Convergence acceleration of modified Fourier series in one or more dimensions

Ben Adcock
DAMTP, Centre for Mathematical Sciences
University of Cambridge
Wilberforce Rd, Cambridge CB3 0WA
United Kingdom

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Abstract

Modified Fourier series have recently been introduced as an adjustment of Fourier series for the approximation of nonperiodic functions defined on d-variate cubes. Such approximations offer a number of advantages, including uniform convergence.

In this paper we extend Eckhoff's method to the convergence acceleration of multivariate modified Fourier expansions. By suitable augmentation of the approximation basis we are able to increase the convergence rate to an arbitrary algebraic order. Moreover, we demonstrate how numerical stability can be greatly improved by the use of appropriate auxiliary functions.

In the univariate setting it is known that Eckhoff's method exhibits an auto-correction phenomenon. We extend this result to the multivariate case. Finally, we demonstrate how a dramatic reduction in the number of approximation coefficients can be achieved by using a hyperbolic cross index set.

Introduction

The modified Fourier basis was introduced in [16, 17] as an adjustment of the Fourier basis for the approximation of smooth, nonperiodic functions defined on $\bar{\Omega}$, where Ω is the d-variate cube $(-1,1)^d$. In the univariate case, the Fourier sine function is replaced by $\sin(n-\frac{1}{2})\pi x$, yielding the basis

$$\{\cos n\pi x, n \in \mathbb{N}\} \cup \{\sin(n-\tfrac{1}{2})\pi x, n \in \mathbb{N}_+\}.$$

The multivariate basis is obtained by Cartesian products. The advantage of this basis is that the modified Fourier expansion of a sufficiently smooth function f converges uniformly on $\bar{\Omega}$. In particular, there is no Gibbs' phenomenon near the boundary, [1, 16, 28].

Unfortunately the convergence rate of the modified Fourier approximation remains relatively slow. If N is a truncation parameter, the uniform error is $\mathcal{O}\left(N^{-1}\right)$ on $\bar{\Omega}$ and $\mathcal{O}\left(N^{-2}\right)$ inside compact subsets of Ω, [1, 16, 28]. Much like the Fourier case, this is due to 'jumps' in certain derivatives of the function at the endpoints $x = \pm 1$ (in the univariate case), [28]. In the multivariate case similar analogues hold, though the jump conditions (otherwise referred to as derivative conditions) are more complicated to express, [1, 15].

In the univariate Fourier setting, provided the values of these jumps are known, there is a effective tool to accelerate convergence, namely the polynomial subtraction device, [19, 21]. This idea was first considered by Krylov, [20], and has been widely studied since then (see [5, 18, 23] and references therein). Polynomial subtraction is readily adapted to modified Fourier expansions, [16] and to the multivariate case, [15, 24, 25, 27].

Unfortunately, such jump values are unknown in general. In typical applications only the modified Fourier coefficients of a given function may be known, and, even if arbitrary pointwise values of the function can be calculated, approximation via finite differences is not recommended for this purpose, [23].

As noted in [10], the previous lack of robust methods for the approximation of jump values is the central reason why the polynomial subtraction technique has not been extensively utilized (see also [23, p.101]).

In this paper, to circumvent the aforementioned problem, we use Eckhoff's method for this task, [8, 9, 10]. This approach is based on the observation that the modified Fourier coefficients themselves contain sufficient information to reconstruct the jump values. This idea was previously considered in [22] (see [8] for further references).

Eckhoff's method was originally presented for the univariate, Fourier case. Analysis for the rate of convergence was carried out in [4]. The the extension to bivariate functions was developed, without analysis, in [24, 25, 27]. The aim of this paper is to extend Eckhoff's method to the modified Fourier approximation of a function defined on the d-variate cube, and to provide analysis therein. The central result we prove demonstrates that approximating the jumps in this manner (as opposed to using their exact values) does not deteriorate the convergence rate.

In fact, using approximate jump values offers at least one significant advantage. It was observed in [26] and proved in the univariate, Fourier case in [29] that Eckhoff's method exhibits an auto-correction phenomenon inside the domain. In other words, the convergence rate of the approximation based on approximate jump values is much faster in compact subsets of Ω than the corresponding approximation based on the exact values. We provide an extension of this result to the multivariate, modified Fourier case.

Polynomial subtraction and Eckhoff's method both rely on the construction of a smooth function to interpolate the jump values. In standard implementations [4, 10, 19, 21] such a function is constructed from a certain set of polynomials (the possibility of using other functions was suggested in [10]). Though this is the most convenient choice, it leads to poor numerical stability. In Section 1 we introduce a set of trigonometric functions that dramatically improve numerical stability.

Standard multivariate approximations using Fourier series involve $\mathcal{O}\left(N^d\right)$ terms. However, it turns out that this figure can be significantly reduced to $\mathcal{O}\left(N(\log N)^{d-1}\right)$ by using a so-called *hyperbolic cross* index set, [3, 33]. The use of such an index set does not deteriorate the convergence rate, aside from possibly a logarithmic factor, [33]. In the modified Fourier case this has been studied in [1, 15]. In the final part of this paper we demonstrate how to incorporate such an index set into Eckhoff's technique. With the aid of numerical examples, we highlight the advantage of this combined approach: using only relatively few terms we are able to produce highly accurate approximations of multivariate functions.

There are numerous devices for convergence acceleration of (univariate) Fourier expansions, including filters [32], Gegenbauer reconstruction [12, 13] and Fourier continuation methods [7] to name but a few. Without doubt, certain methods are more suitable for different applications. However, there are a number of advantages to Eckhoff's approach which warrant detailing. First, as we demonstrate in this paper, the combination of the multivariate version of this technique and hyperbolic cross index sets facilitates the construction of highly accurate approximations comprising only a small number of terms. Second, Eckhoff's method can be incorporated in spectral approximations of boundary value problems (see [10] and references therein for hyperbolic problems and [1, 2] for applications of modified Fourier expansions to second order boundary value problems).

Furthermore, Eckhoff's technique is not restricted to Cartesian product domains. In theory it can be developed for any domain on which suitable orthogonal expansions are known. For example, the modified Fourier basis is known explicitly on the equilateral and right isosceles triangles, [30]. The construction of accurate representation of functions on such domains is typically difficult, and Eckhoff's method may provide an attractive alternative to methods based on orthogonal polynomials. This is an area for future investigation.

The remainder of this paper is organized as follows. In Section 1 we develop and analyse the univariate version of Eckhoff's method for modified Fourier expansions. We then demonstrate how superior numerical results can be obtained by using a particular subtraction basis. Section 2 is devoted to the development and analysis of Eckhoff's method for functions defined on d-variate cubes. In Section 3 we extend the result of [29] concerning the existence of an auto-correction phenomenon to the multivariate form of Eckhoff's method. Finally, in Section 4 we demonstrate, without analysis, how a significant reduction in the number of approximation coefficients can be achieved. Numerical examples are provided.

The main results of this paper, namely the proof of convergence in the multivariate case, the existence of the multivariate auto-correction phenomenon and the use of a particular subtraction basis to improve numerical stability, can be readily adapted to the Fourier setting (with a little care, such results can also be applied to general *Fourier-like* expansions). However, due to the faster convergence rate, we consider modified Fourier approximations throughout.

1 The univariate version of Eckhoff's method

1.1 Definitions and basic properties

Given a function $f \in L^2(-1,1)$ and truncation parameter $N \geq 2$ we define the truncated modified Fourier expansion of f by

$$\mathcal{F}_N[f](x) = \frac{1}{2}\hat{f}_0^{[0]} + \sum_{n=1}^{N-1}\left\{\hat{f}_n^{[0]}\cos n\pi x + \hat{f}_n^{[1]}\sin(n-\tfrac{1}{2})\pi x\right\} = \sum_{i=0}^{1}\sum_{n=0}^{N-1}\hat{f}_n^{[i]}\phi_n^{[i]}(x), \quad x \in [-1,1].$$

Here $\phi_0^{[0]}(x) = \frac{1}{\sqrt{2}}$, $\phi_0^{[1]}(x) = 0$ and $\phi_n^{[0]}(x) = \cos n\pi x$, $\phi_n^{[1]}(x) = \sin(n-\frac{1}{2})\pi x$ otherwise, and

$$\hat{f}_n^{[i]} = \int_{-1}^{1} f(x)\phi_n^{[i]}(x)\, dx, \quad i \in \{0,1\}, \quad n \in \mathbb{N}, \tag{1.1}$$

is the n^{th} modified Fourier cosine ($i = 0$) or sine ($i = 1$) coefficient of f. As demonstrated in [16, 28] this series converges uniformly to f on $[-1,1]$ under some mild regularity assumptions (see also Section 1.2). Indeed, the coefficients $\hat{f}_n^{[i]}$ are $\mathcal{O}\left(n^{-2}\right)$ for large n (in comparison to $\mathcal{O}\left(n^{-1}\right)$ in the Fourier case).

Provided $f \in H^{2k}(-1,1)$, $k \in \mathbb{N}$, simple integration by parts of the right hand side of (1.1) yields

$$\hat{f}_n^{[i]} = \sum_{r=0}^{k-1}\frac{(-1)^{n+i}}{(\mu_n^{[i]})^{r+1}}\mathcal{A}_r^{[i]}[f] + \frac{(-1)^k}{(\mu_n^{[i]})^k}\widehat{f^{(2k)}}_n^{[i]}, \quad i \in \{0,1\}, \quad n \in \mathbb{N}_+, \tag{1.2}$$

where $\mu_n^{[0]} = n^2\pi^2$, $\mu_n^{[1]} = (n-\frac{1}{2})^2\pi^2$ and

$$(-1)^r\mathcal{A}_r^{[i]}[f] = f^{(2r+1)}(1) + (-1)^{i+1}f^{(2r+1)}(-1), \quad i \in \{0,1\}, \quad r \in \mathbb{N}. \tag{1.3}$$

The values $\mathcal{A}_r^{[i]}[f]$ are the requisite jump values for modified Fourier expansions. We say that a function f satisfies the first k derivative conditions if the first k such values vanish:

$$f^{(2r+1)}(1) + (-1)^{i+1}f^{(2r+1)}(-1) = 0, \quad i \in \{0,1\}, \quad r = 0, ..., k-1.$$

Equivalently, the first k odd derivatives of f vanish at the endpoints $x = \pm 1$. In this case the coefficients $\hat{f}_n^{[i]} = \mathcal{O}\left(n^{-2k-2}\right)$ and faster convergence of the approximation $\mathcal{F}_N[f]$ is observed (see Section 1.2).

1.2 Polynomial subtraction

If the first k such jump values are non-zero we seek to interpolate them with a function g_k. Since the function $f - g_k$ satisfies the first k derivative conditions, the new approximation $\mathcal{F}_N[f - g_k] + g_k$ converges at a faster rate to f. This is the principle of the polynomial subtraction process, [19, 21].

To find a suitable function g_k we first introduce (smooth) subtraction functions $p_0^{[i]}, ..., p_{k-1}^{[i]}$, where $p_r^{[i]}$ is even (respectively odd) if $i = 0$ ($i = 1$), that satisfy the conditions

$$\mathcal{A}_r^{[i]}\left[p_s^{[i]}\right] = (-1)^r\left\{(p_s^{[i]})^{(2r+1)}(1) + (-1)^{i+1}(p_s^{[i]})^{(2r+1)}(-1)\right\} = \delta_{r,s}, \quad r, s = 0, ..., k-1, \quad i \in \{0,1\}. \tag{1.4}$$

We say that $p_0^{[i]}, ..., p_{k-1}^{[i]}$ are Cardinal functions for the first k derivative conditions. With this in hand, we define g_k as follows:

$$g_k(x) = \sum_{i=0}^{1}\sum_{r=0}^{k-1}\mathcal{A}_r^{[i]}[f]p_r^{[i]}(x), \quad x \in [-1,1]. \tag{1.5}$$

Construction of appropriate Cardinal functions is commonly achieved by taking linear combinations of standard (smooth) functions $q_0^{[i]}, ..., q_{k-1}^{[i]}$. Such functions must be chosen so that the interpolation problem

$$\text{find} \quad a_s^{[i]}: \quad \sum_{s=0}^{k-1}a_s^{[i]}\left\{(q_s^{[i]})^{2r+1}(1) + (-1)^{i+1}(q_s^{[i]})^{2r+1}(1)\right\} = b_r^{[i]}, \quad i \in \{0,1\}, \quad r = 0, ..., k-1, \tag{1.6}$$

has a unique solution for all choices $b_r^{[i]} \in \mathbb{R}$. We call $\{q_r^{[i]} : i \in \{0,1\}, r = 0, ..., k-1\}$ a subtraction basis.

Usually the Cardinal functions $p_r^{[i]}$ are specified to be polynomials of degree $2r+1$, [4, 10, 21], in which case $q_r^{[i]} = x^{2(r+1)-i}$ and we refer to $p_r^{[i]}$ as Cardinal polynomials. This explains the name 'polynomial subtraction'. However, as we shall demonstrate, a significant advantage is gained by allowing the more general form (an idea which was suggested in [10]).

For later use we mention the following subtraction basis:

$$q_r^{[0]}(x) = \cos(r + \tfrac{1}{2})\pi x, \quad q_r^{[1]}(x) = \sin(r+1)\pi x, \quad r = 0, ..., k-1. \tag{1.7}$$

It is readily demonstrated that the interpolation problem (1.6) has a unique solution in this case. The functions $q_r^{[i]}$ are dual to the modified Fourier basis functions in the sense that the derivative of $\phi_n^{[i]}$ is proportional to $q_{n-1}^{[1-i]}$. This property was exploited in [1, 2] to analyse modified Fourier expansions. In the sequel, we demonstrate a practical use of this dual basis in Eckhoff's method. As we shall observe, it offers a significant numerical advantage over subtraction bases consisting of polynomials.

If g_k is given by (1.5) we define

$$\mathcal{F}_{N,k}[f](x) = \mathcal{F}_N[f - g_k](x) + g_k(x), \quad x \in [-1, 1], \tag{1.8}$$

as the k^{th} polynomial subtraction approximation of f (for convenience we interpret $\mathcal{F}_{N,0}[f]$ as $\mathcal{F}_N[f]$).

Suppose that $\|\cdot\|_\infty$ is the uniform norm on some domain Ω and that $\|\cdot\|_q$ is the $H^q(\Omega)$-norm. Concerning the error of polynomial subtraction we quote, without proof, the following two lemmas, found in [28] and [1] respectively:

Lemma 1. *Suppose that $k \in \mathbb{N}$, $f \in H^{2k+2}(-1,1)$ and that $\mathcal{F}_{N,k}[f]$ is given by (1.8) using exact jump values. Then $\|f^{(q)} - (\mathcal{F}_{N,k}[f])^{(q)}\|_\infty$ is $\mathcal{O}\left(N^{q-2k-1}\right)$ for $q = 0, ..., 2k$. If, additionally, $f \in H^{2k+3}(-1,1)$ then convergence rate of $(\mathcal{F}_{N,k}[f])^{(q)}$ to $f^{(q)}$ is $\mathcal{O}\left(N^{q-2k-2}\right)$ uniformly in compact subsets of $(-1,1)$.*

Note that the final condition in this lemma, namely that $f \in H^{2k+3}(-1,1)$, can be relaxed to the condition that $f \in C^{2k+2}[-1,1]$ and $f^{(2k+2)}$ has bounded variation, [28]. Concerning the error in the standard Sobolev norms $\|\cdot\|_q$ we have:

Lemma 2. *Suppose that $f \in H^{2k+2}(-1,1)$ and that $\mathcal{F}_{N,k}[f]$ is as in Lemma 1. Then $\|f - \mathcal{F}_{N,k}[f]\|_q$ is $\mathcal{O}(N^{q-2k-\frac{3}{2}})$ for $q = 0, ..., 2k+1$.*

When $k = 0$ Lemma 1 also establishes the pointwise and uniform convergence rates of $\mathcal{F}_N[f]$ to f described in the Introduction.

1.3 Eckhoff's method for approximation of jump values

Observe that, due to the definition of the Cardinal functions $p_r^{[i]}$, we may re-write (1.2) as

$$\hat{f}_n^{[i]} = \sum_{r=0}^{k-1} \widehat{p_r}_n^{[i]} \mathcal{A}_r^{[i]}[f] + \mathcal{O}\left(n^{-2k-2}\right), \tag{1.9}$$

where, for ease of notation, we write $\widehat{p_r}_n^{[i]}$ for the modified Fourier coefficient of $p_r^{[i]}$ corresponding to $\phi_n^{[i]}$. Note that, by construction, the coefficient corresponding to $\phi_n^{[1-i]}$ is zero. Due to uniform convergence of $\mathcal{F}_N[f]$ to f, we have

$$f(x) - \mathcal{F}_N[f](x) = \sum_{i=0}^{1} \sum_{r=0}^{k-1} \mathcal{A}_r^{[i]}[f] \left(p_r^{[i]}(x) - \mathcal{F}_N[p_r^{[i]}](x)\right) + \mathcal{O}\left(N^{-2k-1}\right), \quad x \in [-1, 1].$$

Now suppose that the values $\mathcal{A}_r^{[i]}[f]$ are approximated by values $\bar{\mathcal{A}}_r^{[i]}[f]$ and that g_k is constructed as in (1.5) using these approximate values. Then, using (1.8) and the above expression, we obtain

$$f(x) - \mathcal{F}_{N,k}[f](x) = \sum_{i=0}^{1} \sum_{r=0}^{k-1} \left(\mathcal{A}_r^{[i]}[f] - \bar{\mathcal{A}}_r^{[i]}[f]\right) \left(p_r^{[i]}(x) - \mathcal{F}_N[p_r^{[i]}](x)\right) + \mathcal{O}\left(N^{-2k-1}\right).$$

Now consider, for example, the uniform error. Since $\|p_r^{[i]} - \mathcal{F}_N[p_r^{[i]}]\|_\infty = \mathcal{O}\left(N^{-2r-1}\right)$, to obtain an $\mathcal{O}\left(N^{-2k-1}\right)$ uniform error using approximate jump values we require that

$$\bar{\mathcal{A}}_r^{[i]}[f] = \mathcal{A}_r^{[i]}[f] + \mathcal{O}\left(N^{2(r-k)}\right), \quad r = 0, ..., k-1, \quad i \in \{0, 1\}. \tag{1.10}$$

To do so we utilize Eckhoff's method, [8, 9, 10], which we now describe.

