization using Radon transform!

- Action-angle coordinates (n degrees of freedom):

$$\dot{\mathbf{I}} = \mathbf{0}, \qquad \dot{\boldsymbol{\theta}} = \mathbf{I}, \qquad \Omega = \mathbb{R}^n \times [-\pi, \pi]^n_{\mathrm{per}}$$

Explicit diagonalization using Radon transform!

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- g in Schwartz space,

$$g(\mathbf{I}, \boldsymbol{\theta}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} \hat{g}_{\mathbf{k}}(\mathbf{I}) e^{i\mathbf{k}\cdot\boldsymbol{\theta}}, \qquad \hat{g}_{\mathbf{k}}(\mathbf{I}) = \frac{1}{(2\pi)^n} \int_{[-\pi,\pi]^n_{\mathrm{per}}} g(\mathbf{I}, \boldsymbol{\theta})\, e^{-i\mathbf{k}\cdot\boldsymbol{\theta}} \mathrm{d}\boldsymbol{\theta}$$

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$$\hat{g}_{\mathbf{k}}(\mathbf{I}) = \sum_{j=1}^{\infty} \sum_{m=-\infty}^{\infty} \int_{[-\pi,\pi]_{\mathrm{per}}} \left\langle g_\theta^{(\mathbf{k},m,j)^*} | g \right\rangle g_\theta^{(\mathbf{k},m,j)} \mathrm{d}\theta$$

$$g_\theta^{(\mathbf{k},m,j)} = \frac{1}{(2\pi)^n} \delta(\theta + 2\pi m - \Delta t \mathbf{k} \cdot \mathbf{I}) \psi_j^{(k)} e^{i\mathbf{k}\cdot\boldsymbol{\theta}}$$

Generalised eigenfunctions

Explicit diagonalization using Radon transform!

- Action-angle coordinates (n degrees of freedom):

$$\dot{\mathbf{I}} = \mathbf{0}, \qquad \dot{\boldsymbol{\theta}} = \mathbf{I}, \qquad \Omega = \mathbb{R}^n \times [-\pi, \pi]^n_{\mathrm{per}}$$

- g in Schwartz space,

$$g(\mathbf{I}, \boldsymbol{\theta}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} \hat{g}_{\mathbf{k}}(\mathbf{I}) e^{i\mathbf{k}\cdot\boldsymbol{\theta}}, \qquad \hat{g}_{\mathbf{k}}(\mathbf{I}) = \frac{1}{(2\pi)^n} \int_{[-\pi,\pi]^n_{\mathrm{per}}} g(\mathbf{I}, \boldsymbol{\theta})\, e^{-i\mathbf{k}\cdot\boldsymbol{\theta}} \mathrm{d}\boldsymbol{\theta}$$

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$$g_\theta^{(\mathbf{k},m,j)} = \frac{1}{(2\pi)^n} \delta(\theta + 2\pi m - \Delta t \mathbf{k} \cdot \mathbf{I}) \psi_j^{(k)} e^{i\mathbf{k}\cdot\boldsymbol{\theta}}$$

Plane wave

Generalised eigenfunctions

Supported on hyperplane

Orthonormal basis of hyperplane

Gelfand's theorem → diagonalisation

- **Finite matrix:** $B \in \mathbb{C}^{n\times n}$, $B^*B = BB^*$, orthonormal basis of e-vectors $\{v_j\}_{j=1}^n$

$$v = \sum_{j=1}^n ({v_j}^* v) v_j, \qquad Bv = \sum_{j=1}^n \lambda_j ({v_j}^* v) v_j \qquad \forall v \in \mathbb{C}^n$$

- **Infinite dimensions:** Unitary $\mathcal{K}$. Typically, **no basis of eigenfunctions**! Some technical assumptions (can always be realized):

$$g = \int_{[-\pi,\pi]_{\mathrm{per}}} \langle g_\theta^* | g \rangle \, g_\theta \, \mathrm{d}\nu(\theta), \qquad \mathcal{K}g = \int_{[-\pi,\pi]_{\mathrm{per}}} e^{i\theta} \langle g_\theta^* | g \rangle \, g_\theta \, \mathrm{d}\nu(\theta)$$

$g \in S \subset L^2(\Omega, \omega)$

Koopman modes

generalized eigenfunctions
<u>*distributions*</u> $\in \mathcal{S}^*$

$e^{i\theta} = \lambda$

Koopman Mode Decomposition

Rigged DMD: Smoothing

Carathéodory function:

$$F_g(z) = (\mathcal{K} + zI)(\mathcal{K} - zI)^{-1} g = \int_{[-\pi,\pi]_{\mathrm{per}}} \boxed{\frac{e^{i\theta} + z}{e^{i\theta} - z}} \langle g_\theta^* | g \rangle \, g_\theta \, \mathrm{d}\nu(\theta)$$

Rigged DMD: Smoothing

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Let $r = 1 + \varepsilon > 1, \theta_0 \in [-\pi, \pi]_{\mathrm{per}}$,

$$\frac{1}{4\pi}\left[F_g\left(r^{-1}e^{i\theta_0}\right) - F_g\left(re^{i\theta_0}\right)\right]$$

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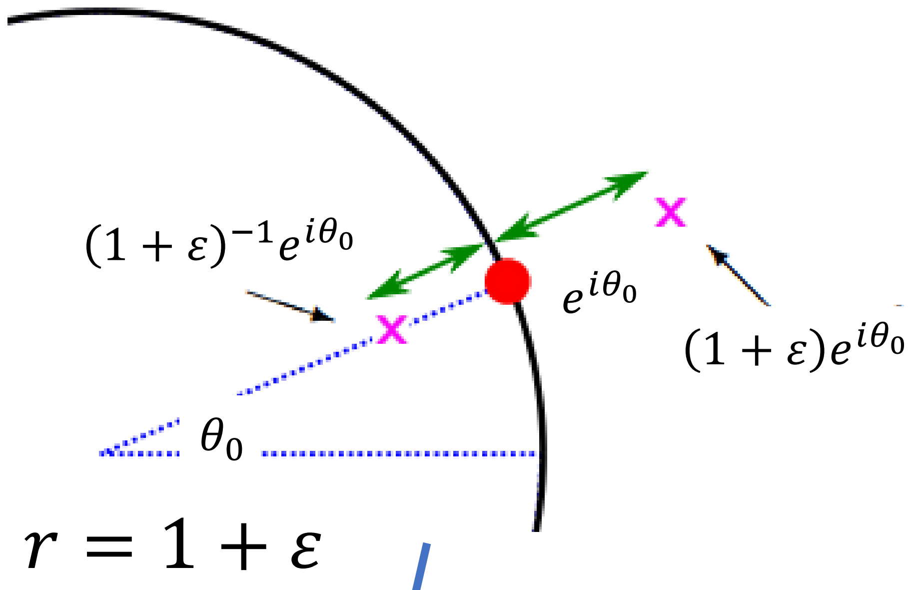
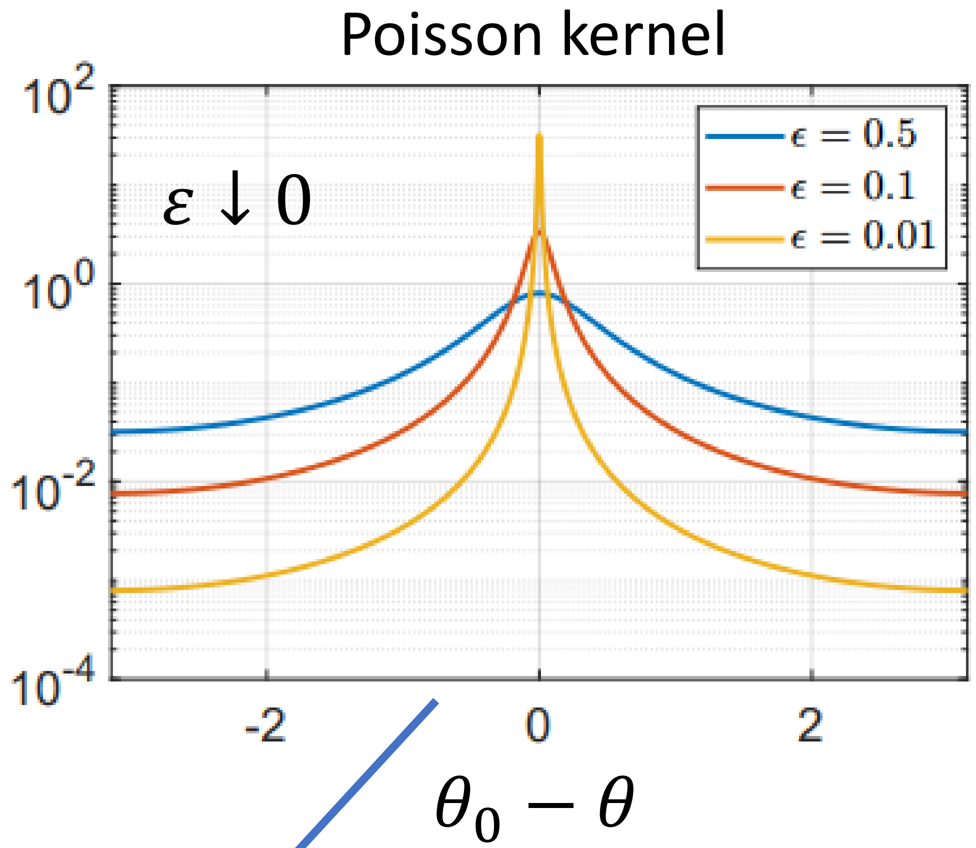
Let $r = 1 + \varepsilon > 1, \theta_0 \in [-\pi, \pi]_{\mathrm{per}}$,

