Critical Phenomena, Phase Transitions and Statistical Field Theory

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1 Introduction

A general problem in physics is to deduce the macroscopic properties of a quantum system from a microscopic description. Such systems can only be described mathematically on a scale much smaller than the scales which are probed experimentally or on which the system naturally interacts with its environment. An obvious reason is that systems consist of particles whose individual behaviour is known and also whose interactions with neighbouring particles is known. On the other hand experimental probes interact only with systems containing large numbers of particles and the apparatus only responds to their large scale average behaviour. Statistical mechanics was developed expressly to deal with this problem but, of course, only provides a framework in which detailed methods of calculation and analysis can be evolved.
These notes are concerned with the physics of phase transitions: the phenomenon that in particular environments, quantified by particular values of external parameters such as temperature, magnetic field etc., many systems exhibit singularities in the thermodynamic variables which best describe the macroscopic state of the system. For example:

(i) the boiling of a liquid. There is a discontinuity in the entropy,

$$\Delta S = \frac{\Delta Q}{T_c}$$

where ΔQ is the latent heat. This is a first order transition;

(ii) the transition from paramagnetic to ferromagnetic behaviour of iron at the Curie temperature. Near the transition the system exhibits large-range cooperative behaviour on a scale much larger than the inter-atomic distance. This is an example of a second order, or continuous, transition. Scattering of radiation by systems at or near such a transition is anomalously large and is called **critical opalescence**. This is because the fluctuations in the atomic positions are correlated on a scale large compared with the spacing between neighbouring atoms, and so the radiation scattered by each atom is in phase and interferes constructively.

Most of the course will be concerned with the analysis of continuous transitions but from time to time the nature of first order transitions will be elucidated. Continuous transitions come under the heading of **critical phenomena**.
Broadly, the discussion will centre on the following area or observations:

(i) the mathematical relationship between the sets of variables which describe the physics of the system on different scales. Each set of variables encodes the properties of the system most naturally on the associated scale. If we know how to relate different sets then we can deduce the large scale properties from the microscopic description. Such mathematical relationships are called, loosely, **renormalisation group** equations, and , even more loosely, the relationship of the physics on one scale with that on another is described by the **renormalisation group**. In fact there is no such thing as **the** renormalisation group, but it is really a shorthand for the set of ideas which implement the ideas stated above and is best understood in the application of these ideas to particular systems. If the description of the system is in terms of a field theory then the renormalisation group approach includes the idea of the **renormalisation** of (quantum) field theories and the construction of **effective** field theories;

(ii) the concept of **universality**. This is the phenomenon that many systems whose microscopic descriptions differ widely nevertheless exhibit the **same** critical behaviour. That is, that near a continuous phase transition the descriptions of their macroscopic properties coincide in essential details. This phenomenon is related to the existence of fixed points of the renormalisation group equations.

(iii) the phenomenon of **scaling**. The relationship between observables and parameters near a phase transition is best described by power-law behaviour. Dimensional analysis gives results of this kind but often the dimensions of the variables are **anomalous**. That is, they are different from the obvious or "engineering" dimensions. This phenomenon occurs particularly in low dimensions and certainly for $d < 4$. For example in a ferromagnet at the Curie temperature T_c we find

$$M \sim h^{\frac{1}{\delta}},$$

where M is the magnetization and h is the external magnetic field. Then the susceptibility, $\chi = \frac{\partial M}{\partial h}$, behaves like

$$\chi \sim h^{\frac{1}{\delta}-1}.$$

Since $\delta > 1$, χ **diverges** as $h \to 0$. The naive prediction for δ is 3. δ is an example of a **critical exponent** which must be calculable in a successful theory. The coefficient of proportionality is the above relations is **not** universal and is not easily calculated. However, in two dimensions the **conformal symmetry** of the theory at the transition point does allow many of these parameters to be calculated as well. We shall not pursue this topic in this course.

Note. The general ideas of relating physical phenomena on widely differing scales by renormalisation group methods is widely applicable in many fields. For example:

(i) general diffusion models such as diffusion on fractal structures and the large scale effects of diffusive transport processes in fluid flow;

(ii) turbulence in fluids. The fluid velocity field, $\mathbf{u}(\mathbf{x}, t)$, of fully developed turbulence has energy density, as a function of wavenumber $\mathbf{k}$,

$$k^{-2}E(\mathbf{k}) \sim \frac{1}{V}\int d\mathbf{x}d\mathbf{y}\mathbf{u}(\mathbf{x},t)\cdot\mathbf{u}(\mathbf{y},t)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$$

with $E(\mathbf{k}) \sim k^{-\frac{5}{3}}$ for large k.

This result is derived by "naive" dimensional analysis (Kolmogorov). However, a full solution of the Navier-Stokes equation can correct the exponent:

$$E(\mathbf{k}) \sim k^{-\frac{5}{3}-\eta},$$

where η is the anomalous term.

2 The Phenomenology of Phase Transitions

Statistical systems in equilibrium are described by macroscopic, thermodynamic, observables which are functions of relevant external parameters, e.g., temperature, T, pressure, P, magnetic field, h. These parameters are **external fields** (they may be $\mathbf{x}, t$ dependent) which influence the system and which are under the control of the experimenter.
the observables **conjugate** to these fields are:

entropy	S	conjugate to temperature	T
volume	V	conjugate to pressure	P
magnetization	M	conjugate to mag. field	h

Of course V and P may be swapped round: either can be viewed as an external field.
More common thermodynamic observables are the specific heats at constant pressure and volume, respectively C_P and C_V; the bulk compressibility, K; and the energy density, ϵ.
Equilibrium for given fixed external fields is described by the minimum of the relevant **thermodynamic potential**:

E	for fixed	S,T
F	for fixed	T,V
Φ	for fixed	T,P
W	for fixed	S,P

A **phase transition** occurs at those values of the external fields for which one or more observables are singular. This singularity may be a discontinuity or a divergence. The transition is classified by the nature of the typical singularity that occur. Different **phases** of a system are separated by phase transitions.
Broadly speaking phase transitions fall into two classes:

(1) **1st order**

  (a) Singularities are discontinuities.

  (b) Latent heat may be non-zero.

  (c) Bodies in two or more different phases may be in equilibrium at the transition point. E.g.,

    (i) the domain structure of a ferromagnet;

    (ii) liquid-solid mixture in a binary alloy: the liquid is richer in one component than is the solid;

  (d) the symmetries of the phases on either side of a transition are unrelated.

(2) **2nd and higher order: continuous transitions**

  (a) Singularities are divergences. An observable itself may be continuous or smooth at the transition point but a sufficiently high derivative with respect to an external field is divergent. C.f., in a ferromagnet at $T = T_c$

$$M \sim h^{\frac{1}{\delta}}, \qquad \chi = \left(\frac{\partial M}{\partial h}\right)_T \sim h^{\frac{1}{\delta}-1}.$$

  (b) There are no discontinuities in quantities which remain finite through the transition and hence the latent heat is zero.

  (c) There can be no mixture of phases at the transition point.

  (d) The symmetry of one phase, usually the low-T one, is a subgroup of the symmetry of the other.

An **order parameter**, Ψ, distinguishes different phases in each of which it takes distinctly different values. Loosely a useful parameter is a collective or long-range coordinate on which the singular variables at the phase transition depend.
In a ferromagnet the spontaneous magnetization at zero field, $\mathbf{M}(T)$, is such an order parameter, i.e.,

$$\mathbf{M}(T) = \lim_{h \to 0+} \mathbf{M}(\mathbf{h}, T)$$

then $|\mathbf{M}(T)| = 0$ for $T \geq T_c$, and $|\mathbf{M}(T)| > 0$ for $T < T_c$.
Note: $\Im(T - T_c)^{\frac{1}{2}}$ will not do.

Ψ is not necessarily a scalar, but in general it is a tensor and is a field of the **effective field theory** which describes the interactions of the system on macroscopic scales (i.e., scales much greater than the lattice spacing). The idea of such effective field theories is common to many areas of physics and is a natural product of renormalisation group strategies.

Examples

(1) **The Ising model in 3 dimensions.**

The theory is defined on the sites of a 3D cubic lattice and the variable on the site labelled by $\mathbf{n}$ is $\sigma_{\mathbf{n}} \in Z_2$. There is a nearest neighbour interaction and an interaction with an external magnetic field, h. Then the energy is written as

$$H = J \sum_{\mathbf{n},\mu} \sigma_{\mathbf{n}} \sigma_{\mathbf{n}+\mu} \; - \; h \sum_{\mathbf{n}} \sigma_{\mathbf{n}},$$

where μ is the lattice vector from a site to its nearest neighbour in the positive direction, i.e.,

$$\mu \in (1,0,0) \; (0,1,0) \; (0,0,1).$$

The order parameter is the magnetization,

$$M = \frac{1}{V} \sum_{\mathbf{n}} \sigma_{\mathbf{n}},$$

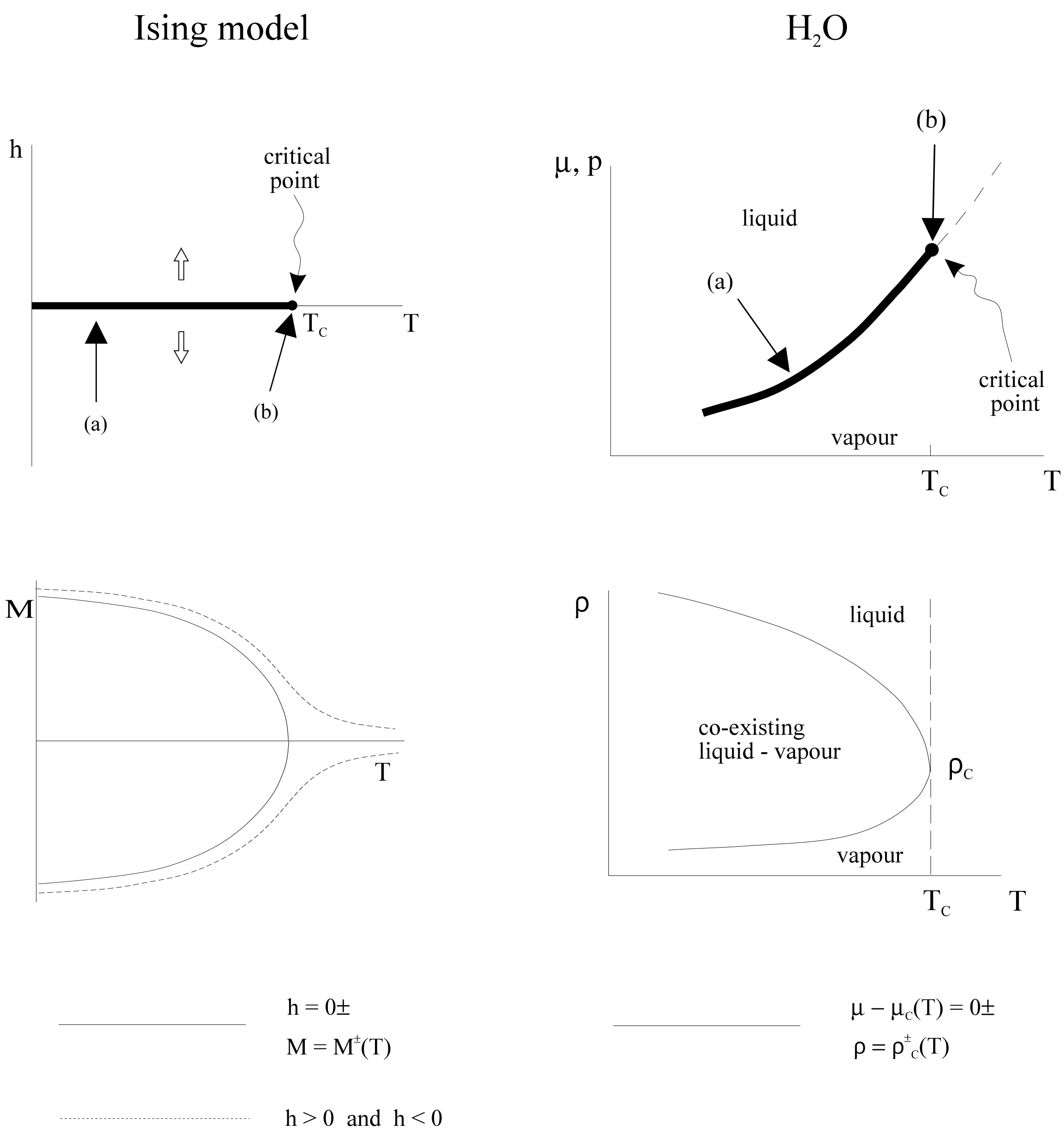
where V is the number of sites in the lattice. Note, that whilst the $\sigma_{\mathbf{n}}$ are discrete, M is a continuous variable, $-1 \leq M \leq 1$ in the limit $V \to \infty$.

(2) $\mathbf{H_2O}$

Look at the two phases of liquid and vapour. The order parameter is the density, ρ, which is large for the liquid phase relative to its value for the vapour phase.

The properties of both systems, their similarities and differences are best exhibited by showing the various phase diagrams.

PHASE DIAGRAMS



μ is the chemical potential, and $\mu = \mu_c(T)$ is the line of first order transitions in the (μ, T) plot for $H_2 0$. It corresponds to the line $h = 0$ in the (h, T) plot for the Ising model.

(i) Approach along (a) gives a 1st order transition whilst approach along (b) through the **critical point** gives a 2nd order transition.

(ii) The order parameters are:

magnetization M	*density* ρ

The conjugate fields are:

magnetic field h	*chemical potential,* μ *or pressure,* P

(iii) The behaviour near $T = T_c$ ($t = ((T - T_c)/T_c)$

(a)

$\underline{t \to 0-,\ h = 0\pm}$	$\underline{t \to 0-,\ \mu - \mu_c(T) = 0\pm}$
$M(T) \sim \|t\|^{\beta}$	$\rho(T) - \rho_c \sim \|t\|^{\beta_+}$ $\rho(T) - \rho_c \sim \|t\|^{\beta_-}$
Clear symmetry in curve	No obvious symmetry but experimentally $\beta_+ = \beta_-$

(b)

$\underline{t \to 0+,\ h = 0}$	$\underline{t \to 0+,\ \mu = \mu_c(T)}$
Susceptibility	
$\chi = \left(\frac{\partial M}{\partial h}\right)_T$	$\chi = \frac{K(T)}{K_0(T)}$ $K(T) = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial P}\right)_T$ $K_0(T)$ is for ideal gas

Then

$$\chi(t) \sim |t|^{-\gamma}$$

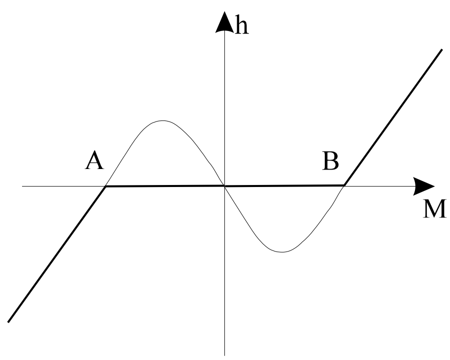
Note that for the Ising model with $t \to 0-,\ h \to 0+$ we find $\chi(T) \sim |t|^{-\gamma\prime}$ with $\gamma\prime = \gamma$. It should be remarked, howver, that $\gamma\prime$ is not defined for all models.

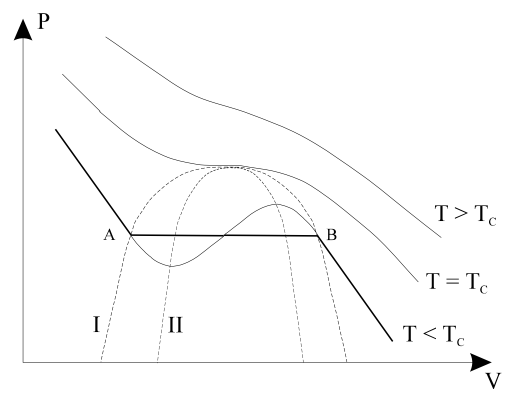
(c) $\underline{t = 0,\ h \to 0+}$ $\underline{t = 0,\ \mu - \mu_c \to 0+}$

$$M \sim h^{\frac{1}{\delta}} \qquad \rho - \rho_c \sim (\mu - \mu_c)^{\frac{1}{\delta}}$$

β, γ, δ are examples of critical exponents.

(iv) Coexisting phases





(a) States between the curves I and II are physical but metastable. They do not violate thermodynamic inequalities. In the PV plot this is equivalent to

$$\left(\frac{\partial P}{\partial V}\right)_T < 0$$

which means that the compressibility is positive. This inequality is derived from entropy being a maximum in equilibrium. Howver, these states are unstable against changing to the mixed system, e.g., domains in the Ising model (or ferromagnet), and liquid-gas mixture for water. The continuous curves shown are the (h, M) and (P, V) curves for a **pure phase**. E.g., the Van-der-Waals equation of state:

$$\left(P + \frac{a}{V}\right)(V - b) = cT$$

(b) The Maxwell construction gives the true equilibrium curve taking into account the formation of the mixed system. The mixture is of the two phases A and B. The rule for finding the interpolation is illustrated in the case of H_2O:

$$P_A = P_B, \quad \mu_A = \mu_B \Rightarrow \mu_A - \mu_B = \int_A^B v\, dP = 0,$$

where $v = V/N$, and N is the number of particles. This is the **equal areas rule** of Maxwell.

What happens is that boundaries between phases or domains form. In each domain the magnetisation is oriented differently and so the bulk average magnetisation can be any value in the range $-|M|$ to $|M|$, where $|M|$ is the magnetization of a pure domain. The walls do increase the energy of the system by $\Delta\epsilon$ and there is also an increase in the entropy, ΔS. since there are many ways of realising the mixed state. However, the resultant change in the free-energy, $\Delta F = \Delta\epsilon - T\Delta S$, depends on the surface area of the walls and is negligible in the limit of very large volume. Of course, the actual way in which domains, or bubbles, form and move is very important (e.g. in the early universe, cosmic string formation etc.) but needs more analysis than the embodied in the Maxwell construction.

2.1 The general structure of phase diagrams

A thermodynamic space,Y, is some region in an s-dimensional real vector space spanned by field variables $y_1, \ldots, y_s$ (e.g., $P, V, T, \mu, \ldots$). In Y there will be points of two, three, etc. phase coexistence (c.f. A and B in H_2O plot above), together with critical points, multicritical points, critical end points, etc.. Q is the totality of such points. The **phase diagram** is the pair (Y, Q).
Points of a given type lie in a smooth manifold, M, say. The **codimension**, κ, of these points is defined by

$$\kappa = \dim(Y) - \dim(M).$$

E.g., two-phase points have $\kappa = 1$; critical points (points that terminate two phase lines) have $\kappa = 2$.
There do not exist any known rules for constructing geometrically all acceptable phase diagrams, (Y, Q): we cannot construct all the phase diagrams which could occur naturally.

2.1.1 Structures in a phase diagram: a description of Q

I assume that there are C components in the system, and hence there are $(c+1)$ external fields: $\mu_1, \ldots, \mu_c, T$. Then $\dim(Y) = (c+1)$.

(a) Manifolds of multiphase coexistence.

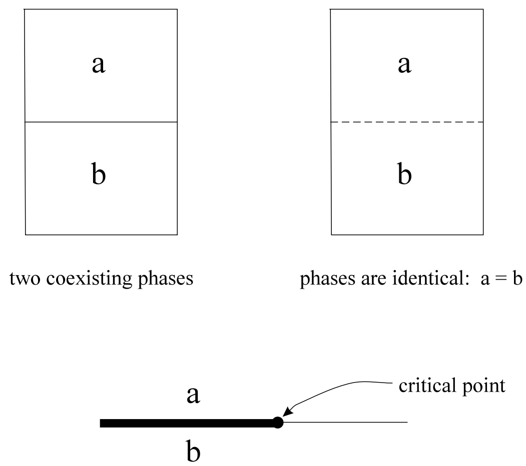
The **Gibbs phase rule** states that the coesistence of m phases in a system with C components has

$$f = c + 2 - m$$

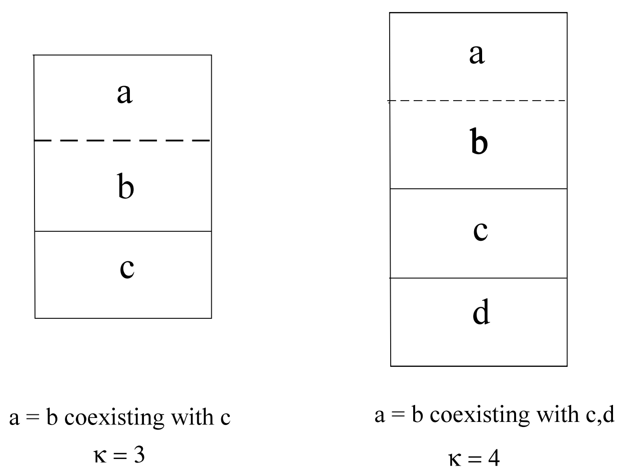
where f is the dimension of the manifold of m-phase coexistence.

proof: $\dim(Y) = (c+1)$ and hence the manifold has codimension $\kappa = (c+1-f)$. But $\kappa = (m-1)$ since κ external fields must be tuned to bring about m-phase coexistence.