Eckhoff's method is based on (1.9). Suppose that $N \leq m(0) < ... < m(k-1) \leq aN$, $m(r) \in \mathbb{N}$ are given values and $a \geq 1$ is constant. Then we define $\bar{\mathcal{A}}_r^{[i]}[f]$ as the solution of the $2k \times 2k$ linear system

$$\sum_{s=0}^{k-1} \widehat{p_s}^{[i]}_{m(r)} \bar{\mathcal{A}}_s^{[i]}[f] = \hat{f}^{[i]}_{m(r)}, \quad r = 0, ..., k-1, \quad i \in \{0, 1\}. \tag{1.11}$$

These linear systems decouple into two $k \times k$ linear systems corresponding to $i = 0$ and $i = 1$, which can be solved in parallel. We write $V^{[i]}$ for the $k \times k$ matrix with $(r, s)^{\text{th}}$ entry $\widehat{p_s}^{[i]}_{m(r)}$. Note that the choice of the values $m(r)$ is essentially arbitrary. However, particular choices lead to better numerical behaviour and the auto-correction phenomenon, [29], as we shall see in the sequel.

Nonsingularity of these linear systems can be immediately guaranteed:

Theorem 1. *For sufficiently large N the linear system (1.11) is nonsingular. In particular, if $p_0^{[i]}, ..., p_{k-1}^{[i]}$ are Cardinal polynomials or arise from the subtraction basis (1.7), then (1.11) is non-singular for all N.*

Proof. Since $m(r) \geq N$, we have $\widehat{p_s}^{[i]}_{m(r)} = (-1)^{m(r)+i}(\mu^{[i]}_{m(r)})^{-s-1} + \mathcal{O}\left(N^{-2k-2}\right)$. Hence $V^{[i]} = D^{[i]}\tilde{V}^{[i]} + \mathcal{O}\left(N^{2k-2}\right)$, where $D^{[i]}$ is a non-singular diagonal matrix and $\tilde{V}^{[i]}$ is a Vandermonde matrix with $(r, s)^{\text{th}}$ entry $\tilde{V}^{[i]}_{r,s} = (\mu^{[i]}_{m(r)})^{-s}$. Since the $m(r)$ are distinct, this matrix is non-singular. Hence we obtain the first result. If $p_r^{[i]}$ are Cardinal polynomials, then $\widehat{p_s}^{[i]}_{m(r)}$ is precisely $(-1)^{m(r)+i}(\mu^{[i]}_{m(r)})^{-s-1}$. Thus non-singularity holds for all N.

Suppose now that the functions $q_r^{[i]}$ are given by (1.7). Then, due to (1.6), it suffices to prove non-singularity of the matrices with $(r, s)^{\text{th}}$ entries

$$\widehat{q_s}^{[0]}_{m(r)} = \frac{2(-1)^{m(r)+s+1}(s+\frac{1}{2})}{\left\{m(r)^2 - (s+\frac{1}{2})^2\right\}\pi}, \quad \widehat{q_s}^{[1]}_{m(r)} = \frac{2(-1)^{m(r)+s+1}s}{\left\{(m(r)-\frac{1}{2})^2 - s^2\right\}\pi}.$$

After appropriate premultiplication by non-singular diagonal matrices, we obtain matrices with $(r, s)^{\text{th}}$ entries

$$\left\{m(r)^2 - (s+\tfrac{1}{2})^2\right\}^{-1}, \quad \left\{(m(r)-\tfrac{1}{2})^2 - s^2\right\}^{-1},$$

respectively. These are Cauchy matrices, hence nonsingularity follows immediately. ☐

The standard construction of Eckhoff's approximation (see [4, 10]), uses the Cardinal functions $p_r^{[i]}$ and values $\bar{\mathcal{A}}_r^{[i]}[f]$ given by (1.11). Indeed, this is the most simple form to consider for analysis. However, for computational purposes it is often more convenient to use the subtraction basis $q_r^{[i]}$, without resorting to Cardinal functions. In this case

$$g_k(x) = \sum_{i=0}^{1} \sum_{r=0}^{k-1} \tilde{\mathcal{A}}_r^{[i]}[f] q_r^{[i]}(x), \quad x \in [-1, 1],$$

and the values $\tilde{\mathcal{A}}_r^{[i]}[f]$ are specified by the linear system

$$\sum_{s=0}^{k-1} \widehat{q_s}^{[i]}_{m(r)} \tilde{\mathcal{A}}_s^{[i]}[f] = \hat{f}^{[i]}_{m(r)}, \quad r = 0, ..., k-1, \quad i \in \{0, 1\}. \tag{1.12}$$

The resulting approximation is identical to the Cardinal function formulation.

1.3.1 Convergence rate of Eckhoff's approximation

Analysis of Eckhoff's method in the univariate, Fourier case was carried out in [4]. Using virtually identical techniques, the following result can be deduced for the modified Fourier setting:

Theorem 2. *Suppose that $m(r) = c(r)N + \mathcal{O}(1)$, where $c(r) \geq 1$ and that at most $l \leq k$ of the $c(r)$ are equal. Suppose further that $2K \geq l+1$ and that $f \in H^{2(k+K)}(-1,1)$. Then the coefficients $\bar{\mathcal{A}}^{[i]}_r[f]$ of Eckhoff's approximation satisfy (1.10).*

This result was originally proved in [4] for the Cardinal basis comprised of polynomials. However, it is easily extended to the general case. Using this result we deduce the following:

Theorem 3. *Suppose that l, K and f are as in Theorem 2, and that $\mathcal{F}_{N,k}[f]$ is the approximation of f with jump values approximated using Eckhoff's method. Then $\|f - \mathcal{F}_{N,k}[f]\|_q$ is $\mathcal{O}(N^{q-2k-\frac{3}{2}})$ for $q = 0, ..., 2k+1$.*

Proof. Suppose that we write $\mathcal{F}^e_{N,k}[f]$ and $\mathcal{F}_{N,k}[f]$ for the approximations based on the exact jump values $\mathcal{A}^{[i]}_r[f]$ and their approximations $\bar{\mathcal{A}}^{[i]}_r[f]$ respectively. In view of Lemma 2 it suffices to consider the difference $\mathcal{F}^e_{N,k}[f] - \mathcal{F}_{N,k}[f]$. We have

$$\|\mathcal{F}^e_{N,k}[f] - \mathcal{F}_{N,k}[f]\|_q \leq \sum_{i=0}^{1} \sum_{r=0}^{k-1} \left|\mathcal{A}^{[i]}_r[f] - \bar{\mathcal{A}}^{[i]}_r[f]\right| \left\|p^{[i]}_r - \mathcal{F}_N[p^{[i]}_r]\right\|_q. \tag{1.13}$$

Now suppose that a smooth function h satisfies the first r derivative conditions. It can be shown that $\left\|h - \mathcal{F}_N[h]\right\|_q = \mathcal{O}(N^{q-2r-\frac{3}{2}})$ for all $q \in \mathbb{N}$. Substituting this result with $h = p^{[i]}_r$ into (1.13) and using Theorem 2 immediately yields the result. □

Theorem 4. *Suppose that f and $\mathcal{F}_{N,k}[f]$ are as in Theorem 3. Then $\|f^{(q)} - (\mathcal{F}_{N,k}[f])^{(q)}\|_\infty$ is $\mathcal{O}\left(N^{q-2k-1}\right)$ for $q = 0, ..., 2k$.*

Proof. This follows immediately from Theorem 3 and the Sobolev inequality

$$\|h\|_\infty \leq c\sqrt{\|h\|\|h\|_1}, \quad \forall h \in H^1(-1,1),$$

where c is a constant independent of h. □

These results, in comparison with those of Section 1.2, demonstrate that Eckhoff's method for approximating jump values does not deteriorate the convergence rate of the approximation. However, as we describe in the Section 3, for certain choices of the values $m(r)$, Eckhoff's approximation offers at least one significant advantage.

The results of Theorems 2, 3 and 4 also demonstrate that, for certain choices of $m(r)$, Eckhoff's method requires additional smoothness to obtain the same convergence rate as the approximation based on the exact jump values. However, whenever the $c(r)$ are distinct, the smoothness requirement is identical.

In [4] the authors also compare the size of the error constants in $\|f - \mathcal{F}^e_{N,k}[f]\|_0$ and $\|f - \mathcal{F}_{N,k}[f]\|_0$. They demonstrate that approximating the jump values in this manner not only leads to the same convergence rate, but also that the error constant is not increased unduly. For this reason we address only the asymptotic order of convergence throughout the remainder of this paper.

1.3.2 Choice of the values $m(r)$

The values $m(r) \geq N$ can be chosen arbitrarily, provided they are distinct and satisfy $m(r) = c(r)N + \mathcal{O}(1)$. Numerous choices are possible, including

$$m(r) = N + r, \quad r = 0, ..., k-1. \tag{1.14}$$

In this case $c(r) = 1$ for all r, so that the function f being approximated must have $H^{3k+1}(-1,1)$ or $H^{3k+2}(-1,1)$-regularity (depending on whether k is odd or even) to ensure convergence. Other choices that require only $H^{2k+2}(-1,1)$-regularity are also possible, including

$$m(r) = (r+1)N, \quad r = 0, ..., k-1, \tag{1.15}$$

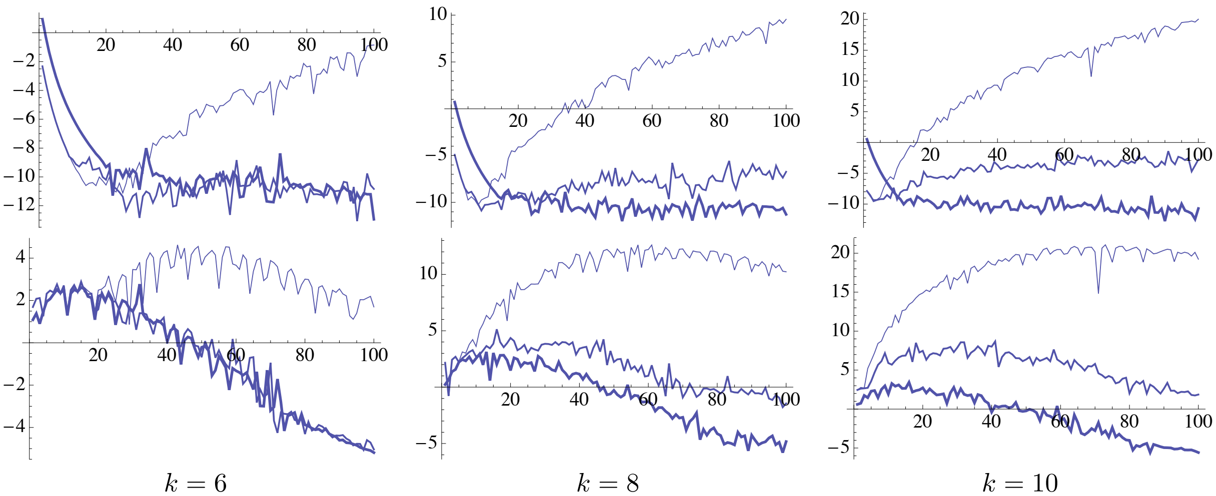


Figure 1: Log error $\log_{10}|f(1) - \mathcal{F}_{N,k}[f](1)|$ against $N = 1, ..., 100$ for Eckhoff's approximation using three different bases: Cardinal polynomial basis (thinnest line), Chebyshev polynomial basis and the dual basis (1.7) (thickest line). Here $f(x) = \cosh 6x$ (top diagrams), $f(x) = 5e^{\cos 5\pi(1-x^2)}$ (bottom diagrams) and $m(r) = N + r$, $r = 0, ..., k-1$. Numerical results obtained in standard precision, using the *LinearSolve* routine in *Mathematica*.

and, given some arbitrary value $\omega = 2, 3, ...$,

$$m(r) = \omega^r N, \quad r = 0, ..., k-1. \tag{1.16}$$

One immediate disadvantage of these choices is they do not lead to an auto-correction phenomenon (see Section 3). Further, the values $\hat{f}^{[i]}_n$, $n = 0, ..., N-1$, $n = m(r)$, $r = 0, ..., k-1$, required to form the approximation are not contiguous for (1.15) and (1.16), in contrast to (1.14)

1.3.3 Practical solution

The matrix $V^{[i]}$ is ill-conditioned. In fact, since $V^{[i]}$ is of the form $D^{[i]}\tilde{V}^{[i]}$, where $\tilde{V}^{[i]}$ is a Vandermonde matrix, the condition number is $\mathcal{O}\left(N^{2k+l-3}\right)$ for any choice of the values $m(r)$, where l is the number of equal values $c(r)$. This can be proved using well-known bounds for the norm of the inverse of a Vandermonde matrix (see [11]). Nonetheless, reasonably accurate numerical results can be obtained using the Björk–Pereyra algorithm, [6]. In this manner the values $\bar{\mathcal{A}}^{[i]}_r[f]$ can be found in $\mathcal{O}\left(k^2\right)$ operations.

However, far better numerical results can be obtained by replacing the Cardinal basis $p^{[i]}_r$ with an appropriately chosen subtraction basis $q^{[i]}_r$. In this case, the linear system to solve, namely (1.12), is often much more mildly conditioned (though asymptotically the same order), leading to better numerical results.

A significant improvement is offered by choosing $q^{[i]}_r$ as the $(2r+i)^{\text{th}}$ Chebyshev polynomial. This is a fairly standard approach, and the underlying matrix of the linear system is a *generalized Vandermonde* matrix (see [14]).

However, this can be further improved upon by using the basis of dual functions (1.7). In Figure 1 we give numerical results for this basis and the Chebyshev and Cardinal polynomial bases applied to several functions. We observe that the approximation based on (1.7) offers the smallest error. Moreover, unlike the Cardinal polynomial basis, the error remains bounded. Note that the functions used here exhibit two features, large derivatives and high oscillation, which make their approximation prone to numerical errors. However, by simply replacing the subtraction basis we are able to obtain vastly superior approximations.

Regardless of the particular problem, the functions (1.7) offer a vast improvement in terms of the condition number of the linear system. As mentioned, the condition number scales like N^{2k+l-3} regardless of the subtraction functions used. However, a vast reduction in the constant occurs when using (1.7). For $k = 10$ and values (1.14), the L^∞ condition number constant is roughly 3×10^{-16} for the linear system based on

N	25	50	100	150	200
$m(r) = N + r$	1.215×10^{24}	1.808×10^{31}	7.398×10^{38}	2.784×10^{43}	5.335×10^{46}
$m(r) = (r+1)N$	8.688×10^{30}	2.147×10^{36}	5.552×10^{41}	8.185×10^{44}	1.451×10^{47}
$m(r) = 2^r N$	2.933×10^{42}	7.206×10^{47}	1.861×10^{53}	2.742×10^{56}	4.859×10^{58}

Table 1: L^∞ condition number of the linear system (1.12) using the functions (1.7) with $k = 10$ and values $m(r)$ given by (1.14)–(1.16). All values to 4 significant figures.

(1.7). In comparison, for the Chebyshev and Cardinal polynomial bases these figures are 1×10^{-3} and 3×10^3 respectively, the latter being roughly 10^{19} times larger.

This effect is perhaps not surprising: the underlying matrix of the linear system (1.12) is a Cauchy matrix (see Theorem 1). Typically such matrices, though ill-conditioned themselves, are less poorly conditioned than Vandermonde matrices (see [14, chapter 22]). Note that such a linear system can also be solved in $\mathcal{O}\left(k^2\right)$ operations.

In all numerical results thus far, we have used the values (1.14). Seemingly, the condition number of the linear system (1.11) can be vastly improved from $\mathcal{O}\left(N^{3(k-1)}\right)$ to $\mathcal{O}\left(N^{2(k-1)}\right)$ by using the values (1.15) or (1.16) instead. However, though true in theory, in practice the constant is so overbearingly large that it nullifies this effect. In Table 1 we give figures for the condition number of this linear system using the values (1.14)–(1.16). In this case, $N > 200$ before the values (1.15) begin to offer an advantage (for the values (1.16) the scenario is much worse). However, since $k = 10$ in this example, any reasonable function will be well-resolved by Eckhoff's approximation for a much smaller value of N.

Numerical results can often be further improved by solving over-determined least squares problems. This approach is fairly standard, [4, 10]. For practical purposes, the least squares systems are solved by singular value decompositions, which can be found to high accuracy for Cauchy matrices, [14, p.515].

2 Eckhoff's method for multivariate expansions

In this section we extend Eckhoff's method to functions defined on the d-variate cube $\bar{\Omega} = [-1, 1]^d$. To do so, it is first necessary to introduce multivariate modified Fourier expansions and the multivariate polynomial subtraction technique. The reader is referred to [1, 17] and [15] for further details.

2.1 Multivariate modified Fourier expansions

Suppose that $x = (x_1, ..., x_d) \in \bar{\Omega}$ and that $f \in L^2(\Omega)$. The N^{th} truncated modified Fourier series of f can be written in the following succinct form:

$$\mathcal{F}_N[f](x) = \sum_{i \in \{0,1\}^d} \sum_{n \in I_N} \hat{f}_n^{[i]} \phi_n^{[i]}(x), \quad x \in \bar{\Omega}. \tag{2.1}$$

Here $i = (i_1, ..., i_d)$, $n = (n_1, ..., n_d)$ and $\phi_n^{[i]}(x) = \phi_{n_1}^{[i_1]}(x_1)...\phi_{n_d}^{[i_d]}(x_d)$. $I_N \subset \mathbb{N}^d$ is some finite index set. Throughout this section we assume that I_N is the full index set

$$I_N = \left\{ n \in \mathbb{N}^d : 0 \le n_1, ..., n_d \le N - 1 \right\}. \tag{2.2}$$

Note that $|I_N| = \mathcal{O}\left(N^d\right)$. In Section 4 we consider a different choice of index set, which greatly reduces the complexity Eckhoff's approximation without unduly affecting the convergence rate.