Poisson kernel

$$\frac{1}{4\pi}\left[F_g\left(r^{-1} e^{i\theta_0}\right) - F_g\left(r e^{i\theta_0}\right)\right] = \frac{1}{2\pi} \int_{[-\pi,\pi]_{\mathrm{per}}} \frac{r^2 - 1}{1 + r^2 - 2r\cos(\theta_0 - \theta)} \langle g_\theta^* | g \rangle \, g_\theta \, \mathrm{d}\nu(\theta)$$

Smoothed generalized eigenfunction





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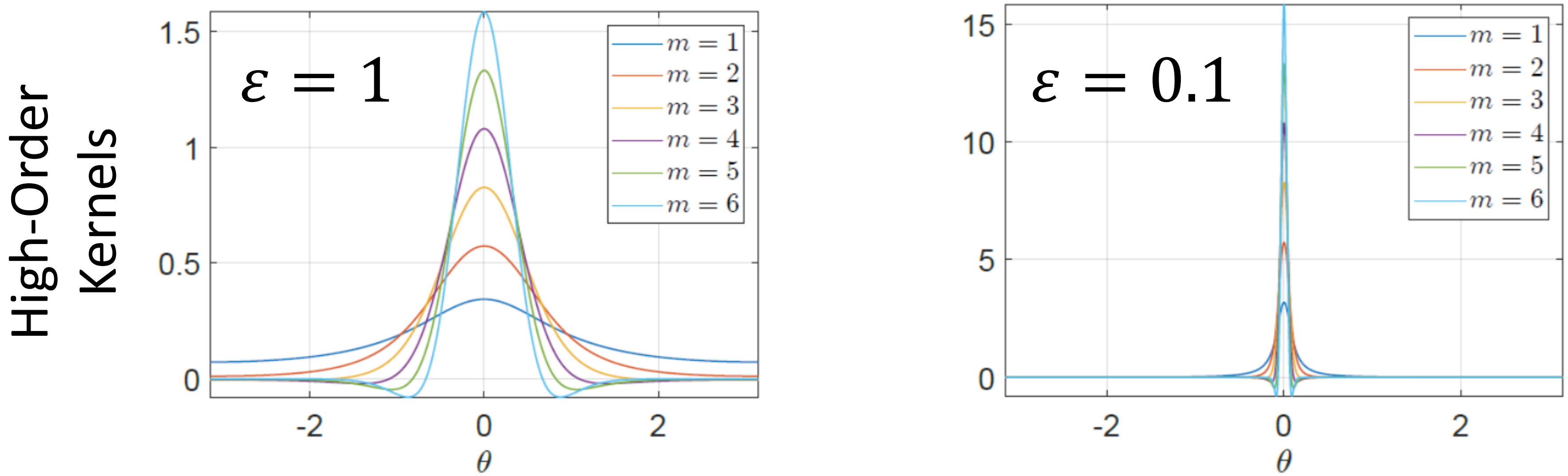
Smoothed generalized eigenfunction

Better smoothing kernels as $\varepsilon \downarrow 0$

- Poisson kernel: **slow** convergence $\mathcal{O}(\varepsilon \log(1/\varepsilon))$.
- Construct high-order kernels using $F_g(z)$.

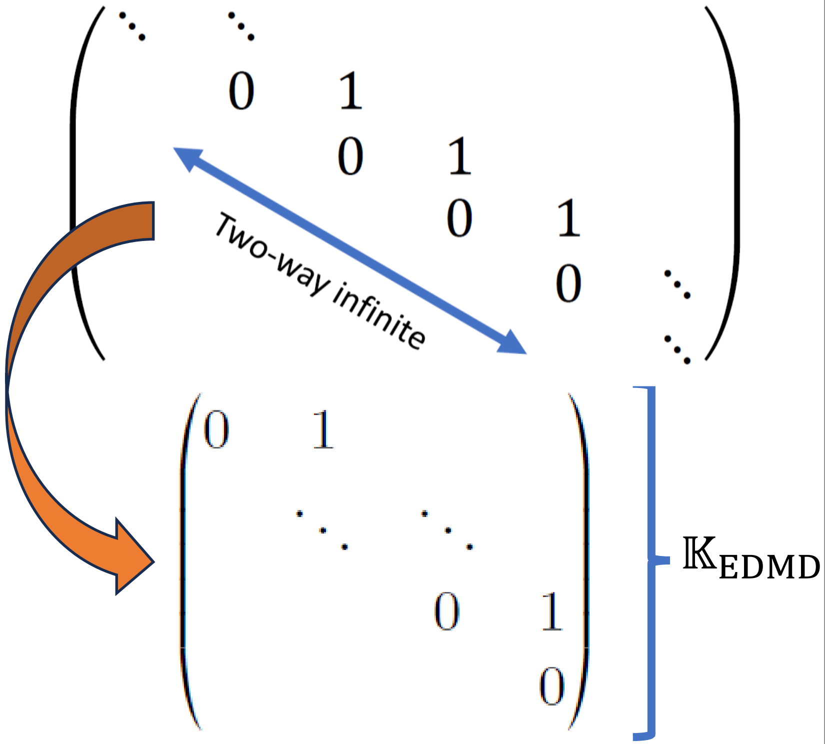
Smaller ε requires more data



- **Theorems: fast** $\mathcal{O}(\varepsilon^m \log(1/\varepsilon))$ convergence for
  - Generalized eigenfunctions (topology of $\mathcal{S}^*$).
  - Spectral measures (traces of generalized eigenfunctions): pointwise, L^p, weak,…
  - Forecasting (i.e., iterating Koopman mode decomposition), coherency etc.

Final ingredient: F_g requires $(\mathcal{K} - zI)^{-1}$

EDMD diverges:



Acts on $\text{span}\{e_{-N}, \dots, e_N\}$

E.g., if $|z| < 1$,

$$(\mathbb{K}_{\text{EDMD}} - zI)^{-1} e_0 = \sum_{j=1}^{N} \frac{-1}{z^j} e_{-j}$$

Exponential blowup as $N \to \infty$.

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$$\begin{pmatrix} \ddots & \ddots & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix}$$

Two-way infinite

$$\underbrace{\begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}}_{\mathbb{K}_{\mathrm{EDMD}}}$$

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mpEDMD converges:

$$\begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

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General method: unitary part of a **polar decomposition** of EDMD!

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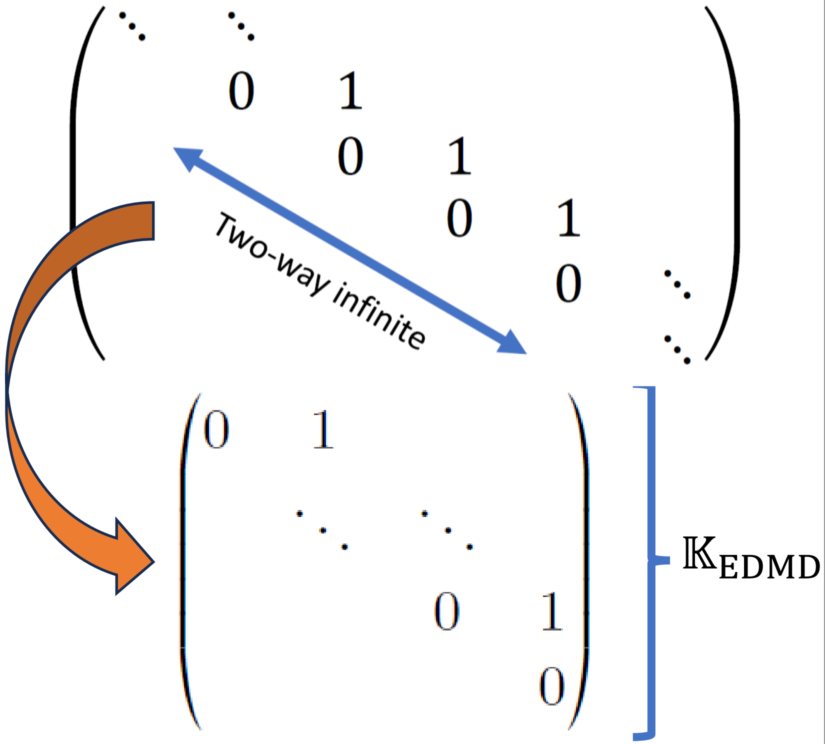