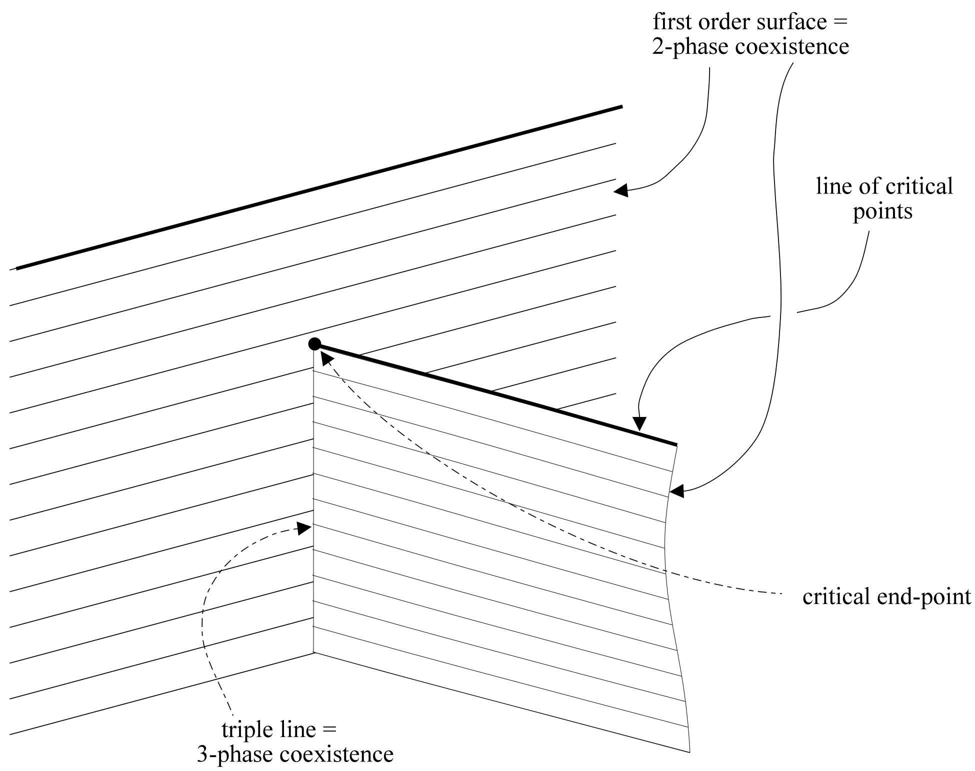
(b) An ordinary **critical point** has $\kappa = 2$ and occurs when two coexisting phases become identical:



(c) A **critical end-point** occurs with codimension κ when two coexisting phases become identical in the presence of $(\kappa - 2)$ other phases:

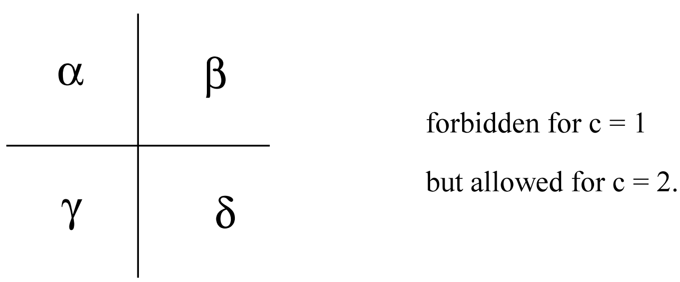


An example of all these structures in a three dimensional phase diagram is shown below.

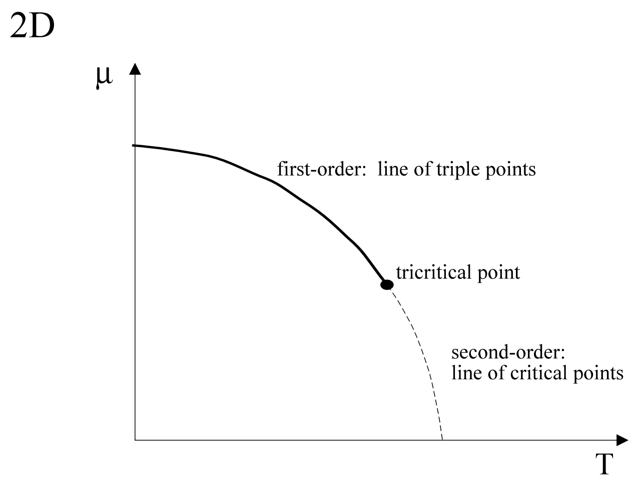


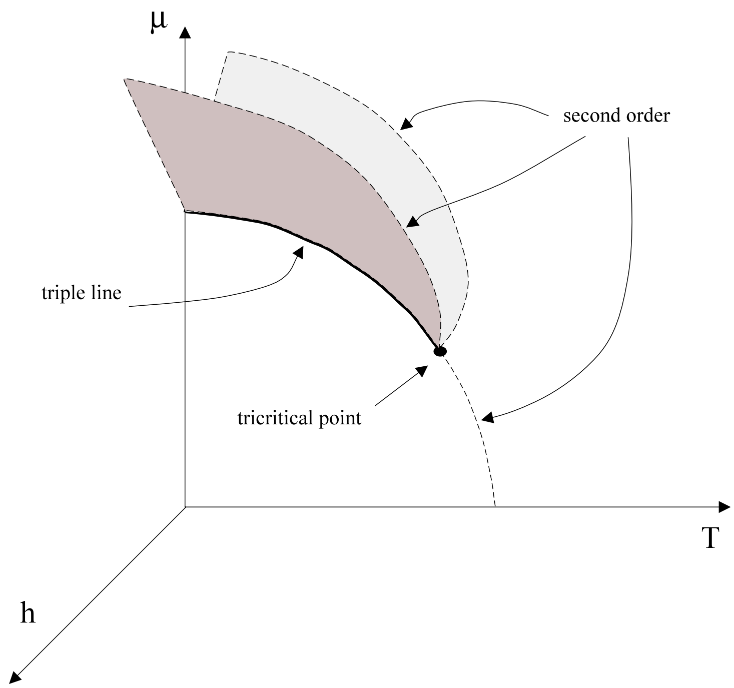
thus the critical end-point terminates a line of critical points and also terminates a line of triple points.

Note that a phase diagram can often only be properly understood if plotted in the space of all relevant parameters. E.g., the Gibbs rule might seem to be violated since too many phases are coexisting at one point. However, if the space is enlarged in dimension this will be seen as a special case which only occurs for a particular cross-section of the enlarged space;



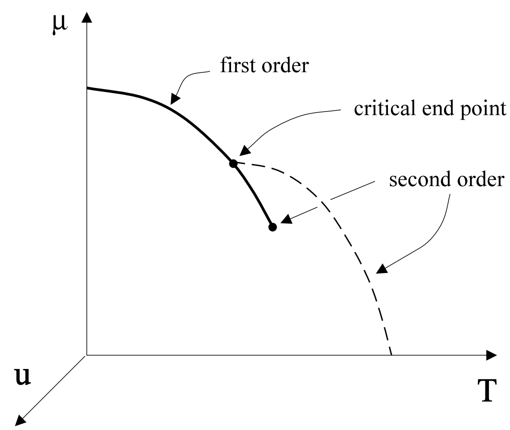
(d) A **tricritical point** has $\kappa = 4$. Its nature is most easily seen first in three dimensions. This is already a special case since we can only be sure it will appear in four dimensions. We suppose we have taken the appropriate cross-section of the 4D space. this often occurs naturally since some of the parameters are naturally set to the special values necessary to show up the tricritical point: e.g., by symmetry considerations.





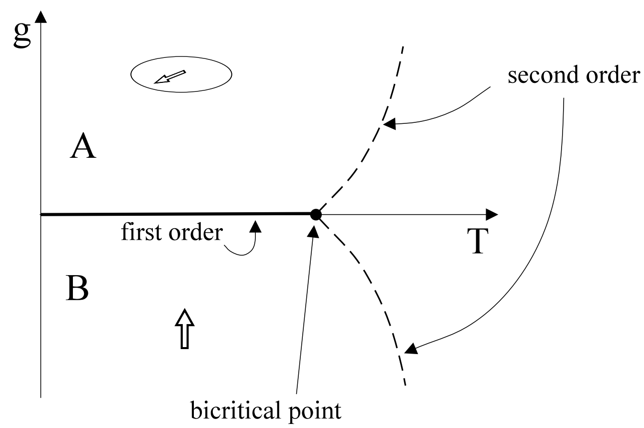
The hatched surfaces are 1st-order surfaces: surfaces of two phase coexistence. Thus the 1st order line in 2D is really a line of triple points (three phase coexistence) in higher dimensions.

A less special 2D cross-section of the same model will be:



Here we have set $h = 0$ (see 3D plot) but have changed the value of a fourth parameter u.

(e) A **bicritical point** is a critical point at which two critical lines terminate. A typical phase diagram is: A model which has a phase diagram like this



is given by the Hamiltonian

$$H = -J \sum_{<ij>} \mathbf{s}_i \cdot \mathbf{s}_j + \frac{1}{2} g \sum_i \left((\mathbf{s}_i^z)^2 - \frac{1}{2}((\mathbf{s}_i^x)^2 + (\mathbf{s}_i^y)^2) \right)$$

$< ij >$ means nearest neighbour pairs, i.e., it labels the links on the lattice. $\mathbf{s}_i$ is a vector at the i-th site with $|\mathbf{s}_i| = 1$.

For **low** T thermal fluctuations can be ignored and it is safe to just find the configuration (i.e., set of values) of spins $\{\mathbf{s}_i\}$ which minimizes H. Since $J > 0$ the first term causes the spins to align with each other to give ferromagnetic ordering.

$g < 0$ Ordering is prefered along z-axis. This is phase B.

$g > 0$	Ordering is prefered in the xy-plane $\perp$ z-axis. This is phase A.
$g = 0$	Neither A nor B prefered: two-phase coexistence.

For high T ordering is absent: it is destroyed by thermal fluctuations. For $g \neq 0$ as T increases we must have a second order transition from ferromagnetic to paramagnetic phases. The surfaces to the low-T side of the lines of critical points are first-order surfaces. This can be seen by imposing a magnetic field $\mathbf{h}$ on the system with components both $\|$ and $\perp$ to the z-axis and adding the magnitude, h, of $\mathbf{h}$ (including sign) as a third orthogonal axis to generate a 3D phase plot of which our 2D plot is the $h = 0$ cross-section. Then as h changes sign the magnetisation, $\mathbf{M}$, changes discontinuously at $h = 0$. This occurs as the surfaces in our 2D phase diagram on the low-T side of the critical lines are punctured, and hence they are in fact first-order surfaces. Of course, because the order parameter is a vector the possible patterns of behaviour and the competition between the effects of the terms governed by the coupling, g, and by h is, in general, complicated. An r-critical point is where r critical lines terminate.

(f) A **critical point** of n-th order has $\kappa = n + 2$ and is complicated.

3 Landau-Ginsberg theory and mean field theory

The Landau-Ginsberg theory is a phenomenological theory describing all types of phase transition which can be derived from the more complete theory. It is a classical approach which breaks down in its simple form for low dimensions. However, it can be used for developing the structure of phase transitions and phase diagrams. Landau theory gives the correct prediction for critical indices in dimensions $d > d_c$, where d_c is a critical dimension which is different for different kinds of critical point. E.g., for an ordinary critical point $d_c = 4$, and for a tricritical point $d_c = 3$.

Mean field theory is a method of analysing systems in which the site variable (spin etc.) is assumed to interact with the **mean field** of the neighbours with which it interacts. In a spin model each of the neighbouring spins has the value of the mean magnetisation per spin, $\mathbf{m}$. The problem now reduces to that of a single spin in an external field and can easily be solved. By demanding that the mean value of the spin in question is $\mathbf{m}$ the solution yields a non-linear equation expressing this assumption of self-consistency and from which $\mathbf{m}$ can be calculated as a function of T. The approximation of the method is that it ignores fluctuations in the spins about their mean. It will turn out that Landau theory suffers from the same deficiency as we shall demonstrate. Mean field theory and Landau theory give the same, classical, predictions for critical exponents.

We shall consider the following example.

Let the order parameter be M, and expand the free energy, A, as

$$A = A_0 + \frac{1}{2}A_2M^2 + \frac{1}{4}A_4M^4 + \frac{1}{6}A_6M^6 + \dots,$$

with $A_2 \propto (T - T_c)$.
There are no terms with odd powers of M. These can be present in principle but can be consistently excluded by symmetry considerations if the microscopic Hamiltonian is invariant under $M \to -M$. If odd powers of M are present then generally the theory has only first order transitions, although higher order transitions cannot be totally excluded. T_c is a complicated function of the couplings in the original, microscopic, Hamiltonian as are the other coefficients, A_{2n}. It is an assumption that A_2 is analytic in T: an assumption that can only be plausibly justified under certain circumstances. This assumption as well as others is wrong if the dimension is low enough.

Equilibrium is given by minimising A:

$$\frac{dA}{dM} = 0.$$

The observable value of the order parameter, $M(T)$, is the solution of this equation. Then

$$|M(T)| = \left|\frac{A_2}{A_4}\right|^{\frac{1}{2}} \left(1 + \frac{1}{2}\frac{A_6A_2}{A_4^2} + \dots\right).$$

Thus as $T \to T_c$

$$|M(T)| \sim |T - T_c|^{\frac{1}{2}} \Rightarrow \beta = \frac{1}{2}$$

We can rewrite the expression for $M(T)$ as

$$M(T) = \left|\frac{A_2}{A_4}\right|^{\frac{1}{2}} m(x), \qquad \text{where} \qquad x = \frac{A_6A_2}{A_4^2}.$$

If we assign to M a dimension of (-1) and dimension d to A then the coefficients A_{2n} have dimension $(d + 2n)$. Thus in this artifical dimensional analysis we find that x is dimensionless and that the critical exponent is predicted on dimensional grounds. The analysis above is only possible if $A_4 > 0$, in which case A_6 only occurs in the correction terms. If $A_4 < 0$ then we require $A_6 > 0$ to stabilize the calculation and the results are different (see below). In the former case since only A_2 and A_4 are important the critical exponent follows uniquely from the dimensional argument.

$\mathbf{A_4 > 0}$

If a field h is applied then the symmetry is broken and

$$A = A_0 - hM + \frac{1}{2}A_2M^2 + \frac{1}{4}A_4M^4.$$

At $T = T_c$ ($A_2 = 0$) the condition for equilibrium is

$$-h + A_4 M^3 = 0 \quad \Rightarrow \quad M \sim h^{\frac{1}{3}} \quad \Rightarrow \quad \delta = 3.$$

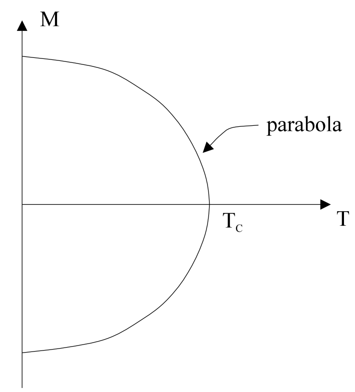
For $T > T_c$ we have ($t = \frac{(T-T_c)}{T_c}$)

$$-h + a_2 T_c t M + A_4 M^3 = 0.$$

Then the susceptibility is given by

$$\chi = \left(\frac{\partial M}{\partial h}\right)_{h=0} = \frac{1}{a_2 T_c} t^{-1} \quad \Rightarrow \quad \gamma = -1.$$

The curve of $\pm M(T)$ vs T is a parabola: For $T < T_c$ we find

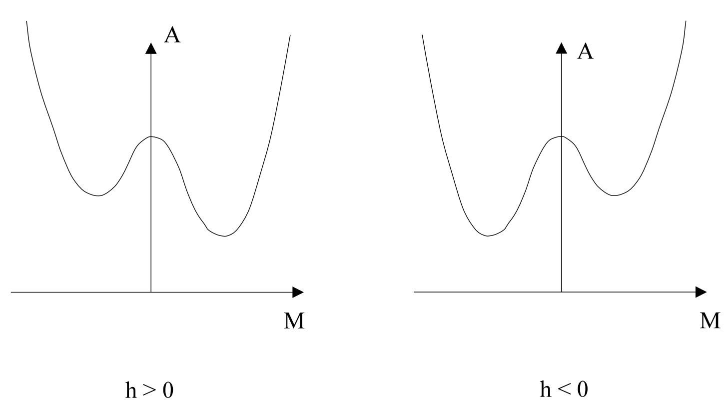


$$h = -a_2 |T - T_c| M + A_4 M^3,$$

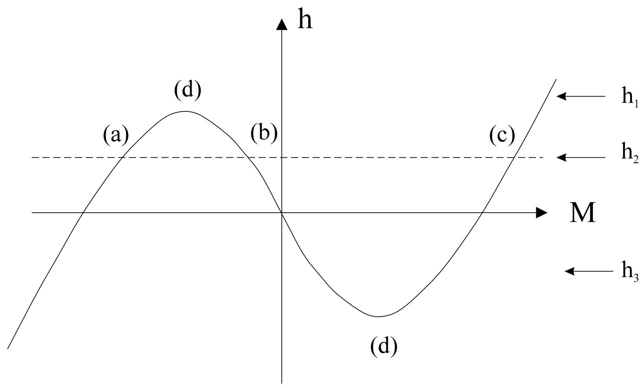
where in general A_4 is a function of T which does not vanish at $T = T_c$. Thus the equation of state has the form

$$h = -a(T)M + \beta(T)M^3 + \ldots$$

Then



If h is tuned from positive to negative the true minimum describing equilibrium changes from one to the other, **but** it takes time to re-establish th equilibrium especially if the intervening barrier is high. Consequently, the system can be in a metastable state corresponding to the local but not global minimum. This is the phenomenon of **hysteresis**. What happens is clear from the equation of state:

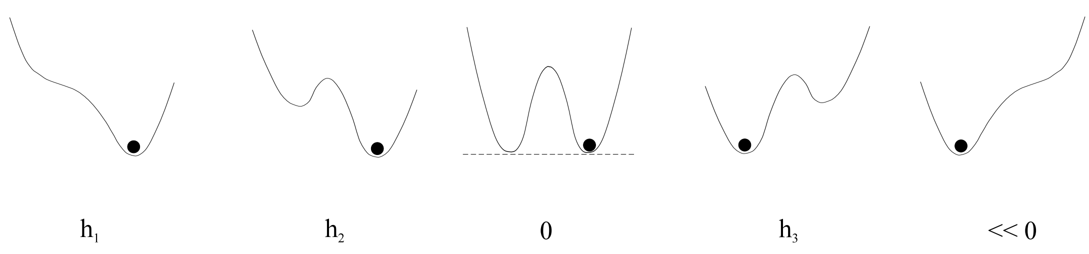


(i) For h_1 there is unique minimum and the state is stable.

(ii) For h_2 there are two minima (a) and (c) and a maximum (b). State (c) is **stable** and (a) is **metastable**, but (b) is **unstable** corresponding to a **maximum** of A and thermodynamic inequalities are violated here.

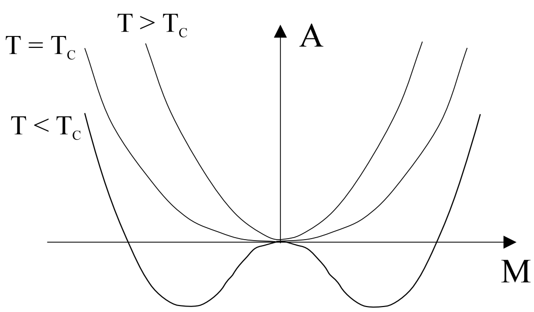
(iii) Follow what happens as h decreases from h_1 through h_2 and h_3 and eventually becomes large and negative: M varies smoothly as a function of h and metastability (supercooling) occurs, then at (d) the state becomes unstable and any fluctuation precipitates the change to the true equilibrium state.