Throughout this section the bivariate case will serve as our primary example. In this setting (2.1) is

$$\mathcal{F}_N[f](x_1,x_2) = \frac{1}{4}\hat{f}^{[0,0]}_{0,0} + \frac{1}{2}\sum_{n_1=1}^{N-1}\left\{\hat{f}^{[0,0]}_{n_1,0}\cos n_1\pi x_1 + \hat{f}^{[1,0]}_{n_1,0}\sin(n_1-\tfrac{1}{2})\pi x_1\right\}$$
$$+\frac{1}{2}\sum_{n_2=0}^{N-1}\left\{\hat{f}^{[0,0]}_{0,n_2}\cos n_2\pi x_2 + \hat{f}^{[0,1]}_{0,n_2}\sin(n_2-\tfrac{1}{2})\pi x_2\right\}$$
$$+\sum_{n_1,n_2=1}^{N-1}\left\{\hat{f}^{[0,0]}_{n_1,n_2}\cos n_1\pi x_1\cos n_2\pi x_2 + \hat{f}^{[0,1]}_{n_1,n_2}\cos n_1\pi x_1\sin(n_2-\tfrac{1}{2})\pi x_2\right.$$
$$\left.+\hat{f}^{[1,0]}_{n_1,n_2}\sin(n_1-\tfrac{1}{2})\pi x_1\cos n_2\pi x_2 + \hat{f}^{[1,1]}_{n_1,n_2}\sin(n_1-\tfrac{1}{2})\pi x_1\sin(n_2-\tfrac{1}{2})\pi x_2\right\}.$$

2.1.1 Expansion of multivariate modified Fourier coefficients

The multivariate coefficients $\hat{f}^{[i]}_n$, given by

$$\hat{f}^{[i]}_n = \int_\Omega f(x)\phi^{[i]}_n(x)\,dx, \quad i\in\{0,1\}^d, \quad n\in\mathbb{N}^d,$$

are $\mathcal{O}\left(n^{-2}\right)$ for large n, where $n^{-2}=(n_1...n_d)^{-2}$. In fact

$$|\hat{f}^{[i]}_n| \lesssim (\bar{n}_1...\bar{n}_d)^{-2} = \bar{n}^{-2},$$

where $\bar{m}=\max\{m,1\}$ for $m\in\mathbb{N}$. Here, and for the remainder of this paper, we use the symbol $A\lesssim B$ to mean that there exists a constant c independent of N such that $A\le cB$.

The coefficients $\hat{f}^{[i]}_n$ admit an expansion similar to that of the univariate coefficients given in (1.2). For this we need some additional notation. Suppose that $[d]$ is the set of ordered tuples of length at most d with entries in $\{1,...,d\}$. We define $[d]^*=[d]\cup\{\emptyset\}$. For $t\in[d]$ we write $|t|$ for the length (number of elements) in t, so that $t=(t_1,...,t_{|t|})$. We also write $\bar{t}\in[d]$ for the tuple of length $d-|t|$ of elements not in t. For $j\in\{1,...,d\}$ we say that $j\in t$ if $j=t_l$ for some $l=1,...,|t|$. Given $x=(x_1,...,x_d)$ we also define $x_t=(x_{t_1},...,x_{t_{|t|}})$.

For a multi-index $\alpha=(\alpha_1,...,\alpha_d)\in\mathbb{N}^d$, we define $|\alpha|$ and $|\alpha|_\infty$ by

$$|\alpha| = \sum_{j=1}^d \alpha_j, \quad |\alpha|_\infty = \max_{j=1,...,d}\alpha_j,$$

and the differentiation operator D^α by

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}_{x_1}...\partial^{\alpha_d}_{x_d}} = \partial^{\alpha_1}_{x_1}...\partial^{\alpha_d}_{x_d}.$$

If $\alpha=(r,r,...,r)$, $r\in\mathbb{N}$, we also write D^r. If $t\in[d]$ and $r\in\mathbb{N}$ we define $D^r_t=\partial^r_{x_{t_1}}...\partial^r_{x_{t_{|t|}}}$.

Given $j=1,...,d$, $r_j=0,...,k-1$ and $i_j\in\{0,1\}$ we define $\mathcal{B}^{[i_j]}_{r_j}[f]$ by

$$(-1)^{r_j}\mathcal{B}^{[i_j]}_{r_j}[f](x_1,...,x_{j-1},x_{j+1},...,x_d) = \partial^{2r_j+1}_{x_j}f(x_1,...,x_{j-1},1,x_{j+1},...,x_d)$$
$$+(-1)^{i_j+1}\partial^{2r_j+1}_{x_j}f(x_1,...,x_{j-1},-1,x_{j+1},...,x_d).$$

For $t\in[d]^*$, $r_t=(r_{t_1},...,r_{t_{|t|}})\in\mathbb{N}^{|t|}$ and $i_t=(i_{t_1},...,i_{t_{|t|}})\in\{0,1\}^{|t|}$ we define $\mathcal{B}^{[i_t]}_{r_t}[f]$ as the composition

$$\mathcal{B}^{[i_t]}_{r_t}[f](x_{\bar{t}}) = \mathcal{B}^{[i_{t_1}]}_{r_{t_1}}\left[\mathcal{B}^{[i_{t_2}]}_{r_{t_2}}\left[...\left[\mathcal{B}^{[i_{t_{|t|}}]}_{r_{t_{|t|}}}[f]\right]...\right]\right].$$

Note that these operators commute with each other and with differentiation in the variable $x_{\bar{t}}$.

Finally, given $t \in [d]^*$, $r_t \in \mathbb{N}^{|t|}$, $i \in \{0,1\}^d$ and $n_{\bar{t}} = (n_{\bar{t}_1}, ..., n_{\bar{t}_{|\bar{t}|}}) \in \mathbb{N}^{|\bar{t}|}$ we define $\mathcal{A}^{[i]}_{r_t,n_{\bar{t}}}[f] \in \mathbb{R}$ by

$$\mathcal{A}^{[i]}_{r_t,n_{\bar{t}}}[f] = (-1)^{k|\bar{t}|} \prod_{j \notin t} \left(\mu^{[i_j]}_{n_j}\right)^{-k} \int \mathcal{B}^{[i_t]}_{r_t}[D^{2k}_{\bar{t}} f](x_{\bar{t}}) \phi^{[i_{\bar{t}}]}_{n_{\bar{t}}}(x_{\bar{t}}) \, dx_{\bar{t}}. \tag{2.3}$$

Note that the final integral is just the modified Fourier coefficient of the function $\mathcal{B}^{[i_t]}_{r_t}[D^{2k}_{\bar{t}} f](x_{\bar{t}})$ corresponding to indices $i_{\bar{t}}$ and $n_{\bar{t}}$. For this reason, we have the bound

$$\left|\mathcal{A}^{[i]}_{r_t,n_{\bar{t}}}[f]\right| \lesssim \prod_{j \notin t} \bar{n}_j^{-2k-2} = \bar{n}_{\bar{t}}^{-2k-2}, \quad \forall n_{\bar{t}} \in \mathbb{N}^{|\bar{t}|}, \quad i \in \{0,1\}^d. \tag{2.4}$$

We are now able to derive an expansion for $\hat{f}^{[i]}_n$. After k integrations by parts in each variable, we obtain

$$\hat{f}^{[i]}_n = \sum_{t \in [d]^*} \sum_{|r_t|_\infty = 0}^{k-1} \mathcal{A}^{[i]}_{r_t,n_{\bar{t}}}[f] (-1)^{|n_t|+|i_t|} \prod_{j \in t} \left(\mu^{[i_j]}_{n_j}\right)^{-(r_j+1)}. \tag{2.5}$$

As we establish in the sequel, the values $\mathcal{A}^{[i]}_{r_t,n_{\bar{t}}}[f]$, $t \in [d]$, are the appropriate generalization of the univariate 'jumps' $\mathcal{A}^{[i]}_r[f]$ given in (1.3). The task of approximating these values to sufficient accuracy is the content of the remainder of this paper.

Suppose that $p^{[i]}_0, ..., p^{[i]}_{k-1}$ are the Cardinal functions introduced in Section 1.2. Given $t \in [d]$, $i_t \in \{0,1\}^{|t|}$ and $r_t \in \{0, ..., k-1\}^{|t|}$ we define

$$p^{[i_t]}_{r_t}(x_t) = \prod_{j \in t} p^{[i_j]}_{r_j}(x_j).$$

With this in hand, we may rewrite (2.5) as

$$\hat{f}^{[i]}_n = \sum_{t \in [d]^*} \sum_{|r_t|_\infty = 0}^{k-1} \mathcal{A}^{[i]}_{r_t,n_{\bar{t}}}[f] \widehat{p_{r_t}}^{[i_t]}_{n_t} + \mathcal{O}\left(n^{-2k-2}\right). \tag{2.6}$$

If the functions $p^{[i]}_r$ are Cardinal polynomials then the final term of (2.6) vanishes. To simplify matters, throughout the remainder of this paper, unless specified otherwise, we assume that this is the case.

We have not yet specified the smoothness required for the expansion (2.5)–(2.6) to be valid. Appropriate conditions can be established upon introduction of the following spaces:

2.1.2 Sobolev spaces of dominating mixed smoothness

As described in greater detail in [1], modified Fourier expansions are best studied in so-called Sobolev spaces of *dominating mixed smoothness*, [31, 33]. Given $q \in \mathbb{N}$, we define the q^{th} such space by

$$H^q_{\text{mix}}(\Omega) = \{f : D^\alpha f \in L^2(\Omega), \ \forall \alpha \in \mathbb{N}^d : |\alpha|_\infty \le q\},$$

with norm

$$\|f\|^2_{q,\text{mix}} = \sum_{|\alpha|_\infty \le q} \|D^\alpha f\|^2.$$

The importance of such spaces in the study of modified Fourier expansions is immediately emphasized by the observation that $\mathcal{F}_N[f]$ converges uniformly to f on $\bar{\Omega}$ provided $f \in H^1_{\text{mix}}(\Omega)$, [1]. Returning to the expansions derived in the previous section, it is readily seen that (2.5)–(2.6) are valid for functions $f \in H^{2k}_{\text{mix}}(\Omega)$.

We shall not discuss such spaces in greater detail. We refer to [31, 33] for further reading, and to [1] for use of such spaces in the study of multivariate modified Fourier expansions.

2.2 Multivariate polynomial subtraction

As described in greater detail in [15], to intepolate the jump values $\mathcal{A}^{[i]}_{r_t,n_{\bar t}}[f]$ it suffices to interpolate the exact Neumann data of the function f on the boundary. In other words, given $k \in \mathbb{N}_+$, we seek a function g_k such that

$$\partial^{2r+1}_{x_j} g_k\big|_{x_j=\pm 1} = \partial^{2r+1}_{x_j} f\big|_{x_j=\pm 1}, \quad \forall j = 1, ..., d, \quad r = 0, ..., k-1.$$

In the notation of the previous section

$$\mathcal{B}^{[i_j]}_{r_j}[g_k] = \mathcal{B}^{[i_j]}_{r_j}[f], \quad i_j \in \{0, 1\}, \quad r_j = 0, ..., k-1, \quad j = 1, ..., d. \tag{2.7}$$

As in the univariate case, the function $f - g_k$ satisfies homogeneous Neumann boundary conditions up to order k, and (as we shall observe) this guarantees faster convergence of the approximation $\mathcal{F}_{N,k}[f]$ given by

$$\mathcal{F}_{N,k}[f] = \mathcal{F}_N[f - g_k] + g_k.$$

We refer to $\mathcal{F}_{N,k}[f]$ as the k^{th} polynomial subtraction approximation of f.

A suitable function g_k is given by the following lemma:

Lemma 3. *Suppose that* $f \in H^{2k}(\Omega)$ *and that*

$$g_k(x) = \sum_{t \in [d]} \sum_{i_t \in \{0,1\}^{|t|}} \sum_{|r_t|_\infty = 0}^{k-1} (-1)^{|t|+1} \mathcal{B}^{[i_t]}_{r_t}[f](x_{\bar t}) p^{[i_t]}_{r_t}(x_t), \quad x \in \bar\Omega. \tag{2.8}$$

Then g_k *satisfies (2.7).*

Proof. It suffices to prove that g_k satisfies (2.7) with $j = 1$, $i_j = 0$ and $r_j = s$. We split the terms of (2.8) corresponding to different $t \in [d]$ into the three following cases: (i) $t = (1)$, (ii) $t = (1, u)$, where $u \in [d]$, $1 \notin u$, and (iii) $t = u$, where $1 \notin u$.

Consider case (i). The contribution of the corresponding term to $\mathcal{B}^{[0]}_s[g_k]$ is

$$\sum_{i_1=0}^{1} \sum_{r_1=0}^{k-1} \mathcal{B}^{[0]}_s \left[\mathcal{B}^{[i_1]}_{r_1}[f](x_2, ..., x_d) p^{[i_1]}_{r_1}(x_1) \right] (x_2, ..., x_d) = \mathcal{B}^{[0]}_s[f](x_2, ..., x_d).$$

Here the second equality follows directly from the properties of the Cardinal functions $p^{[i]}_r$. It now suffices to prove that the contributions corresponding to cases (ii) and (iii) cancel. For case (ii) the contribution is

$$\sum_{i_u \in \{0,1\}^{|u|}} \sum_{|r_u|_\infty = 0}^{k-1} \sum_{i_1=0}^{1} \sum_{r_1=0}^{k-1} (-1)^{|u|} \mathcal{B}^{[0]}_s \left[\mathcal{B}^{[i_t]}_{r_t}[f](x_{\bar t}) p^{[i_t]}_{r_t}(x_t) \right] (x_2, ..., x_d)$$
$$= \sum_{i_u \in \{0,1\}^{|u|}} \sum_{|r_u|_\infty = 0}^{k-1} (-1)^{|u|} \mathcal{B}^{[(0,i_u)]}_{(s,r_u)}[f](x_{\bar u}) p^{[i_u]}_{r_u}(x_u),$$

where $(0, i_u) = (0, i_{u_1}, ..., i_{u_{|u|}})$ and $(s, r_u) = (s, r_{u_1}, ..., r_{u_{|u|}})$. It is readily seen that this is precisely the negative of the contribution of case (iii). □

Concerning the error of polynomial subtraction we have the following result, proved in [1]:

Theorem 5. *Suppose that* $f \in H^{2k+2}_{mix}(\Omega)$ *and that* $\mathcal{F}_{N,k}[f]$ *is the* k^{th} *polynomial subtraction approximation to* f. *Then* $\|f - \mathcal{F}_{N,k}[f]\|_\infty$ *is* $\mathcal{O}\left(N^{-2k-1}\right)$ *and* $\|f - \mathcal{F}_{N,k}[f]\|_q$ *is* $\mathcal{O}(N^{q-2k-\frac{3}{2}})$ *for* $q = 0, ..., 2k+1$. *If, additionally,* $f \in H^{2k+3}_{mix}(\Omega)$ *then* $f(x) - \mathcal{F}_{N,k}[f](x)$ *is* $\mathcal{O}\left(N^{-2k-2}\right)$ *uniformly in compact subsets of* Ω.

As in the univariate case, we interpret $\mathcal{F}_{N,0}[f]$ as just $\mathcal{F}_N[f]$. When $k = 0$ this theorem also establishes the rate of convergence of the multivariate modified Fourier expansion $\mathcal{F}_N[f]$.

For $d=2$, the function g_k is given by

$$g_k(x) = \sum_{i_1=0}^{1}\sum_{r_1=0}^{k-1} p_{r_1}^{[i_1]}(x_1)\mathcal{B}_{r_1}^{[i_1]}[f](x_2) + \sum_{i_2=0}^{1}\sum_{r_2=0}^{k-1} \mathcal{B}_{r_2}^{[i_2]}[f](x_1)p_{r_2}^{[i_2]}(x_2) - \sum_{i_1,i_2=0}^{1}\sum_{r_1,r_2=0}^{k-1} \mathcal{B}_{r_1}^{[i_1]}\left[\mathcal{B}_{r_2}^{[i_2]}[f]\right] p_{r_1}^{[i_1]}(x_1)p_{r_2}^{[i_2]}(x_2).$$

For $d \geq 2$ the problem with the practical computation of g_k, as given by (2.8), is that it requires knowledge of the exact derivatives of the function f over $(d-1)$-dimensional subsets of the boundary. One approach to alleviate this problem, which we now introduce since it will be used in the sequel, is to approximate these lower dimensional functions using polynomial subtraction. To do so, requires knowledge of functions over $(d-2)$-dimensional subsets of the boundary. However, we may repeat the same process, replacing exact functions by polynomial subtraction approximations, until we obtain an approximation that uses only derivative values over the 0-dimensional subsets of the boundary consisting of the vertices $(\pm 1, \pm 1, ..., \pm 1)$.

To differentiate between the two approaches we refer to the approximation based on (2.8) as *exact* polynomial subtraction and the approximation obtain by the above process as *approximate* polynomial subtraction. We write g_k^e, $\mathcal{F}_{N,k}^e[f]$ and g_k^a, $\mathcal{F}_{N,k}^a[f]$ respectively. Note that for $d=1$ both approximations coincide.