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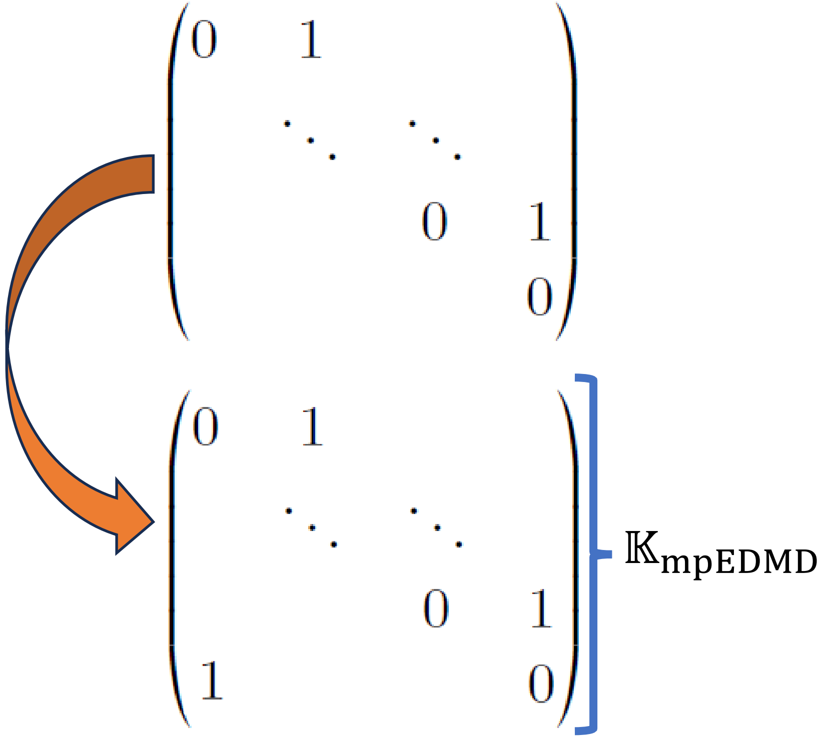
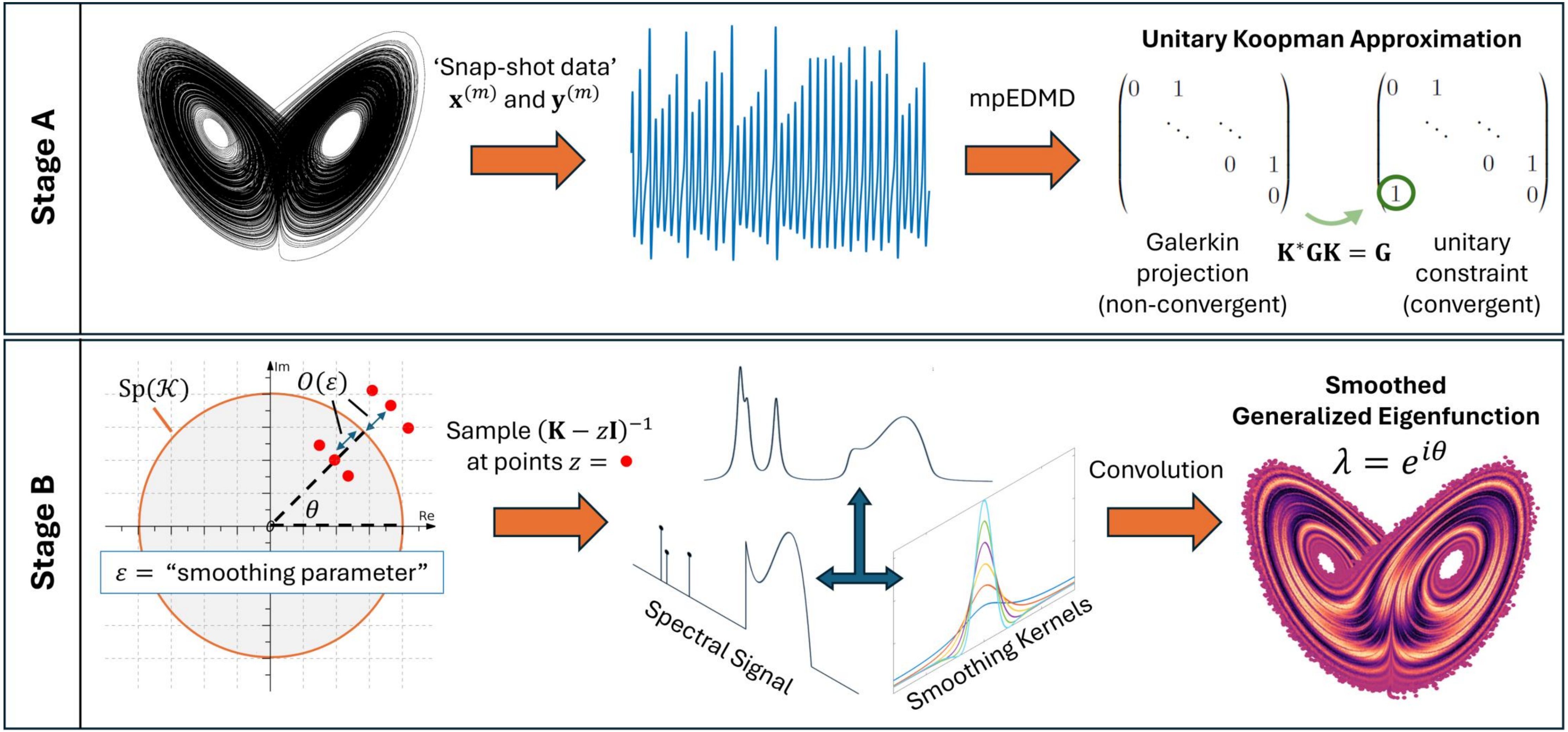


General method: unitary part of a **polar decomposition** of EDMD!

Rigged DMD converges:

- For general $\mathcal{K}$: $\left(\mathbb{K}_{\text{mpEDMD}} - zI\right)^{-1}\mathbf{g}$ converges to $(\mathcal{K} - zI)^{-1} g$ as $\lim_{N\to\infty} \lim_{M\to\infty}$
- Hence, Rigged DMD converges as $\lim_{\varepsilon\downarrow 0} \lim_{N\to\infty} \lim_{M\to\infty}$
- ResDMD allows us to select $\varepsilon = \varepsilon(N)$ adaptively (convergence in **2 limits**)

Rigged DMD

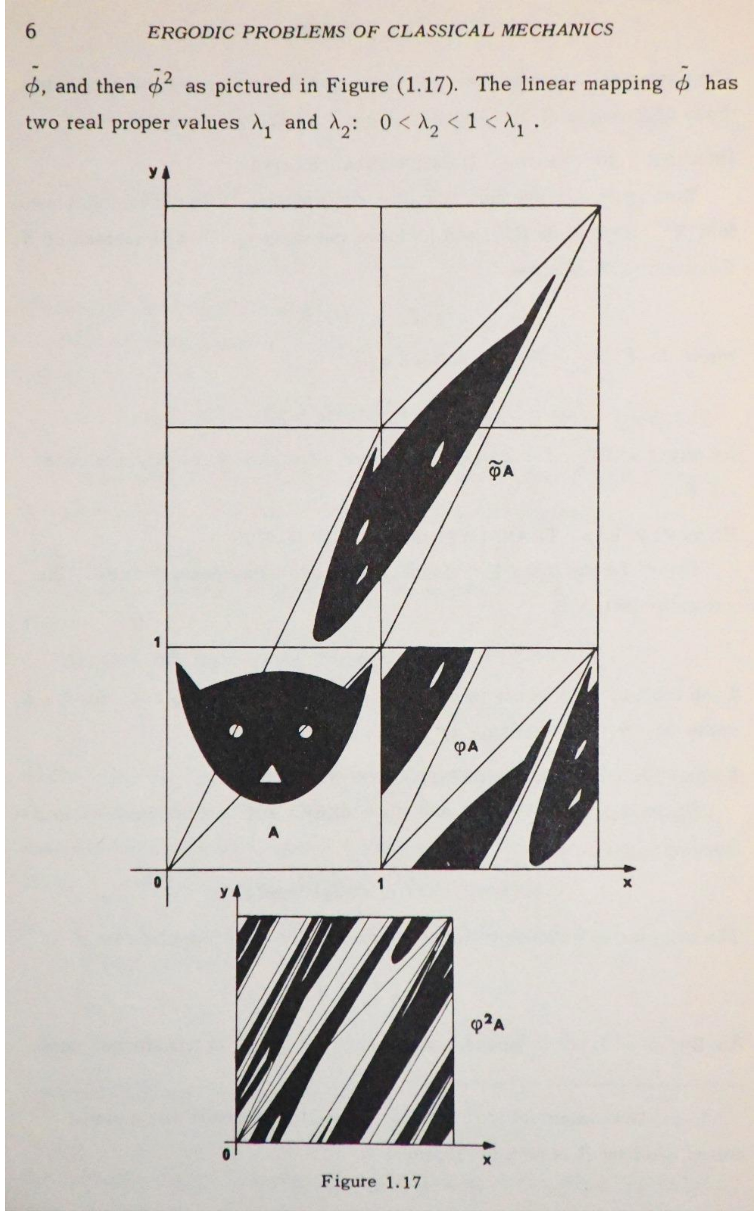


- C., Drysdale, Horning, *"Rigged Dynamic Mode Decomposition: Data-Driven Generalized Eigenfunction Decompositions for Koopman Operators"*, arxiv preprint.
- Code*:* https://github.com/MColbrook/Rigged-Dynamic-Mode-Decomposition

Example: Arnold's cat map

$F(x, y) = (2x + y, x + y) \mod 2\pi$

$\Omega = [-\pi, \pi]^2_{\mathrm{per}}, \quad \omega =$ Lebesgue measure


6 ERGODIC PROBLEMS OF CLASSICAL MECHANICS

$\bar{\phi}$, and then $\bar{\phi}^2$ as pictured in Figure (1.17). The linear mapping $\bar{\phi}$ has two real proper values λ_1 and λ_2: $0 < \lambda_2 < 1 < \lambda_1$.