Plots of A vs M for the different h values make this interpretation clear:

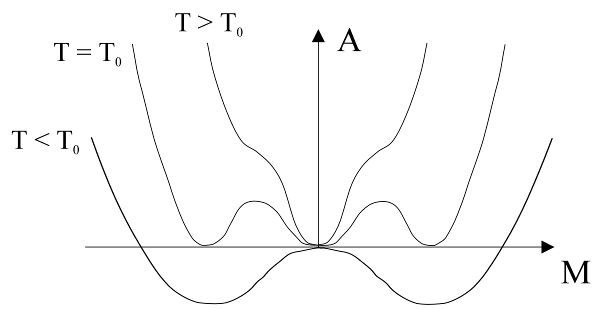


The **Maxwell construction** tells us that the stable state is not most generally characterised by a constant M. **Locally** M is a constant but it can change **globally** giving a domain structure. The Maxwell construction corresponds to the situation when $h = 0$ and there are two degenerate minima associated with the two different domains that can co-exist. Thus at $h = 0$ any value of M between these minima is possible and corresponds to an appropriate mixture of domains. The above analysis relies on the smoothness and differentiability of all functions A, h, M and can never directly address the mixed-phase system.

$\mathbf{A_4 \leq 0}$
As T decreases $A(M)$ behaves qualitatively differently depending on whether $A_4 > 0$ or $A_4 \leq 0$:



second-order transition at T = T_C



first-order phase transition at T_0 , T_0 > T_C

Hence the system passes from a second-order transition to a first-order transition as A_4 changes sign and becomes negative.
The stationary points are at $M = 0$ and at

$$M^2 = \left[-A_4 \pm \left(A_4^2 - 4A_2A_6\right)^{\frac{1}{2}}\right]/2A_6 \equiv M_{\pm}^2.$$

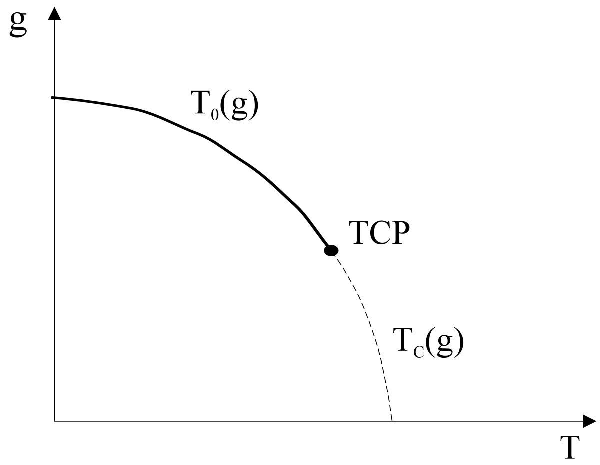
The $+$ sign gives the minima and the $-$ sign the maxima.
T_0 is determined by $A(M) = 0$ having a double root at $M = \pm M_+$ (note that A_0 is set to zero so that $A(0) = 0$ is the minimum for $T > T_0$). The solution is

$$A_2 = \frac{3}{16}\frac{A_4^2}{A_6},$$

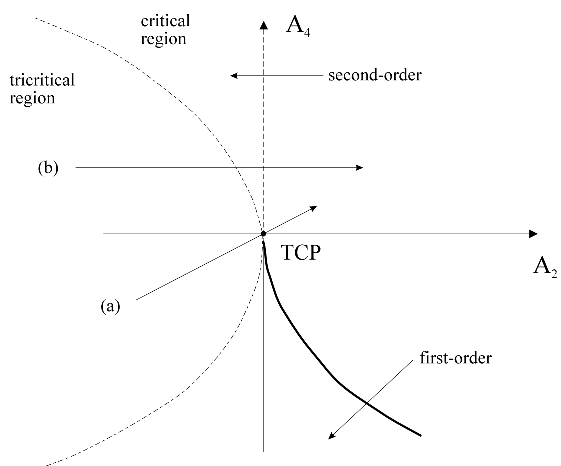
and at the transition

$$\left(\Delta M^2\right)^2 = M_+^2 = -\frac{3}{4}\frac{A_4}{A_6}$$

Thus the point $T = T_c, A_4 = 0$ separates the first-order line from the second-order line: this is a **tricritical point**. To see the tricritical point these two parameters have to take these special values and this requires tuning two external fields in the phase diagram.
In the space of physical fields, denoted by T and g (e.g., g can be identified with a chemical potential controlling the relative abundances in a two component system), the phase diagram has the form:



Alternaltively the phase diagram in (A_2, A_4) space takes the form: vskip 5 truemm



(i) **Both** trajectories (a) and (b) exhibit a second-order transition.

(ii) Trajectory (a) passes through the TCP and lies entirely within the tricritical region. The transition is characterised entirely by the properties of the TCP, and all critical exponents are tricritical ones.

(iii) Trajectory (b) exhibits an **ordinary** second order transition. However, it starts in the tricritical region and so initially the divergence of the relevant quantities is controlled by the TCP. Eventually it passes into the critical region and the transition is characterised by the line of ordinary critical points and the critical exponents that are given above.

In other words we only see a transition controlled by the TCP when we apporoach along a trajectory lying in the tricritical region. For trajectories that pass from one region to another we see a change in the critical behaivour. This change is characterised by **crossover exponents**.

At the TCP $A_4 = 0$, and so we have

$$|M| = \left|\frac{A_2}{A_6}\right|^{\beta_t} \quad \text{with} \quad \beta_t = \frac{1}{4}.$$

New exponents can be defined:

$$\begin{aligned}(T_0 - T_c) &\sim (-A_4)^{\frac{1}{\psi}} \quad \text{with} \quad \psi = \frac{1}{2} \\ \Delta M &\sim |A_4|^{\beta_u} \quad \text{with} \quad \beta_u = \frac{1}{2},\end{aligned}$$

where ΔM is the discontinuity in M across the first-order line ($A_4 < 0$). then for small A_4 we can write

$$\begin{aligned}|M| &= \left|\frac{A_2}{A_6}\right|^{\beta_t} m(x), \\ x &= \frac{A_4}{2|A_2|^{\frac{1}{2}} A_6^{\frac{1}{2}}}\end{aligned}$$

To see tricritical behaviour along a trajectory we clearly need x small, i.e.,

$$A_4^2 \ll 4|A_2|A_6.$$

This defines a parabola in the (A_4, A_6) plane separating the tricritical from critical regions. This is shown on the figure. In the space of physical parameters it translates into a similar shaped curve defining the two regions controlled by the TCP and ordinary critical points respectively.

The general theory of continuous phase transitions can be encoded in terms of scaling functions and relies on dimensional analysis together with some assumptions about the behaviour of the scaling functions for small argument. If naive or engineering dimensions are used this is generally a recoding of Landau theory but is often used to describe the behaviour of the relevant thermodynamic variables as a function of the actual external fields and hence parametrises the experimental observations.
Add a magnetic field, h, with contribution to the free energy of $-hM$. Then we can always write

$$A = \frac{|A_2|^{\frac{3}{2}}}{A_6^{\frac{1}{2}}} F\left(\frac{A_4}{2|A_2|^{\frac{1}{2}} A_6^{\frac{1}{2}}}, \frac{hA_6^{\frac{1}{4}}}{|A_2|^{\frac{5}{4}}}\right).$$

The point is that the **equilibrium** free energy, A, can always be written in terms of dimensionless ratios in this way. As before assign dimension (-1) to M and dimension d to A, and then A_n has dimension $(d+n)$. The above expression is then a general way of writing the dependence of A at equilibrium on the coefficients A_{2n} in terms of a **scaling function**, F. Note that since A_6 is always taken as positive it causes no problem in the denominators.

We now compare with a standard form parametrising A near the TCP:

$$A = |T - T_c(\tilde{g})|^{2-\alpha}\ F\left(\frac{\tilde{g}}{|T - T_c(\tilde{g})|^{\phi}}, \frac{h}{|T - T_c(\tilde{g})|^{\Delta}}\right),$$

where $\tilde{g} \propto A_4$, and thus measures the distance from the TCP along the tangent to the critical line at the TCP. Note that $\tilde{g}$ has been substituted for the field g as the second independent external field: the critical line is thus parametrised as $T_c(\tilde{g})$. The TCP then is at position $(0, T_t)$ in the $(\tilde{g}, T)$ plane where $T_t = T_c(0)$. Labelling the critical exponents at the TCP by suffix, t, we clearly have

$$\alpha_t = \frac{1}{2}, \qquad \phi_t = \frac{1}{2}, \qquad \Delta_t = \frac{5}{4}.$$

The following examples clarify the interpretation.

(i) $h = 0,\ \tilde{g} = 0,\ T \to T_t$ such that $\frac{\tilde{g}}{|T-T_t|^{\phi_t}} \equiv x$ is fixed. Then

$$A = |T - T_t|^{\frac{3}{2}} F(x, 0) \quad \text{with} \quad F(0,0) \quad \textbf{finite}.$$

We see tricritical behaviour and since $\tilde{g} \sim |T - T_t|^{\phi_t}$ the trajectory lies in the tricritical region. ϕ_t is the **cros-over exponent**.

(ii) $h \to 0,\ \tilde{g}$ fixed, $T \to T_c$.

$$A = |T - T_c|^{\frac{3}{2}}\ F\left(\frac{\tilde{g}}{|T - T_c|^{\frac{1}{2}}}, 0\right).$$

the argument of F is not under control and so we rearrange the expression:

$$A = \frac{|T - T_c|^2}{\tilde{g}}\ G\left(\frac{|T - T_c|^{\frac{1}{2}}}{\tilde{g}}, 0\right),$$

where $G(z, 0) = zF(\frac{1}{z}, 0)$ and $G(0,0)$ is finite and non-zero. This property of G is an assumption in the general theory and could be violated. It **does** follow from the standard Landau analysis and hence if it turned out to be false in an experiment it would signal a breakdown of the Landau theory. The goal then would be to rescue the dimensional analysis approach by assigning values to the dimensions of the parameters different from the naive ones but which render the scaling functions F and G well behaved for small argument.

In this case we find that A shows the normal critical behaviour associated with an ordinary critical point, namely

$$A \sim |T - T_c|^{2-\alpha} \quad \text{with} \quad \alpha = 0.$$

(iii) $T = T_t,\ \tilde{g} \neq 0,\ h \neq 0$

$$A = \lim_{T \to T_t} |T - T_t|^{2-\alpha_t}\ F\left(\frac{\tilde{g}}{|T - T_t|^{\phi_t}}, \frac{h}{|T - T_t|^{\Delta_t}}\right),$$

but A remains non-zero for $h \neq 0$ and we must be able to rewrite A in the limit as

$$A = h^{\frac{2-\alpha_t}{\Delta_t}}\ J\left(\frac{\tilde{g}}{h^{\frac{\phi_t}{\Delta_t}}}\right),$$

i.e., F must have a singularity that cancels the external factor of $|T-T_t|^{2-\alpha_t}$. Thus

$$A = h^{\frac{6}{5}}\ J\left(\frac{\tilde{g}}{h^{\frac{2}{5}}}\right).$$

If we assume $J(0)$ is non-zero and finite then at the TCP

$$A \sim h^{\frac{6}{5}} \quad \text{and} \quad M = -\frac{\partial A}{\partial h} \sim h^{\frac{1}{5}} \Rightarrow \delta_t = 5.$$

This result can be derived much more directly from the Landau theory: at $T = T_t$ both A_2 and A_4 are zero and so we can write

$$A = -hM + A_6 M^6.$$

Thus we find that

$$\frac{\partial A}{\partial M} = 0 \quad \Rightarrow \quad M \sim h^{\frac{1}{5}}.$$

However, it is important to see how the parametrisation in terms of scaling functions works, and, as has already been remarked, this form of parametrisation is more general than the naive Landau theory: the assumption that $F(0,0)$ etc. are non-zero and/or finite breaks down when Landau theory ceases to be valid.

A summary of the critical indices is

	$A \sim \lvert t\rvert^{2-\alpha}$ α	$M \sim (-t)^{\beta}$ β	$\chi \sim \lvert t\rvert^{-\gamma}$ γ	$M \sim h^{\frac{1}{\delta}}$ δ	cross-over ϕ_t
CP	0	$\frac{1}{2}$	-1	3	$\frac{1}{2}$
TCP	$\frac{1}{2}$	$\frac{1}{4}$	-1	5	$\frac{1}{2}$

$$\begin{aligned} (T_0 - T_c) &\sim |A_4|^{\frac{1}{\psi}} \qquad \psi = \frac{1}{2} \\ \Delta M &\sim |A_4|^{\beta_u} \qquad \beta_u = \frac{1}{2}. \end{aligned}$$

To finish this section we look in general at the Landau model with one parameter.

Let the thermodynamic potential be Ψ and the order parameter x. Then

$$\Psi = a_0 + a_1 x + a_2 x^2 + \ldots x^{2q}.$$

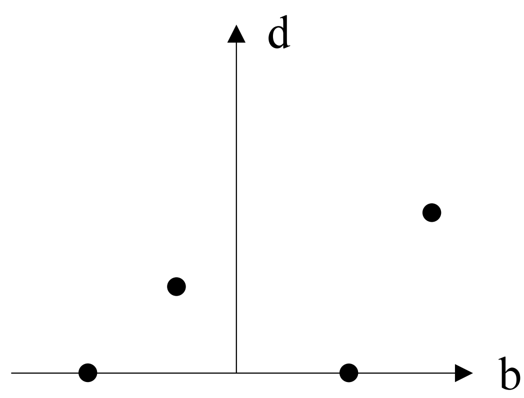
We assume that the coefficient of the highest order term is 1. This means that higher order terms will not affect the phase structure since the coefficient of this term will not be allowed to vanish at any point in that part of the phase space in which we are interested. In addition this highest power is even. This ensures stability, i.e., Ψ has a global minimum characterising equilibrium.
The a_i are thermodynamic field variables which span the phase space, Y, and for a given choice of the a_i the stable thermodynamic state is identified with the value of x which minimises Ψ. Choose a_0 such that this minimum is always at $\Psi = 0$: this simply fixes the origin of Ψ. Then we can write

$$\Psi = \prod_{j=1}^{q} \left[(x - b_j)^2 + d_j \right],$$

where $\forall j$ the b_j are real and the d_j are real and non-negative. However, at least one of the d_j is zero since a_0 has been chosen appropriately. Then Ψ can be represented by the configuration of a set of points (b_j, d_j) in the upper half of the (b, d) plane. Consider the following examples.

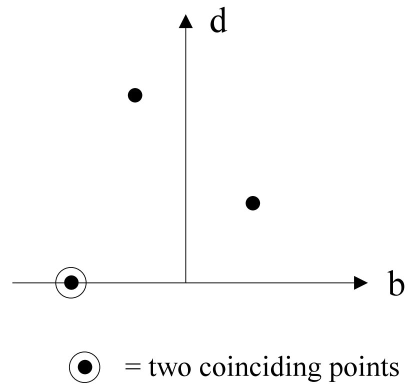
(i) ***m*-phase coexistence**

The minimum of Ψ occurs for m **distinct** values of x. These are identified with the m coexisting phases and it corresponds to m of the d_j being zero. In general this requires $(m-1)$ constraints on the d_j and hence the manifold of m-phase coexistence has codimension $\kappa = m - 1$. For $m = 2$ the configuration takes the form:



(ii) **A critical point**

A critical point occurs if two of the d_j are zero and the corresponding b_j coincide:



For this configuration

$$\Psi = (x-b)^4 + \alpha_5 (x-b)^5 + \dots .$$

There is no quadratic term $(x-b)^2$ and hence have a second order transition. This is a more general situation than before where we imposed symmetry under $x \to -x$ for simplicity.

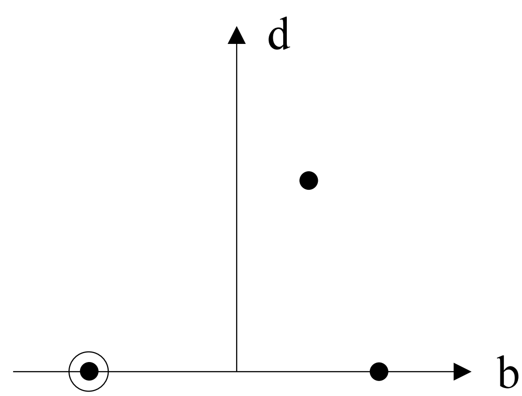
The additional constraint on the b_j means that κ is one more than for two phase coexistence, i.e., $\kappa = 2$.

Remarks

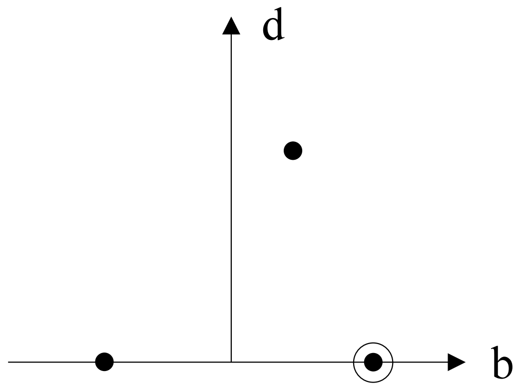
(a) All points of m-phase coexistence can be continuously transformed into each other: inspect diagrams of the kind shown in (i) above. Hence they all lie on one manifold.

(b) There is only one manifold of critical points. This is clear from (ii) above.

(iii) **Critical end-points**

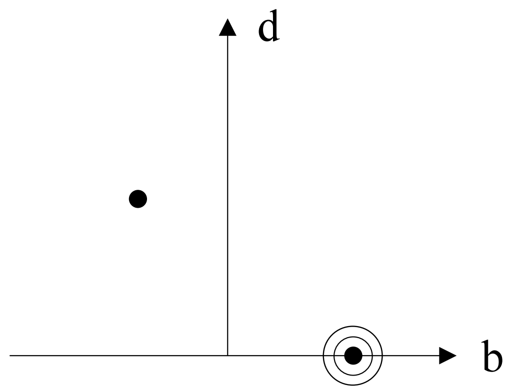


or alternatively



It is not possible to pass continuously between these two diagrams and hence there are two distinct manifolds of critical end-points separated by a

(iv) **Tricritical point**



This point has $\kappa = 4$: two constraints occur because there are three coexisting phases (i.e., $m = 3 \Rightarrow (m-1)$ constraints), and two further constraints arise because of the requirement that the relevant b_j coincide. Howver, κ is just the number of such constraints.

(v) **Multicritical points of order r**

r of the d_j are zero and the corresponding b_j coincide. thus

$$\kappa = (r-1) + (r-1) = 2r - 2.$$

clearly we can determine how the manifolds fit together by following the flow of points in (b, d) space as the a_i vary.

4 The Partition Function and Field Theory

The partition function $\mathcal{Z}$ is defined by

$$\mathcal{Z} = \int \{d\phi\} e^{-\beta H(\phi)}$$

where $\phi(\mathbf{x})$ is a generic field degree of freedom and $\beta = 1/kT$. $H(\phi)$ is the Hamiltonian given by

$$H(\phi) = \int_{\Lambda^{-1}} d\mathbf{x}\ \mathcal{H}(\phi(\mathbf{x}))$$

where $\mathcal{H}(\phi(\mathbf{x}))$ is the Hamiltonian density.Λ is the large momentum cut-off which, for a lattice of spacing a, is $\Lambda = 2\pi/a$. In this case the integral will be a sum over all sites of a discrete Hamiltonian density. The **crucial** point is that there will, in general, be a cut-off of some kind.
We shall assume that the coefficients in H depend only on the volume, V, of the system. Then, for a given temperature T and volume V of a subsystem the equilibrium probability density for finding the subsystem with field configuration $\phi(\mathbf{x})$ is

$$p(\phi) = \frac{1}{\mathcal{Z}} e^{-\beta H(\phi)}.$$

Then the entropy S is given by

$$\begin{aligned} S &= -k \int \{d\phi\} p(\phi) \log(p(\phi)) \\ &= -k \int \{d\phi\} \frac{1}{\mathcal{Z}} e^{-\beta H(\phi)} \left(-\beta H - \log\ \mathcal{Z}\ \right) \\ &= k \left(\beta U + \log\ \mathcal{Z}\ \right), \end{aligned}$$

where

$$U = \frac{1}{\mathcal{Z}} \int \{d\phi\} H(\phi) e^{-\beta H} \quad \equiv \quad \text{internal energy.}$$

Thus

$$kT \log\ \mathcal{Z}\ = -U + TS = -VF,$$

and hence

$$F = -\frac{1}{\beta V} \log\ \mathcal{Z}\ .$$

F is the thermodynamic potential appropriate for T and V as independent variables.