In the $d=2$ case we merely replace the univariate functions $\mathcal{B}_{r_1}^{[i_1]}[f]$ and $\mathcal{B}_{r_2}^{[i_2]}[f]$ by their k^{th} polynomial subtraction approximation. This yields the new function g_k^a given by

$$g_k^a(x) = \sum_{i_1=0}^{1}\sum_{r_1=0}^{k-1} p_{r_1}^{[i_1]}(x_1)\mathcal{F}_{N,k}\left[\mathcal{B}_{r_1}^{[i_1]}[f]\right](x_2) + \sum_{i_2=0}^{1}\sum_{r_2=0}^{k-1} \mathcal{F}_{N,k}\left[\mathcal{B}_{r_2}^{[i_2]}[f]\right](x_1)p_{r_2}^{[i_2]}(x_2) - \sum_{i_1,i_2=0}^{1}\sum_{r_1,r_2=0}^{k-1} \mathcal{B}_{r_1}^{[i_1]}\left[\mathcal{B}_{r_2}^{[i_2]}[f]\right] p_{r_1}^{[i_1]}(x_1)p_{r_2}^{[i_2]}(x_2).$$

For $d \geq 3$ we define the new approximation inductively. If $\mathcal{F}_{N,k}^a[\cdot]$ has been obtained for $d-1$, we define the d-variate approximate polynomial subtraction function g_k^a by

$$g_k^a(x) = \sum_{t\in[d]}\sum_{i_t\in\{0,1\}^{|t|}}\sum_{|r_t|_\infty=0}^{k-1} (-1)^{|t|+1}\mathcal{F}_{N,k}^a\left[\mathcal{B}_{r_t}^{[i_t]}[f]\right](x_{\bar{t}})p_{r_t}^{[i_t]}(x_t), \quad x\in\bar{\Omega}. \tag{2.9}$$

We now prove the following lemma, which demonstrates that replacing (2.8) by (2.9) does not deteriorate the convergence rate of the approximation:

Lemma 4. *Suppose that $f \in H_{mix}^{2k+2}(\Omega)$ and that $\mathcal{F}_{N,k}^a[f]$ is the k^{th} approximate polynomial subtraction approximation of f. Then $\|f-\mathcal{F}_{N,k}^a[f]\|_\infty$ is $\mathcal{O}\left(N^{-2k-1}\right)$ and $\|f-\mathcal{F}_{N,k}^a[f]\|_q$ is $\mathcal{O}(N^{q-2k-\frac{3}{2}})$ for $q=0,...,2k+1$. If, additionally, $f\in H_{mix}^{2k+3}(\Omega)$ then $f(x)-\mathcal{F}_{N,k}^a[f](x)$ is $\mathcal{O}\left(N^{-2k-2}\right)$ uniformly in compact subsets of Ω.*

Proof. By Theorem 5 it suffices to consider the difference $\mathcal{F}_{N,k}^e[f]-\mathcal{F}_{N,k}^a[f]$. We use induction on d. For $d=1$ there is nothing to prove. Now suppose that the result holds for $d-1$. We have

$$\mathcal{F}_{N,k}^e[f](x)-\mathcal{F}_{N,k}^a[f](x) = g_k^e(x)-g_k^a(x)-\mathcal{F}_N\left[g_k^e-g_k^a\right](x).$$

Since $\widehat{\mathcal{F}_{N,k}^a[h]}_n^{[i]} = \widehat{\mathcal{F}_{N,k}^e[h]}_n^{[i]} = \hat{h}_n^{[i]}$ for all $i\in\{0,1\}^d$, $n\in I_N$ and arbitrary functions h, it is follows that $\mathcal{F}_N[g_k^e-g_k^a]=0$. Hence

$$\mathcal{F}_{N,k}^e[f](x)-\mathcal{F}_{N,k}^a[f](x) = g_k^e(x)-g_k^a(x) = \sum_{t\in[d]}\sum_{i_t\in\{0,1\}^{|t|}}\sum_{|r_t|_\infty=0}^{k-1}(-1)^{|t|+1}\left(\mathcal{B}_{r_t}^{[i_t]}[f](x_{\bar{t}})-\mathcal{F}_{N,k}^a\left[\mathcal{B}_{r_t}^{[i_t]}[f]\right](x_{\bar{t}})\right)p_{r_t}^{[i_t]}(x_t).$$

If $f \in H^{2k+2}_{\text{mix}}(\Omega)$ then it can be shown that $\mathcal{B}^{[i_t]}_{r_t}[f] \in H^{2k+2}_{\text{mix}}(-1,1)^{|\bar{t}|}$, [1]. Since $|t| \geq 1$, we may use the induction hypothesis on each such term to obtain the result. $\square$

For practical purposes we need an explicit expression for g^a_k. This is given by the following lemma:

Lemma 5. *The approximate polynomial subtraction function g^a_k is given by*

$$g^a_k(x) = \sum_{i\in\{0,1\}^d} \sum_{t\in[d]} \sum_{|r_t|_\infty=0}^{k-1} \sum_{|n_{\bar{t}}|_\infty=0}^{N-1} \mathcal{A}^{[i]}_{r_t,n_{\bar{t}}}[f] p^{[i_t]}_{r_t}(x_t) \phi^{[i_{\bar{t}}]}_{n_{\bar{t}}}(x_{\bar{t}}), \tag{2.10}$$

where the coefficients $\mathcal{A}^{[i]}_{r_t,n_{\bar{t}}}[f]$ are defined in (2.3).

To prove this lemma we need the following notation. Given $t \in [d]$ we write $[t]$ for the set of tuples $u \in [d]$ with $u \subseteq t$. We write $[t]^* = [t] \cup \{\emptyset\}$ and $\bar{u} \in [t]^*$ for the tuple of elements in t but not in u. Further, given $t, u \in [d]^*$ we write $t \cup u \in [d]$ for the ordered tuple of elements $j = 1, ..., d$ in t or in u and $t \cap u$ for the tuple of elements in both t and u.

Proof of Lemma 5. We prove this result by induction on d. For $d = 1$, since $g^a_k = g^e_k$ and $\mathcal{A}^{[i]}_r[f] = \mathcal{B}^{[i]}_r[f]$, there is nothing to prove. Now assume that the result holds for $d-1$. Then, by definition

$$g^a_k(x) = \sum_{t\in[d]} \sum_{i_t\in\{0,1\}^{|t|}} \sum_{|r_t|_\infty=0}^{k-1} (-1)^{|t|+1} \mathcal{F}^a_{N,k}\left[\mathcal{B}^{[i_t]}_{r_t}[f]\right](x_{\bar{t}}) p^{[i_t]}_{r_t}(x_t). \tag{2.11}$$

Since $\mathcal{B}^{[i_t]}_{r_t}[f]$ is a function of at most $(d-1)$ variables, we may use the induction hypothesis to derive an expression for $\mathcal{F}^a_{N,k}\left[\mathcal{B}^{[i_t]}_{r_t}[f]\right](x_{\bar{t}})$. To do so, we first note the following:

$$\mathcal{A}^{[i_{\bar{t}}]}_{r_u,n_{\bar{u}}}\left[\mathcal{B}^{[i_t]}_{r_t}[f]\right] = (-1)^{k|\bar{u}|} \prod_{j\in\bar{u}} (\mu^{[i_j]}_{n_j})^{-k} \int \mathcal{B}^{[i_u]}_{r_u}\left[D^{2k}_{\bar{u}} \mathcal{B}^{[i_t]}_{r_t}[f]\right] \phi^{[i_{\bar{u}}]}_{n_{\bar{u}}}(x_{\bar{u}})\, dx_{\bar{u}}, \quad \forall u \in [\bar{t}]^*.$$

Since the operators $\mathcal{B}^{[i_u]}_{r_u}$ and $\mathcal{B}^{[i_t]}_{r_t}$ commute with each other and with differentiation in the independent variables, this gives

$$\mathcal{A}^{[i_{\bar{t}}]}_{r_u,n_{\bar{u}}}\left[\mathcal{B}^{[i_t]}_{r_t}[f]\right] = (-1)^{k|\bar{u}|} \prod_{j\in\bar{u}} (\mu^{[i_j]}_{n_j})^{-k} \int \mathcal{B}^{[i_{t\cup u}]}_{r_{t\cup u}}\left[D^{2k}_{\bar{u}} f\right] \phi^{[i_{\bar{u}}]}_{n_{\bar{u}}}(x_{\bar{u}})\, dx_{\bar{u}}.$$

Since $u \in [\bar{t}]^*$, $\bar{u} = \bar{t}\backslash u = \overline{t\cup u}$. Hence $\mathcal{A}^{[i_{\bar{t}}]}_{r_u,n_{\bar{u}}}\left[\mathcal{B}^{[i_t]}_{r_t}[f]\right] = \mathcal{A}^{[i]}_{r_{t\cup u},n_{\overline{t\cup u}}}[f]$. Our next observation is the following. If h is a function of at most $(d-1)$ variables, and g^a_k is the approximate polynomial subtraction function for h, then

$$\mathcal{F}_N[h - g^a_k](x) = \sum_{i\in\{0,1\}^{d-1}} \sum_{|n|_\infty=0}^{N-1} \mathcal{A}^{[i]}_n[h] \phi^{[i]}_n(x), \quad x \in [-1,1]^{d-1},$$

where $\mathcal{A}^{[i]}_n[h]$ is the value $\mathcal{A}^{[i]}_{r_t,n_{\bar{t}}}[h]$ given by (2.3) with $t = \emptyset$. This follows immediately from the induction hypothesis and equations (2.5) and (2.10).

Returning to $\mathcal{B}^{[i_t]}_{r_t}[f]$ and using these observations we obtain

$$\begin{aligned}\mathcal{F}^a_{N,k}\left[\mathcal{B}^{[i_t]}_{r_t}[f]\right](x_{\bar{t}}) = &\sum_{i_{\bar{t}}\in\{0,1\}^{|\bar{t}|}} \sum_{|n_{\bar{t}}|_\infty=0}^{N-1} \mathcal{A}^{[i]}_{r_t,n_{\bar{t}}}[f] \phi^{[i_{\bar{t}}]}_{n_{\bar{t}}}(x_{\bar{t}}) \\ &+ \sum_{i_{\bar{t}}\in\{0,1\}^{|\bar{t}|}} \sum_{u\in[\bar{t}]} \sum_{|r_u|_\infty=0}^{k-1} \sum_{|n_{\bar{u}}|_\infty=0}^{N-1} \mathcal{A}^{[i]}_{r_{t\cup u},n_{\overline{t\cup u}}}[f] p^{[i_u]}_{r_u}(x_u) \phi^{[i_{\bar{u}}]}_{n_{\bar{u}}}(x_{\bar{u}}).\end{aligned}$$

Substituting this into (2.11) gives

$$g_k^a(x) = \sum_{i\in\{0,1\}^d}\sum_{t\in[d]}(-1)^{|t|+1}\Bigg\{\sum_{|r_t|_\infty=0}^{k-1}\mathcal{A}^{[i]}_{r_t,n_{\bar t}}[f]p^{[i_t]}_{r_t}(x_t)\phi^{[i_{\bar t}]}_{n_{\bar t}}(x_{\bar t}) + \sum_{u\in[\bar t]}\sum_{|r_{t\cup u}|_\infty=0}^{k-1}\sum_{|n_{\bar u}|_\infty=0}^{N-1}\mathcal{A}^{[i]}_{r_{t\cup u},n_{\overline{t\cup u}}}[f]p^{[i_{t\cup u}]}_{r_{t\cup u}}(x_{t\cup u})\phi^{[i_{\bar u}]}_{n_{\bar u}}(x_{\bar u})\Bigg\}. \quad (2.12)$$

To complete the proof it suffices to show that, for any $v\in[d]$, the coefficient of $\mathcal{A}^{[i]}_{r_v,n_{\bar v}}[f]p^{[i_v]}_{r_v}(x_v)\phi^{[i_{\bar v}]}_{n_{\bar v}}(x_{\bar v})$ in (2.12) is precisely 1. The first term of (2.12) gives a contribution of $(-1)^{|v|+1}$. For the second, the terms that give contributions satisfy $t\cup u = v$. Since $t,u\neq\emptyset$ and there are $\begin{pmatrix}|v|\\ l\end{pmatrix}$ possible choices of u with $|u|=l$, the contribution of the second term is

$$(-1)^{|v|}\begin{pmatrix}|v|\\ 1\end{pmatrix} + ... + \begin{pmatrix}|v|\\ |v|-1\end{pmatrix} = (-1)^{|v|+1}\sum_{l=1}^{|v|-1}\begin{pmatrix}|v|\\ l\end{pmatrix}(-1)^l = 1-(-1)^{|v|+1}.$$

Summing together the contributions of the two terms now yields the result. □

The result of this lemma justifies the statement made in the previous section: in the multivariate case, to accelerate convergence it suffices to approximate the jump values $\mathcal{A}^{[i]}_{r_t,n_{\bar t}}[f]$. This indicates the appropriate generalization of Eckhoff's method, which we consider in the next section.

Though the approximate polynomial subtraction process achieves a significant improvement over exact polynomial subtraction, it still requires knowledge of the values $\mathcal{A}^{[i]}_{r_t,n_{\bar t}}[f]$. In general these are unknown. Since there are

$$2^d\sum_{j=1}^{d}\begin{pmatrix}d\\ j\end{pmatrix}k^jN^{d-j} = 2^d\left\{(k+N)^d-k^d\right\} = \mathcal{O}\left(kN^{d-1}\right),\quad k\ll N,$$

such values, the need for a method of approximation becomes more vital as d increases.

2.3 The multivariate version of Eckhoff's method

We now extend Eckhoff's method to the multivariate setting. The bivariate version of this method was originally developed, without analysis, in [24, 25, 27]. In this section we first provide an extension for general d and then provide pertinent analysis.

We seek to approximate the exact jump values $\mathcal{A}^{[i]}_{r_t,n_{\bar t}}[f]$ with values $\bar{\mathcal{A}}^{[i]}_{r_t,n_{\bar t}}[f]$. Suppose that we define the subtraction function

$$g_k(x) = \sum_{i\in\{0,1\}^d}\sum_{t\in[d]}\sum_{|r_t|_\infty=0}^{k-1}\sum_{|n_{\bar t}|_\infty=0}^{N-1}\bar{\mathcal{A}}^{[i]}_{r_t,n_{\bar t}}[f]p^{[i_t]}_{r_t}(x_t)\phi^{[i_{\bar t}]}_{n_{\bar t}}(x_{\bar t}), \quad (2.13)$$

and the approximation $\mathcal{F}_{N,k}[f]=\mathcal{F}_N[f-g_k]+g_k$. In the univariate setting it follows from (1.11) that the function g_k satisfies the condition

$$\widehat{g_k}^{[i]}_n = \hat{f}^{[i]}_n,\quad n=m(0),...,m(k-1),\quad i\in\{0,1\}. \quad (2.14)$$

For the d-variate extension we enforce a similar condition. Suppose that we define the finite index set $M_k\subset\mathbb{N}^d$ by

$$M_k = \bigcup_{t\in[d]}\left\{n=(n_1,...,n_d)\in\mathbb{N}^d : n_j=m(r_j),\ r_j=0,...,k-1 \text{ if } j\in t,\ n_j=0,...,N-1,\text{ otherwise}\right\}. \quad (2.15)$$

We now impose the condition

$$\widehat{g_k}^{[i]}_n = \hat{f}^{[i]}_n,\quad \forall n\in M_k,\quad i\in\{0,1\}^d. \quad (2.16)$$

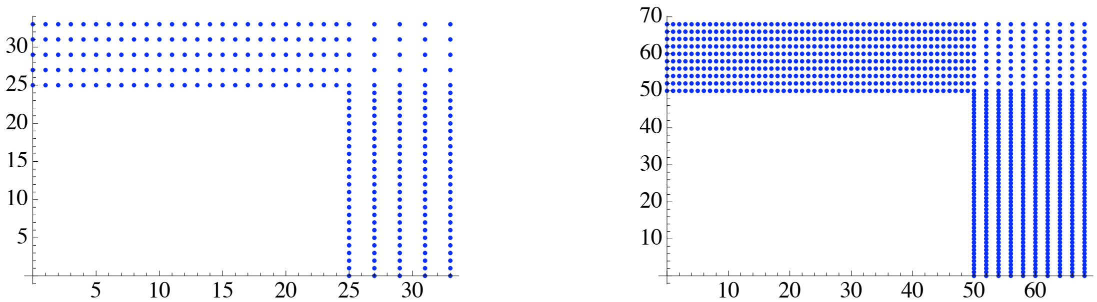


Figure 2: (left) Index set M_5 with $N=25$ and $m(r)=N+2r$. (right) Index set M_{10} with $N=50$ and $m(r)=N+2r$.

For $d=1$ (2.16) reduces to (2.14). For $d=2$ we obtain the following system of equations

$$\begin{aligned}\widehat{g_k}^{[i]}_{m(r_1),m(r_2)} &= \widehat{f}^{[i]}_{m(r_1),m(r_2)}, \quad r_1,r_2=0,...,k-1, \quad i\in\{0,1\}^2,\\ \widehat{g_k}^{[i]}_{m(r_1),n_2} &= \widehat{f}^{[i]}_{m(r_1),n_2}, \quad r_1=0,...,k-1, \quad n_2=0,...,N-1, \quad i\in\{0,1\}^2,\\ \widehat{g_k}^{[i]}_{n_1,m(r_2)} &= \widehat{f}^{[i]}_{n_1,m(r_2)}, \quad n_1=0,...,N-1, \quad r_2=0,...,k-1, \quad i\in\{0,1\}^2.\end{aligned} \tag{2.17}$$

Figure 2 shows typical form of the index set M_k for $d=2$. Note that, as in the univariate case, the system of equations (2.16) completely decouples for different values of $i\in\{0,1\}^d$.

For practical and analytical purposes, we need to expand the left hand side of (2.16). Given $u\in[d]$, $s_u\in\{0,...,k-1\}^{|u|}$ and $n_{\bar{u}}\in\{0,...,N-1\}^{|\bar{u}|}$, the corresponding term in g_k is

$$\bar{\mathcal{A}}^{[i]}_{s_u,n_{\bar{u}}}[f]p^{[i_u]}_{s_u}(x_u)\phi^{[i_{\bar{u}}]}_{n_{\bar{u}}}(x_{\bar{u}}) = \bar{\mathcal{A}}^{[i]}_{r_u,n_{\bar{u}}}[f]\prod_{j\in u}p^{[i_j]}_{s_j}(x_j)\prod_{j\notin u}\phi^{[i_j]}_{n_j}(x_j).$$

For this term to give a non-zero contribution to the left hand side of (2.16) we require that $t\subseteq u$, where $t\in[d]$ is the tuple corresponding to $n\in M_k$. Hence

$$\widehat{g_k}^{[i]}_n = \sum_{\substack{u\in[d]\\ t\subseteq u}}\sum_{|s_u|_\infty=0}^{k-1}\bar{\mathcal{A}}^{[i]}_{s_u,n_{\bar{u}}}[f]\prod_{j\in u}\widehat{p_{s_j}}^{[i_j]}_{n_j} = \sum_{|s_t|_\infty=0}^{k-1}\prod_{j\in t}V^{[i_j]}_{r_j,s_j}\left\{\sum_{t\subseteq u}\sum_{|s_{u\setminus t}|_\infty=0}^{k-1}\bar{\mathcal{A}}^{[i]}_{s_u,n_{\bar{u}}}[f]\prod_{j\in u\setminus t}\widehat{p_{s_j}}^{[i_j]}_{n_j}\right\}, \tag{2.18}$$

where $u\backslash t\in[d]^*$ is the ordered $|u|-|t|$ tuple of elements $j=1,..,d$ that are in u but not t. Here r_j, $j\in t$ is the index used in the definition of M_k, (2.15), and $V^{[i]}$ is the matrix introduced in Section 1.3.