Figure 1.17

Arnold's "Ergodic Problems of Classical Mechanics"

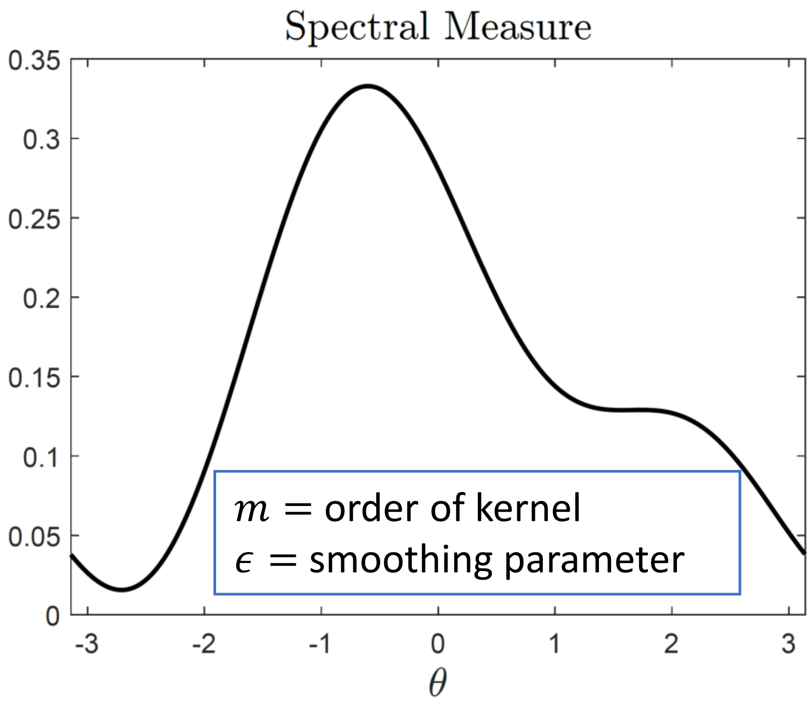
Example: Arnold's cat map

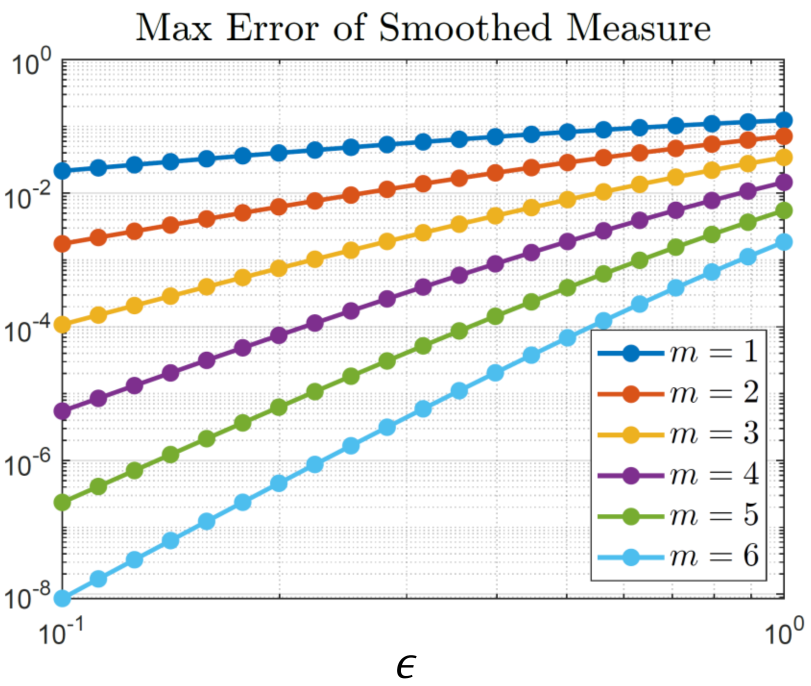
Experimental details
Length-one trajectories, $M = 50 \times 50, N = 500$
$g(x,y) = \sin(x) + \frac{1}{2}\sin(2x+y) + \frac{i}{4}\sin(5x+3y)$
Krylov subspace: $V_N = \{g, \mathcal{K}g, \dots, \mathcal{K}^{N-1}g\}$

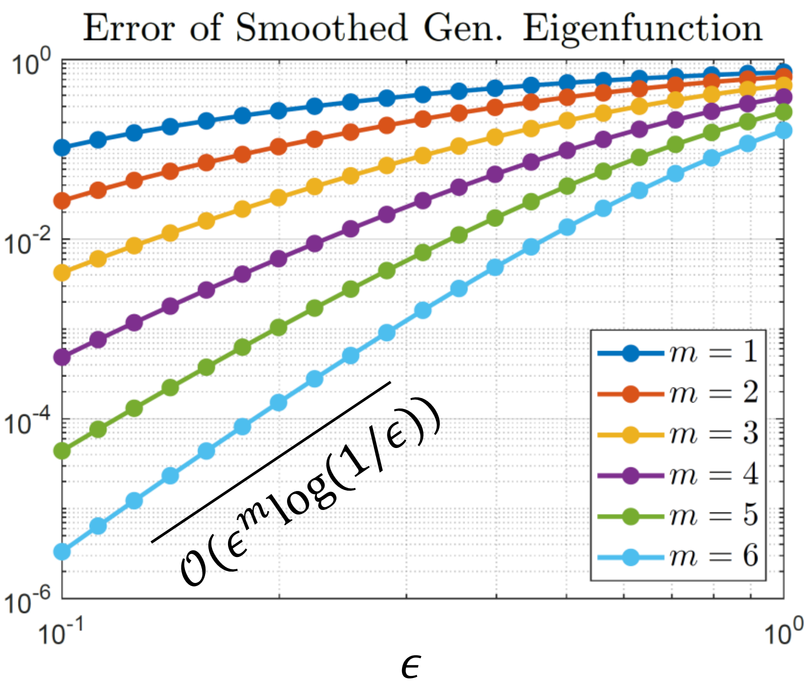
$F(x,y) = (2x+y, x+y) \mod 2\pi$

$\Omega = [-\pi,\pi]^2_{\text{per}}, \quad \omega =$ Lebesgue measure

Explicit formula: g_θ become more oscillatory as $\epsilon \downarrow 0$ (non-decaying Fourier series)