We are interested in the macroscopic properties of the system and so we re-express $\mathcal{Z}$ in terms of a **macroscopic** variable $\hat{\phi}(\mathbf{x})$. This is our "guess" for the order parameter, but it could be that it will not reveal all the possible phases of the

system. We are trying to pick out the collective coordinates on which the long-range/low-momentum physics depends. For a scalar field system this is not too difficult but when the field has internal degrees of freedom it is much harder.For example, both the phenomena of superconductivity and He_3 superfluidity have spin-$\frac{1}{2}$ fundamental fields which pair to form spin-0 bosons which condense. **BUT** in the superconductor this pairing is in the $^1S_0$ state whereas in He_3 it is in the $^3P_0$ state. Hence different choices must be made in these two cases.
We can choose

$$\hat{\phi}(\mathbf{x}) = \frac{1}{L^D}\int_v d\mathbf{x}'\ \phi(\mathbf{x}'),$$

where v is a volume centred at $\mathbf{x}$ with $v = L^D \gg a^D$.
Alternatively

$$\hat{\phi}(\mathbf{x}) = \int_{|\mathbf{p}|\leq\tilde{\Lambda}\ll\Lambda} \frac{d^Dp}{(2\pi)^D}\ e^{i\mathbf{px}}\ \tilde{\phi}(\mathbf{p}).$$

Here $\hat{\phi}(\mathbf{x})$ is composed of the low-momentum degrees of freedom only : $\tilde{\Lambda} = 2\pi/L$. The cut-off is now Λ/L and hence it is possible to define the field theory for $\hat{\phi}$ on a lattice of spacing L. Then

$$\begin{aligned} e^{-\beta VF} &= \int d\phi d\hat{\phi}\ \delta\left(\hat{\phi} - \hat{\phi}(\phi)\right)\ e^{-\beta H(\phi)} \\ &= \int d\hat{\phi}\ e^{-\beta H(L,\hat{\phi})}. \end{aligned}$$

Suppose, for example,

$$\mathcal{H}(\phi(\mathbf{x})) = \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m_0^2\phi^2 + \frac{1}{4!}g_0\phi^4\ \ldots + \frac{1}{n!}g_n\phi^n,$$

where n is taken as even and where m_0^2, g_n are the bare or microscopic coupling constants; i.e., they parameterize field interactions on a lattice of spacing $a = 2\pi/\Lambda$. $H(L,\hat{\phi})$ is much more appropriate to our needs since the unit of length is now $L \gg a$. Write

$$\mathcal{H}(L,\hat{\phi}) = \frac{1}{2}Z^{-1}(L,T)(\nabla\hat{\phi})^2 + \frac{1}{2}m^2(L,T)\hat{\phi}^2 + \frac{1}{4!}g(L,T)\hat{\phi}^4\ + \ldots + (\nabla\hat{\phi})^4 + \ldots$$

Note that the kinetic term no longer has unit coefficient. Generally at this stage it is convenient to rescale $\hat{\phi}$ to make this particular coefficient unity:

$$\hat{\phi} \to Z^{\frac{1}{2}}(L,T)\hat{\phi}.$$

However, we will not do this rescaling for the moment.
Typically as L becomes large enough

$$Z^{-1}(L)(\nabla\hat{\phi})^2 \sim \frac{Z^{-1}(L)}{L^2}\hat{\phi}^2 \ll m^2\hat{\phi}^2.$$

Hence the **effective** Hamiltonian is insensitive to $\nabla\hat{\phi}$ and so $\hat{\phi}$ can be treated as a constant. Then

$$\begin{aligned} e^{-\beta VF} &= \int d\hat{\phi}\; e^{-V\beta H(\infty,\hat{\phi})}, \\ H &= \frac{1}{2}m^2(\infty,T)\hat{\phi}^2 \;+\; \frac{1}{4!}g(\infty,T)\hat{\phi}^4 \;+\; \ldots \end{aligned}$$

We have integrated out over all scales and produced **renormalized** coupling constants and an **effective** Hamiltonian depending on $\hat{\phi}$.
Now

$$\begin{aligned} \int d\mathbf{x}\; e^{-VS(\mathbf{x})} &= e^{-VS(\mathbf{x}_0)} \int d\xi\; e^{-V[\frac{1}{2}S''(\mathbf{x}_0)\xi^2 \;+\; \ldots]} \\ &= e^{-VS(\mathbf{x}_0)} \int d\xi\; e^{-\frac{1}{2}VS''(\mathbf{x}_0)\xi^2}\; [1 - \frac{1}{4!}VS^{(4)}\xi^4 \;+\; \ldots], \\ &= e^{-VS(\mathbf{x}_0)} \left(\frac{2\pi}{VS''}\right)^{\frac{1}{2}} \left[1 - \frac{1}{4V}\frac{S^{(4)}}{(S'')^2} \;+\; \ldots\right], \\ \text{where}\;\; S'(\mathbf{x}_0) &= 0 \end{aligned}$$

Hence

$$F = H(\infty,\hat{\phi}_0) \;+\; O\left(\frac{\log V}{V}\right),$$

where $\hat{\phi}_0$ is the **global minimum** of $H(\infty,\hat{\phi})$. This is **Landau's** method with $H(\infty,\hat{\phi})$ as the free energy. function **BUT** this procedure requires the limit $L \to \infty$ to be under control. In particular Landau assumes that $m^2(\infty,T)$ is **analytic** in T. Then all that the integration over scales has done is to fix the value of T_c. This assumption is **wrong** for $D \leq D_c$ and Landau's method fails. Note that $\hat{\phi}_0$ is the analogue of the **magnetization**: it is the order parameter and can be measured in an experiment.

Since it is the limit $L \to \infty$ which causes the trouble let us investigate calculating F with L finite and identify the source of the problem.
Let $S(\hat{\phi}) = \beta H(L,\hat{\phi})$ and define $\hat{\phi}_0$ by

$$\left(\frac{\partial S(\hat{\phi})}{\partial\hat{\phi}(\mathbf{x})}\right)_{\hat{\phi}=\hat{\phi}_0} = 0.$$

Then

$$e^{-\beta VF} = e^{-S(\hat{\phi}_0)} \int d\psi\; e^{-\frac{1}{2}S''\psi^2 \;+\; \cdots},$$

where

$$S''\psi^2 \equiv \int d\mathbf{y}\; d\mathbf{z}\; \left(\frac{\partial^2 S}{\partial\hat{\phi}(\mathbf{y})\hat{\phi}(\mathbf{z})}\right)_{\hat{\phi}=\hat{\phi}_0} \psi(\mathbf{y})\psi(\mathbf{z}).$$

Assuming $\hat{\phi}_0$ is a constant independent of position and taking logs we find

$$F = \mathcal{H}(L, \hat{\phi}_0) \ + \ \log \int d\psi e^{-\frac{1}{2}S''\psi^2 \ + \ \cdots} \quad .$$

In $\mathcal{H}$ the mass and couplings are **analytic** in T for fixed L. (This is certainly plausible and is a reasonable assumption). The integral is calculated using the **loop expansion**. Landau assumes that these loops just change T_c and that for L **large enough** they may be neglected keeping only the first term in F above. To illustrate how problems occur consider the effect of an external constant field J.

$$H_J(\phi) = H(\phi) - J \int \ d\mathbf{x} \ \phi(\mathbf{x}) \quad .$$

The induced magnetization is

$$< \phi >_J = \frac{1}{\mathcal{Z}_J} \int d\phi \ \phi(0) e^{-\beta H_J(\phi)} \quad ,$$

and the susceptibility is

$$\chi = \left(\frac{\partial < \phi >_J}{\partial J} \right)_{J=0} = \beta \int d\mathbf{x} \ \ < \phi(0)\phi(\mathbf{x}) >_c .$$

That is, the integral over the **connected** two-point function

$$< \phi(0)\phi(\mathbf{x}) >_c = < \phi(0)\phi(\mathbf{x}) > \ - \ < \phi(0) >< \phi(\mathbf{x}) > \quad .$$

The second term arises from differentiation of $\mathcal{Z}_J$ in the denominator.
We now define the smoothed field $\hat{\phi}(L, x)$ including a renormalization to keep the coefficient of $\frac{1}{2}(\nabla\phi)^2$ in $H(L, \hat{\phi})$ to be unity. Of course, we must remember this rescaling if we re-express results in terms of the actual magnetization. Then

$$\begin{aligned} \hat{\phi}(\mathbf{x}) &= \frac{Z_1^{-\frac{1}{2}}(L)}{L^D} \int_L \ d\mathbf{x}' \ \phi(\mathbf{x}'), \\ \chi &= \beta Z_1(L) \int \ d\mathbf{x} \ \ < \hat{\phi}(0)\hat{\phi}(\mathbf{x}) >_c \quad . \end{aligned}$$

We define

$$\begin{aligned} \hat{\mathcal{G}}_L(\mathbf{p}) &= \int \ d\mathbf{x} \ \ < \hat{\phi}(0)\hat{\phi}(\mathbf{x}) >_c \ e^{-i\mathbf{p}\mathbf{x}} \quad , \\ \hat{\Gamma}_L(\mathbf{p}) &= \hat{\mathcal{G}}_L(\mathbf{p})^{-1}. \end{aligned}$$

$\hat{\Gamma}_L$ is the **truncated two-point function**.
[In general, the n-point function $G_n(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ is defined by

$$G_n(\mathbf{p}_1, \ldots, \mathbf{p}_n) = \int d\mathbf{x}_1 \ldots d\mathbf{x}_n \ \ < \phi(\mathbf{x}_1)\phi(\mathbf{x}_2) \ldots \phi(\mathbf{x}_n) >_c \ e^{-i\mathbf{p}_i \cdot \mathbf{x}_i} \quad ,$$

and the truncated n-point function is defined by

$$\Gamma_n(\mathbf{p}_1, \ldots, \mathbf{p}_n) = \frac{G_n(\mathbf{p}_1, \ldots, \mathbf{p}_n)}{G(\mathbf{p}_1)G(\mathbf{p}_2)\ldots G(\mathbf{p}_n)} \quad .$$

G_n contains the denominator of the RHS as a factor and Γ_n is often a more useful quantity with which to work.]
Then we have

$$\chi^{-1} = \frac{1}{\beta \mathcal{Z}_1(L)} \hat{\Gamma}_L(0) \quad .$$

From now on we will absorb a factor of $\beta^{\frac{1}{2}}$ into $\hat{\phi}$, i.e., replace

$$\hat{\phi} \rightarrow \beta^{-\frac{1}{2}} \hat{\phi} \quad .$$

then $m^2 \rightarrow m^2$, $g \rightarrow \beta g$ etc. and

$$\chi^{-1} = \frac{\hat{\Gamma}_L(0)}{\mathcal{Z}_1(L)} \quad .$$

To calculate $\hat{\Gamma}_L$ we need to use perturbation theory.

4.1 The Perturbation Expansion

We start with the definition

$$< \phi(0)\phi(\mathbf{x}) >= \frac{1}{\mathcal{Z}} \int d\phi \ \phi(0)\phi(\mathbf{x}) \ e^{-S(\phi)} \ .$$

Let

$$S(\phi) = \int d\mathbf{x} \ \frac{1}{2}(\nabla\phi)^2 \ + \ \frac{1}{2} m^2\phi^2 \ + \ \frac{1}{4!} g\phi^4 \quad ,$$

where m^2 can be negative. The situation $m^2 < 0$ corresponds to spontaneous breaking of the symmetry $\phi \rightarrow -\phi$ in the bare Hamiltonian. However, we shall see that the **effective** mass is positive once loop corrections are included.
In order to be able to perform the loop expansion we write

$$\begin{aligned} S(\phi) &= \int d\mathbf{x} \ \frac{1}{2}(\nabla\phi)^2 \ + \ \frac{1}{2}\mu^2\phi^2 \ + \ \frac{1}{2}(m^2-\mu^2)\phi^2 \ + \ \frac{1}{4!}g\phi^4 \\ S_0(\phi) &= \int d\mathbf{x} \ \frac{1}{2}(\nabla\phi)^2 \ + \ \frac{1}{2}\mu^2\phi^2, \end{aligned}$$

and **choose** $\mu^2 > 0$.
Then with

$$\mathcal{Z} = \int d\phi \ e^{-S_0(\phi)} \ e^{-\int d\mathbf{x} \ \frac{1}{2}(m^2-\mu^2)\phi^2 \ + \ \frac{1}{4!}g\phi^4} \quad ,$$

we have

$$\begin{aligned} &< \phi(0)\phi(\mathbf{x}) > \\ &= \frac{1}{\mathcal{Z}} \int d\phi \ \phi(0)\phi(\mathbf{x}) \ e^{-S_0(\phi)} \ e^{-\int d\mathbf{x} \ \frac{1}{2}(m^2-\mu^2)\phi^2 \ + \ \frac{1}{4!}g\phi^4} \\ &= \frac{1}{\mathcal{Z}} \int d\phi \ \phi(0)\phi(\mathbf{x}) \left[1 - \int d\mathbf{x}\frac{1}{2}(m^2-\mu^2)\phi^2 \ + \ \frac{1}{4!}g\phi^4 \ + \ \ldots\right] \ e^{S_0(\phi)} \end{aligned}$$

This consists of a sequence of gaussian integrals which are best done in momentum space. they take the form

$$\int d\phi\ e^{-\int \mathbf{dydz}\ \phi(\mathbf{y})Q(\mathbf{y}-\mathbf{z})\phi(\mathbf{z})}\ \phi(0)\phi(\mathbf{x}) = A\,(\det Q)^{-\frac{1}{2}}\ Q^{-1}(x)$$

and

$$\int d\phi\ e^{-\int\ \phi Q\phi}\ \phi(0)\phi(\mathbf{x})\ \int d\mathbf{z}\ g\phi^4(\mathbf{z}) = gB\,(\det Q)^{-\frac{1}{2}}\ \int d\mathbf{z}Q^{-1}(\mathbf{z})Q^{-1}(\mathbf{x}-\mathbf{z})Q^{-1}(0)\ .$$

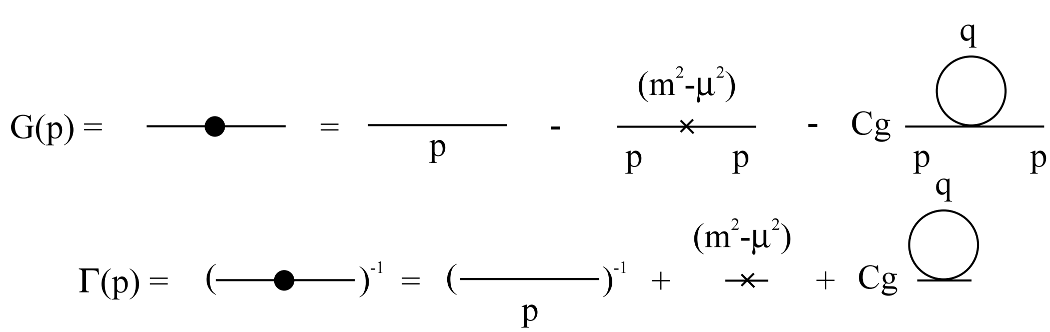
The denominator cancels the $(\det Q)^{-\frac{1}{2}}$ leaving a calculable combinatorial coefficient. In momentum space we have the diagram expansion as follows

$$\begin{aligned} G(\mathbf{p}) &= \int d\mathbf{x} < \phi(0)\phi(\mathbf{x}) >_c\ e^{-i\mathbf{px}} \\ &= \tilde{Q}^{-1}(\mathbf{p}) - \tilde{Q}^{-1}(\mathbf{p})(m^2-\mu^2)\tilde{Q}^{-1}(\mathbf{p}) - Cg\tilde{Q}^{-1}(\mathbf{p})\int \frac{d^Dq}{(2\pi)^D}\ \tilde{Q}^{-1}(\mathbf{q})\tilde{Q}^{-1}(\mathbf{p})\ + \ldots \end{aligned}$$

or

$$\Gamma(\mathbf{p}) = \tilde{Q}^{-1}(\mathbf{p}) + (m^2-\mu^2) + Cg\int \frac{d^Dq}{(2\pi)^D}\ \tilde{Q}^{-1}(\mathbf{q})\ + \ldots$$

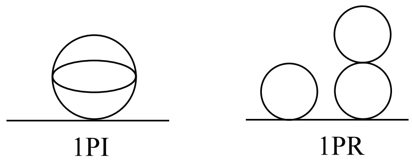
$\tilde{Q}^{-1}(\mathbf{p})$ is the **bare** Feynman propagator: $\tilde{Q}^{-1}(\mathbf{p}) = (\mathbf{p}^2+\mu^2)^{-1}$.
Then



The expansion shown for Γ is exact to $O(g)$ but corresponds to a sum over selected **1-particle irreducible** graphs in G. this can be seen as follows.

$$G = \frac{1}{\Gamma} = \frac{1}{Q+\delta m^2+\Sigma}\ ,$$

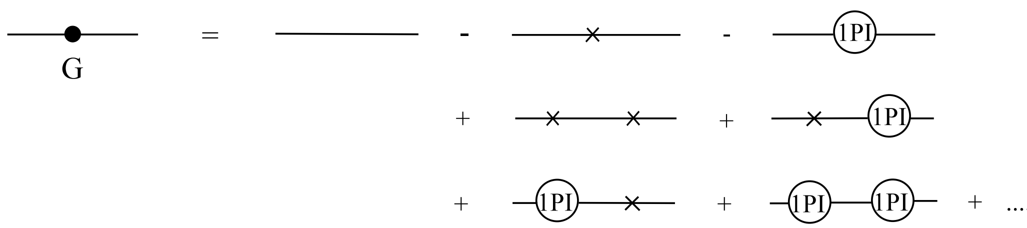
where Σ represents **1-particle irreducible** (1PI) diagrams. This means that they cannot be separated into two pieces by cutting one line only. E.g., Expand

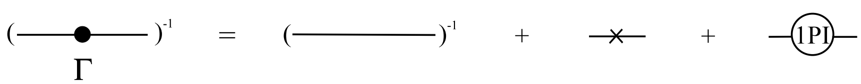


the above expression to give

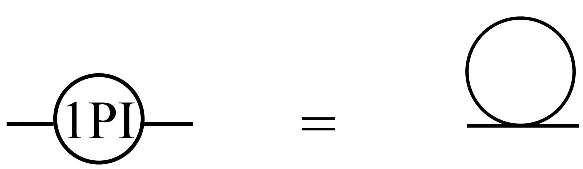
$$G = \frac{1}{Q} \quad - \quad \frac{1}{Q}(\delta m^2 + \Sigma)\frac{1}{Q}$$
$$\quad + \quad \frac{1}{Q}(\delta m^2 + \Sigma)\frac{1}{Q}(\delta m^2 + \Sigma)\frac{1}{Q} \quad + \ldots$$

Diagrammatically we have



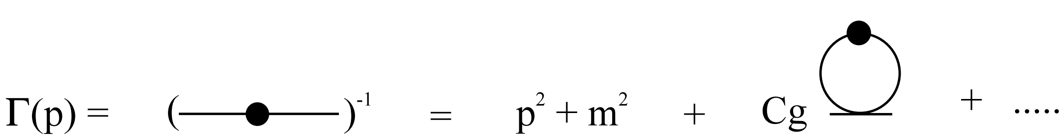


To $O(g)$:



Note that Γ is **truncated** because it contains no propagator factors, $\tilde{Q}^{-1}$, for external legs.