For $d=2$, we may expand the system of equations (2.17) using (2.18) to give the following linear system:

$$\sum_{s_1,s_2=0}^{k-1}V^{[i_1]}_{r_1,s_1}V^{[i_2]}_{r_2,s_2}\bar{\mathcal{A}}^{[i]}_{s_1,s_2}[f] = \hat{f}^{[i]}_{m(r_1),m(r_2)}, \quad r_1,r_2=0,...,k-1, \quad i\in\{0,1\}^2,$$

$$\sum_{s_1=0}^{k-1}V^{[i_1]}_{r_1,s_1}\left\{\bar{\mathcal{A}}^{[i]}_{s_1,n_2}[f]+\sum_{s_2=0}^{k-1}\bar{\mathcal{A}}^{[i]}_{s_1,s_2}[f]\widehat{p_{r_2}}^{[i_2]}_{n_2}\right\} = \hat{f}^{[i]}_{m(r_1),n_2}, \quad r_1=0,...,k-1, \quad n_2=0,...,N-1, \quad i\in\{0,1\}^2,$$

$$\sum_{s_2=0}^{k-1}V^{[i_2]}_{r_2,s_2}\left\{\bar{\mathcal{A}}^{[i]}_{s_2,n_1}[f]+\sum_{s_1=0}^{k-1}\bar{\mathcal{A}}^{[i]}_{s_1,s_2}[f]\widehat{p_{r_1}}^{[i_1]}_{n_1}\right\} = \hat{f}^{[i]}_{n_1,m(r_2)}, \quad n_1=0,...,N-1, \quad r_2=0,...,k-1, \quad i\in\{0,1\}^2.$$

In this case, it is obvious how to solve these equations. We first obtain $\bar{\mathcal{A}}^{[i]}_{r_1,r_2}[f]$ from the first equation, then use this to find $\bar{\mathcal{A}}^{[i]}_{r_1,n_2}[f]$ and $\bar{\mathcal{A}}^{[i]}_{r_2,n_1}[f]$ explicitly. At each stage we have to solve linear systems involving the matrix $V^{[i]}$, and, since we need to do this repeatedly, it is easiest to find $(V^{[i]})^{-1}$ first.

The same can be done in $d\geq 3$ dimensions. Starting with the equation corresponding to $t=(1,2,...,d)$, we find $\bar{\mathcal{A}}^{[i]}_{r_t}[f]$. Using this we solve the d equations corresponding to $|t|=d-1$. Continuing in this manner we obtain all the coefficients $\bar{\mathcal{A}}^{[i]}_{r_t,n_{\bar{t}}}[f]$. Though straightforward in theory, this process becomes increasingly complicated to implement for large d.

Note that existence and uniqueness of a solution to these linear systems is completely determined by the non-singularity of the matrix $V^{[i]}$ (see Theorem 1).

In the univariate case, the complexity of forming Eckhoff's approximation is $\mathcal{O}\left(\max\{k^2, kN\}\right)$. In the multivariate setting it is readily seen that this figure is

$$\mathcal{O}\left(\max\{k^{d+1}, k^d N^d\}\right).$$

Typically $k \ll N$ so this figure reduces to $k^d N^d$. In comparison, forming the approximation $\mathcal{F}_N[f]$ involves $\mathcal{O}\left(N^d\right)$ operations, so the increase in complexity is relatively mild for moderate values of k. Nonetheless, the value N^d grows exponentially with d. In Section 4 how it can be reduced dramatically without affecting the convergence rate of $\mathcal{F}_{N,k}[f]$ unduly.

2.4 Analysis of Eckhoff's method

To commence our analysis we require the following two lemmas, the first of which is a generalization of Theorem 2:

Lemma 6. *Suppose that $h \in H^{2(k+K)}_{mix}(\Omega)$, where $2K \geq l+1$ and l is the number of equal values $c(r)$, and that $t \in [d]$. Suppose further that*

$$\mathcal{B}^{[i_j]}_{r_j}[h] = 0, \quad r_j = 0, ..., k-1, \quad i_j \in \{0,1\}, \quad j \in t,$$

and that the values $\mathcal{E}^{[i_t]}_{r_t,n_{\bar{t}}}$ satisfy

$$\sum_{|s_t|_\infty=0}^{k-1} \prod_{j\in t} V^{[i_j]}_{r_j,s_j} \mathcal{E}^{[i_t]}_{r_t,n_{\bar{t}}} = \hat{h}^{[i]}_n,$$

for all $n \in \mathbb{N}^d$ such that $n_j = m(r_j)$, $r_j = 0, ..., k-1$ when $j \in t$ and $n_j \in \mathbb{N}$ otherwise. Then we have

$$\left|\mathcal{E}^{[i_t]}_{r_t,n_{\bar{t}}}\right| \lesssim N^{2(|r_t|_\infty - k)} \bar{n}_{\bar{t}}^{-2}.$$

Proof. For each $j \in t$ we may expand $\hat{h}^{[i]}_n$ $(k+K)$ times with respect to n_j using the univariate expansion (1.2). We now apply $(V^{[i_j]}_{r_j,s_j})^{-1}$ to the result and use Theorem 2. ☐

Lemma 7. *Suppose that $t \in [d]$, $r_t \in \{0, ..., k-1\}^{|t|}$, $n_{\bar{t}} \in \{0, ..., N-1\}^{|\bar{t}|}$ and*

$$\mathcal{E}^{[i]}_{r_t,n_{\bar{t}}}[f] = \sum_{\substack{u\in[d]\\ t\subseteq u}} \sum_{|s_{u\setminus t}|_\infty=0}^{k-1} \left(\mathcal{A}^{[i]}_{s_u,n_{\bar{u}}}[f] - \bar{\mathcal{A}}^{[i]}_{s_u,n_{\bar{u}}}[f]\right) \prod_{j\in u\setminus t} \widehat{p_{s_j}}^{[i_j]}_{n_j}. \tag{2.19}$$

Then

$$\left|\mathcal{E}^{[i]}_{r_t,n_{\bar{t}}}[f]\right| \lesssim N^{2(|r_t|_\infty - k)} \bar{n}_{\bar{t}}^{-2}.$$

Proof. Consider the right hand side of (2.16). Using the expansion (2.6) gives

$$\hat{f}^{[i]}_n = \sum_{\substack{u\in[d]\\ t\subseteq u}} \sum_{|s_u|_\infty=0}^{k-1} \mathcal{A}^{[i]}_{s_u,n_{\bar{u}}}[f] \widehat{p_{s_u}}^{[i_u]}_{n_u} + \sum_{\substack{u\in[d]^*\\ t\not\subseteq u}} \sum_{|s_u|_\infty=0}^{k-1} \mathcal{A}^{[i]}_{s_u,n_{\bar{u}}}[f] \widehat{p_{s_u}}^{[i_u]}_{n_u}, \quad n \in M_k, \quad i \in \{0,1\}^d.$$

Equating this with (2.18) and rearranging gives

$$\sum_{|s_t|_\infty=0}^{k-1} \prod_{j\in t} V^{[i_j]}_{r_j,s_j} \mathcal{E}^{[i]}_{r_t,n_{\bar{t}}}[f] = -\sum_{\substack{u\in[d]^*\\ t\not\subseteq u}} \sum_{|s_u|_\infty=0}^{k-1} \mathcal{A}^{[i]}_{s_u,n_{\bar{u}}}[f] \widehat{p_{s_u}}^{[i_u]}_{n_u}, \quad n \in M_k, \quad i \in \{0,1\}^d. \tag{2.20}$$

We now split the terms in the right hand side into two cases. If $u \in [d]^*$ and $t \not\subseteq u$ the either $t \cap u = \emptyset$ or $t \cap u \neq \emptyset$. Suppose first that $t \cap u \neq \emptyset$. We have

$$\widehat{p_{s_u}}^{[i_u]}_{n_u} = \prod_{j \in u} \widehat{p_{s_j}}^{[i_j]}_{n_j} = \prod_{j \in t \cap u} V^{[i_j]}_{r_j, s_j} \prod_{j \in u \setminus t} \widehat{p_{s_j}}^{[i_j]}_{n_j}.$$

Substituting this back into (2.20) we obtain

$$\sum_{|s_t|_\infty = 0}^{k-1} \prod_{j \in t} V^{[i_j]}_{r_j, s_j} \mathcal{E}^{[i]}_{r_t, n_{\bar{t}}}[f] = - \sum_{\substack{t \not\subseteq u \\ t \cap u \neq \emptyset}} \sum_{|s_u|_\infty = 0}^{k-1} \mathcal{A}^{[i]}_{s_u, n_{\bar{u}}}[f] \prod_{j \in t \cap u} V^{[i_j]}_{r_j, s_j} \prod_{j \in u \setminus t} \widehat{p_{s_j}}^{[i_j]}_{n_j} - \sum_{\substack{t \not\subseteq u \\ t \cap u = \emptyset}} \sum_{|s_u|_\infty = 0}^{k-1} \mathcal{A}^{[i]}_{s_u, n_{\bar{u}}}[f] \widehat{p_{s_u}}^{[i_u]}_{n_u}. \tag{2.21}$$

We now use Lemma 6 on each term of the right hand side. Consider first terms in the latter sum. The values $\mathcal{A}^{[i]}_{s_u, n_{\bar{u}}}[f] \widehat{p_{s_u}}^{[i_u]}_{n_u}$ are the modified Fourier coefficients of a d-variate function h that satisfies

$$\mathcal{B}^{[i_j]}_{r_j}[h] = 0, \quad r_j = 0, ..., k-1, \quad j \in \bar{u}.$$

Since $t \subseteq \bar{u}$, an application of Lemma 6 now yields the required bound for such terms.

For terms in the first sum, we essentially repeat the same process, but taking the term $\prod_{j \in t \cap u} V^{[i_j]}_{r_j, s_j}$ into account where necessary. □

Due to Lemma 4, to estimate the convergence rate of the multivariate Eckhoff approximation $\mathcal{F}_{N,k}[f]$, it suffices to consider the difference $\mathcal{F}^a_{N,k}[f] - \mathcal{F}_{N,k}[f]$, where $\mathcal{F}^a_{N,k}[f]$ is the approximate polynomial subtraction approximation introduced in Section 2.2. For this we need the following lemma, which demonstrates the importance of the quantity (2.19) in the analysis of Eckhoff's approximation:

Lemma 8. *We have*

$$\mathcal{F}^a_{N,k}[f](x) - \mathcal{F}_{N,k}[f](x) = \sum_{i \in \{0,1\}^d} \sum_{t \in [d]} \sum_{|r_t|_\infty = 0}^{k-1} \sum_{|n_{\bar{t}}|_\infty = 0}^{N-1} \mathcal{E}^{[i]}_{r_t, n_{\bar{t}}}[f] \phi^{[i]}_{n_{\bar{t}}}(x_{\bar{t}}) \prod_{j \in t} \left(p^{[i_j]}_{r_j}(x_j) - \mathcal{F}_N[p^{[i_j]}_{r_j}](x_j) \right). \tag{2.22}$$

Proof. We may write

$$\mathcal{F}^a_{N,k}[f](x) - \mathcal{F}_{N,k}[f](x) = h_k(x) - \mathcal{F}_N[h_k](x), \tag{2.23}$$

where h_k is the smooth function

$$h_k(x) = \sum_{i \in \{0,1\}^d} \sum_{t \in [d]} \sum_{|r_t|_\infty = 0}^{k-1} \sum_{|n_{\bar{t}}|_\infty = 0}^{N-1} \left(\mathcal{A}^{[i]}_{r_t, n_{\bar{t}}}[f] - \bar{\mathcal{A}}^{[i]}_{r_t, n_{\bar{t}}}[f] \right) p^{[i_t]}_{r_t}(x_t) \phi^{[i_{\bar{t}}]}_{n_{\bar{t}}}(x_{\bar{t}}).$$

To prove the result it suffices to demonstrate that the right hand sides of (2.22) and (2.23) have equal modified Fourier coefficients for all indices $i \in \{0,1\}^d$ and $n \in \mathbb{N}^d$. For coefficients with $n \in I_N$ it is readily shown that both sides give zero, hence we consider $n \notin I_N$. In this case, there is some $u \in [d]$ such that $n_j \geq N$ whenever $j \in u$ and $n_j = 0, ..., N-1$ otherwise. By identical arguments to those used to obtain (2.18) we have that the coefficient the right hand side of (2.23), namely $\widehat{h_k}^{[i]}_n$, is

$$\widehat{h_k}^{[i]}_n = \sum_{|r_u|_\infty = 0}^{k-1} \widehat{p_{r_u}}^{[i_u]}_{n_u} \mathcal{E}^{[i]}_{r_u, n_{\bar{u}}}[f]. \tag{2.24}$$

We now consider the corresponding coefficient of (2.22). For each $t \in [d]$, due to the function $\phi^{[i_{\bar{t}}]}_{n_{\bar{t}}}$, we must have that $u \subseteq t$ otherwise the corresponding term vanishes. However, due to the product, we must also have that $t \subseteq u$ for a non-zero contribution. Hence, $t = u$ and the modified Fourier coefficient of (2.22) reduces to (2.24), completing the proof. □

With this in hand we are able to deduce the main result of this section:

Theorem 6. *Suppose that* $f \in H^{2(k+K)}_{mix}(\Omega)$, *where* $2K \geq l+1$ *and* l *is the number of equal* $c(r)$, *and that* $\mathcal{F}_{N,k}[f]$ *is the multivariate Eckhoff approximation of* f. *Then* $\|f - \mathcal{F}_{N,k}[f]\|_\infty$ *is* $\mathcal{O}\left(N^{-2k-1}\right)$ *and* $\|f - \mathcal{F}_{N,k}[f]\|_q$ *is* $\mathcal{O}(N^{q-2k-\frac{3}{2}})$ *for* $q = 0, ..., 2k+1$.

Proof. It suffices to consider the difference $\mathcal{F}^a_{N,k}[f] - \mathcal{F}_{N,k}[f]$. Using Lemma 8, the bound derived in Lemma 7 and the fact that $\|p^{[i]}_r - \mathcal{F}_N[p^{[i]}_r]\|_\infty = \mathcal{O}\left(N^{-2r-1}\right)$, $r \in \mathbb{N}$, we deduce that

$$\begin{aligned}\|\mathcal{F}^a_{N,k}[f] - \mathcal{F}_{N,k}[f]\|_\infty &\lesssim \sum_{i\in\{0,1\}^d} \sum_{t\in[d]} \sum_{|r_t|_\infty=0}^{k-1} \sum_{|n_{\bar{t}}|_\infty=0}^{N-1} \left|\mathcal{E}^{[i]}_{r_t,n_{\bar{t}}}[f]\right| \prod_{j\in t} \left|p^{[i_j]}_{r_j}(x_j) - \mathcal{F}_N[p^{[i_j]}_{r_j}](x_j)\right| \\ &\lesssim \sum_{t\in[d]} \sum_{|r_t|_\infty=0}^{k-1} \sum_{|n_{\bar{t}}|_\infty=0}^{N-1} \bar{n}_{\bar{t}}^{-2} N^{2(|r_t|_\infty-k)} \prod_{j\in t} N^{-2r_j-1} \lesssim N^{-2k-1},\end{aligned}$$

which gives the result for the uniform error. Now suppose that $\alpha \in \mathbb{N}^d$ is a multi-index with $|\alpha| \leq q$. Then

$$\begin{aligned}&D^\alpha\left(\mathcal{F}^a_{N,k}[f] - \mathcal{F}_{N,k}[f]\right) \\ &\quad = \sum_{i\in\{0,1\}^d} \sum_{t\in[d]} \sum_{|r_t|_\infty=0}^{k-1} \sum_{|n_{\bar{t}}|_\infty=0}^{N-1} \mathcal{E}^{[i]}_{r_t,n_{\bar{t}}}[f] D^{\alpha_{\bar{t}}}\phi^{[i]}_{n_{\bar{t}}}(x_{\bar{t}}) \prod_{j\in t}\left(\partial^{\alpha_j}_{x_j} p^{[i_j]}_{r_j}(x_j) - \partial^{\alpha_j}_{x_j}\mathcal{F}_N[p^{[i_j]}_{r_j}](x_j)\right).\end{aligned}$$

Note that

$$\|D^{\alpha_{\bar{t}}}\phi^{[i_{\bar{t}}]}_{n_{\bar{t}}}\| \lesssim \bar{n}_{\bar{t}}^{\alpha_{\bar{t}}} = \prod_{j\notin t} \bar{n}_j^{\alpha_j},$$

and that $\|p^{[i]}_r - \mathcal{F}_N[p^{[i]}_r]\|_q = \mathcal{O}(N^{q-2r-\frac{3}{2}})$. Using these observations, we obtain

$$\begin{aligned}\|D^\alpha\left(\mathcal{F}^a_{N,k}[f] - \mathcal{F}_{N,k}[f]\right)\| &\lesssim \sum_{i\in\{0,1\}^d} \sum_{t\in[d]} \sum_{|r_t|_\infty=0}^{k-1} \sum_{|n_{\bar{t}}|_\infty=0}^{N-1} \bar{n}_{\bar{t}}^{\alpha_{\bar{t}}-2} N^{2(|r_t|_\infty-k)} \prod_{j\in t} N^{\alpha_j-2r_j-\frac{3}{2}} \\ &\lesssim N^{|\alpha|-2k-\frac{3}{2}} \leq N^{q-2k-\frac{3}{2}}.\end{aligned}$$

Summing over all $|\alpha| \leq q$ now gives the result for the H^q-norm error. ☐

As in the univariate case, additional smoothness is required for the multivariate version of Eckhoff's method over approximation by polynomial subtraction unless the values $c(r)$, $r = 0, ..., k-1$, are distinct. However, as we now consider, there is an advantage to choosing equal values $c(r)$, namely a much faster convergence rate inside the domain Ω.

3 The auto-correction phenomenon

As demonstrated in Lemmas 1 and 4, the approximation based on exact jump values has a convergence rate one power of N faster inside the domain than on the boundary. It turns out that for the particular choice of the values $m(r) = N + r$, Eckhoff's approximation possesses the much faster convergence rate of $\mathcal{O}\left(N^{-3k-2}\right)$ away from the boundary; a full $\mathcal{O}\left(N^{k+1}\right)$ faster. This auto-correction phenomenon was observed numerically in [26] and proved in the univariate, Fourier case in [29]. The aim of this section is to extend this result to the multivariate modified Fourier setting.

In previous sections we observed that Eckhoff's approximation decouples into terms corresponding to each paticular value of i. The analysis of each such term can be handled separately, and, since the analysis is virtually identical, it suffices to consider only one particular value. For the remainder of this section we assume that f only has non-zero modified Fourier coefficients when $i = (0, 0, ..., 0)$. Accordingly, we drop the $[i]$ superscript.