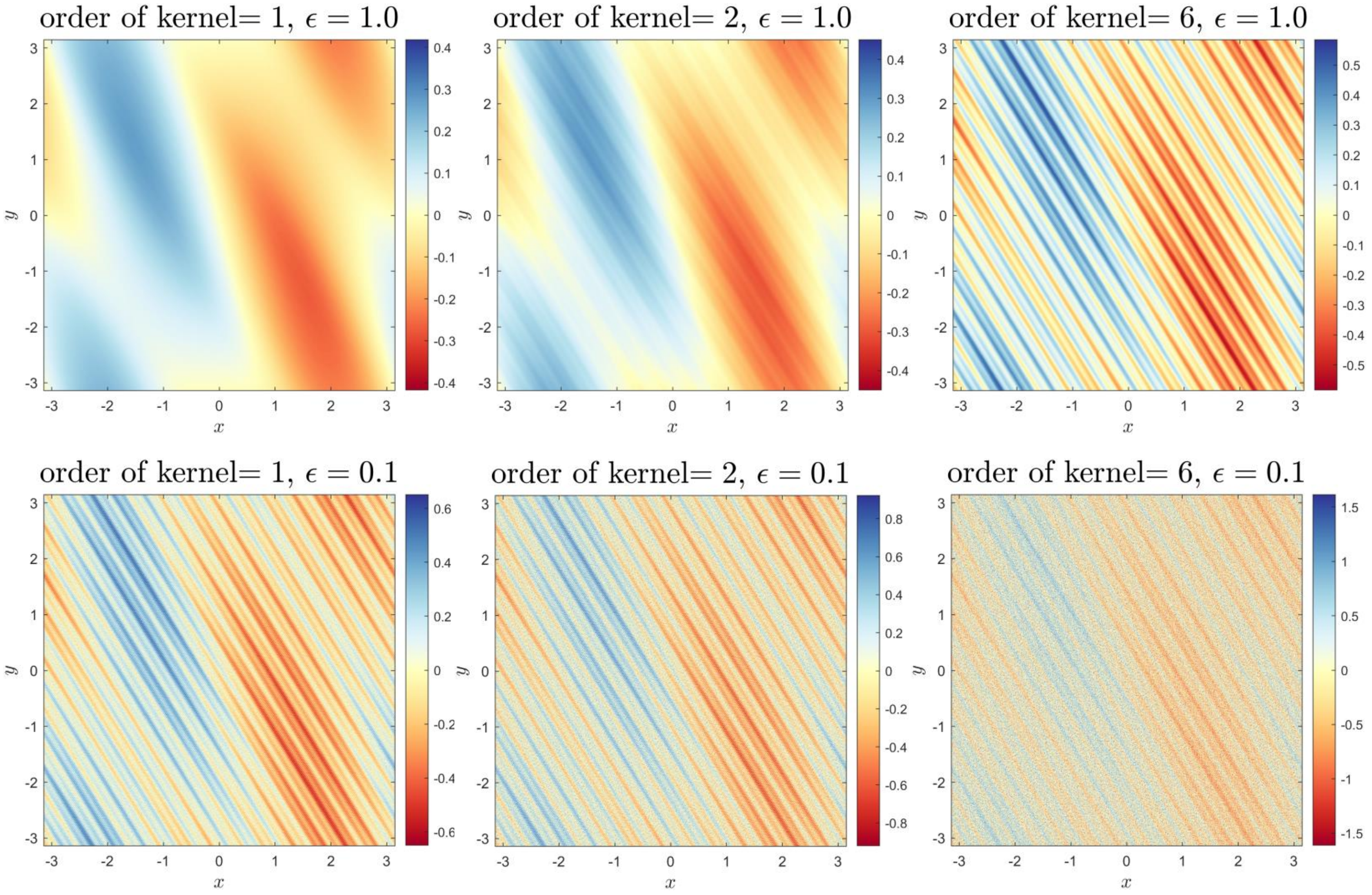






Higher kernel order (accuracy)

Higher resolution ($\varepsilon \downarrow 0$)



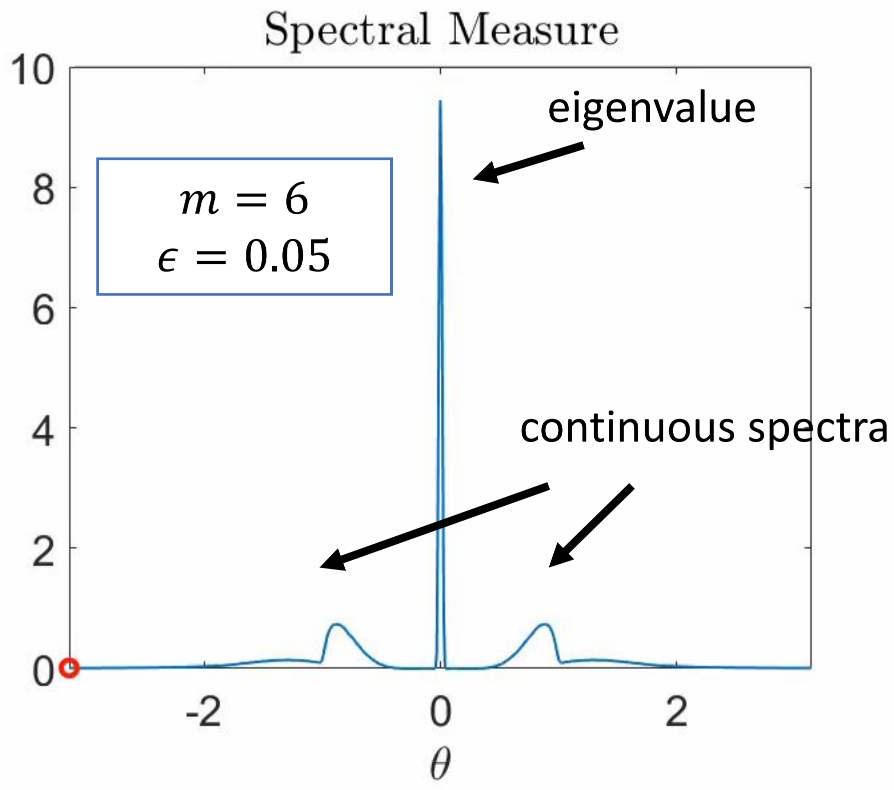
Increased oscillations of generalized eigenfunction

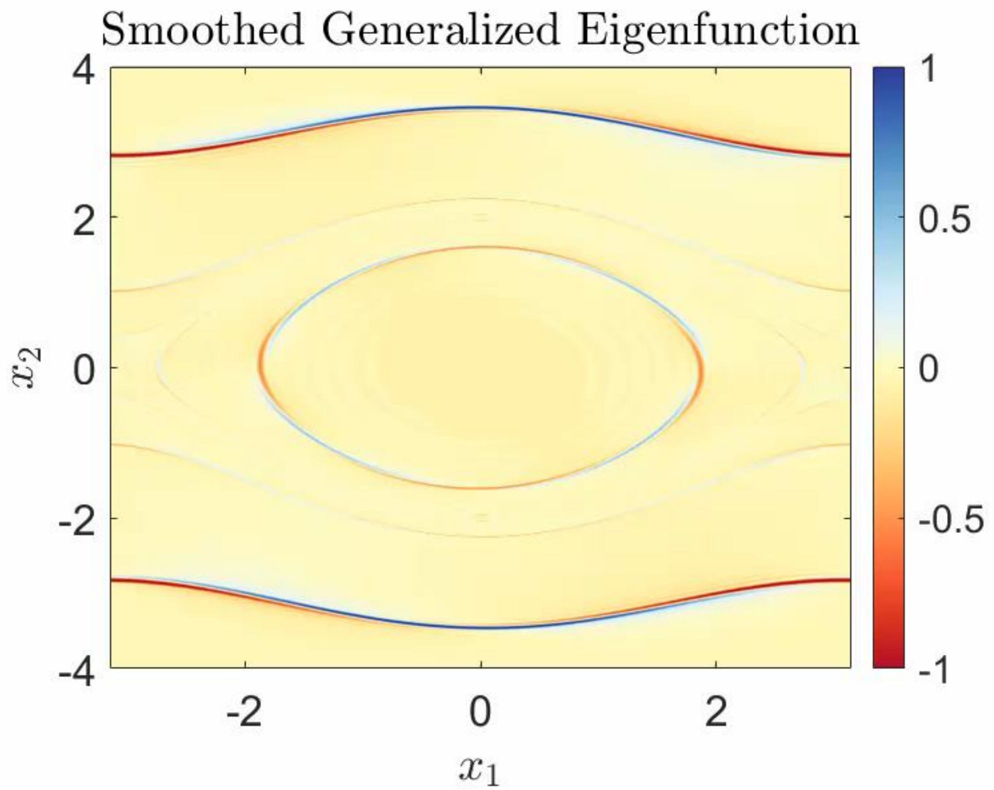
Example: Nonlinear pendulum

Experimental Details
Length-one trajectories over grid
$M = 500 \times 500, N = 300$
$g(x_1, x_2) = \exp(ix_1)/\cosh(x_2)$
Krylov subspace: $V_N = \{g, \mathcal{K}g, \dots, \mathcal{K}^{N-1}g\}$

$\dot{x}_1 = x_2,\ \dot{x}_2 = -\sin(x_1)$, $\Omega = [-\pi, \pi]_{\text{per}} \times \mathbb{R}$, $\Delta_t = 1$, $\omega =$ Lebesgue measure

Explicit formula: g_θ become plane waves concentrated on unions of lines of constant energy as $\epsilon \downarrow 0$.





Interlude: Can we always find an $\mathcal{S}$?

- If $\mathcal{K}$ is represented by an infinite matrix with finitely many non-zero entries in each column, can build $\mathcal{S}$ using weighted sequence spaces.
- Always possible using time-delay embedding:

$$\{\text{Unions (different } g\text{) of spaces } \operatorname{span}\{g, \mathcal{K}g, \ldots, \mathcal{K}^{N-1}g, \ldots\}\} \subset \mathcal{S}$$

- Generalises shift example but in coefficient space w.r.t. Krylov subspaces.