The expansion for $\Gamma(\mathbf{p})$ can be improved by summing over all **bubble insertions** on the internal loop. Then

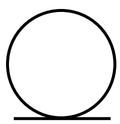


The Feynman rules for constructing the diagrams are

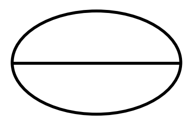
$\frac{1}{\mathbf{p}^2+m^2}$	for each propagator line
$\int \frac{d^Dq}{(2\pi)^D}$	for each closed loop
$\delta^D(\sum \mathbf{p}_i)$	momentum conservation at each vertex

factor $\quad C = \frac{1}{S}$

Where S is the number of symmetry operations that leave the graph unchanged, for example

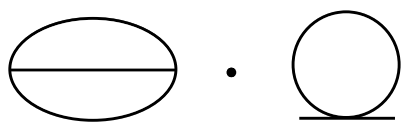


S = 2



S = 6

Note that **disconnected** contributions to the numerator, e.g.,



are cancelled by terms in the denominator. Hence, only get **connected** graphs for expansion of G. For our smoothed theory we have

$$m^2 \equiv m^2(L,T).$$

Remember that m^2 is not necessarily positive. Then

$$= \frac{Z_L(\mathbf{p}^2,T)}{\mathbf{p}^2 + m^2(\infty,T)} \ .$$

Here $m^2(\infty,T) = \xi^{-2}(T)$ and is positive. The loop correction **renormalizes** m^2 additively and the resulting effective mass is positive. Now

$$\chi^{-1} = \frac{\hat{\Gamma}_L(0)}{Z_1(L)} = \frac{m^2(\infty,T)}{Z_1(L)Z_L(0,T)}$$

χ^{-1} vanishes at $T = T_0$ which is given by the vanishing of the renormalized mass:

$$\lim_{T\to T_c} m^2(\infty,T) = 0.$$

Note that in general $\lim_{T\to T_c} Z_L(0,T) = 0$ as well but that the numerator dominates. In fact for our present discussion

$$Z_L = 1 + O(g^2) \ ,$$

and hence to $O(g)$ $Z_L = 1$.
Then

$$m^2(\infty,T) = m^2(L,T) + \frac{1}{2}g\int \frac{d^Dp}{(2\pi)^D}\,\frac{1}{\mathbf{p}^2 + m^2(\infty,T)} \quad + \ldots \quad .$$

Note that the bubble summation removed all reference to μ^2. In fact we can see that by choosing $\mu^2 = m^2(\infty,T)$ in the first place we automatically get the improved formula from the one-loop expression. the coefficient of $\frac{1}{2}$ comes from the symmetry factor as described above. Using $m^2(\infty,T_c) = 0$ we write

$$\begin{aligned} m^2(\infty,T) &= m^2(L,T) - m^2(L,T_c) + \frac{1}{2}g\int^{\frac{2\pi}{L}} \frac{d^Dp}{(2\pi)^D}\left[\frac{1}{\mathbf{p}^2 + m^2(\infty,T)} - \frac{1}{\mathbf{p}^2}\right] \\ &= At - \frac{1}{2}gm^2(\infty,T)\int^{\frac{2\pi}{L}} \frac{d^Dp}{(2\pi)^D}\,\frac{1}{(\mathbf{p}^2 + m^2(\infty,T))\mathbf{p}^2} \quad + \ldots \end{aligned}$$

Where $t = (T - T_c)/T_c$.
We consider two cases:

$D > 4$ The integral is dominated by large $\mathbf{p}$ and is **finite** in the limit $m^2 \to 0$. Then

$$m^2(\infty,T) = At + Bm^2(\infty,T).$$

This is consistent with the Landau behaviour

$$m^2(\infty,T) = Ct$$

$D < 4$ The integral is sensitive to low $\mathbf{p}$ and is **Infra-Red divergent** unless $m^2 > 0$:

$$m^2(\infty,T) = At + Bm^{D-2}(\infty,T)$$

This is clearly inconsistent with the Landau assumption: the integral **is** important. IR divergences invalidate Landau's assumption.

$D = 4$ A marginal case with IR logarithmic corrections

$$m^2(\infty,T) = At + Bm^2\log m^2.$$

Get Landau behaviour modified by logs.

Hence $D_c = 4$ and for $D \leq 4$ IR effects destroy mean-field predictions.
This theory applies to an **ordinary** critical point in Landau's theory: the coefficient of ϕ^2 vanishes. If, for some reason, the ϕ^4 were absent and the interaction started at the $g\phi^6$ term then the loop contribution would be

$$\sim \int^{\frac{2\pi}{L}} \frac{d^Dp}{(2\pi)^D}\,\frac{1}{\mathbf{p}^2 + m^2(\infty,T)}\int^{\frac{2\pi}{L}} \frac{d^Dq}{(2\pi)^D}\,\frac{1}{\mathbf{q}^2 + m^2(\infty,T)} \quad .$$

Subtracting off the value at $m^2 = 0$ and isolating the most IR divergent integral gives

$$\int^{\frac{2\pi}{L}} \frac{d^D p}{(2\pi)^D} \int^{\frac{2\pi}{L}} \frac{d^D q}{(2\pi)^D} \frac{1}{\mathbf{p}^2(\mathbf{p}^2 + m^2(\infty, T))(\mathbf{q}^2 + m^2(\infty, T))} \quad .$$

There are no IR problems for $2D - 6 > 0$, i.e., $D > 3$. Hence in this case $D_c = 3$. In Landau's theory this situation corresponds to a **tricritical** point: the coefficients of ϕ^2 and ϕ^4 vanish.
In general, for interaction ϕ^{2n} we find

$$(n-1)D_c - 2n = 0,$$

or

$$D_c = \frac{2n}{n-1}.$$

This interaction is relevant only for critical points of order n. For $D = 2$ **all** critical points show anomalous behaviour but for $D = 3$ only critical and tricritical points are anomalous, and even then the corrections to mean-field predictions at the tricritical point are only logarithmic.

It is important to understand heuristically or phenomenologically what is happening before actually doing detailed calculations. First we define the correlation length, ξ, a bit more explicitly than before.
For $x \gg a$

$$< \phi(0)\phi(\mathbf{x}) >_c \sim e^{-|\mathbf{x}|/\xi} \quad .$$

In fact if $\mathbf{x} = (x_0, \ldots, x_{D-1})$ then

$$\int dx_1 \ldots dx_{D-1} \;\; < \phi(0)\phi(\mathbf{x}) >_c \to C e^{-x_0/\xi} \; ,$$

as x_0 becomes large. This is how we might calculate ξ on a computer.
[Note

$$\int d\mathbf{p} \frac{e^{i\mathbf{p}\mathbf{x}}}{\mathbf{p}^2 + m^2} = A e^{-m|\mathbf{x}|} \quad .$$

]
Now, $\hat{\phi}(L)$ averages ϕ over blocks of size L^D, **but** if $D \leq D_c$ $M(L) = Z_1^{\frac{1}{2}}(L)\hat{\phi}(L)$ **cannot** be identified with the magnetization of the bulk volume, V, since there are **still** fluctuations in the system of wavelengths greater than L which cause these **blocks** to interact. In other words a block of size L^D is not big enough to be a good model of a larger system. Only for $L \gg \xi$ will this be true. For $D \leq D_c$ there are important fluctuations on all scales $L < \xi$, and only for $D > D_c$ are these fluctuations suppressed and Landau's theory works.

(1) In the theory with $L = \xi$ we do not expect significant loop corrections. Hence

$$< \phi(0)\phi(\mathbf{x}) >_c \sim e^{-m(\xi,T)|\mathbf{x}|} ,$$

which identifies $\xi^{-1} = m(\xi, T)$.

If we postulate that $m(\xi, T) \sim \xi^\sigma \cdot (T - T_c)$, then we find

$$\xi \sim t^{-\nu} , \qquad \nu = \frac{1}{1+\sigma} .$$

Also

$$\chi^{-1} = \Gamma(\mathbf{p} = 0) \sim \frac{1}{Z_1(\xi)} m^2(\xi, T) \quad .$$

Then if $Z_1(\xi) \sim \xi^\rho \equiv m^{-\rho}$ we get

$$\chi \sim t^{-\alpha} , \qquad \alpha = \nu(\rho + 2) .$$

(2) In general we have

$$\begin{aligned} G(\mathbf{x}) &\equiv < \phi(0)\phi(\mathbf{x}) >_c \\ &= C(a)\frac{\xi^\rho}{(|\mathbf{x}|\,\xi)^{(D-1)/2}}\, e^{-|\mathbf{x}|/\xi} , && a \ll \xi \ll |\mathbf{x}| , \\ &= C'(a)\frac{1}{|\mathbf{x}|^{2\Delta_\phi}} , \quad \Delta_\phi = (D - 1 - \rho)/2 , && a \ll |\mathbf{x}| \ll \xi , \end{aligned}$$

where A is the lattice spacing, i.e., $\Lambda = 2\pi/a$ is the UV cut- off. In all we do Λ is finite **but** $\Lambda \gg m$ and for all momenta $\mathbf{p}$ of interest $\Lambda \gg |\mathbf{p}|$ (i.e., $a \ll |\mathbf{x}|$). Hence we are always working in the limit of **very large cut-off**.

(a) It is important to realize that ξ sets the scale and **not** a, otherwise the correlators would be badly behaved as $\xi/a \to \infty$ and/or $|\mathbf{x}|/a \to \infty$. This is the basic assumption of the **scaling hypothesis**.

(b) "Naive" dimensional analysis must work.

$$\left[\int d\mathbf{x}\ (\nabla\phi)^2\right] = 0 \Rightarrow [\phi] = \frac{1}{2}(D-2) .$$

Where [] signifies the dimension of the enclosed quantity in units of momentum. Hence

$$[< \phi\phi >] = D - 2 .$$

Thus in the above expression for $G(\mathbf{x})$ we must have

$$C(a) \sim a^{2\eta} , \qquad \eta = \Delta_\phi - \frac{1}{2}(D-2) .$$

η is the **anomalous** dimension of the field ϕ, and Δ_ϕ is the **scaling** dimension of ϕ.

Now

$$\begin{aligned}\chi &= \int d\mathbf{x}\, G(\mathbf{x}) \sim \int d\mathbf{x}\, \frac{e^{-|\mathbf{x}|/\xi}}{|\mathbf{x}|^{2\Delta_\phi}} \\ \Rightarrow \chi &\sim \xi^{D-2\Delta_\phi}\ .\end{aligned}$$

Comparing with the alternative expression for the behaviour of ξ in (1) above we find the scaling relation

$$\alpha = \nu(D - 2\Delta_\phi)\ .$$

(3) The results of (2) can be seen in a different way. We have the general parameterization

$$G(\mathbf{p}) = \frac{Z(\mathbf{p},\xi)}{\mathbf{p}^2 + m^2(\xi,T)}\ .$$

We were concerned with $\mathbf{p} = 0$ earlier ($\chi = G(\mathbf{p} = 0)$), but now instead consider $\Lambda \gg |\mathbf{p}| \gg m$. Because Z is dimensionless we have

$$Z \sim (ap)^{-\rho'} f\left(\frac{m}{p}\right) \qquad p \gg m \equiv \xi^{-1}\ ,$$

where $p \equiv \mathbf{p}$. Also from above in (1) we have

$$Z \sim (am)^{-\rho} g\left(\frac{p}{m}\right) \qquad p \ll m \equiv \xi^{-1}\ .$$

This is because $Z \sim Z_1(\xi) \sim \xi^\rho$ for $p = 0$.

Note that f and g do not depend explicitly on (ap) or (am) for the reasons outlined in (2a) above: because of the scaling hypothesis. Then

(a) $Z(p = 0, m)$ is non-zero and finite for $m \neq 0$.
$Z(p, m = 0)$ is non-zero and finite for $p \neq 0$.

(b) the scaling hypothesis asserts that m (i.e. ξ) sets the scale and not Λ (i.e. not a). Hence the above scaling forms must be valid with $f(0)$ and $g(0)$ non-zero and finite. It then follows that

$$\begin{aligned}\rho &= \rho'\ , \\ g(z) &= \left(\frac{1}{z}\right)^\rho f\left(\frac{1}{z}\right)\ .\end{aligned}$$

By comparing powers of a we also get

$$\rho = -2\eta.$$

Substituting ρ in terms of α from part (1) gives the scaling relation derived in (2).

The important thing to notice in all of this is that each quantity in which we are interested depends on a in a simple and special way. Namely, a appears raised to a power as a **multiplicative** factor only: a does not appear as part of an argument of any of the functions. These function depend only on the **long range** observables. Since a has dimensions it follows that whilst "naive" dimensional analysis holds when including the factor depending on a, the dimension of the function it multiplies is not constrained. This means that if we concentrate on the dimensions of this function (e.g., f or g above) the contributions to its dimension from the observable quantities can add up to something other than the "naive" dimension, i.e. the **anomalous** dimension. Since, in any given system a is fixed the anomalous dimension is the most natural dimension with which to be concerned. After all, we can vary ξ by varying T but a is, of course, unchanged. It is the object of most analyses of critical phenomena to calculate these **anomalous** dimensions.

4.2 The Effective Potential and the Legendre Transform

Consider the partition function in the presence of an external source J:

$$\mathcal{Z}(J) = \int d\phi \; e^{-S(\phi)+J\phi} \; .$$

Denote $S(\phi, J) = S(\phi) - J\phi$.
Then

$$\mathcal{Z}(J) = e^{-S(\phi(J),J)} \int d\psi \; e^{-\frac{1}{2}S_J^{(2)}\psi^2 \; +\dots} \; ,$$

where

$$\left(\frac{\delta S}{\delta \phi}\right)(\phi, J) = 0 \quad \text{when} \quad \phi = \phi(J) \equiv \phi(\mathbf{x}; J),$$

and

$$\begin{aligned} S_J^{(2)}\psi^2 &= \int d\mathbf{y}d\mathbf{z} \left(\frac{\delta^2 S}{\delta\phi(\mathbf{y})\delta\phi(\mathbf{z})}\right)_{\phi=\phi(J)} \psi(\mathbf{y})\psi(\mathbf{z}) \; , \\ \psi(\mathbf{y}) &= \phi(\mathbf{y}) - \phi(\mathbf{y}; J). \end{aligned}$$

Space arguments will be suppressed wherever possible.
Thus

$$\log \mathcal{Z}(J) = -S(\phi(J), J) \; + \; \log \int d\psi \; e^{-\frac{1}{2}S_J^{(2)}\psi^2 \; +\dots} \; .$$

The integral generates **loop** corrections: this is the **loop expansion**. It is essentially the same as the perturbation expansion we saw before except that here the expansion is about the minimum of $S(\phi, J)$. Then we have terms of the form

$$\begin{aligned} \int d\psi \; e^{-\frac{1}{2}S_J^{(2)}\psi^2} \; e^{-\frac{1}{3!}S^{(3)}(J)\psi^3 \; -\dots} &= \int d\psi \; e^{-\frac{1}{2}S_J^{(2)}\psi^2} \left(1 - \frac{1}{3!}S^{(3)}(J)\psi^3 \; - \dots\right) \\ &= A\left[\det(S_J^{(2)})\right]^{-\frac{1}{2}} (1 \; + \; \dots). \end{aligned}$$

We get a set of gaussian integrals which generate the loops. For example, the 1-loop contribution to $\log \mathcal{Z}(J)$ is

$$-\frac{1}{2}\left[\det(S_J^{(2)})\right] .$$

In general

$$< \phi(\mathbf{x}_1)\ldots\phi(\mathbf{x}_n) >_c = \left(\frac{\delta^n \log \mathcal{Z}(J)}{\delta J(\mathbf{x}_1)\ldots\delta J(\mathbf{x}_n)}\right)_{J=0} .$$

(See problem sheet 2a for an explicit example of the 1-loop calculation.)
The **tree** diagrams are generated by $-S(\phi(J),J)$. Once we understand how this works we can verify the preceding statements. Consider the **approximation**

$$W(J) \equiv \log \mathcal{Z}(J) \approx W_T(J) = -S(\phi(J),J) .$$

$\phi(J)$ satisfies

$$\left(\frac{\delta S}{\delta \phi}\right)_{\phi=\phi(J)} = 0 .$$

This is the field equation and $\phi(J)$ is the **classical** solution. Then

$$\begin{aligned}\frac{\delta W_T(J)}{\delta J} &= -\frac{\delta S}{\delta\phi(J)}\frac{\delta\phi(J)}{\delta J} + J\frac{\delta\phi(J)}{\delta J} + \phi(J)\\ &= \phi(J).\end{aligned}$$

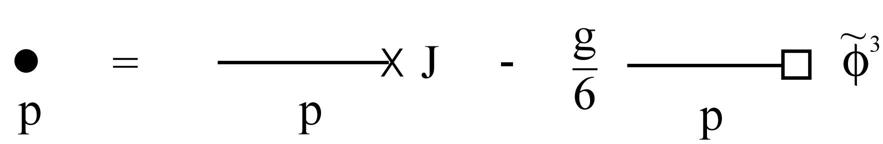
This field equation is explicitly of the form

$$(-\nabla^2 + m^2)\phi(\mathbf{x};J) + \frac{1}{6}g\phi^3(\mathbf{x};J) = J(\mathbf{x}) .$$

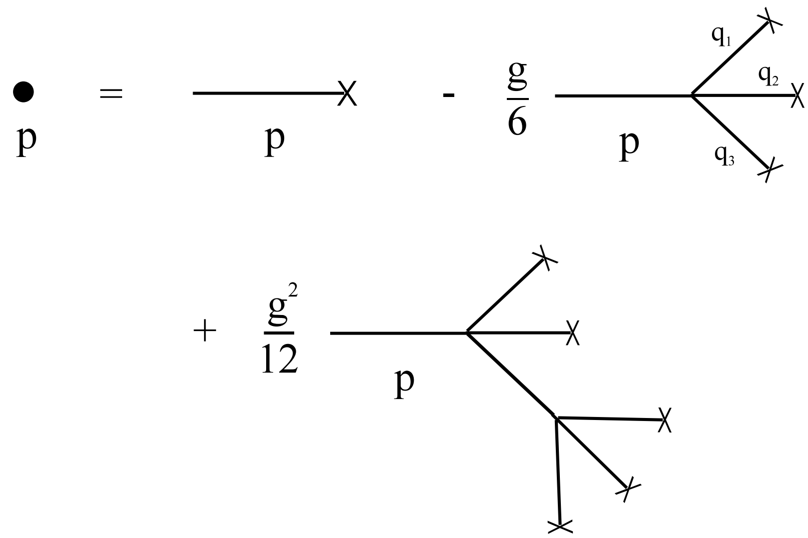
Clearly **all** derivatives of W with respect to J can be generated from the previous equation once this **classical** equation has been solved. In perturbation theory we have, in momentum space

$$\phi(\mathbf{p};J) = \frac{J(\mathbf{p})}{\mathbf{p}^2+m^2} - \frac{g}{6(\mathbf{p}^2+m^2)}\int d\mathbf{x}\ \phi^3(\mathbf{x};J)\ e^{-i\mathbf{p}\mathbf{x}} .$$

Or, diagrammatically



Iterating this equation gives



All momenta are integrated with momentum conserved at each vertex. This is just the **tree** expansion of $W(J)$. It is **generated** by the **classical field solution**, $\phi(J)$: i.e., there are **no** $\hbar$ factors.
E.g.,

$$\left(\frac{\delta^4 W_T(J)}{\delta J(\mathbf{p})\delta J(\mathbf{q}_1)\delta J(\mathbf{q}_2)\delta J(\mathbf{q}_3)}\right)_{J=0} = g\delta^4(\mathbf{p}+\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3) \ ,$$

This is the tree contribution to the 4-point vertex.
Now, two important points

(1) If $W_T(J)$ is the tree approximation to $W(J)$ as outlined above, then $W_T(J)$ is the **Legendre transform** of $S(\phi)$. That is, it follows directly from the above discussion that

$$W_T(J) = \max_\phi (J\phi - S(\phi)) \ ,$$

which **is** the Legendre transform of $S\phi)$.