Since uniform convergence of Eckhoff's approximation on $\bar{\Omega}$ is guaranteed by Theorem 6 we may write

$$f(x) - \mathcal{F}_{N,k}[f](x) = \sum_{n\notin I_N} \hat{v}_n\phi_n(x) = \sum_{t\in[d]} \sum_{|n_t|_\infty \geq N} \sum_{|n_{\bar{t}}|_\infty=0}^{N-1} \hat{v}_n\phi_n(x), \quad x \in \bar{\Omega}, \tag{3.1}$$

where $v(x) = f(x) - g_k(x)$ and g_k is given by (2.13). Following the same method of proof as in [29] we seek to expand the right hand side of (3.1) using the so-called *Abel transformation*. Given a sequence $a_m \in \mathbb{R}$, $m \in \mathbb{N}$, we define the operator $\triangle_{r,n}$, $r, n \in \mathbb{N}$, by

$$\triangle_{0,n}[a_m] = a_n, \quad \triangle_{r+1,n}[a_m] = \triangle_{r,n}[a_m] + \triangle_{r,n+1}[a_m], \quad r, n \in \mathbb{N}.$$

It is easily seen that

$$\triangle_{r,n}[a_m] = \sum_{s=0}^{r} \begin{pmatrix} r \\ s \end{pmatrix} a_{n+s}, \quad r, n \in \mathbb{N}. \tag{3.2}$$

Now suppose that $a_m \in \mathbb{R}$, $m \in \mathbb{N}^d$. We write $\triangle^j_{r,n}$, $j = 1, .., d$, for the above operator acting on the j^{th} entry of n. Further, given $t \in [d]$, $r \in \mathbb{N}^{|t|}$ and $n \in \mathbb{N}^{|t|}$ we define $\triangle^t_{r,n}$ by the composition of $|t|$ such operators:

$$\triangle^t_{r,n}[a_m] = \triangle^{t_1}_{r_{t_1},n_{t_1}} \left[\triangle^{t_2}_{r_{t_2},n_{t_2}} \left[... \triangle^{t_{|t|}}_{r_{t_{|t|}},n_{t_{|t|}}} [a_m] \right] \right].$$

It follows from (3.2) that

$$\triangle^t_{r,n}[a_m] = \sum_{s_{t_1}=0}^{r_{t_1}} .... \sum_{s_{t_{|t|}}=0}^{r_{t_{|t|}}} \begin{pmatrix} r_{t_1} \\ s_{t_1} \end{pmatrix} ... \begin{pmatrix} r_{t_{|t|}} \\ s_{t_{|t|}} \end{pmatrix} a_{(n+s,m;t)}, \tag{3.3}$$

where $(n + s, m; t)$ has j^{th} entry $n_j + s_j$ if $j \in t$ and m_j otherwise.

Before using this transform, we need some additional notation. Given $x, y \in \mathbb{R}^d$ we write $x.y = x_1 y_1 + ... x_d y_d$, and if $y = (c, c, ..., c)$ has equal entries, just $x.c$. Moreover, given $u \in [t]^*$, $r_u \in \mathbb{N}^{|u|}$ and $k \in \mathbb{N}$ we define $(r_u; k) \in \mathbb{N}^{|t|}$ by the condition that the j^{th} entry of $(r_u; k)$, which we write $(r_u; k)_j$, takes value r_j if $j \in u$ and k otherwise.

Lemma 9. *Suppose that $g \in H^1_{mix}(\Omega)$, $t \in [d]$ and that $x \in \Omega$. Then, for $k \in \mathbb{N}$, we have*

$$\sum_{|n_t|_\infty \geq N} \hat{g}_n \phi_{n_t}(x_t) =$$
$$\Re \left\{ \sum_{u \in [t]^*} \sum_{|r_u|_\infty = 0}^{k} e^{i\pi x_u . (N-1)} \prod_{j \in t} (1 + e^{-i\pi x_j})^{-(r_u;k)_j - 1} \sum_{|n_{\bar{u}}|_\infty \geq N} \triangle^t_{(r_u;k+1),(n_{\bar{u}};N)} [\hat{g}_m] e^{i\pi n_{\bar{u}} . x_{\bar{u}}} \right\}.$$

Proof. We proceed by induction on $|t|$. Suppose first that $|t| = 1$ and, without loss of generality, that $d = 1$. The verification of the lemma in this case is very standard (see also [29]). We have

$$\sum_{n \geq N} \hat{g}_n e^{in\pi x} = \sum_{n \geq N} \left(\triangle_{1,n}[\hat{g}_m] - \hat{g}_{n+1} \right) e^{in\pi x} = \sum_{n \geq N} \triangle_{1,n}[\hat{g}_m] e^{in\pi x} - e^{-i\pi x} \sum_{n \geq N} \hat{g}_n e^{in\pi x} + \hat{g}_N e^{i(N-1)\pi x}.$$

Rearranging gives

$$\sum_{n \geq N} \hat{g}_n e^{in\pi x} = \frac{e^{i(N-1)\pi x}}{1 + e^{-i\pi x}} \hat{g}_N + \frac{1}{1 + e^{-i\pi x}} \sum_{n \geq N} \triangle_{1,n}[\hat{g}_m] e^{in\pi x},$$

which provides the result for $k = 0$. Iterating this process yields the result for general k.

Now let $t \in [d]$ be of length $|t| \geq 2$. Write $t = (t_1, \tau)$, where $\tau \in [d]$ and $|t| = |\tau| + 1$. We have

$$\sum_{|n_t|_\infty \geq N} \hat{g}_n \phi_{n_t}(x_t) = \sum_{n_{t_1} \geq N} \phi_{n_{t_1}}(x_{t_1}) \sum_{|n_\tau|_\infty \geq N} \hat{g}_n \phi_{n_\tau}(x_\tau).$$

Using the induction hypothesis we obtain

$$\sum_{|n_t|_\infty \geq N} \hat{g}_n \phi_{n_t}(x_t) = \Re \sum_{n_{t_1} \geq N} e^{i n_{t_1} \pi x_{t_1}} \Bigg\{ \sum_{u \in [\tau]^*} e^{i\pi x_u.(N-1)} \sum_{|r_u|_\infty = 0}^{k} \prod_{j \in t} (1 + e^{-i\pi x_j})^{-(r_u;k)_j - 1} \times \sum_{|n_{\bar{u}}|_\infty \geq N} \triangle^{\tau}_{(r_u;k+1),(n_{\bar{u}};N)} [\hat{g}_m] e^{i\pi n_{\bar{u}}.x_{\bar{u}}} \Bigg\}$$

$$= \Re \sum_{u \in [\tau]^*} e^{i\pi x_u.(N-1)} \sum_{|r_u|_\infty = 0}^{k} \prod_{j \in t} (1 + e^{-i\pi x_j})^{-(r_u;k)_j - 1} \times \sum_{|n_{\bar{u}}|_\infty \geq N} e^{i\pi n_{\bar{u}}.x_{\bar{u}}} \sum_{n_{t_1} \geq N} e^{i n_{t_1} \pi x_{t_1}} \triangle^{\tau}_{(r_u;k+1),(n_{\bar{u}};N)} [\hat{g}_m]. \tag{3.4}$$

Using the result for $|t| = 1$ yields

$$\sum_{n_{t_1} \geq N} e^{i n_{t_1} \pi x_{t_1}} \triangle^{\tau}_{(r_u;k+1),(n_{\bar{u}};N)} [\hat{g}_n] = \sum_{r_{t_1}=0}^{k} e^{i\pi x_{t_1}(N-1)} (1 + e^{i\pi x_{t_1}})^{-r_{t_1} - 1} \triangle_{r_{t_1},N} \left[\triangle^{\tau}_{(r_u;k+1),(n_{\bar{u}};N)} [\hat{g}_m] \right] + \sum_{n_{t_1} \geq N} (1 + e^{-i\pi x_{t_1}})^{-k-1} \triangle_{k+1,n_{t_1}} \left[\triangle^{\tau}_{(r_u;k+1),(n_{\bar{u}};N)} [\hat{g}_m] \right]. \tag{3.5}$$

If we substitute (3.5) into (3.4) we obtain the result. Note that if $v \in [t]^*$ then either $v = (t_1, u)$ for some $u \in [\tau]^*$ or $v \in [\tau]^*$. The two terms of (3.5) correspond respectively to these scenarios. □

The crux of the auto-correction phenomenon is the following trivial observation:

Lemma 10. *Suppose that $v = f - g_k$, where g_k is given by (2.13), and that the values $m(r) = N + r$, $r = 0, ..., k-1$. Then $\triangle^t_{r_t,n_t}[\hat{v}_m] = 0$ for all $|r_t|_\infty \leq k-1$, $|n_t|_\infty \leq N$ and $t \in [d]$.*

Proof. By construction $\hat{v}_n = 0$ for $|n|_\infty \leq N + k - 1$. We now use (3.3) to obtain the result. □

We may now re-write (3.1) as follows:

$$f(x) - \mathcal{F}_{N,k}[f](x) = \sum_{t \in [d]} \sum_{|n_{\bar{t}}|_\infty = 0}^{N-1} h_{n_{\bar{t}}}(x_t) \phi_{n_{\bar{t}}}(x_{\bar{t}}). \tag{3.6}$$

Here $h_{n_{\bar{t}}}(x_t)$ is obtained from the expansion derived in Lemma 9. This gives

$$h_{n_{\bar{t}}}(x_t) = \Re \Bigg\{ \sum_{u \in [t]^*} \sum_{|r_u|_\infty = 0}^{k} e^{i\pi x_u.(N-1)} \prod_{j \in t} (1 + e^{-i\pi x_j})^{-(r_u;k)_j - 1} \sum_{|n_{\bar{u}}|_\infty \geq N} \triangle^{t}_{(r_u;k+1),(n_{\bar{u}};N)} [\hat{v}_m] e^{i\pi n_{\bar{u}}.x_{\bar{u}}} \Bigg\}.$$

Consider the term of $h_{n_{\bar{t}}}$ corresponding to $u = t$ separately. This is

$$e^{i\pi x_t.(N-1)} \sum_{|r_t|_\infty = 0}^{k} \prod_{j \in t} (1 + e^{-i\pi x_j})^{-r_j - 1} \triangle^t_{r_t,N}[\hat{v}_m].$$

Using Lemma 10, all terms of this expression where $|r_t|_\infty < k$ are zero. Hence, we define

$$H_{n_{\bar{t}}}(x_t) = e^{i\pi x_t.(N-1)} \sum_{|r_t|_\infty = k} \prod_{j \in t} (1 + e^{-i\pi x_j})^{-r_j - 1} \triangle^t_{r_t,N}[\hat{v}_m], \tag{3.7}$$

and

$$G_{n_{\bar{t}}}(x_t) = \sum_{\substack{u \in [t]^* \\ u \neq t}} \sum_{|r_u|_\infty = 0}^{k} e^{i\pi x_u.(N-1)} \prod_{j \in t} (1 + e^{-i\pi x_j})^{-(r_u;k) - 1} \sum_{|n_{\bar{u}}|_\infty \geq N} \triangle^{t}_{(r_u;k+1),(n_{\bar{u}};N)} [\hat{v}_m] e^{i\pi n_{\bar{u}}.x_{\bar{u}}}, \tag{3.8}$$

so that the function h_{n_t} may be expressed as $h_{n_{\bar{t}}}(x_t) = \Re\{G_{n_{\bar{t}}}(x_t) + H_{n_{\bar{t}}}(x_t)\}$. To derive an estimate for the error $f(x) - \mathcal{F}_{N,k}[f](x)$ we need bounds for the functions $G_{n_{\bar{t}}}$ and $H_{n_{\bar{t}}}$, which we present in the sequel. First, however, it is useful to consider the case $d = 1$ to demonstrate elements of the multivariate proof. This is given in a similar form in [29]:

3.1 The case $d = 1$

For $d = 1$, using (3.1) and the characterization given in Lemma 9 with $t = (1)$ we may write

$$f(x) - \mathcal{F}_{N,k}[f](x) = \sum_{n \geq N} \hat{v}_n \phi_n(x) = \Re\left\{ \sum_{r=0}^{k} \frac{e^{i(N-1)\pi x}}{(1 + e^{-i\pi x})^{-r-1}} \triangle_{r,N}[\hat{v}_m] + \frac{1}{(1 + e^{-i\pi x})^{k+1}} \sum_{n \geq N} \triangle_{k+1,n}[\hat{v}_m] e^{in\pi x} \right\}.$$

In light of Lemma 10, $\triangle_{r,N}[\hat{v}_m] = 0$ for $r = 0, ..., k-1$, so this reduces to

$$\begin{aligned} f(x) - \mathcal{F}_{N,k}[f](x) &= \Re\left\{ \frac{e^{i(N-1)\pi x}}{(1 + e^{-i\pi x})^{-k-1}} \triangle_{k,N}[\hat{v}_m] + \frac{1}{(1 + e^{-i\pi x})^{k+1}} \sum_{n \geq N} \triangle_{k+1,n}[\hat{v}_m] e^{in\pi x} \right\} \\ &= \Re\{H(x) + G(x)\}, \end{aligned} \tag{3.9}$$

where $G(x)$ and $H(x)$ are the univariate forms of $G_{n_{\bar{t}}}$ and $H_{n_{\bar{t}}}$. Note that for $d = 1$ there is only one $t \in [d]$, namely $t = (1)$, and trivially $\bar{t} = \emptyset$. We now seek bounds for G and H.

To do so, we require the following lemma, which is given in a similar form in [29]:

Lemma 11. *Suppose that p_s, $s = 0, ..., k-1$ are the univariate Cardinal polynomials for the first k derivative conditions. Then*

$$\triangle_{r,n}[\widehat{p_s}_m] = \widehat{p_s}_n \frac{(2s+r+1)!(-1)^r}{(2s+1)!n^r} + \mathcal{O}\left(n^{-2s-r-3}\right), \quad \forall r \in \mathbb{N}, \quad s = 0, ..., k-1, \quad n \to \infty.$$

Proof. By construction, $\widehat{p_s}_m = (-1)^m (m\pi)^{-2(s+1)}$. Using (3.2) we obtain

$$\begin{aligned} \triangle_{r,n}[\widehat{p_s}_m] &= \sum_{l=0}^{r} \binom{r}{l} \frac{(-1)^{n+l}}{((n+l)\pi)^{2(s+1)}} = \frac{(-1)^n}{(n\pi)^{2(s+1)}} \sum_{l=0}^{r} \binom{r}{l} \frac{(-1)^l}{(1+\frac{l}{n})^{2(s+1)}} \\ &= \frac{(-1)^n}{(n\pi)^{2(s+1)}} \sum_{l=0}^{r} (-1)^l \binom{r}{l} \left\{ \sum_{p=0}^{r} \left(\frac{l}{n}\right)^p \binom{2s+p+1}{p} + \mathcal{O}\left(n^{-r-1}\right) \right\} \\ &= \widehat{p_s}_n \sum_{p=0}^{r} n^{-p} \binom{2s+p+1}{p} \sum_{l=0}^{r} (-1)^l \binom{r}{l} l^p + \mathcal{O}\left(n^{-2s-r-3}\right). \end{aligned}$$

It is readily seen that

$$\sum_{l=0}^{r} (-1)^l \binom{r}{l} l^p = \begin{cases} 0 & p = 0, ..., r-1, \\ (-1)^r r! & p = r. \end{cases}$$

Substituting this into the previous expression now gives the result. ☐

We now consider the coefficients $\hat{v}_m$. Using the asymptotic expansion (1.2) and the form of the univariate function g_k, we obtain

$$\hat{v}_n = \sum_{r=0}^{k-1} \left(\mathcal{A}_r[f] - \bar{\mathcal{A}}_r[f]\right) \widehat{p_r}_n + \sum_{r=k}^{k+K-1} \mathcal{A}_r[f] \widehat{p_r}_n + \mathcal{O}\left(n^{-2(k+K+1)}\right),$$

provided $f \in H^{2(k+K+1)}(-1, 1)$. In particular

$$\triangle_{s,n}[\hat{v}_m] = \sum_{r=0}^{k-1} \left(\mathcal{A}_r[f] - \bar{\mathcal{A}}_r[f]\right) \triangle_{s,n}[\widehat{p_r}_m] + \sum_{r=k}^{k+K-1} \mathcal{A}_r[f] \triangle_{s,n}[\widehat{p_r}_m] + \mathcal{O}\left(n^{-2(k+K+1)}\right).$$

Using Theorem 2 and Lemma 11 we obtain

$$|\triangle_{s,n}[\hat{v}_m]| \lesssim \sum_{r=0}^{k-1} N^{2(r-k)}\bar{n}^{-2r-s-2} + \bar{n}^{-2k-s-2} + \bar{n}^{-2(k+K+1)}.$$

In particular, provided $2K \geq k+1$, we have

$$|\triangle_{k,N}[\hat{v}_m]| \lesssim N^{-3k-2}, \quad |\triangle_{k+1,n}[\hat{v}_m]| \lesssim N^{-3k-1}n^{-2}.$$

Recalling the definitions of G and H given in (3.9), this yields

$$|G(x)| \lesssim N^{-3k-2}, \quad |H(x)| \lesssim N^{-3k-2}, \quad x \in (-1,1),$$

provided $f \in H^{3(k+1)}(-1,1)$. From this we immediately obtain the univariate result:

Theorem 7. *Suppose that $\mathcal{F}_{N,k}[f]$ is the univariate Eckhoff approximation of $f \in H^{3(k+1)}(\Omega)$ using the values $m(r) = N+r$, $r = 0,...,k-1$. Then $f(x) - \mathcal{F}_{N,k}[f](x)$ is $\mathcal{O}\left(N^{-3k-2}\right)$ uniformly for x in compact subsets of Ω.*

3.2 Bounds for $G_{n_{\bar{t}}}$ and $H_{n_{\bar{t}}}$

For the extension of the proof of the auto-correction phenomenon to the multivariate setting we first require bounds for the functions $G_{n_{\bar{t}}}$ and $H_{n_{\bar{t}}}$. For this we need the following preliminary result:

Lemma 12. *Suppose that $t \in [d]$, $r_t \in \mathbb{N}^{|t|}$, $2K \geq k+1$ and that the function $h \in H^{2(k+K)+1}_{mix}(\Omega)$, satisfies the first k derivative conditions. Then*

$$\left|\triangle_{r_t,n_t}[\hat{h}_n]\right| \lesssim \bar{n}^{-2k-2} \prod_{j\in t} \bar{n}_j^{-2r_j} = \bar{n}^{-2k-2}\bar{n}_t^{-2r_t}.$$