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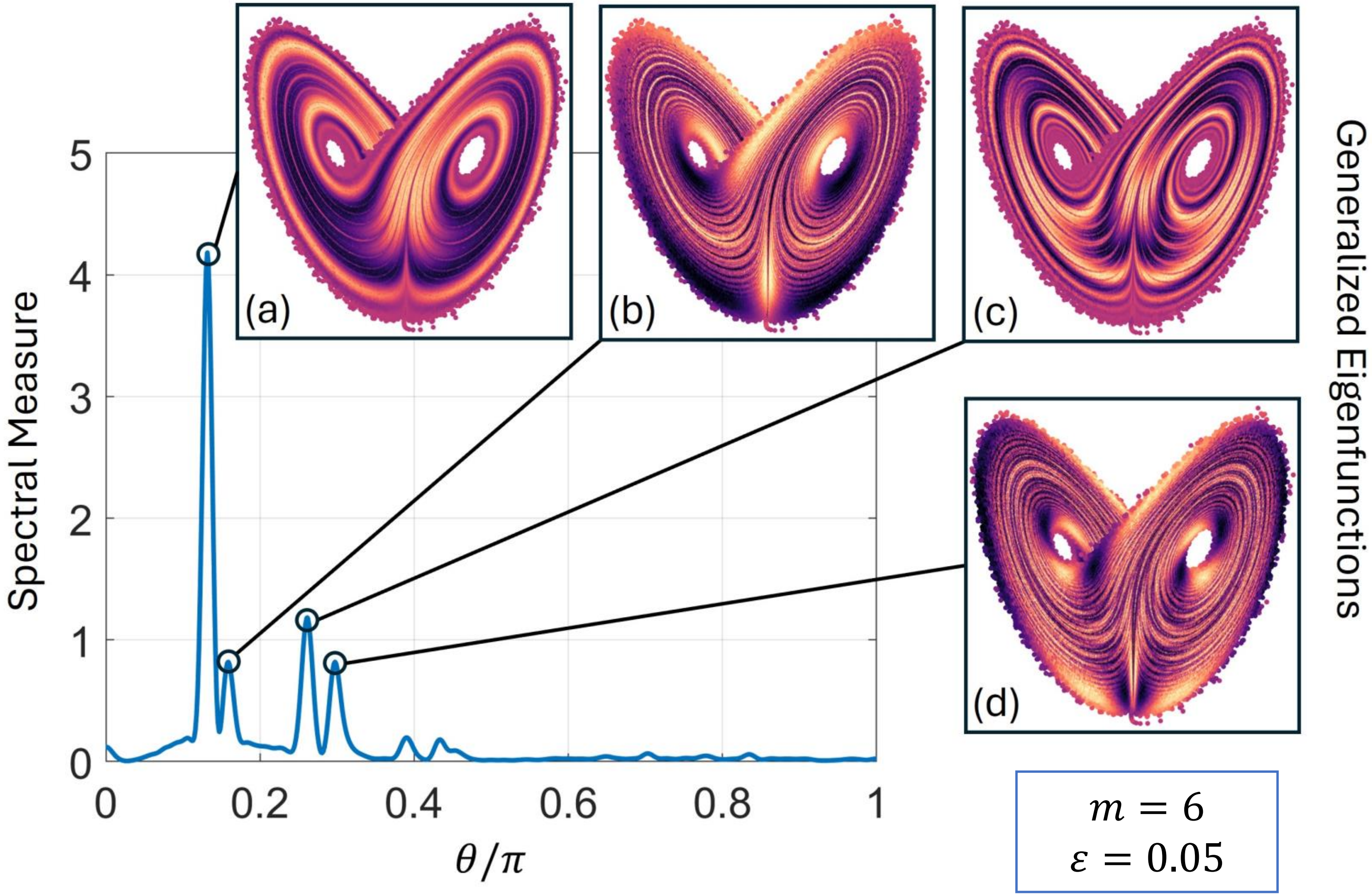
- Generalises shift example but in coefficient space w.r.t. Krylov subspaces.

Let’s do this for Lorenz…

Example: Lorenz system

$\dot{x}_1 = 10(x_2 - x_1),\ \dot{x}_2 = x_1(28 - x_3) - x_2,\ \dot{x}_3 = x_1x_2 - 8/3\, x_3,\quad \Delta_t = 0.05,\ \ \Omega = \text{attractor},\ \ \omega = \text{SRB measure}$



$m = 6$
$\varepsilon = 0.05$

No formula for
generalized eigenfunctions!!

Experimental Details

Single trajectory (ergodic system)

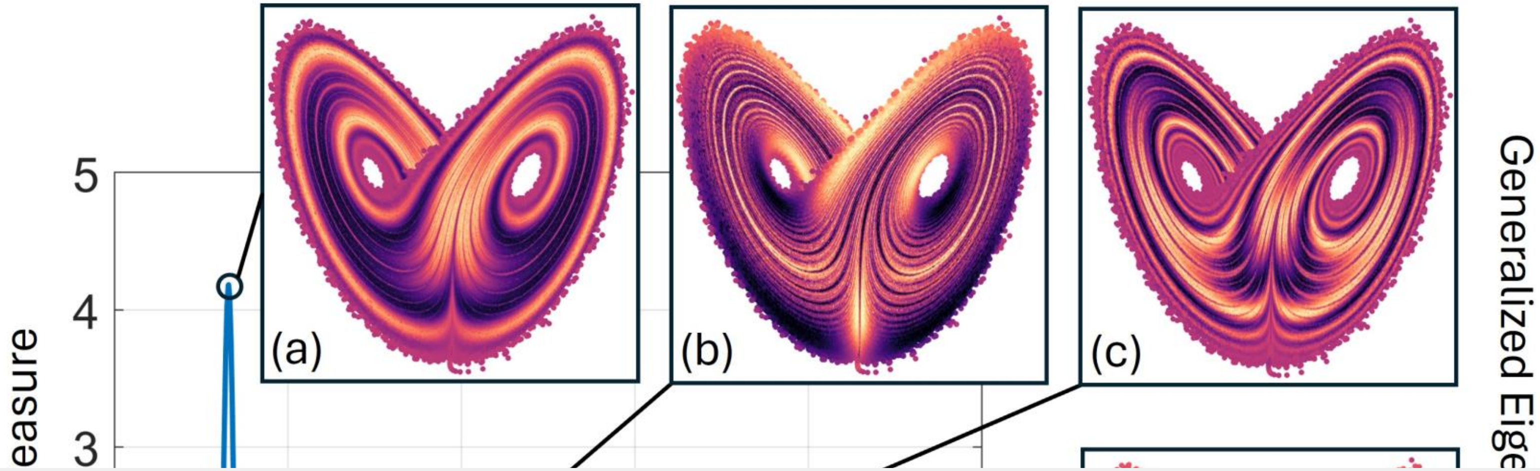
$M = 10000, N = 1000$

$$g(x_1, x_2, x_3) = \tanh\left(\frac{x_1x_2 - 5x_3}{10}\right) - c$$

Krylov subspace: $V_N = \{g, \mathcal{K}g, \dots, \mathcal{K}^{N-1}g\}$

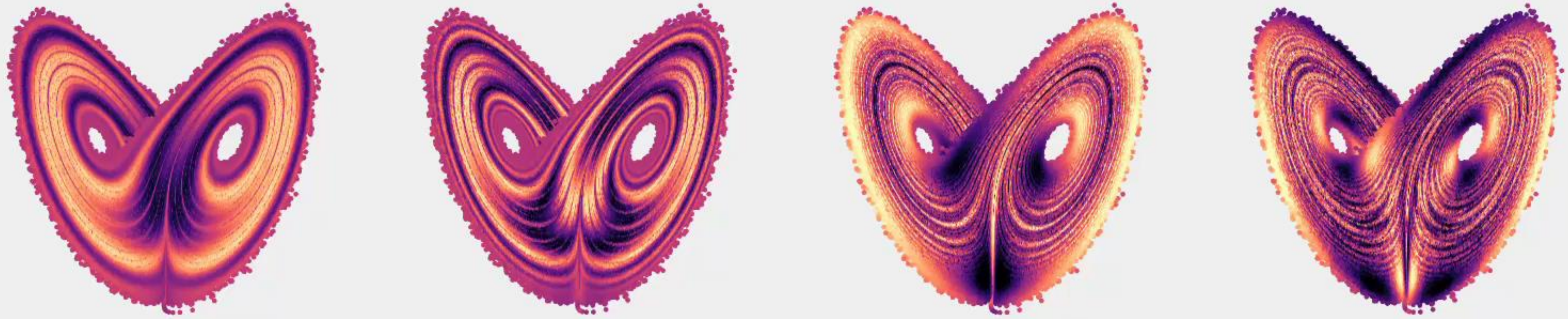
Example: Lorenz system

$\dot{x}_1 = 10(x_2 - x_1),\ \dot{x}_2 = x_1(28 - x_3) - x_2,\ \dot{x}_3 = x_1 x_2 - 8/3\, x_3, \quad \Delta_t = 0.05, \ \ \Omega = \text{attractor}, \ \omega = \text{SRB measure}$