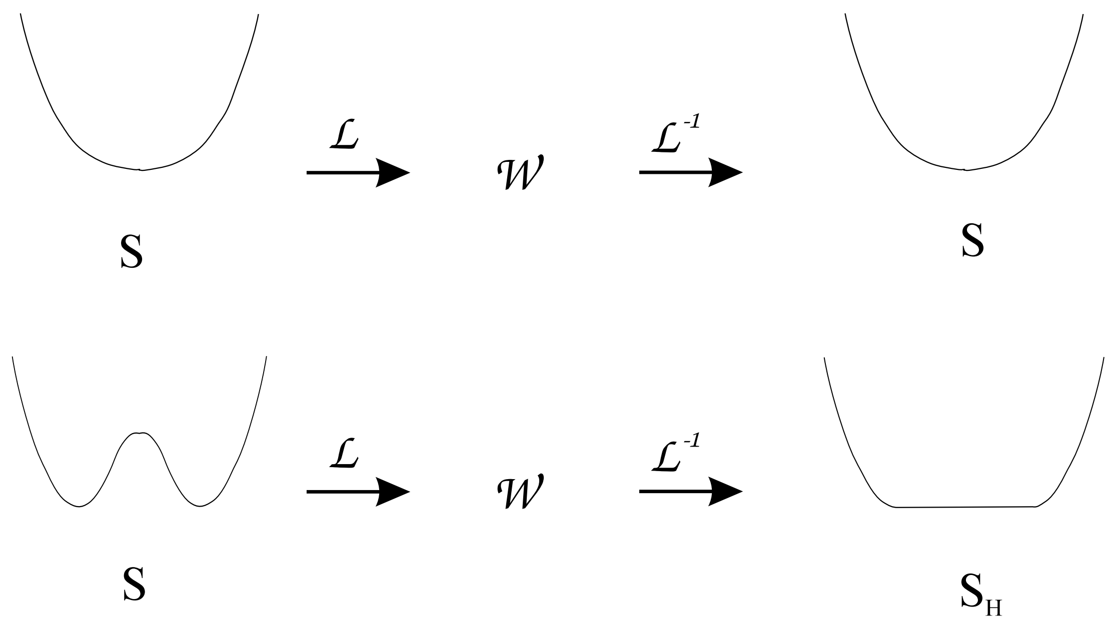
Then **define**

$$S_H(\phi) = \max_J (J\phi - W_T(J)) \ .$$

Clearly $S_H(\phi)$ is **not** a function of J, i.e.,

$$\frac{\partial S_H}{\partial J} = 0 \ .$$

If $S(\phi)$ is convex then $S_H(\phi) = S(\phi)$, otherwise $S_H(\phi)$ is the **convex hull** of $S(\phi)$:



This subtlety has a lot to do with the Maxwell construction and domains associated with a first order transition. It applies to a complete study of the symmetry broken phase but we shall not pursue it further here.

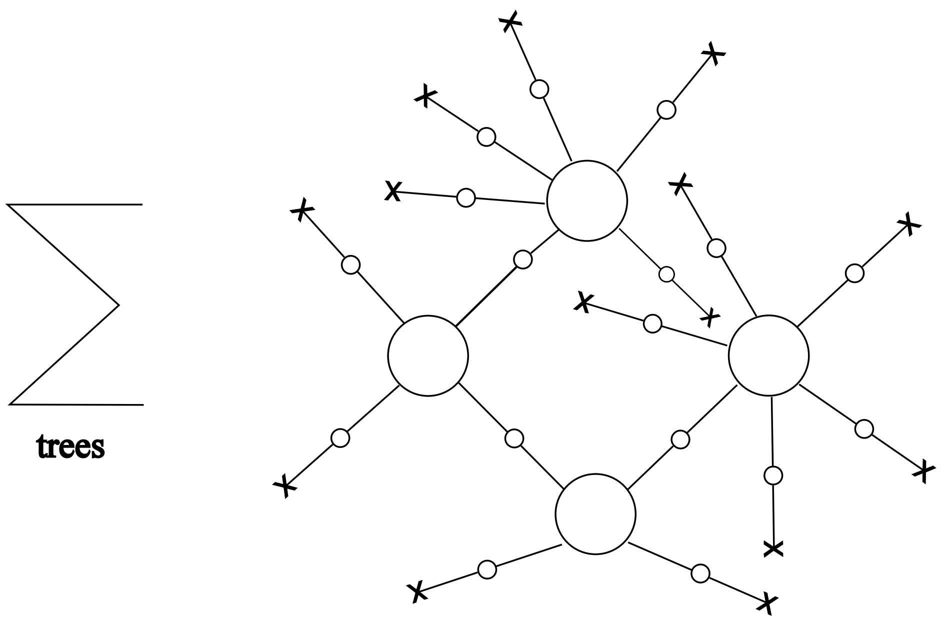
(2) For $\phi = M$, a constant, $S(M)$ was the **effective potential** which is minimized to give Landau's approximation to the free energy. Remember the **form** of $S(M)$ with an assumption of analyticity of the coefficients in T is what was needed: loops just modify T_c. Hence Landau's method is **classical**: it corresponds in **form** to tree diagrams. In general, $S_H(\phi)$ is the **effective** action of the **classical** or Landau theory. Note that $S(\phi)$ is **exactly** the action which appears in the expression for $\mathcal{Z}$

$$\mathcal{Z} = \int d\phi \; e^{-S(\phi)} \; .$$

That is, this $S(\phi)$ **is** the Legendre transform of $W_T(J)$. There are some niceties concerning whether one should use S or S_H here, but we will not discuss this further.

$W(J)$ generates the **connected** diagrams. Then schematically it must have the diagrammatic form

$W(J) \;=$



The vertices of the trees are the 1PI diagrams of the theory, i.e., they contain sums over all loop contributions but that cannot be separated into two disjoint pieces by cutting one line only.
We can generate these tree diagrams with an **appropriately chosen effective action**. This is the **effective action** of the **full** theory. It is the **Legendre transform** of $W(J)$:

$$S_E(\phi) \;=\; \max_J (J\phi - W(J)) \;,$$
$$S_E(\phi) \;=\; \sum_n \frac{1}{n!} \int d\mathbf{x}_1 \ldots d\mathbf{x}_n \; \Gamma_n(\mathbf{x}_1 \ldots \mathbf{x}_n)\phi(\mathbf{x}_1) \ldots \phi(\mathbf{x}_n) \;.$$

Hence $S_E(\phi)$ generates the 1PI diagrams of the theory.
$S_E(\phi)$ is now treated as a **classical** action since it is all that is needed to generate the whole theory by its tree expansion. The equilibrium free energy is

$$F = \frac{-W(0)}{\beta V} \;,$$

and

$$W(0) \;=\; \max_\phi (J\phi - S_E(\phi))_{J=0} \;,$$
$$\;=\; \max_\phi (-S_E(\phi)) \;.$$

Hence the free energy function $F(\phi)$ is identified as

$$F(\phi) = \frac{1}{\beta V} S_E(\phi) \;,$$

and equilibrium corresponds to minimizing $F(\phi)$ with respect to ϕ. This is like the Landau method **but** we cannot assume that the effective coupling constants in S_E are **analytic** in T: they are only for $D > D_c$.
Note

(1) $S_E(\phi)$ is always convex and, in fact, is the correct **effective** action describing the system for all phases.

(2) If J is a constant function then the magnetization is given by

$$\begin{aligned} M &= \frac{dW}{dJ} \quad \Rightarrow \\ \phi(J) &= M \ . \end{aligned}$$

In general, $J(\mathbf{x})$ and $M(\mathbf{x}) = \frac{dW}{dJ(\mathbf{x})}$ are **conjugate** variables. This is also shown by how they appear in the Legendre transformation. This is, of course, the same relationship that they have in thermodynamics.

4.3 The Ising model

The Hamiltonian is given by

$$-\beta H = S_i V_{ij} S_j + J_i S_i \ .$$

S_i is that variable at the i-th site with $S_i = \pm 1$. J_i is the external field and V_{ij} contains the couplings between neighbours: it need not be nearest neighbour. Introduce $X_i \in \mathcal{R}$ and note the identity

$$\begin{aligned} &\int \prod dX_i \ e^{-\frac{1}{4} X_i V_{ij}^{-1} X_j + S_i X_i} \\ &\qquad = \text{Const} \cdot e^{S_i V_{ij} S_j} \ . \end{aligned}$$

Then

$$\mathcal{Z}(J) = \sum_{S_i} \int \prod dX_i e^{-\frac{1}{4}(X_i - J_i) V_{ij}^{-1} (X_j - J_j) + S_i X_i} \ .$$

Now sum on S_i

$$\sum_{S=\pm 1} e^{SX} = 2 e^{\left(\log \cosh X\right)} \equiv 2 e^{A(X)} \ .$$

Hence, ignoring all irrelevant multiplicative constants

$$\mathcal{Z}(J) = \int \prod dX_i \ e^{-\frac{1}{4} X_i V_{ij}^{-1} X_j + \sum_i A(X_i)} \ .$$

We calculate the free energy by steepest descent (i.e. by expanding about the minimum of the exponent as we have done before) and keeping only the first term is equivalent to Landau theory: that is, to the tree approximation. Minimizing the action gives

$$\frac{1}{2}V_{ij}^{-1}(\bar{X}_j - J_j) = \frac{\partial A(\bar{(X_i)})}{\partial X_i} \ .$$

This defines $\bar{X}_i$ which minimizes the action. Thus in this approximation

$$\begin{aligned} W_T(J) &= \log \mathcal{Z}(J) \\ &= \frac{1}{4}(\bar{X}_i - J_i)V_{ij}^{-1}(\bar{X}_j - J_j) \ + \ \sum_i A(\bar{X}_i) \ . \end{aligned}$$

But

$$\begin{aligned} M_i &= \frac{\delta \log \mathcal{Z}(J)}{\delta J_i} \\ &= -\frac{1}{2}V_{ij}^{-1}(\bar{X}_j - J_j) \ , \end{aligned}$$

and using the defining equation for $\bar{X}$ above we get

$$M_i = \frac{\delta A}{\delta X_i}(\bar{X}_i) = \tanh(\bar{X}_i) \ .$$

(Remember, $\bar{X}$ and M are both functions of J.)
To find the free energy function $F(M)$ we take the Legendre transform of $W_T(J)$

$$-\beta N F(M) = \max_J \left(\mathbf{J}\cdot\mathbf{M} - W_T(J)\right) \ ,$$

where N is the number of sites. Then

$$\beta N F(M) = -V_{ij}M_iM_j \ + \ \frac{1}{2}\sum_i \left[(1+M_i)\log(1+M_i) \ + \ (1-M_i)\log(1-M_i)\right] \ .$$

Note that $\beta NF(M)$ in this approximation **is** the **correct** quantum action to be used in the full theory which includes all loops. This is because it **defines** the tree diagrams of the theory and hence gives **all** the **bare vertices**. the loops are built up in the usual way from **these** vertices.
For constant magnetization $M_i = M$ and M small

$$\beta F(M) = -VM^2 + \frac{(1+M)}{2}\left((M - \frac{M^2}{2} + \frac{M^3}{3} + \ldots\right) \ + \ \frac{(1-M)}{2}\left((-M - \frac{M^2}{2} - \frac{M^3}{3} + \ldots\right) \ .$$

F is the free energy per site and $V = \sum_j V_{0j}$. Then

$$\beta F(M) = (\frac{1}{2} - V)M^2 + \frac{1}{12}M^4 + \ldots$$

Since $V = V(\beta)$ we have a second order transition at $\beta = \beta_c$ given by $V(\beta_c) = \frac{1}{2}$.
In the simplest case

$$V_{ij} = \beta\kappa_{ij\Rightarrow} \ \ \beta_c = \frac{1}{2\kappa} \ , \quad \kappa = \sum_j \kappa_{0j}.$$

Notes

(1) This is the mean-field approximation and we know that the critical exponents are analytic only correct for $D > D_c$.

(2) The external field method together with the Legendre transform enabled us to calculate the free energy as a function of the **relevant** order parameter, $\mathbf{M}$. Near the transition the Ising model can be described by a scalar field theory with the full theory based on the action $F(\phi)$. F is the function defined above by $F(\mathbf{M})$ for general (i.e. **not** constant) $\mathbf{M}$ (ϕ is identified with $\mathbf{M}$ to make the correspondence clear). Thus we have found the action for a scalar field theory which is, in every way, equivalent to the Ising model near a continuous phase transition. This is the phenomenon of **universality**.

Universality: many models which have disparate descriptions on the microscopic scale exhibit the **same** critical properties with the **same** critical exponents. Near critical points these models are described by the **same** field theory and belong to the same **universality class**. Here we have found that the Ising model and scalar field theory belong to the same universality class.

(3) Even for $D > D_c$ loop contributions will give corrections to β_c. The 1-loop correction to the tree-level effective action is given by $\frac{1}{2}\det F^{(2)}(\phi)$. This contribution is the subject of example sheet 2a.

4.4 Calculation of the Critical Index

This section is concerned with the calculation of the critical index ν in the Ising model. We work with ϕ^4 field theory in D dimensions.

$$\begin{aligned} S(\phi) &= \int d\mathbf{x}\, \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}g\phi^4 \;+\ldots \\ \left[m^2\right] &= 2 \qquad [g] = 4 - D \equiv \epsilon\ . \end{aligned}$$

We may imagine that we have obtained this action by integrating out the short wavelength modes. In general all types of interaction are present. Clearly ϕ has been renormalized to give the canonical form to the kinetic term. Alternatively, we can accept that $S(\phi)$ has been generated from the Ising model by the transformation described in the previous section. We note for interest that the Ising model is identical to ϕ^4 field theory in the limit $m^2 \to -\infty,\ \ g \to \infty$ such that $6|m^2|/g = 1$.
Suppose that we have integrated over all momemta $p > \Lambda$ and that the effective parameters are $m^2(\Lambda, T)$ and $g(\Lambda, T)$. We shall also allow for multiplicative field renormalization constant $Z(\Lambda, T)$. Now integrate over the next momentum slice. Write

$$\phi(\mathbf{x}) = \phi_>(\mathbf{x}) + \phi_<(\mathbf{x})\ .$$

$\phi_>$ contains contributions from momenta in the range $\Lambda - \delta\Lambda \leq p < \Lambda$.

$\phi_<$ contains contributions from momenta in the range $0 \leq p < \Lambda - \delta\Lambda$.

Then

$$\mathcal{Z} = \int d\phi_< d\phi_> \; e^{-S(\phi_> + \phi_<)} \quad .$$

We do the integrals over $\phi_>$ to get an action as a function of $\phi_<$ only.

$$\begin{aligned} S(\phi_< + \phi_>) = S(\phi_<) + S(\phi_>) \quad &+ \quad \int d\mathbf{x} \left[\nabla\phi_> \nabla\phi_< + m^2 \phi_< \phi_> \right] \\ &+ \quad \frac{g}{12} \int d\mathbf{x} \left[2\phi_<^3 \phi_> + 3\phi_<^2 \phi_>^2 + 2\phi_< \phi_>^3 \right] \; . \end{aligned}$$

The term quadratic in the fields vanishes identically. Since in p-space $\phi_>$ and $\phi_<$ have disjoint support

$$\int d\mathbf{x} \phi_>(\mathbf{x}) \phi_<(\mathbf{x}) = \int d\mathbf{p} \tilde{\phi}_<(\mathbf{p}) \tilde{\phi}_>(\mathbf{p}) = 0 \; .$$

Then

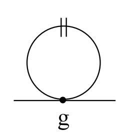
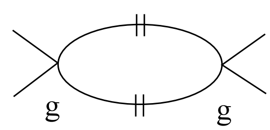
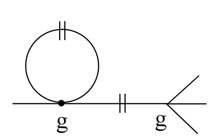
$$\mathcal{Z} = \int d\phi_< \; e^{-S(\phi_<)} \; \int d\phi_> \; e^{-S(\phi_>)} \; e^{-\frac{g}{12} \int d\mathbf{x} \left[2\phi_<^3 \phi_> + 3\phi_<^2 \phi_>^2 + 2\phi_< \phi_>^3 \right]} \; .$$

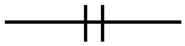
As before we expand all exponentials of non-quadratic terms

$$\begin{aligned} \mathcal{Z} \;\; = \; & \\ & \int d\phi_< \; e^{-S(\phi_<)} \; \int d\phi_> \; e^{-\frac{1}{2}(\nabla\phi_>^2 + \frac{1}{2} m^2 \phi_>^2)} \cdot \left[1 - \frac{g}{4!} \phi_>^4 + \ldots \right] \cdot \\ & \left[1 - \frac{g}{12} (2\phi_<^3 \phi_> + 3\phi_<^2 \phi_>^2 + 2\phi_< \phi_>^3) + \frac{g^2}{288} (2\phi_<^3 \phi_> + 3\phi_<^2 \phi_>^2 + 2\phi_< \phi_>^3)^2 - \ldots \right] \; . \end{aligned}$$

The integrals over odd functions of $\phi_<$ vanish and so will be omitted from now on.
After the $d\phi_>$ integrals have been done we are left with a **polynomial** in $\phi_<$ and its derivatives. We gather these terms up and absorb them into $S(\phi_<)$ by a redefinition or **renormalization** of the coupling constants.

The important one-loop terms and their diagrammatic representation are

$-\frac{g}{4} \int d\mathbf{x} \phi_>^2 \phi_<^2$			$\frac{1}{4} \; \phi_<^2(\mathbf{x})$
$\frac{g^2}{32} \int d\mathbf{x} \phi_>^2 \phi_<^2 \int d\mathbf{y} \phi_>^2 \phi_<^2$	$\frac{1}{16} \phi_<^2(\mathbf{x})$		$\phi_<^2(\mathbf{y})$
$\frac{g^2}{36} \int d\mathbf{x} \phi_> \phi_<^3 \int d\mathbf{y} \phi_>^3 \phi_<$	$\frac{1}{12} \phi_<(\mathbf{x})$		$\phi_<^3(\mathbf{y})$

 is the propagator of $\phi_>$ and has support in p-space only for $\Lambda - \delta\Lambda \le p < \Lambda$. The last of the terms above is zero since $\phi_<$ has disjoint p-space support from that of the intermediate propagator.
We can now read off the renormalizations of $m^2(\Lambda, T)$ and $g(\Lambda, T)$ to 1-loop

$$m^2(\Lambda - \delta\Lambda, T) = m^2(\Lambda, T) + \frac{g(\Lambda, T)}{2} \int_{\Lambda-\delta\Lambda}^{\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{\mathbf{p}^2 + m^2(\Lambda, T)} \; .$$

In momentum space we have for the 4-point function

$$\frac{g^2}{16} \int \frac{d^D q}{(2\pi)^D} \tilde{\phi}_<^2(\mathbf{q}) \; \Gamma(\mathbf{q}) \; \tilde{\phi}_<^2(-\mathbf{q}) \; ,$$

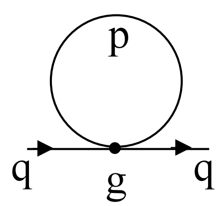
where

$$\Gamma(\mathbf{q}) = \int_{\Lambda-\delta\Lambda}^{\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{\mathbf{p}^2 + m^2(\Lambda, T)} \frac{1}{(\mathbf{p}+\mathbf{q})^2 + m^2(\Lambda, T)} \quad ,$$

with the restriction $\Lambda - \delta\Lambda \le |\mathbf{p} + \mathbf{q}| < \Lambda$.
We can expand $\Gamma(\mathbf{q}) = \Gamma(0) + q^2 \Gamma'(0) + \ldots$ and then the operator generated is

$$\frac{g^2}{16} \left[\Gamma(0) \int d\mathbf{x} \phi_<^4 + 4\Gamma'(0) \int d\mathbf{x} (\phi_< \nabla \phi_<)^2 + \ldots \right] \; .$$

Note that **new** operators are generated as well as ones we are interested in. It should be remarked at this point that the analysis of the higher dimension operators generated in this way is not easy and is better approached by other renormalization group techniques which will be touched upon later. However, there does exist work that has pushed the above style of analysis, i.e., momentum thinning, successfully to the study of the role of such operators. To 1-loop we do not encounter these difficulties. Also note that because the graph



is independent of $\mathbf{q}$ we do not generate the operator $(\nabla \phi_<)^2$ to 1-loop. More on this later.
Then we find the equation for g

$$g(\Lambda - \delta\Lambda, T) = g(\Lambda, T) - \frac{3}{2} g^2(\Lambda, T) \int_{\Lambda-\delta\Lambda}^{\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{(\mathbf{p}^2 + m^2(\Lambda, T))^2} \; .$$

We note that from the equation for m^2 the renormalization is **positive**. Hence if we want

$$\lim_{\Lambda \to 0} m^2(\Lambda, T_c) = 0 \; ,$$

we require $m^2(\Lambda, T) < 0$: the "bare" mass is **negative**. The expansion is still fine since it is only necessary that the gaussian exponent gives convergent integrals. This demands only that

$$\Lambda^2 > -m^2(\Lambda, T) \qquad \forall\Lambda \ .$$

It can be verified in what follows that this inequality is always respected.
We now rewrite the two renormalization equations in differential form in terms of **dimensionless** quantities.

