Mechanics Lecture Notes

1 Lecture 6: Newton's Laws: dynamics of particles

1.1 Introduction

This lecture deals with the application of Newton's second law to single particles and to simple systems (e.g. particles in contact or joined by strings). For particles or bodies in contact, Newton's third law plays a big part.

The worked examples are one-dimensional, and need to be understood before the vector versions are tackled in the Dynamics and Relativity course.

It hardly needs to be said that the application of Newton's laws is fundamental to all of theoretical physics, either directly (as for example in basic fluid dynamics) or indirectly as a model for more sophisticated theories (such as General Relativity).

1.2 Key concepts

- Newton's second law, in the form 'force equals mass times acceleration'.

- Newton's third law: 'to every action there is an equal and opposite reaction'.

1.3 Newton's Laws

Newton's Laws of motion were formulated in the mid-seventeenth century, in Latin and before some of the terms we are familiar with were current. There is therefore some ambiguity in their interpretation, which need not concern us. They are (in colloquial English)

(N1) A body persists in a state of rest or of uniform motion unless acted upon by an external force.

(N2) Force equals mass times acceleration ($F = ma$).

(N3) To every action there is an equal and opposite reaction.

For our purposes, N1 can be ignored; or, rather, regarded as a special case of N2.[1]

N2 is given here in its simplest form, applicable when the mass of the body is fixed; if the mass is variable (for example, a rocket or a particle in the theory of Special Relativity) the right hand side must be replaced by rate of change of momentum.[2]

It is not stated, though it is fundamental, that the law applies in inertial frames[3]: not, for example, in rotating or accelerating frames.

N3 seems natural and obvious: if you and I stand firm and I push you, then I feel you pushing me equally hard. It is less obvious when dealing with, say, magnets that act at a distance. But it seems that N3 holds in all situations and for all known forces.

1.4 Application of N2 to a single particle

Applied to a single particle of mass m with position vector $\mathbf{r}$ (a function of time t), we have

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} \tag{1}$$

where the force $\mathbf{F}$ could depend on t, on position $\mathbf{r}$ (a particle on a spring, for example), on velocity $\dot{\mathbf{r}}$ (air resistance, for example), on the mass m, or on any combination of these[4].

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[1]There is a different interpretation, which does not assume that Newton was so dim as to formulate a law which was just a special case of another law — and, further, to formulate it many years *after* he formulated the second law! However, one could just assume that he wished to emphasise this special case because it is so surprising and counter-intuitive, given that in every familiar case moving bodies do in fact slow down.

[2]A rocket, together with its fuel, can be regarded as a system of particles, the mass of each particle being fixed, and so can be dealt with using $\mathbf{F} = m\mathbf{a}$; for a particle in Special Relativity there is no way of avoiding the 'force equals rate of change of momentum' formulation.

[3]The question of how to define an inertial frame then arises. One possibility would be to say that it is the set of frames (i.e. coordinate axes) in which N1 holds; or one could define an inertial frame to be one that is fixed (or moves with constant velocity) with respect to distant stars/galaxies/etc. Let's not worry too much about this now.

[4]Or on a multitude of other quantities such as electric field strength.

If **F** is constant, or a function just of t, then equation (1) can be integrated twice (at least in principle), taking each component separately. Physical problems normally come with initial conditions (the initial position and the initial velocity, normally), or boundary conditions, which can be applied at each integration. Altogether 6 conditions are required for what is essentially three second-order differential equations.[5]

In one dimension, if F is a function of x only (not t) we can integrate (1) by means of the chain rule using the change of variable discussed in lecture 5:

$$F = m\frac{dv}{dt} = m\frac{dx}{dt}\frac{dv}{dx} = mv\frac{dv}{dx} = \tfrac{1}{2}m\frac{d(v^2)}{dx}$$

so

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2 = \int_0^x F(s)ds$$

where u is the speed of the particle at $x = 0$.[6] Having done the integral on the right hand side, for a given function F, we can integrate again to find $x(t)$ by setting $v = dx/dt$.

1.5 Worked examples

The general plan for solving problems always starts with writing down the equations of motion[7], here N2. For a system of particles or body, we can apply N2 to the whole, or to each particle individually remembering to include the forces between particles.

(i) Two bodies of masses m and M are joined together and move in contact, subject to a single force **F**. *Find the acceleration of the bodies.*

A small variation of this problem would be a model of a car towing a caravan.
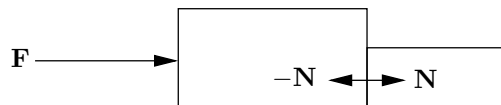
First we apply N2 to the whole system

$$F = (M + m)a$$

where F is the magnitude of **F** and a is the acceleration of the two bodies. Thus $a = F/(M + m)$.

We can obtain the same result by applying N2 to the bodies individually, using N3. We have to include all the forces, including internal forces, on the left hand side of N2: each body experiences a force of reaction from the other body, which is internal to the system, and N3 says that these force are equal and opposite. Thus

$$(F - N) = Ma \quad \text{and} \quad N = ma$$

Adding these gives the previous result for a.

(ii) Two particles of masses m and M are attached to a light inextensible string that passes over a light frictionless fixed pulley. Find the accelerations of each particle and the force on the axle of the pulley.

As always, we start with a good picture and then write down the equations of motion. The equation of motion in this case is N2, again taking into account all the forces on each particle.

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[5]Rather surprisingly, until you learn about Green's functions in Part IB Methods, the solution corresponding (for example) to a particle initially at rest at the origin can be written as a *single* integral:

$$x(t) = \int_0^t (t - \tau)F_x(\tau)d\tau$$

where F_x is the x-component of the force, and similar formulae hold for y and z. You can fairly easily verify that this satisfies the differential equation, and the initial conditions.

[6]We will see in the next lecture that this equation relates the change of kinetic energy of a particle to the work done by the force on the particle.

[7]Or first integrals such as energy and momentum conservation — see lectures 7 and 8.
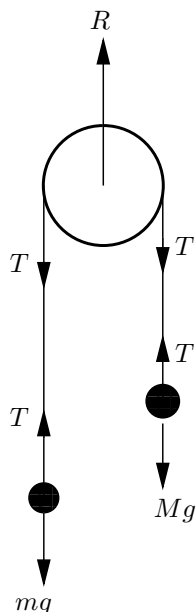
In the diagram, T is the tension in the string, which is the same on both sides of the pulley, because the string is light and the pulley is light and moves without friction.[8] R is the reaction of the axle on the pulley (i.e. the force that stops the pulley falling).

Note that the accelerations of the two particles are equal and opposite, because the string is inextensible.

The equations of motion are as follows.

$$\text{For the mass } m \quad T - mg = ma; \tag{2}$$

$$\text{For the mass } M \quad T - Mg = -Ma; \tag{3}$$

$$\text{For the pulley} \quad 2T - R = 0. \tag{4}$$

Eliminating T, we find[9]

$$a = \frac{M - m}{m + M} g.$$

Eliminating a gives

$$T = \frac{2mM}{m + M} g.$$

Thus

$$R = 2T = \frac{4mM}{m + M} g.$$

[8]If, for example, the pulley had friction, then there would be an extra equation of motion relating the difference between the tensions on the two sides to the frictional force.

[9]This result makes sense. If we write it in the form $(M - m)g = (M + m)a$ and compare with N2, we see that the force is provided by gravity acting on what could be thought of as the gravitational mass of the system, namely $M - m$, whereas $M + m$ is the inertial mass of the system (it all needs to be accelerated).