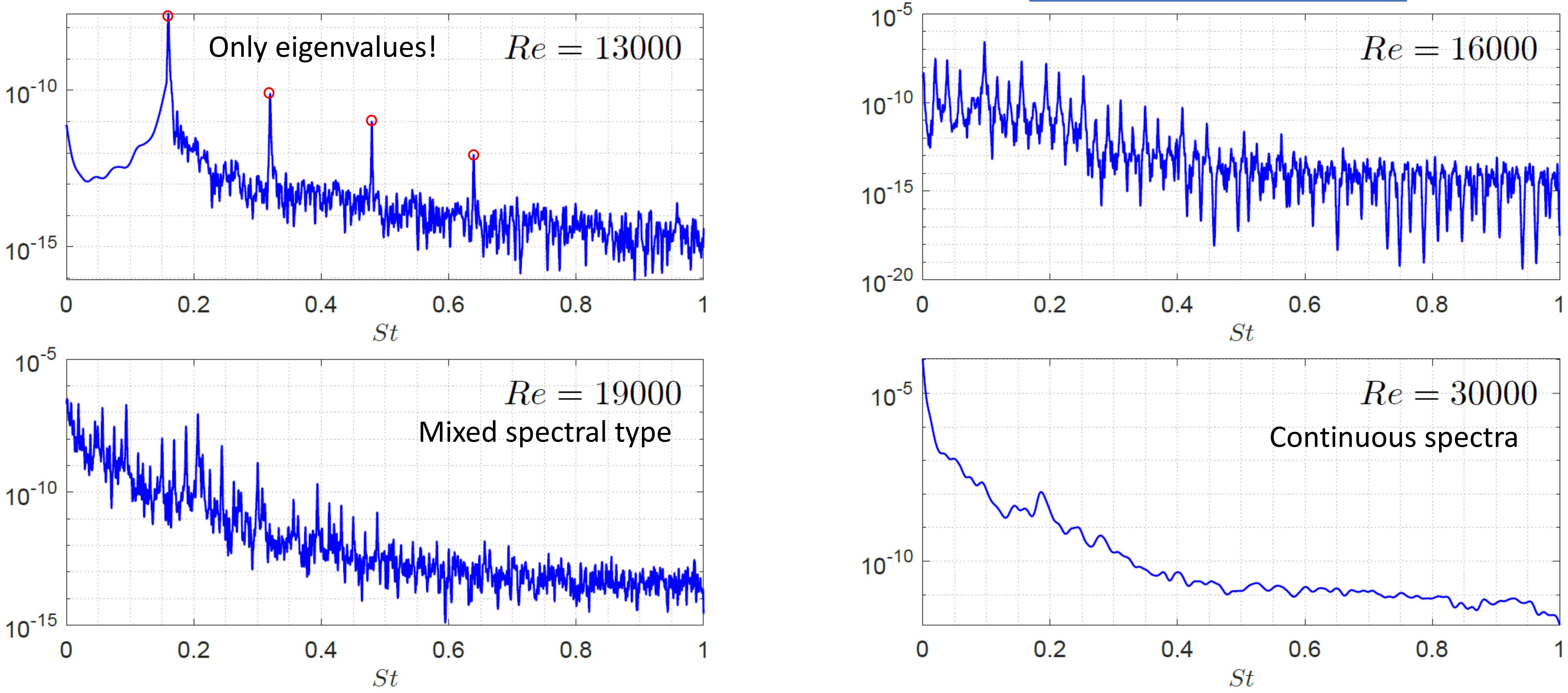
No formula for
generalized eigenfunctions!!

Experimental Details



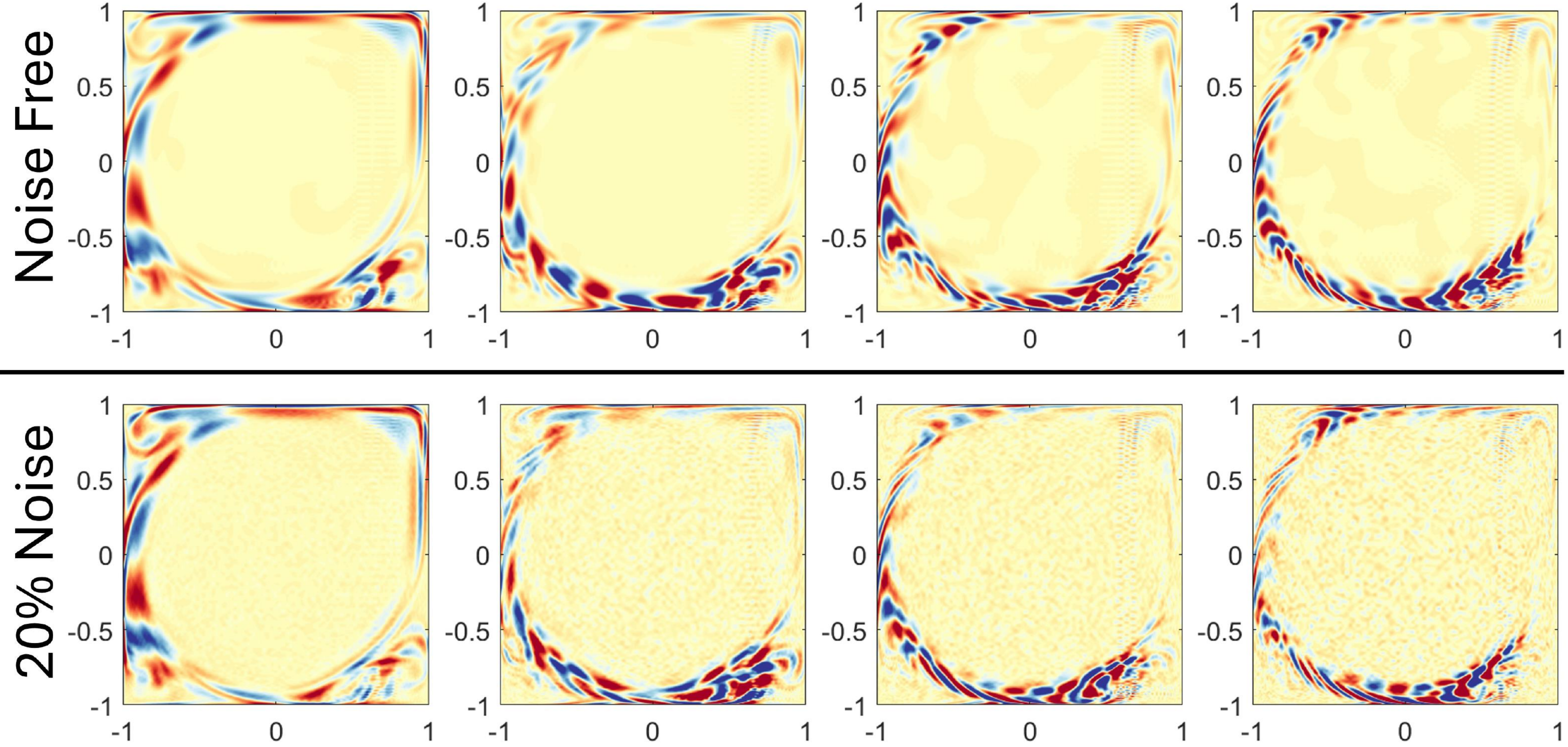
Example: Noisy cavity flow (spectral measures)

Single trajectory
$M = 10000, N$ varies
Basis: POD modes
20% Gaussian noise
*Raw measurements provided
Arbabi and Mezić (PRF 2017)



Example: Noisy cavity flow (generalized Koopman modes)

Re=30000
Deep in the continuous spectrum!!!



Outline

- General systems:
  - Residual Dynamic Mode Decomposition.
- Measure-preserving systems:
  - Rigged Dynamic Mode Decomposition
  - Measure-Preserving Extended Dynamic Mode Decomposition.
- The **Solvability Complexity Index** – *classification* of problems and *optimality* of algorithms.

Wider program: Solvability Complexity Index

- ResDMD: convergence as $\lim_{N\to\infty}\lim_{M\to\infty}$
- Rigged DMD: convergence ($\varepsilon = \varepsilon(N)$) as $\lim_{N\to\infty}\lim_{M\to\infty}$

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Convergence in **multiple successive limits**

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- **FACT:** Unless you have very strong assumptions (e.g., uniform ergodicity, finite-dimensional invariant subspace etc.), *EVERY* convergent Koopman algorithm to date needs multiple limits. These limits can be different things.

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- Solvability Complexity Index (SCI): smallest number k for which we can solve problem with $\lim_{n_k\to\infty} \dots \lim_{n_1\to\infty}$ via an algorithm ($n_1, \dots, n_k$ can be ***anything***).

- C., Hansen, "*The foundations of spectral computations via the solvability complexity index hierarchy,*" **J. Eur. Math. Soc.**, 2022.
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$\Longrightarrow$ Classification of problems, optimality of algorithms.