$$\begin{aligned} u^2(b,T) &= \Lambda^{-2} m^2(\Lambda, T) \\ \lambda(b,T) &= \Lambda^{-\epsilon} g(\Lambda, T) \ , \end{aligned}$$

where $b = \log(\Lambda_0/\Lambda)$ and $\epsilon = 4 - D$. Λ_0 is the cut-off for the original theory. Then we have the evolution equations

$$\begin{aligned} \frac{du^2}{db} &= 2u^2 + \frac{\Omega_D}{2(2\pi)^D}\frac{\lambda}{1+u^2} \\ \frac{d\lambda}{db} &= \epsilon\lambda - \frac{3\Omega_D}{2(2\pi)^D}\frac{\lambda^2}{(1+u^2)^2} , \end{aligned}$$

Ω_D is the surface area of a unit sphere in D dimensions: $\Omega_4 = 2\pi^2$.
In order to integrate these equations the propagator for $\phi_<$ must take the canonical form $1/(\mathbf{p}^2 + m^2)$. In general, the renormalized action, $S_R(\phi_<)$ will be of the form

$$\begin{aligned} S_R(\phi_<) &= S(\phi_<) + \frac{1}{2}\delta m^2 \phi_<^2 + \frac{1}{4!}\delta g \phi_<^4 \\ &- \frac{1}{2}\delta(\log\ Z)(\nabla\phi_<)^2 + \frac{1}{4!}\delta g'(\phi_< \nabla\phi_<)^2 + \ldots \end{aligned}$$

Thus we must rescale the field to give the **canonical** quadratic part. Thus the new field for the next iteration of the procedure is

$$\phi(\mathbf{x}) = (1 - \frac{1}{2}\frac{d\log\ Z}{db}\delta b)\phi_<(\mathbf{x}) \ .$$

This **defines** the renormalized field ϕ for finite b

$$\phi(\mathbf{x}, b) = Z^{-\frac{1}{2}}(b,T)\phi(\mathbf{x}, 0) \ .$$

Now we have **additional** renormalizations of u^2 and λ

$$u_R^2(b,T) = Z(b,T)u^2(b,T) \qquad \lambda_R(b,T) = Z^2(b,T)\lambda(b,T) \ ,$$

and then it follows that

$$\begin{aligned} \frac{du_R^2}{db} &= \frac{d(\log\ Z)}{db} u_R^2 + Z^{-1}\left(\frac{du^2}{db}\right)_{Z=1} \\ \frac{d\lambda_R}{db} &= 2\frac{d(\log\ Z)}{db}\lambda_R + Z^{-2}\left(\frac{d\lambda}{db}\right)_{Z=1} \ . \end{aligned}$$

However, we have noted that to the order in which we are working,(1-loop), $Z = 1$. Since we shall be expanding in ϵ this means that $Z = 1$ to $O(\epsilon)$ and so $d(\log\ Z)/db \sim O(\epsilon^2)$. Henceforth, we set $Z = 1$ above, and identify (u_R, λ_R) with (u, λ).
It is important to remark at this stage that the renormalization choices I am making (e.g. defining the coefficient of the kinetic term to be unity), are not forced on me by physics: they are convenient for the perturbative-style analysis. Other choices may be necessary in more complex situations in order to reveal the structure of more complex phase transitions in the most effective way.
The transformation takes the form

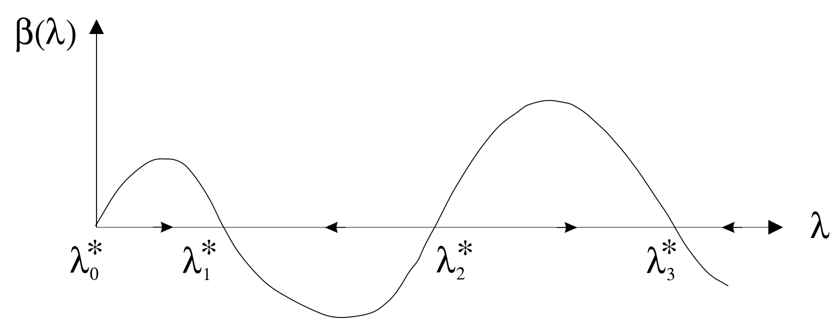
$$\begin{array}{ccc} S(\phi, m^2, g_n, \Lambda_0) & \stackrel{\text{integration}}{\longrightarrow} & S(Z^{-\frac{1}{2}}\phi, m^2(b), g_n(b), e^{-b}\Lambda_0) \\ & \stackrel{\text{field renorm.}}{\longrightarrow} & S(\phi, m^2(b), g_n(b), e^{-b}\Lambda_0)\ . \end{array}$$

Here $Z \equiv Z(b)$ and g_n is the coupling associated with field monomial of order n. This forms the major part of the renormalization group transformation to be discussed later. The derived equations are a form of the **renormalization group** flow equations for (u^2, λ). Note that by dealing with **dimensionless quantities** all explicit reference to the cut-off Λ disappears.

Consider a flow equation of the form

$$\frac{d\lambda}{db} = \beta(\lambda)\ .$$

This equation has fixed points λ_p^* where $\beta(\lambda_p^*) = 0$, e.g.,



If	$\lambda_0^* < \lambda(0,T) < \lambda_2^*$	then	$\lim_{b\to\infty} \lambda(b,T) = \lambda_1^*$
If	$\lambda_2^* < \lambda(0,T) < \lambda_4^*$	then	$\lim_{b\to\infty} \lambda(b,T) = \lambda_3^*$

Fixed points λ_p^* where $\beta'(\lambda_p^*) < 0$ are **Infra-Red** attractive, and alternatively if $\beta'(\lambda_p^*) > 0$ the points are **Ultra-Violet** attractive.
Note that

(1) In **statistical mechanics** we are **given** the UV or **bare** coupling $\lambda(0, T)$, and the IR or **renormalized** coupling is an IR point of $\beta(\lambda)$.

(2) In **quantum field theory** we are **given** the IR or **renormalized** coupling $\lambda(\infty, T)$: it is a renormalization condition from experiment or other **low energy** condition. Since $\lambda(\infty, T)$ is not generally a fixed point of $\beta(\lambda)$ it must be that the UV or **bare** coupling is a UV fixed point of β. (It should be noted that although we have worked exclusively in Euclidean space it is believed that the divergences are the same as for the Minkowski version of the theory. This belief can be demonstrated in perturbation theory.)

In our calculation we find the fixed points (u^{*2}, λ^*)

$\epsilon < 0$	$(0 \;,\; 0)$	trivial f.p. - IR attractive
$\epsilon > 0$	$(0 \;,\; 0)$	trivial f.p. - UV attractive
	$(-\frac{\epsilon}{6} \;,\; \frac{16\pi^2}{3}\epsilon)$	non-trivial f.p. - IR attractive

It is believed that there are no others, not even at $\lambda = \infty$. This means for quantum field theory that there is no UV fixed point to be associated with the bare coupling **except** the one at $(0, 0)$ when $\epsilon < 0$. Hence, it follow that

$$\begin{array}{lll} \epsilon < 0 & \lambda_R & = 0 \\ \epsilon \geq 0 & \lambda_R & \leq \frac{16\pi^2}{3}\epsilon, \end{array}$$

where $\lambda^R = \lambda(\infty, T)$. This is the statement of triviality for ϕ^4 quantum field theory. It means that in dimension four or greater the only consistent renormalized ϕ^4 theory in the limit of infinite cut-off is a **free** theory: $\lambda_R = 0$. Of course, we are not required to take the cut-off to infinity, rather we should keep it much larger than the physical momentum scales on which the theory is being applied. Even then, for **finite** Λ the analysis presented gives an upper limit to λ_R. Such upper limits have been used to set upper bounds on the Higgs mass in the most common version of the standard model.

Since ϵ is small we can get a reliable expansion in ϵ for both u^{*2} and λ^*. This is the epsilon expansion

$$\lambda^* = \sum_{n=1}^{\infty} a_n \epsilon^n \;.$$

It is hoped that the radius of convergence is greater than one, thus including results for $D = 3$.
Write the equation for u^2 as

$$\frac{du^2}{db} = f(u^2) \;.$$

We now show how to derive the postulated scaling behaviour from the flow equations.
For $\Lambda\xi \ll 1$ all parameters have reached their **low-energy renormalized** values.
Let

$$A = \frac{1}{\Lambda\xi} \gg 1 \quad \Rightarrow \quad b = \log(A\xi\Lambda_0) \;.$$

We have

$$u(b,T) = \frac{m(b,T)}{\Lambda} = \frac{1}{\Lambda\xi} = A \ .$$

This follows since for $A \gg 1$, $m(b,T) = \xi^{-1}$. Then

$$\begin{aligned} \int_{u^2(\bar{b},T)}^{A^2} \frac{du^2}{f(u^2)} &= b - \bar{b} \\ &= \log(A\xi\bar{\Lambda}) \\ &= \log(\xi) + C \ , \end{aligned}$$

with $\bar{\Lambda} = e^{-\bar{b}}\Lambda_0$ and $\xi^{-1} \ll \bar{\Lambda} \ll \Lambda_0$, but such that

$$\lambda(\bar{b},T) \approx \lambda^* \ .$$

This will always be possible if ξ is **big** enough. The key point is that we consider the flow first from Λ_0 to $\bar{\Lambda}$ so that λ flows very close to its IR attractive fixed point. We still have the condition that $\bar{\Lambda} \gg \xi$. We then consider the flow to $\Lambda \ll \xi^{-1}$. For this part of the flow λ is **fixed** at its fixed point value λ^*. It is from **this** latter part of the flow that the scaling form for ξ as a function of t follows. The sequence of flows can be summarized thus

$$\begin{array}{ccccccc} \Lambda_0 & \gg & \bar{\Lambda} & \gg & \xi^{-1} & \gg \Lambda & \\ & & & & & & \\ \lambda(0,T) & & \lambda(\bar{b},T) \approx \lambda^* & & & \lambda(\bar{b},T) \approx \lambda^* & \\ m^2(0,T) & & m^2(\bar{b},T) & & & m^2(b,T) = \xi^{-2} & \\ 0 & \rightarrow & \bar{b} & & \rightarrow & b & \end{array}$$

Mow, for $T = T_c$, $\quad \xi = \infty$. Hence it follow from above that

$$\int_{u^2(\bar{b},T_c)}^{A^2} \frac{du^2}{f(u^2)} = \infty \ .$$

Thus $u(\bar{b},T_c) = u^*$ and the integral has a logarithmic singularity. We write

$$u^2(\bar{b},T_c) = u^{*2} + Bt, \qquad t = \frac{T - T_c}{T_c} \ .$$

Then

$$\begin{aligned} \int_{u^{*2}+Bt}^{A^2} \frac{du^2}{f(u^2)} &= \log\ \xi(t) + C \ \Rightarrow \\ \frac{d\xi(t)}{\xi(t)} &= \frac{Bdt}{f(u^{*2}+Bt)} \\ &= -\frac{1}{f'(u^{*2})}\frac{dt}{t} \ . \end{aligned}$$

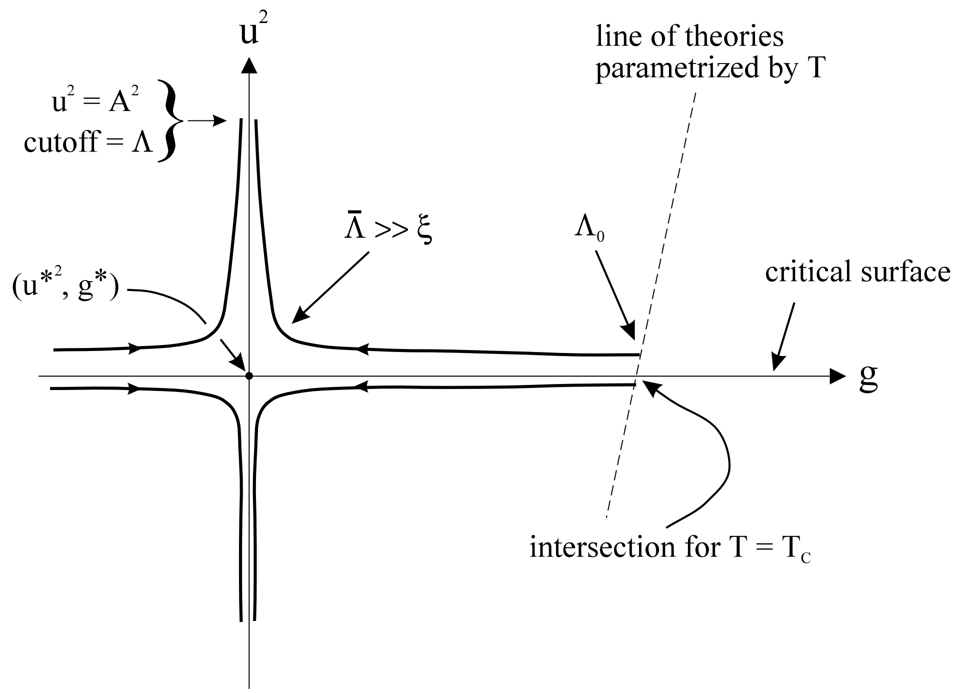
From above we have that

$$f'(u^{*2}) = \left(\frac{\partial f}{\partial u^2}\right)_{(u^{*2},\lambda^*)} = 2 - \frac{1}{3}\epsilon \ .$$

Hence we obtain the prediction that

$$\xi(t) = Dt^{-\nu} \quad \text{with} \quad \nu = \frac{1}{2} + \frac{1}{12}\epsilon \ .$$

The solution of the full flow equations looks like



We remark the following points

(1) In a general description, the theory for some intermediate cut-off will be given by the values of a complete set of coupling constants associated with all possible field monomials in the action. These couplings, denoted h_i, will be functions of Λ and T, and they flow in such a way that the low-energy predictions of the theory are independent of Λ. We would expect the flow equations to take the form

$$\frac{dh_i}{db} = \beta_i(\mathbf{h}) \ .$$

The diagram of the flow is then, in principle, an ∞-dimensional space.

(2) The **critical surface** is the manifold of all theories in coupling constant space, $\mathbf{h}$, which flow to a fixed point. If there is more than one fixed point then the critical surface is divided into **domains of attraction** each associated with one fixed point. All theories in the critical surface have $\xi = \infty$, i.e., they are at a continuous transition. This follows because it requires $b \to \infty$ to flow to $u^2 = A^2$ which is the reference theory. Hence for all **finite** b we must have $\Lambda \equiv b\Lambda_0 > \xi^{-1}$ which implies that $\xi = \infty$.

(3) As T changes the theory follows a trajectory (dotted line) which intersects the critical surface at $T = T_c$. As shown above the critical exponents are given by the behaviour of trajectories in the **neighbourhood** of the IR fixed point. Hence we only need to calculate $\beta_i(\mathbf{h})$ in this neighbourhood.

(4) The number of parameters we need to tune in order to intercept the critical surface is clearly equal to the number of **unstable** directions at the fixed point. In our calculation the number of such unstable parameters is one, namely the temperature T (i.e., $u^2(\bar{b}, T)$ which is controlled by T). Thus the transition corresponds to an ordinary critical point.

A fixed point with two unstable directions requires two external variables to be tuned e.g., T, P, and thus corresponds to a tricritical or bicritical point.

(5) The critical surface separates the two phases i.e., the symmetric phase from the broken phase.

(6) We expand $\beta_i(\mathbf{h})$ about the fixed point, $\mathbf{h}^*$. This linearizes the flow equations. Writing $\Delta_i = h_i - h_i^*$

$$\frac{d\Delta_i}{db} = R_{ij}\Delta_i + O(\Delta^2) \ .$$

The eigenvectors, $\mathbf{e}_p$, of R_{ij} define the flow directions and the associated eigenvalues, μ_p determine the stability properties

$$\mu_p < 0 \ \text{ stable}, \quad \mu_p > 0 \ \text{ unstable}, \quad \mu_p = 0 \ \text{ tricky} \ .$$

(7) For operator $g_n\phi^{2n}$ we have $[g_n] \equiv d_n = n(D-2) - D$. Then $\beta_n(\mathbf{h})$ has the linear term

$$\beta_n(\mathbf{h}) = -d_n h_n \ + \ldots$$

(The h_n are the dimensionless versions of the g_n.) This coupling will be **stable** for $d_n > 0$ unless there are any very large renormalizations from other terms. Although large corrections cannot be ruled out it is expected that since, in practical cases, the d_n are integers we find that the operators are stable if

$$n > \frac{D}{D-2} \ .$$

Such operators are said to be **irrelevant** since their couplings automatically flow onto the critical surface and do not need to be "tuned".

$D = 4$	$n > 2$	$\Rightarrow$	$\phi^6, \ \phi^8, \ldots$	irrelevant
$D = 3$	$n > 3$	$\Rightarrow$	$\phi^8, \ldots$	irrelevant

This reflects the result of our earlier **perturbative** calculation of D_c and the criteria for the validity of Landau theory.

(7) The flow equations are a coupled non-linear set. The h_n of different field monomials get mixed. This reflects the operator mixing induced by renormalization. In the linearized version the eigenvalues of R_{ij} define eigenoperators: linear combinations of field monomials which do not mix with each other. This can be generalized to the full non-linear version of the flow equations.

(8) In the Landau theory we associate the coefficient of M^4 with the **low-momentum** dimensionful coupling

$$g(T) = \Lambda_0^\epsilon \lim_{b\to\infty} \; e^{-\epsilon b}\lambda(b,T) \; .$$

This is because $g(T)$ is the coefficient of M^4 in the **effective** action derived by the Legendre transform. ($Z = 1$ to $O(\epsilon)$ otherwise we would need to include Z^2 in the definition of $g(T)$).

Let

$$g(b,T) = e^{-\epsilon b}\lambda(b,T) \; .$$

Then the equation for the flow of λ in the neighbourhood of the fixed point implies

$$\frac{dg}{db} = -\frac{3}{16\pi^2} \; e^{\epsilon b} g^2 \; ,$$

or

$$\frac{1}{g(\bar{b})} - \frac{1}{g(b)} = -\frac{3}{16\pi^2} \; e^{\epsilon\bar{b}} \left(e^{\epsilon(b-\bar{b})} - 1 \right) \; ,$$

and thus

$$\begin{aligned} g(b) &= \frac{g(\bar{b})}{1 + \frac{a(\epsilon)}{\epsilon}\left(e^{\epsilon(b-\bar{b})} - 1 \right)} \; . \\ a(\epsilon) &= \frac{3}{16\pi^2} \; e^{\epsilon\bar{b}} \; . \end{aligned}$$

Denoting $g(T) \equiv g(\infty, T)$ we have the following results

$\epsilon > 0:$ $g(T) = 0$. Hence Landau's method fails and the relevant physics is encoded in **how** $g(b,T) \to 0$ as $b \to \infty$.

$\epsilon < 0:$ We have

$$g(T) = \frac{g(\bar{b})}{1 + \frac{a(\epsilon)}{|\epsilon|}} \; .$$

Since here $g(T)$ is finite and non-zero Landau's method works. As before we are assuming that $\bar{b}$ is large enough so that the equations we integrated to get this result are valid. In this case $g(T)$ is actually independent of $\bar{b}$.

$\epsilon = 0 :\ g(T) = 0$. Landau theory fails but

$$
\begin{aligned}
g(b) &= \frac{g(\bar{b})}{1 + \frac{3}{16\pi^2}(b - \bar{b})} \\
g(b) &= \frac{g(\bar{b})}{1 + \frac{3}{16\pi^2}\log\frac{\bar{\Lambda}}{\Lambda}} .
\end{aligned}
$$

I.e., only logarithmic violations of Landau theory.