Proof. It suffices to consider $t = (1,...,d)$ and use induction on d. Consider $d = 1$ and a univariate function h. Since h obeys the first k derivative conditions, we have

$$\hat{h}_n = \sum_{s=k}^{k+K-1} \mathcal{A}_s[h]\widehat{p_s}_n + \mathcal{O}\left(n^{-2(k+K)-1}\right),$$

Hence, using Lemma 11, we obtain

$$\triangle_{r,n}[\hat{h}_n] = \sum_{s=k}^{k+K-1} \mathcal{A}_s[h]\triangle_{r,n}[\widehat{p_s}_n] + \mathcal{O}\left(n^{-2(k+K+1)}\right) \lesssim \bar{n}^{-r-2k-2} + \bar{n}^{-2(k+K)-1}.$$

This gives the result for $d = 1$. Now assume that the result holds for all functions of at most $(d-1)$ variables. Then, if h is function of d variables and $t = (1,...,d)$, we have

$$\begin{aligned}\triangle^t_{r_t,n_t}[\hat{h}_n] &= \sum_{u\in[d]} \sum_{|s_u|_\infty=k}^{k+K-1} \triangle^t_{r_t,n_t}\left[\mathcal{A}_{s_u,n_{\bar{u}}}[h]\widehat{p_{s_u}}_{n_u}\right] + \mathcal{O}\left(n^{-2(k+K)-1}\right) \\ &= \sum_{u\in[d]} \sum_{|s_u|_\infty=k}^{k+K-1} \triangle^{\bar{u}}_{r_{\bar{u}},n_{\bar{u}}}\left[\mathcal{A}_{s_u,n_{\bar{u}}}[h]\right] \triangle^u_{r_u,n_u}\left[\widehat{p_{s_u}}_{n_u}\right] + \mathcal{O}\left(n^{-2(k+K)-1}\right).\end{aligned}$$

Using Lemma 11, we deduce that

$$\left|\triangle^u_{r_u,n_u}\left[\widehat{p_{s_u}}_{n_u}\right]\right| \lesssim \prod_{j\in u}\left|\triangle^j_{r_j,n_j}[\widehat{p_{s_j}}_{n_j}]\right| \lesssim \bar{n}_u^{-r_u-2s_u-2}. \tag{3.10}$$

Furthermore $\mathcal{A}_{s_u,n_{\bar{u}}}[h]$ is the modified Fourier coefficient of a function of $x_{\bar{u}}$ that satisfies the first k derivative conditions. Since $|\bar{u}| < d$, we may use the induction hypothesis and (3.10) to give

$$\left|\triangle^t_{r_t,n_t}[\hat{h}_n]\right| \lesssim \sum_{u\in[d]} \sum_{|s_u|_\infty=k}^{k+K-1} \bar{n}_{\bar{u}}^{-r_{\bar{u}}-2k-2} \bar{n}_u^{-r_u-2s_u-2} + \bar{n}^{-2(k+K)-1} \lesssim \bar{n}_t^{-r_t} \bar{n}^{-2k-2},$$

as required. □

With this in hand we may estimate the functions $G_{n_{\bar{t}}}$ and $H_{n_{\bar{t}}}$. We have:

Lemma 13. *Suppose that $f \in H^{3(k+1)}_{mix}(\Omega)$. Then the function $H_{n_{\bar{t}}}$ defined by (3.7) satisfies the bound*

$$|H_{n_{\bar{t}}}(x_t)| \lesssim N^{-3k-2} \bar{n}_{\bar{t}}^{-2},$$

uniformly for x_t in compact subsets of $(-1,1)^{|t|}$.

Proof. We first observe that, for $n \in \mathbb{N}^d$ such that $n_j \geq N$ whenever $j \in t$ and $n_j = 0, ..., N-1$ otherwise, $\hat{v}_n$ satisfies

$$\hat{v}_n = \sum_{|s_t|_\infty=0}^{k-1} \widehat{p_{s_t n_t}} \mathcal{E}_{s_t,n_{\bar{t}}}[f] + \sum_{\substack{v\in[d]^* \\ t\not\subseteq v}} \sum_{|s_v|_\infty=0}^{k-1} \mathcal{A}_{s_v,n_{\bar{v}}}[f] \widehat{p_{s_v n_v}}. \tag{3.11}$$

We now substitute the two terms of (3.11) into the definition of $H_{n_{\bar{t}}}$ given in (3.7) and consider them separately. For the first term we observe that

$$\triangle^t_{r_t,N}\left[\widehat{p_{s_t n_t}} \mathcal{E}_{s_t,n_{\bar{t}}}[f]\right] = \triangle^t_{r_t,N}\left[\widehat{p_{s_t n_t}}\right] \mathcal{E}_{s_t,n_{\bar{t}}}[f] = \mathcal{E}_{s_t,n_{\bar{t}}}[f] \prod_{j\in t} \triangle^j_{r_j,N}[\widehat{p_{s_j n_j}}].$$

Using Lemma 7 and (3.10) we obtain the bound

$$\left|\triangle^t_{r_t,N}\left[\widehat{p_{s_t n_t}} \mathcal{E}_{s_t,n_{\bar{t}}}[f]\right]\right| \lesssim N^{2(|s_t|_\infty-k)} \prod_{j\in t} N^{-2s_j-r_j-2} \bar{n}_{\bar{t}}^{-2} \lesssim N^{-2k-|r_t|-2|t|} \bar{n}_{\bar{t}}^{-2}.$$

Since $|r_t| \geq |r_t|_\infty = k$ and $|t| \geq 1$, we obtain the required bound for the first term.

Now consider the second term of (3.11) substituted into (3.7). For $v \in [d]^*$ with $t \not\subseteq v$ either (i) $v \cap t \neq \emptyset$ or (ii) $v \cap t = \emptyset$. Consider case (i) first. We have

$$\triangle^t_{r_t,N}\left[\mathcal{A}_{s_v,n_{\bar{v}}}[f] \widehat{p_{s_v n_v}}\right] = \triangle^{t\cap v}_{r_{t\cap v},N}[\widehat{p_{s_v n_v}}] \triangle^{t\backslash v}_{r_{t\backslash v},N}\left[\mathcal{A}_{s_v,n_{\bar{v}}}[f]\right].$$

Since $\mathcal{A}_{s_v,n_{\bar{v}}}[f] = \hat{h}_{n_{\bar{v}}}$, where h is a function of $x_{\bar{v}}$ that obeys the first k derivative conditions, we may apply Lemma 12 to give

$$\begin{aligned}\left|\triangle^t_{r_t,N}\left[\mathcal{A}_{s_v,n_{\bar{v}}}[f] \widehat{p_{s_v n_v}}\right]\right| &\lesssim \prod_{j\in t\cap v} N^{-2s_j-r_j-2} \prod_{j\in t\backslash v} N^{-2k-r_j-2} \bar{n}_{v\backslash t}^{-2s_{v\backslash t}-2} \bar{n}_{\overline{t\cup v}}^{-2k-2} \\ &\lesssim N^{-|r_t|-2|t\cap v|-2(k+1)(|t\backslash v|)} \bar{n}_{\bar{t}}^{-2} \lesssim N^{-3k-2} \bar{n}_{\bar{t}}^{-2}.\end{aligned}$$

Here the final inequality follows since, by assumption, $|t\cap v|, |t\backslash v| \geq 1$. Now consider case (ii). Since $t\cap v = \emptyset$, we have

$$\triangle^t_{r_t,N}\left[\mathcal{A}_{s_v,n_{\bar{v}}}[f] \widehat{p_{s_v n_v}}\right] = \triangle^t_{r_t,N}\left[\mathcal{A}_{s_v,n_{\bar{v}}}[f]\right] \widehat{p_{s_v n_v}}.$$

Using Lemma 12 and (3.10) we obtain

$$\left|\triangle^t_{r_t,N}\left[\mathcal{A}_{s_v,n_{\bar{v}}}[f] \widehat{p_{s_v n_v}}\right]\right| \lesssim \prod_{j\in t} N^{-r_j-2k-2} \prod_{j\notin v\cup t} \bar{n}_j^{-2k-2} \prod_{j\in v} \bar{n}_j^{-2s_j-2} \lesssim N^{-|r_t|_\infty-2k-2} \bar{n}_{\bar{t}}^{-2} \lesssim N^{-3k-2} \bar{n}_{\bar{t}}^{-2},$$

since $|r_t|_\infty = k$. This completes the proof. □

We now derive a bound for $G_{n_{\bar{t}}}$:

Lemma 14. *Suppose that $f \in H^{3(k+1)}_{mix}(\Omega)$. Then the function $G_{n_{\bar{t}}}$ defined by (3.8) satisfies the bound*

$$|G_{n_{\bar{t}}}(x_t)| \lesssim N^{-3k-2} \bar{n}_{\bar{t}}^{-2}, \tag{3.12}$$

uniformly for x_t in compact subsets of $(-1,1)^{|t|}$.

Proof. Since $x_t \in (-1,1)^{|t|}$ it suffices to bound

$$\sum_{\substack{u \in [t]^* \\ u \neq t}} \sum_{|r_u|_\infty = 0}^{k} \sum_{|n_{\bar{u}}|_\infty \geq N} \left| \triangle^t_{(r_u;k+1),(n_{\bar{u}};N)}[\hat{v}_m] \right|, \tag{3.13}$$

by the right hand side of (3.12). To do so we substitute the two terms of (3.11) into (3.13) and consider them separately. For the first term we have

$$\sum_{\substack{u \in [t]^* \\ u \neq t}} \sum_{|r_u|_\infty = 0}^{k} \sum_{|n_{\bar{u}}|_\infty \geq N} \sum_{|s_t|_\infty = 0}^{k-1} \left| \triangle^t_{(r_u;k+1),(n_{\bar{u}};N)} \left[\widehat{p_{s_t}}_{n_t} \mathcal{E}_{s_t,n_{\bar{t}}}[f] \right] \right|. \tag{3.14}$$

Since $u \subseteq t$, we observe that

$$\triangle^t_{(r_u;k+1),(n_{\bar{u}};N)} \left[\widehat{p_{s_t}}_{n_t} \mathcal{E}_{s_t,n_{\bar{t}}}[f] \right] = \mathcal{E}_{s_t,n_{\bar{t}}}[f] \prod_{j \in u} \triangle^j_{r_j,N} [\widehat{p_{s_j}}_{n_j}] \prod_{j \in t \setminus u} \triangle^j_{k+1,n_j} [\widehat{p_{s_j}}_{n_j}].$$

Using Lemmas 7 and (3.10) we deduce that

$$\left| \triangle^t_{(r_u;k+1),(n_{\bar{u}};N)} \left[\widehat{p_{s_t}}_{n_t} \mathcal{E}_{s_t,n_{\bar{t}}}[f] \right] \right| \lesssim N^{2(|s_t|_\infty - k)} \prod_{j \in u} N^{-2s_j - r_j - 2} \bar{n}_{\bar{t}}^{-2} \bar{n}_{\bar{u}}^{-2s_{\bar{u}} - k - 3}.$$

Substituting this into (3.14) we obtain

$$\begin{aligned}
&\sum_{\substack{u \in [t]^* \\ u \neq t}} \sum_{|r_u|_\infty = 0}^{k} \sum_{|n_{\bar{u}}|_\infty \geq N} \sum_{|s_t|_\infty = 0}^{k-1} \left| \triangle^t_{(r_u;k+1),(n_{\bar{u}};N)} \left[\widehat{p_{s_t}}_{n_t} \mathcal{E}_{s_t,n_{\bar{t}}}[f] \right] \right| \\
&\quad \lesssim \sum_{\substack{u \in [t]^* \\ u \neq t}} \sum_{|r_u|_\infty = 0}^{k} \sum_{|n_{\bar{u}}|_\infty \geq N} \sum_{|s_t|_\infty = 0}^{k-1} N^{2(|s_t|_\infty - k)} \prod_{j \in u} N^{-2s_j - r_j - 2} \bar{n}_{\bar{t}}^{-2} \bar{n}_{\bar{u}}^{-2s_{\bar{u}} - k - 3} \\
&\quad \lesssim \bar{n}_{\bar{t}}^{-2} \sum_{\substack{u \in [t]^* \\ u \neq t}} \sum_{|r_u|_\infty = 0}^{k} \sum_{|s_t|_\infty = 0}^{k-1} N^{2(|s_t|_\infty - k)} \prod_{j \in u} N^{-2s_j - r_j - 2} \prod_{j \in t \setminus u} N^{-2s_j - k - 2} \\
&\quad \lesssim \bar{n}_{\bar{t}}^{-2} \sum_{\substack{u \in [t]^* \\ u \neq t}} \sum_{|r_u|_\infty = 0}^{k} N^{-2k - |r_u| - 2|u| - (k+2)(|t| - |u|)} \lesssim N^{-3k-2} \bar{n}_{\bar{t}}^{-2}.
\end{aligned}$$

Here the last inequality follows by noting that $|t| -| u| \geq 1$.

We now consider the second term of (3.11) substituted into (3.13):

$$\sum_{\substack{v \in [d]^* \\ t \not\subseteq v}} \sum_{|s_v|_\infty = 0}^{k-1} \sum_{\substack{u \in [t]^* \\ u \neq t}} \sum_{|r_u|_\infty = 0}^{k} \sum_{|n_{\bar{u}}|_\infty \geq N} \left| \triangle^t_{(r_u;k+1),(n_{\bar{u}};N)} \left[\mathcal{A}_{s_v,n_{\bar{v}}}[f] \widehat{p_{s_v}}_{n_v} \right] \right|. \tag{3.15}$$

As in the proof of Lemma 13 we split this into two cases: either (i) $v \cap t \neq \emptyset$ or (ii) $v \cap t = \emptyset$. Suppose that we consider case (i). Since $v \cap t \neq \emptyset$ we have

$$\triangle^{t}_{(r_u;k+1),(n_{\bar{u}};N)} \left[\mathcal{A}_{s_v,n_{\bar{v}}}[f]\widehat{p_{s_v n_v}}\right] = \triangle^{t\cap v}_{(r_{u\cap v};k+1),(n_{\bar{u}\cap v};N)} \left[\widehat{p_{s_v n_v}}\right] \triangle^{t\cap \bar{v}}_{(r_{u\cap \bar{v}};k+1),(n_{\bar{u}\cap \bar{v}};N)} \left[\mathcal{A}_{s_v,n_{\bar{v}}}[f]\right].$$

We have

$$\left|\triangle^{t\cap v}_{(r_{u\cap v};k+1),(n_{\bar{u}\cap v};N)} \left[\widehat{p_{s_v n_v}}\right]\right| \lesssim \prod_{j\in u\cap v} N^{-2s_j-r_j-2} \bar{n}_{\bar{u}\cap v}^{-2s_{\bar{u}\cap v}-k-3} \bar{n}_{v\backslash t}^{-2s_{v\backslash t}-2}.$$

Furthermore

$$\left|\triangle^{t\cap \bar{v}}_{(r_{u\cap \bar{v}};k+1),(n_{\bar{u}\cap \bar{v}};N)} \left[\mathcal{A}_{s_v,n_{\bar{v}}}[f]\right]\right| \lesssim \prod_{j\in u\cap \bar{v}} N^{-2k-r_j-2} \bar{n}_{\bar{u}\cap \bar{v}}^{-3k-3} \bar{n}_{\overline{t\cup v}}^{-2k-2}.$$

Combining these two estimates we obtain

$$\left|\triangle^{t}_{(r_u;k+1),(n_{\bar{u}};N)} \left[\mathcal{A}_{s_v,n_{\bar{v}}}[f]\widehat{p_{s_v n_v}}\right]\right| \lesssim \prod_{j\in u\cap v} N^{-2s_j-r_j-2} \prod_{j\in u\cap \bar{v}} N^{-2k-r_j-2} \bar{n}_{\bar{u}\cap v}^{-2s_{\bar{u}\cap v}-k-3} \bar{n}_{\bar{u}\cap \bar{v}}^{-3k-3} \bar{n}_{\bar{t}}^{-2}.$$

Hence

$$\begin{aligned}
&\left| \sum_{|s_v|_\infty=0}^{k-1} \sum_{\substack{u\in[t]^* \\ u\neq t}} \sum_{|r_u|_\infty=0}^{k} \sum_{|n_{\bar{u}}|_\infty\geq N} \triangle^{t}_{(r_u;k+1),(n_{\bar{u}};N)} \left[\mathcal{A}_{s_v,n_{\bar{v}}}[f]\widehat{p_{s_v n_v}}\right] \right| \\
&\qquad \lesssim \sum_{|s_v|_\infty=0}^{k-1} \sum_{\substack{u\in[t]^* \\ u\neq t}} \sum_{|r_u|_\infty=0}^{k} \prod_{j\in u\cap v} N^{-2s_j-r_j-2} \prod_{j\in u\cap \bar{v}} N^{-2k-r_j-2} \prod_{j\in \bar{u}\cap v} N^{-2s_j-k-2} \prod_{j\in \bar{u}\cap \bar{v}} N^{-3k-2} \bar{n}_{\bar{t}}^{-2} \\
&\qquad \lesssim \sum_{\substack{u\in[t]^* \\ u\neq t}} N^{-2(k+1)|u\cap \bar{v}|} N^{-(k+2)|\bar{u}\cap v|} N^{-(3k+2)|\bar{u}\cap \bar{v}|} \bar{n}_{\bar{t}}^{-2}.
\end{aligned}$$

We claim that this term is $\lesssim N^{-3k-2}\bar{n}_{\bar{t}}^{-2}$. We have two possibilities: either $\bar{u}\cap\bar{v} \neq \emptyset$ or $\bar{u}\cap\bar{v} = \emptyset$. If $\bar{u}\cap\bar{v} \neq \emptyset$ then the result follows immediately. Suppose that $\bar{u}\cap\bar{v} = \emptyset$. In this case, it follows that $u\cap\bar{v} \neq \emptyset$ and $\bar{u}\cap v \neq \emptyset$. Hence we also obtain the result. This completes case (i).