Steve Smale

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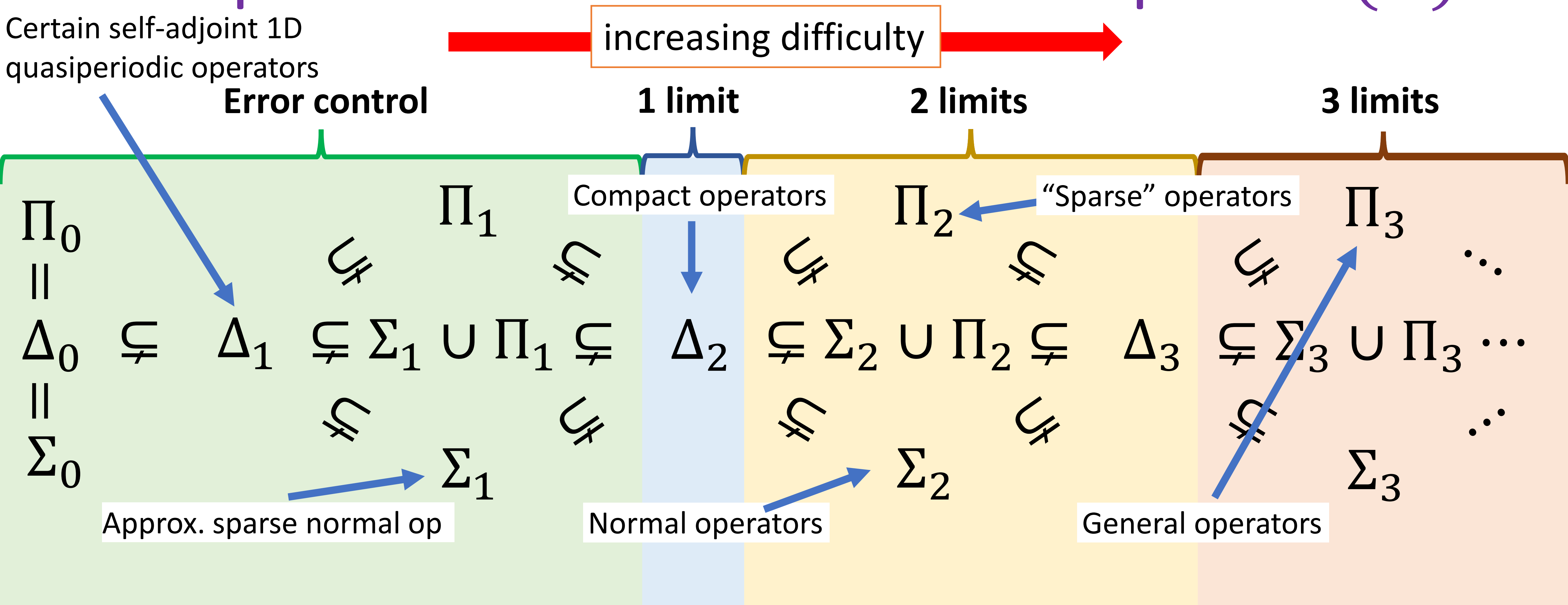
Sample: some results for bounded op. on $l^2(\mathbb{N})$



Zoo of problems: spectral type (pure point, absolutely continuous, singularly continuous), Lebesgue measure and fractal dimensions of spectra, discrete spectra, essential spectra, eigenspaces + multiplicity, spectral radii, essential numerical ranges, geometric features of spectrum (e.g., capacity), spectral gap problem, resonances ...

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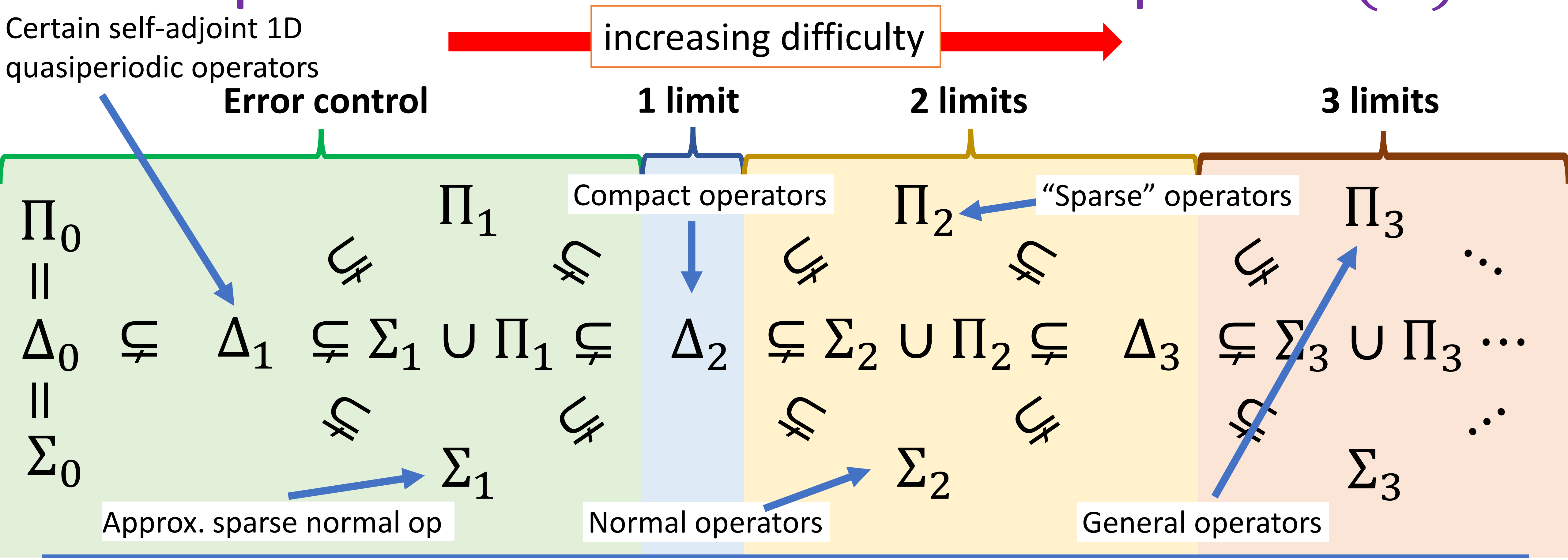

- C., *"The foundations of infinite-dimensional spectral computations,"* **PhD diss.**, University of Cambridge, 2020.
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- C., Horning, Townsend, *"Computing spectral measures of self-adjoint operators,"* **SIAM Rev.**, 2021.
- Ben-Artzi, Marletta, Rösler, *"Computing the sound of the sea in a seashell,"* **Found. Comput. Math.**, 2022.
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- C., *"On the computation of geometric features of spectra of linear operators on Hilbert spaces,"* **Found. Comput. Math.**, 2022.
- Webb, Olver, *"Spectra of Jacobi operators via connection coefficient matrices,"* **Commun. Math. Phys.**, 2021.
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- Rösler, Tretter, *"Computing Klein-Gordon Spectra,"* prepint.

Coming soon… SCI for Koopman (with Igor Mezić)

- General systems (computing spectrum).
- Measure-preserving systems (spectrum, spectral type etc.)

Bottom line:

- Many problems are impossible in one limit, even with perfect and unlimited snapshots, probabilistic algorithms, nice smooth F on compact manifolds.

  E.g., computing spectrum (as a set) of smooth measure-preserving systems on unit disc.

- Problems can be tackled in multiple limits under very general conditions.

$\Longrightarrow$ New program on foundations and classification for Koopman.

Summary: Infinities matter

Practical + theoretical guarantees

- A complete picture has emerged on $L^2(\Omega, \omega)$
  - *General systems:* **ResDMD** compute spectral properties with error control. **CONTROL INFINITE-DIMENSIONAL RESIDUALS**
  - *Measure-preserving systems:* **mpEDMD** structure-preserving

  Rigged DMD continuous spectra (and generalized eigenfunctions)
  SMOOTHING KERNELS and the **RESOLVENT**.

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Future work

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- What about other function spaces?
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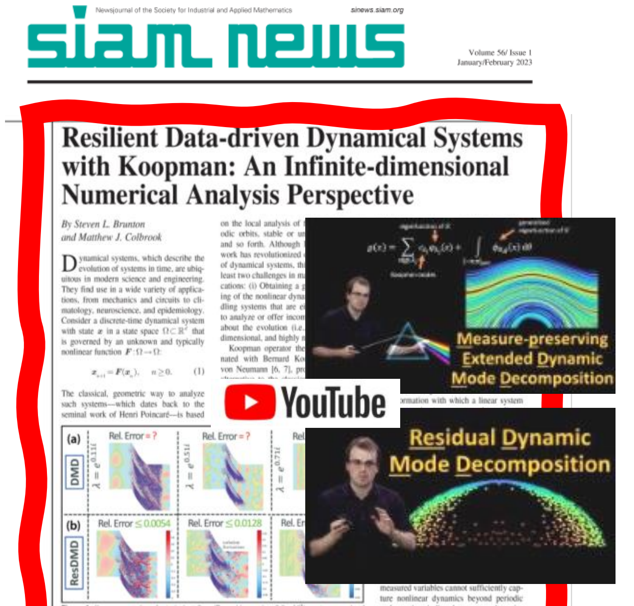

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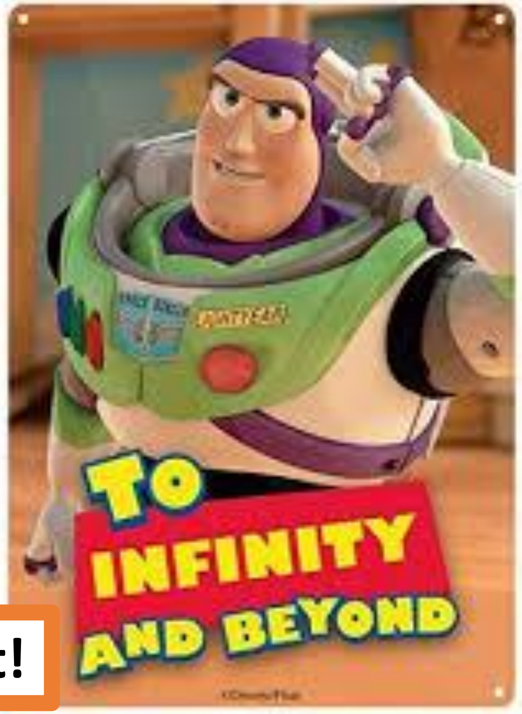

Buzz was right!

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