Next consider case (ii). Since $v \cap t = \emptyset$ we have

$$\triangle^{t}_{(r_u;k+1),(n_{\bar{u}};N)} \left[\mathcal{A}_{s_v,n_{\bar{v}}}[f]\widehat{p_{s_v n_v}}\right] = \triangle^{t}_{(r_u;k+1),(n_{\bar{u}};N)} \left[\mathcal{A}_{s_v,n_{\bar{v}}}[f]\right] \widehat{p_{s_v n_v}}.$$

In the standard manner we obtain

$$\left|\triangle^{t}_{(r_u;k+1),(n_{\bar{u}};N)} \left[\mathcal{A}_{s_v,n_{\bar{v}}}[f]\widehat{p_{s_v n_v}}\right]\right| \lesssim \prod_{j\in u} N^{-2k-r_j-2} \bar{n}_{\bar{u}}^{-3k-3} \bar{n}_{\bar{v}\backslash t}^{-2k-2} \bar{n}_{v}^{-2s_v-2} \lesssim N^{-2(k+1)|u|-|r_u|_\infty} \bar{n}_{\bar{u}}^{-3k-3} \bar{n}_{\bar{t}}^{-2}.$$

Hence, in this case

$$\sum_{|s_v|_\infty=0}^{k-1} \sum_{\substack{u\in[t]^* \\ u\neq t}} \sum_{|r_u|_\infty=0}^{k} \sum_{|n_{\bar{u}}|_\infty\geq N} \left|\triangle^{t}_{(r_u;k+1),(n_{\bar{u}};N)} \left[\mathcal{A}_{s_v,n_{\bar{v}}}[f]\widehat{p_{s_v n_v}}\right]\right| \lesssim \prod_{j\in\bar{u}} N^{-3k-2}\bar{n}_{\bar{t}}^{-2} \lesssim N^{-3k-2}\bar{n}_{\bar{t}}^{-2},$$

where the final inequality follows since $|\bar{u}| \geq 1$. This completes the proof. □

3.3 Analysis of the auto-correction phenomenon and numerical results

We may now prove the key result of this section:

Theorem 8. *Suppose that $\mathcal{F}_{N,k}[f]$ is the multivariate Eckhoff approximation of $f \in H^{3(k+1)}_{mix}(\Omega)$ using the values $m(r) = N + r$, $r = 0, ..., k-1$. Then $f(x) - \mathcal{F}_{N,k}[f](x)$ is $\mathcal{O}\left(N^{-3k-2}\right)$ uniformly for x in compact subsets of Ω.*

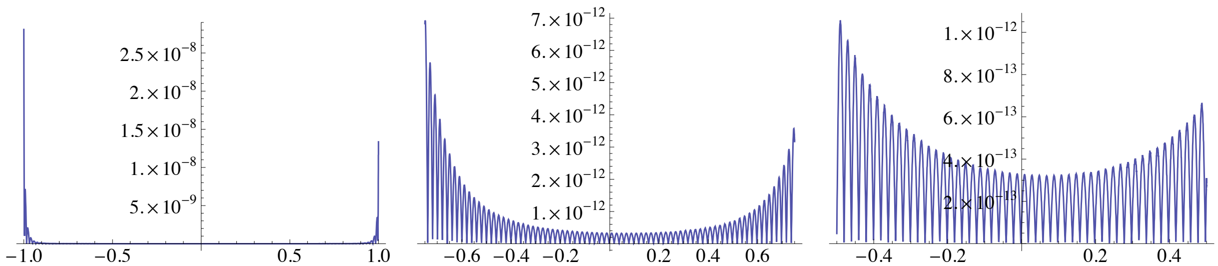


Figure 3: Graphs of $|f(x) - \mathcal{F}_{N,k}[f](x)|$ for $-1 \leq x \leq 1$ (left), $-0.75 \leq x \leq 0.75$ (middle) and $-0.5 \leq x \leq 0.5$ (right), where $N = 50$, $k = 2$ and $f(x) = x^2 \sin 5x + \cos 6x$.

(x_1, x_2)	$N = 10$	$N = 20$	$N = 30$	$N = 40$	$N = 50$
$(1, 1)$	4.958×10^{-8}	1.307×10^{-10}	3.799×10^{-12}	3.022×10^{-13}	4.202×10^{-14}
$(-1, -1)$	6.341×10^{-8}	1.372×10^{-10}	3.723×10^{-12}	2.861×10^{-13}	3.898×10^{-14}
$(\frac{1}{2}, \frac{2}{3})$	1.189×10^{-12}	4.293×10^{-15}	2.039×10^{-19}	4.673×10^{-19}	1.485×10^{-20}
$(0, 0)$	9.542×10^{-13}	1.885×10^{-16}	9.473×10^{-19}	2.037×10^{-20}	1.002×10^{-21}

Table 2: Pointwise error $|f(x_1, x_2) - \mathcal{F}_{N,k}[f](x_1, x_2)|$ for various values of (x_1, x_2) and N, where $k = 4$ and $f(x_1, x_2) = \left(e^{3x_1} + e^{-4x_1}\right)\left(\sin 5x_2 + \frac{1}{2}\right)$. Results to 4 significant figures.

Proof. Substituting the bounds derived in Lemmas 13 and 14 into the expansion (3.6) immediately yields the result. □

Note that for the auto-correction phenomenon we require $f \in H^{3(k+1)}_{\text{mix}}(\Omega)$, rather than just $f \in H^{3k+1}_{\text{mix}}(\Omega)$ or $f \in H^{3k+2}_{\text{mix}}(\Omega)$ for uniform convergence (see Theorem 6). This extra smoothness condition is also present for polynomial subtraction: here $H^{2k+3}_{\text{mix}}(\Omega)$-regularity is required to obtain an $\mathcal{O}\left(N^{-2k-2}\right)$ error away from the boundary, rather than just $H^{2k+2}_{\text{mix}}(\Omega)$-regularity for uniform convergence (see [1]). In [29] the author demonstrates that slightly different smoothness assumptions can be imposed depending on whether k is even or odd. For simplicity we do not make this distinction.

For values general values $m(r)$ it can be shown using similar methods that an auto-correction phenomenon is present provided the first $l \leq k$ values are chosen so that $m(r) = N + r$, $r = 0, ..., l-1$. In this case the convergence rate away from the boundary is $\mathcal{O}\left(N^{-2k-l-2}\right)$. In particular if $m(0) = N$, as is the case with the choices (1.15) and (1.16), then the convergence rate is $\mathcal{O}\left(N^{-2k-3}\right)$.

The auto-correction phenomenon is also exhibited by the error $f - \mathcal{F}_{N,k}[f]$ measured in the $L^2(\Omega')$ norm, where Ω' is some set compactly contained in Ω. This has been studied in the univariate, Fourier case in [29] and the extension to the multivariate, modified Fourier case is straightforward.

Though the analysis in this section was carried out for the approximation based on Cardinal polynomials, it is a simple exercise to extend it to the general subtraction bases described in Section 1. Hence we have established the existence of an auto-correction phenomenon for arbitrary dimension d and arbitrary subtraction basis $q^{[i]}_r$.

In Figure 3 we demonstrate the univariate auto-correction phenomenon. For the particular choice of function and parameters the error at the endpoints is roughly 10^{-8}, whereas in the interval $[-0.5, 0.5]$ this figure is 5×10^{-13}.

In Table 2 we present numerical results for the auto-correction phenomenon in the bivariate setting. Once more we observe that the error inside the domain is much smaller than on the boundary.

4 Hyperbolic cross index sets and Eckhoff's method

Thus far the approximation $\mathcal{F}_{N,k}[f]$ has been based on the full index set (2.2). This is arguably the most natural index set to consider. However, it turns out that the truncated expansion $\mathcal{F}_N[f]$ of a function f

based on this index set includes a large number of terms that have a insignificant contribution to the overall sum.

In view of this, an alternative approach to define I_N is to include only those terms in $\mathcal{F}_N[f]$ that are greater in absolute value than some tolerance ϵ. This is the idea of hyperbolic cross index sets, [3, 33]. In many applications, modified Fourier series included, such an approach leads to a greatly reduced index set of size $|I_N| = \mathcal{O}\left(N(\log N)^{d-1}\right)$. Moreover the approximation $\mathcal{F}_N[f]$ converges to f at a comparable rate. In this section we consider the use of such a set in Eckhoff's approximation.

4.1 A hyperbolic cross for modified Fourier coefficients

To develop a hyperbolic cross index set for modified Fourier coefficients we need an estimate for $|\hat{f}^{[i]}_n|$. This is provided by the bound $|\hat{f}^{[i]}_n| \lesssim (\bar{n}_1...\bar{n}_d)^{-2}$. If we set $\epsilon = N^{-2}$, then the term $\hat{f}^{[i]}_n$ is included in $\mathcal{F}_N[f]$ only if $\bar{n}_1...\bar{n}_d < N$. This leads to a hyperbolic cross index set:

$$I_N = \left\{ n \in \mathbb{N}^d : \bar{n}_1...\bar{n}_d < N \right\}. \tag{4.1}$$

This set, in conjunction with modified Fourier series, has been investigated in [1, 15]. It is lementary to show that $|I_N| = \mathcal{O}\left(N(\log N)^{d-1}\right)$; a vast reduction over the full index set (2.2) for which this value is $\mathcal{O}\left(N^d\right)$. Furthermore, we have the following result proved in [1]:

Theorem 9. *Suppose that $f \in H^{2k+2}_{mix}(-1,1)^2$ and that $\mathcal{F}^e_{N,k}[f]$ is the exact polynomial subtraction approximation to f based on the hyperbolic cross index set (4.1). Then*

$$\|f - \mathcal{F}^e_{N,k}[f]\|_\infty = \mathcal{O}\left(N^{-2k-1}(\log N)^{d-1}\right), \quad \|f - \mathcal{F}^e_{N,k}[f]\|_0 = \mathcal{O}\left(N^{-2k-\frac{3}{2}}(\log N)^{\frac{d-1}{2}}\right),$$
$$\|f - \mathcal{F}^e_{N,k}[f]\|_q = \mathcal{O}\left(N^{q-2k-\frac{3}{2}}\right), \quad q = 1, ..., 2k+1.$$

If, additionally, $f \in H^{2k+3}_{mix}(\Omega)$ then $f(x) - \mathcal{F}^e_{N,k}[f](x)$ is $\mathcal{O}\left(N^{-2k-2}(\log N)^{d-1}\right)$ uniformly in compact subsets of Ω.

In view of Theorem 5 we conclude that replacing the full index set (2.2) by (4.1) does not affect the convergence rate of the approximation aside from possibly a logarithmic factor (note that setting $k = 0$ in the above theorem establishes the convergence rate of $\mathcal{F}_N[f]$ to f). This, combined with the significant reduction in number of expansion terms, makes hyperbolic cross index sets greatly beneficial.

4.2 The hyperbolic cross version of Eckhoff method

Given $n \in \mathbb{N}^d$ we define $|n|_0 = \bar{n}_1...\bar{n}_d$ so that the hyperbolic cross index set (4.1) includes only those n with $|n|_0 < N$. To adapt the multivariate version of Eckhoff's method to use hyperbolic cross index sets we first replace the function g_k given in (2.13) by

$$g_k(x) = \sum_{i \in \{0,1\}^d} \sum_{t \in [d]} \sum_{|r_t|_\infty = 0}^{k-1} \sum_{|n_{\bar{t}}|_0 = 0}^{N-1} \bar{\mathcal{A}}^{[i]}_{r_t, n_{\bar{t}}}[f] p^{[i_t]}_{r_t}(x_t) \phi^{[i_{\bar{t}}]}_{n_{\bar{t}}}(x_{\bar{t}}). \tag{4.2}$$

The new function g_k satisfies the conditions

$$\widehat{g_k}^{[i]}_n = \hat{f}^{[i]}_n, \quad n \in M_k,$$

where M_k is the index set

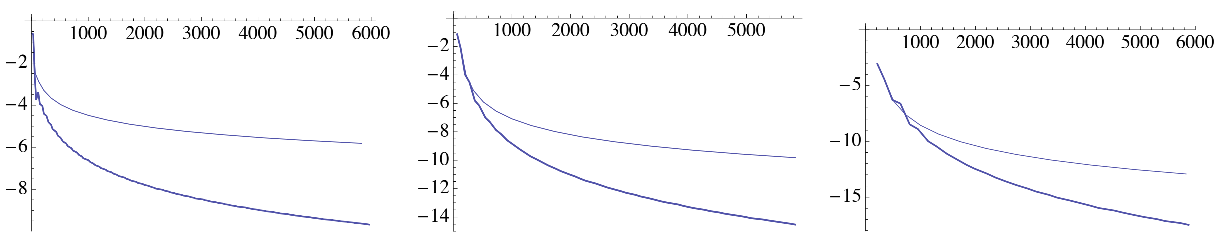
$$M_k = \bigcup_{t \in [d]} \left\{ n = (n_1, ..., n_d) \in \mathbb{N}^d : n_j = m(r_j),\ r_j = 0, ..., k-1 \text{ if } j \in t,\ |n_{\bar{t}}|_0 < N,\ \text{otherwise} \right\}.$$

Note that the only difference in the definitions of g_k and M_k is the replacement of $|n_{\bar{t}}|_\infty$ by $|n_{\bar{t}}|_0$. We now define the approximation $\mathcal{F}_{N,k}[f]$ in the standard manner:

$$\mathcal{F}_{N,k}[f] = \mathcal{F}_N[f - g_k] + g_k.$$

k	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}	10^{-14}	10^{-16}
1	121	1521	31329	—	—	—	—	—
	89	513	3053	17461	97241	—	—	—
2	49	121	561	1849	10201	60025	—	—
	49	105	297	841	2269	6269	17501	48485
3	81	121	169	441	1225	3969	13689	47089
	81	117	193	353	697	1333	2773	5585
4	81	121	169	289	529	1089	2401	5929
	81	121	165	257	397	593	1005	1649
5	121	121	169	289	361	625	1089	2025
	121	121	169	273	329	493	789	1145

Table 3: Number of terms in the full (top value) and hyperbolic cross (bottom value) index set versions of Eckhoff's approximation applied to the function $f(x,y) = e^{2x}(\cos 3y + \sin 2y)$ required to obtain an accuracy of $|\mathcal{F}_{N,k}[f](1,1) - f(1,1)| < 10^{-2j}$ for $j = 1, 2, ..., 8$ (the dash indicates where more than 100,000 terms are required to obtain the prescribed tolerance).



$k = 2$ $k = 4$ $k = 6$

Figure 4: Log error $\log_{10} |f(1,1,1) - \mathcal{F}_{N,k}[f](1,1,1)|$ against number of approximation terms for the full (thin line) and hyperbolic cross (thick line) versions of Eckhoff's method applied to (4.3).

Here $\mathcal{F}_N[h]$ is the truncated modified Fourier series of a function h based on the index set (4.1).

For $d = 2$ there is no difference between the functions given in (2.13) and (4.2). The only difference between the two resulting approximations arises from index set used in $\mathcal{F}_N[\cdot]$. However, for $d \geq 3$, the functions (2.13) and (4.2) are distinct.

It is readily seen that the operational cost of forming the hyperbolic cross version of Eckhoff's approximation is $\mathcal{O}\left(\max\{k^{d+1}, k^d N (\log N)^{d-1}\}\right)$. For $k \ll N$ this represents a significant reduction over the full index set version, where the corresponding figure is $\mathcal{O}\left(\max\{k^{d+1}, k^d N^d\}\right)$ (see Section 2.3).

In Table 3 we demonstrate the improvement offered by this approximation. As an example, we observe that for $k = 3$ to obtain an error of less than 10^{-16} requires around 50,000 terms for the full index set version of Eckhoff approximation, but only around 5,500 for its hyperbolic cross counterpart.

For $d = 3$ the improvement offered is more substantial. In Figure 4 we compare the error of the full and hyperbolic cross versions Eckhoff's method applied to the function

$$f(x_1, x_2, x_3) = \left(x_1^2 \cos 5x_1 + \tfrac{46}{125} \sin 5 - \tfrac{4}{25} \cos 5\right)(\cosh 2x_2 - \cosh 1 \sinh 1)\left(x_3 \sin 2x_3 + \tfrac{1}{2}\cos 2 - \tfrac{1}{4}\sin 2\right). \tag{4.3}$$

For $k = 4$, using roughly 5000 terms, the hyperbolic cross version offers an error roughly 10^5 times smaller than the full version. For $k = 6$, the hyperbolic cross approximation obtains machine epsilon using roughly 4000 terms. The full index set approximation will not reach this value until the number of terms exceeds 20,000.

Figure 4 also demonstrates the advantage offered by the combination of Eckhoff's method and the hyperbolic cross. To obtain an accuracy of 10^{-10} with $k = 6$ requires less than 2000 terms, whereas to do the same with the original ($k = 0$) modified Fourier approximation $\mathcal{F}_N[f]$ requires in excess of 10^{12} terms.

The analysis of this approximation is beyond the scope of this paper. Numerical results indicate that the uniform convergence rate remains $\mathcal{O}\left(N^{-2k-1}(\log N)^{d-1}\right)$. This is a subject of further investigation.

Unfortunately, numerical results also demonstrate that there is no auto-correction phenomenon for the hyperbolic cross version of Eckhoff's method. Away from the boundary the approximation converges at the same rate as exact polynomial subtraction. In other words, the error is $\mathcal{O}\left(N^{-2k-2}(\log N)^{d-1}\right)$.

Conclusions and future work

The aim of this paper was the convergence acceleration of multivariate modified Fourier expansions. To do so we have generalized Eckhoff's method to multivariate expansions, and proved that this approach yields not only faster uniform convergence but also an auto-correction phenomenon inside the domain. We have then considered two improvements. First, we have greatly increased numerical stability by using a particular subtraction basis. Second, we have demonstrated how a significant reduction in the number of approximation coefficients can be achieved by using a hyperbolic cross index set. The combination of Eckhoff's method and such index sets yields accurate approximations comprising only a relatively small number of terms.

There are a number of areas for future investigation. First, as mentioned in the Introduction, Eckhoff's method can be extended to non-Cartesian product domains, provided suitable orthogonal expansions are known. Due to their applications in spectral elements, the equilateral and right isosceles triangles are two important examples which warrant future consideration.

In [1, 2] the author considers the application of modified Fourier series to the spectral approximation of second order boundary value problems. The method possesses a number of advantages including mild conditioning of the discretization matrix and the availability of an optimal, diagonal preconditioner. However, the convergence rate is only cubic in the truncation parameter. Accelerating convergence is a subject of current investigation, including the incorporation of the methods developed in this paper into such approximations.

Finally, there are several open problems relating to this paper itself. First, as mentioned, the analysis of the hyperbolic cross version of Eckhoff's method has not yet been carried out. We intend to address this in a future paper. Second, we have demonstrated numerically the advantage offered by the subtraction basis (1.7). However, we are yet to explain theoretically why this is the case.

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