4.5 The Renormalization Group in General

Our procedure so far consists of "thinning" the degrees of freedom by integrating out the high momentum modes. We can define the transformation ($\lambda < 1$)

$$S(\lambda)H(\phi) = H(\lambda, \phi, \lambda\Lambda) ,$$

with $H(\phi) \equiv H(1, \phi, \Lambda)$ as the **bare** or **microscopic** Hamiltonian. The explicit λ dependence is there because the couplings have changed in a λ-dependent way. The $S(\lambda)$ form a semi-group

$$S(\lambda_1)S(\lambda_2) = S(\lambda_1\lambda_2) .$$

There is no inverse because of the existence of fixed points: information lost cannot be retrieved.
It is **convenient** to perform a **rescaling** at this stage

$\mathbf{x} \to \mathbf{x}' = \lambda\mathbf{x}$		length rescaling
$\mathbf{q} \to \mathbf{q}' = \lambda^{-1}\mathbf{q}$		momentum rescaling
$\phi \to \phi'$	$\phi'(\mathbf{q}') = A^{-\frac{1}{2}}(\lambda)\phi(\mathbf{q})$	field rescaling
$\lambda\Lambda \to \Lambda$		cut-off rescaling

The cut-off is **rescaled** back to its original value. The new fields ϕ' have support $0 \leq \mathbf{q}' \leq \Lambda$ in momentum space, i.e., the same support as the original fields before thinning. $A(\lambda)$ is chosen by **convenience**. It may be chosen so that the coefficient of $\frac{1}{2}(\nabla\phi)^2$ in the new Hamiltonian is one, but other choices can turn out to be more appropriate in more complex theories.
We then define the **renormalized** Hamiltonian $H_R(\phi, \Lambda)$ by

$$\mathcal{H}_R(\phi', \Lambda) = H(\lambda, \phi, \lambda\Lambda) .$$

For example consider the kinetic term in H

$$Z^{-1}(\lambda)\int_0^{\lambda\Lambda} \frac{d^Dq}{(2\pi)^D}\ \mathbf{q}^2|\tilde{\phi}(\mathbf{q})|^2 .$$

Change variables to $\mathbf{q}' = \lambda^{-1}\mathbf{q}$

$$
\begin{aligned}
&= Z^{-1}(\lambda)\lambda^{D+2}\int_0^{\Lambda} \frac{d^Dq}{(2\pi)^D}{}'\ \mathbf{q}'^2|\tilde{\phi}(\mathbf{q})|^2 \\
&= Z^{-1}(\lambda)\lambda^{D+2}\int_0^{\Lambda} \frac{d^Dq}{(2\pi)^D}{}'\ \mathbf{q}'^2 A(\lambda)|\tilde{\phi}'(\mathbf{q}')|^2 \\
&= A(\lambda)Z^{-1}(\lambda)\lambda^{D+2}\int_0^{\Lambda} \frac{d^Dq}{(2\pi)^D}\ \mathbf{q}^2|\tilde{\phi}'(\mathbf{q})|^2 .
\end{aligned}
$$

If we choose $A(\lambda) = Z(\lambda)\lambda^{-(D+2)}$, then the kinetic term in $H_R(\phi, \Lambda)$ has the canonical form.

The renormalization group transformation is $R(\lambda)$ where

$$R(\lambda)H(\phi, \Lambda) = H_R(\phi, \Lambda) \ .$$

H_R is defined on the same **lattice** as is H, **but** all **dimensionful** observables are scaled in the appropriate manner.
Define

$$G(\mathbf{q}_1, \ldots, \mathbf{q}_n) =< \phi(\mathbf{q}_1) \ldots \phi(\mathbf{q}_n) > \ .$$

Then we must have

$$\begin{array}{ccc} < \phi(\mathbf{q}_1) \ldots \phi(\mathbf{q}_n) > & = & A^{\frac{n}{2}}(\lambda) < \phi(\frac{\mathbf{q}_1}{\lambda}) \ldots \phi(\frac{\mathbf{q}_n}{\lambda}) >_R \\ \text{computed with } H & & \text{computed with } H_R \\ G(\mathbf{q}_1, \ldots, \mathbf{q}_n) & = & A^{\frac{n}{2}}(\lambda) G_R(\frac{\mathbf{q}_1}{\lambda}, \ldots, \frac{\mathbf{q}_n}{\lambda}) \end{array}$$

With the choice for $A(\lambda)$ above, H_R is a function of the set of **dimensionless** couplings which we denote here by $\mathbf{g}(\lambda)$. then

$$H_R(\phi, \Lambda) = H(\mathbf{g}(\lambda), \phi, \Lambda) \ ,$$

and

$$R(\lambda)H(\mathbf{g}_0, \phi, \Lambda) = H(\mathbf{g}(\lambda), \phi, \Lambda) \ , \qquad \mathbf{g}_0 = \mathbf{g}(1) \ .$$

This **defines** the **RG transformation** on the couplings

$$R(\lambda) \ \mathbf{g}_0 = \mathbf{g}(\lambda)$$

and since $R(\lambda_1)R(\lambda_2) = R(\lambda_1\lambda_2)$ we have

$$R(\lambda_2) \ \mathbf{g}(\lambda_1) = \mathbf{g}(\lambda_1\lambda_2) \ .$$

Thus

$$\begin{aligned} R(1-\alpha) \ \mathbf{g}(\lambda) &= \mathbf{g}(\lambda) - \alpha\lambda\frac{d}{d\lambda}\mathbf{g}(\lambda) \ + \ldots \quad \Rightarrow \\ \beta_i(\mathbf{g}) &= -\frac{dg_i(\lambda)}{d\log\lambda} \\ &= \lim_{\alpha\to 0}\frac{1}{\alpha}\left[R(1-\alpha)g_i(\lambda) - g_i(\lambda)\right] \ . \end{aligned}$$

This defines the function $\beta_i(\mathbf{g})$.

(1) β_i has no explicit dependence on Λ if $\mathbf{g}$ is a vector of dimensionless couplings. The g_i correspond to the dimensionful couplings scaled by Λ.

(2) β_i does not depend explicitly on λ: the only λ-dependence is through $\mathbf{g}(\lambda)$ in H. This follows since $R(1-\alpha)$ is independent of λ. The next step along the flow in coupling constant space **cannot** depend on λ since λ tells us how **far** we have already evolved, or flowed, along the trajectory. However, the next step depends only on the couplings at that point on the flow and **not** on the flow history. This is guaranteed by the semi-group property of the RG transformation.

Setting $b = -\log\lambda$ we have

$$\frac{dg_i}{db} = \beta_i(\mathbf{g}) \ .$$

A fixed point is determined by $\beta_i(\mathbf{g}^*) = 0, \ \forall i$. Then

$$R(\lambda)H(\mathbf{g}^*, \phi, \Lambda) = H(\mathbf{g}^*, \Lambda) \ ,$$

and hence from above, when $\mathbf{g} = \mathbf{g}^*$

$$\begin{aligned} G(\mathbf{q}_1, \ldots; \mathbf{q}_n) &= A^{\frac{n}{2}}(\lambda) G(\frac{\mathbf{q}_1}{\lambda}, \ldots; \frac{\mathbf{q}_n}{\lambda}) \\ A(\lambda) &= \lambda^{-(D+2)} Z(\lambda) \ . \end{aligned}$$

Then writing $Z(\lambda) = \lambda^{2\eta_\phi}$ we have

$$A(\lambda) = \lambda^{-2(D-\Delta_\phi)} \ , \qquad \Delta_\phi = \left(\frac{D}{2} - 1\right) + \eta_\phi \ ,$$

and thus

$$G(\mathbf{q}_1, \ldots; \mathbf{q}_n) = \lambda^{-n(D-\Delta_\phi)} G(\frac{\mathbf{q}_1}{\lambda}, \ldots; \frac{\mathbf{q}_n}{\lambda})$$

This is the scaling law for Green functions at the fixed point.

Δ_ϕ	is the **scaling dimension** of	$\phi(\mathbf{x})$
$\left(\frac{D}{2} - 1\right)$	is the **engineering dimension** of	$\phi(\mathbf{x})$
η_ϕ	is the **anomalous dimension** of	$\phi(\mathbf{x})$.

The full renormalization group thus consists of two stages.

Momentum space

$$\begin{array}{ccccc} \Lambda & & \lambda\Lambda & & \Lambda \\ \mathbf{q} & & \mathbf{q} & & \lambda^{-1}\mathbf{q} \\ & \xrightarrow{S(\lambda)} & & \xrightarrow{rescale} & \\ m & & m & & \lambda^{-1}m \\ \tilde{\phi}(\mathbf{q}) & & \tilde{\phi}(\mathbf{q}) & & \tilde{\phi}'(\mathbf{q}') = A^{-\frac{1}{2}}\tilde{\phi}(\mathbf{q}) \\ & & & & \qquad = \lambda^{(D-\Delta_\phi)}\phi(\mathbf{q}) \end{array}$$

Coordinate space

$$\begin{array}{ccccc} a & & a/\lambda & & a \\ \mathbf{x} & & \mathbf{x} & & \lambda\mathbf{x} \\ & \xrightarrow{S(\lambda)} & & \xrightarrow{rescale} & \\ \xi & & \xi & & \lambda\xi \\ \phi(\mathbf{x}) & & \phi(\mathbf{x}) & & \phi'(\mathbf{x}') = \lambda^{-D}A^{-\frac{1}{2}}\phi(\mathbf{x}) \\ & & & & = \lambda^{-\Delta_\phi}\phi(\mathbf{x}) \end{array}$$

(1) The special form for ϕ' applies at the fixed point.

(2) Since the full transformation, $R(\lambda)$, leaves Λ,or a **unchanged** the overall effect is a **rescaling** of the **physical parameters** and **observables** only. Λ plays no role in the dimensional analysis based on **this** rescaling.

(3) A general field combination $f(\phi(\mathbf{x}))$ generally "renormalizes" in a complicated way

$$f(\phi) \xrightarrow{R} f'(\phi) ,$$

but for special combinations, the **scaling fields**, a dimension, Δ_f, can be assigned

$$f'(\phi(\mathbf{x})) = \lambda^{\Delta_f} f(\phi'(\mathbf{x}')) .$$

This is possible only near a fixed point.

[(*) *for information only – not directly part of course*

(i) Before thinning have term in action of the form

$$\int_a d^Dx\; hf(\phi(\mathbf{x})) ,$$

where h is a coupling or external field. After thinning this term becomes

$$\int_{a/\lambda} d^Dx\; hf'(\phi(\mathbf{x})) .$$

Near to the fixed point we can write this term alternatively as

$$\int_{a/\lambda} d^Dx\; h\lambda^{\Delta_f} f(\phi'(\mathbf{x}')) , \qquad \mathbf{x}' = \lambda\mathbf{x} .$$

$$= \int_a d^Dx'\; h\lambda^{\Delta_f - D} f(\phi'(\mathbf{x}')) .$$

But from the RG we are told that this term in the rescaled action must be

$$\int_a d^Dx\; h\lambda^{-\Delta_h} f(\phi(\mathbf{x})) .$$

and hence we verify the existence of a scaling dimension for the operator function f with dimension Δ_f which, comparing the last two expressions, is given by $\Delta_f = D - \Delta_h$.

(ii) An alternate way of seeing this is to use the scaling rules for Green functions near to a fixed point. These rules are derived in section 5 and you should refer back to here after reading it. We have

$$\frac{\partial}{\partial h} G(\mathbf{x}_1, \ldots, \mathbf{x}_n; h) \;=\; \int d^D y\, G_f(\mathbf{x}_1, \ldots, \mathbf{x}_n, y; h) \;,$$

where G_f means an insertion of $f(\phi)$ into the expectation value. By the association of a dimension Δ_f with f we use the scaling rules to write this in the alternative form:

$$= \int d^D y\, \lambda^{n\Delta_\phi + \Delta_f}\, G_f(\lambda\mathbf{x}_1, \ldots, \lambda\mathbf{x}_n, \lambda y; \lambda^{-\Delta_h} h) \;.$$

However, this last expression must also be given by

$$\begin{aligned} & \frac{\partial}{\partial h}\left[\lambda^{n\Delta_\phi} G(\lambda\mathbf{x}_1, \ldots, \lambda\mathbf{x}_n; \lambda^{-\Delta_h} h)\right] \\ = & \int d^D y\, \lambda^{n\Delta_\phi}\lambda^{-\Delta_h}\, G_f(\lambda\mathbf{x}_1, \ldots, \lambda\mathbf{x}_n, y; \lambda^{-\Delta_h} h) \\ = & \;\lambda^D \int d^D y'\, \lambda^{n\Delta_\phi}\lambda^{-\Delta_h}\, G_f(\lambda\mathbf{x}_1, \ldots, \lambda\mathbf{x}_n, \lambda y'; \lambda^{-\Delta_h} h) \;. \end{aligned}$$

Comparing these two versions we find $\Delta_f = \Delta - \Delta_h$ as before.]

(4) The anomalous dimension, η_ϕ, is computed from $Z(\lambda)$, the field renormalization constant.

Now consider the flow in the neighbourhood of a fixed point. Write

$$\mathbf{g} = \mathbf{g}^* + \mathbf{v} \;.$$

Then

$$\frac{dv_i}{db} = K_{ij}(\mathbf{g}^*) v_j \;.$$

The eigenvalues of K_{ij} are Δ_α with eigenvectors $\mathbf{e}_a$.
Then we find

$$\begin{aligned} \mathbf{v} &= \sum_\alpha h_\alpha \mathbf{e}_\alpha \quad \Rightarrow \\ \frac{dh_\alpha}{db} &= \Delta_\alpha(g^*) h_\alpha \quad \text{no sum on } \alpha, \quad \Rightarrow \\ h_\alpha &= h_\alpha^{(0)} \lambda^{-\Delta_\alpha} \;. \end{aligned}$$

Remember that $b = \log\lambda \;, \quad \lambda = \Lambda/\Lambda_0 < 1$.
If $\Delta_\alpha > 0$ the eigencoupling h_α is unstable and is repelled by the fixed point and vice-versa if $\Delta_\alpha < 0$. We have recovered the result that we stated before in a slightly different context namely that every fixed point corresponds to a continuous transition and can be labelled by the number r of repulsive (i.e.,

positive) eigenvalues. then r is the number of **external** fields that must be tuned so that we see the transition controlled by that fixed point.

In general the Hamiltonian is

$$H = \int d\mathbf{x} \sum_n g_n A_n(\mathbf{x}) = H^* + \int d\mathbf{x} \sum v_n A_n(\mathbf{x}) ,$$

where the A_n are general operators and are functions of the fields. Near a fixed point we can write instead

$$H = H^* + \int d\mathbf{x} \sum_\alpha h_\alpha A_\alpha(\mathbf{x}) ,$$

where

$$\begin{array}{lr} A_\alpha = \sum_n (\mathbf{e}_\alpha)_n A_n(\mathbf{x}) & \text{scaling fields} \\ h_\alpha = (\mathbf{u}_\alpha)_n g_n & \text{scaling couplings.} \end{array}$$

Here we have defined the vectors $\mathbf{u}_\alpha$ by

$$\mathbf{e}_\alpha \cdot \mathbf{u}_\beta = \delta_{\alpha\beta} \quad \Rightarrow \quad (\mathbf{e}_\alpha)_n(\mathbf{u}_\alpha)_m = \delta_{nm} .$$

Now we define $[A_\alpha] = \Delta_{A_\alpha}$ and we have that $[h_\alpha] = \Delta_\alpha$. But H is **dimensionless** and so we must have

$$\Delta_{A_\alpha} = D - \Delta_\alpha .$$

Δ_α is an **eigenvalue** of the linearized RG transformation.
Denote Green functions by

$$G(\mathbf{x}_1, \ldots, \mathbf{x}_n; \mathbf{h}) =< \phi(\mathbf{x}_1 \ldots \phi(\mathbf{x}_n) >_c ,$$

where $\mathbf{h}$ is the vector of eigencouplings. Near a fixed point we have

$$G(\mathbf{x}_1, \ldots, \mathbf{x}_n; \mathbf{h}) = \bar{A}^{\frac{n}{2}}(\lambda) G(\lambda\mathbf{x}_1, \ldots, \lambda\mathbf{x}_n; \lambda^{-\Delta_\alpha}\mathbf{h}) ,$$

where $\bar{A}^{\frac{1}{2}} = \lambda^D A^{\frac{1}{2}}$. The extra factor of λ^D occurs when G is fourier transformed from momentum space to coordinate space. But we have

$$\bar{A}^{\frac{1}{2}} = \lambda^D A^{\frac{1}{2}} = \lambda^{\Delta_\phi} ,$$

and hence it follows that

$$G(\mathbf{x}_1, \ldots, \mathbf{x}_n; \mathbf{h}) = (\lambda)^{n\Delta_\phi} G(\lambda\mathbf{x}_1, \ldots, \lambda\mathbf{x}_n; \lambda^{-\Delta_\alpha}\mathbf{h}) .$$

This is the full scaling result for Green functions evaluated near to a fixed point. In the next section we shall derive the scaling theory which demonstrates that such results can be obtained even when the bare theory is not near to a fixed point **as long as** it is sufficiently close to a critical surface, i.e., near a continuous phase transition.

Note for example that the "magnetic field", h, has dimension

$$\Delta_h = D - \Delta_\phi \ ,$$

since the relevant term in the Hamiltonian is $-\int d\mathbf{x}\ h\phi$ which must be dimensionless.
If A_α is a relevant operator then we must have

$$\Delta_{h_\alpha} = D - \Delta_{A_\alpha} > 0 \quad \Rightarrow \quad \Delta_{A_\alpha} < D \ .$$

So, for example, ϕ is a relevant operator.
We can see an immediate application of these ideas as follows. Let $B(\mathbf{x})$ have dimension Δ_B which does not necessarily satisfy $\Delta_B < D$, i.e., B need not be a relevant operator. Let A be a relevant operator conjugate to external field h: $\Delta_A = D - \Delta_h$ and $\Delta_A < D$. All other parameters **except** h are fixed at their critical values, and so as $h \to 0$ ξ diverges. The corresponding generalized susceptibility is

$$\frac{\partial^n \chi}{\partial h^n} = \int \prod d\mathbf{x}_i \ < B(0)A(\mathbf{x}_1)\ldots A(\mathbf{x}_n) >_h \ .$$

[This follows since interesting term in H is $-h\int d\mathbf{x}\ A(\mathbf{x})$ and we have

$$< B(0) > = \int d\phi\ e^{-H}\ B(\phi(0)).$$

]
The only length scale in the problem is ξ and hence on dimensional grounds

$$\frac{\partial^n \chi}{\partial h^n} \sim \xi^{(nD-\Delta_B - n\Delta_A)} \ ,$$

and for n large enough this will diverge as $h \to 0$, $\xi \to \infty$ if

$$n(D - \Delta_A) > \Delta_B \ .$$

This will occur for some n since A is **relevant** and hence $\Delta_A < D$.
Thus singular thermodynamic quantities are associated with **relevant operators** and consequently with the **unstable** or **repulsive** eigenvalues of the linearized RG transformation.

5 Scaling theory

Consider the Green function for a field $\phi(\mathbf{x})$ which has dimension Δ_ϕ

$$G(\mathbf{x}_1,\ldots,\mathbf{x}_n;\mathbf{g}|\xi) = < \phi(\mathbf{x}_1)\ldots\phi(\mathbf{x}_n) >_c \ .$$

G has been labelled by ξ which will be assumed to be large. ξ is a dynamical variable determined by $\mathbf{g}$, the couplings in dimensionless form. From the RG we have, in general

$$G(\mathbf{x}_1,\ldots,\mathbf{x}_n;\mathbf{g}|\xi) = \bar{A}^{\frac{n}{2}}(\lambda) G(\lambda\mathbf{x}_1,\ldots,\lambda\mathbf{x}_n;\mathbf{g}(\lambda)|\lambda\xi) \ .$$

Now consider a particular value, $\bar{\lambda}$, of λ such that for cases where $\xi \gg 1$

$$\bar{\lambda} \ll 1 \quad \text{and} \quad \bar{\lambda}\xi \gg 1 \quad \text{and} \quad \mathbf{g}(\bar{\lambda}) = \mathbf{g}^* \ .$$

This will always be possible if ξ is large enough, although how large it must be for this to be true is a question we have not addressed. Then we can define the function F by

$$G(\mathbf{x}_1, \ldots, \mathbf{x}_n; \mathbf{g}|\xi) = F(\mathbf{x}_1, \ldots, \mathbf{x}_n|\xi) \ .$$

Which, from above, means that

$$F(\mathbf{x}_1, \ldots, \mathbf{x}_n|\xi) = \bar{A}^{\frac{n}{2}}(\bar{\lambda})G(\bar{\lambda}\mathbf{x}_1, \ldots, \bar{\lambda}\mathbf{x}_n; \mathbf{g}^*|\bar{\lambda}\xi) \ .$$

Then a **further** renormalization gives

$$\begin{aligned} F(\mathbf{x}_1, \ldots, \mathbf{x}_n|\xi) &= \bar{A}^{\frac{n}{2}}(\lambda)F(\lambda\mathbf{x}_1, \ldots, \lambda\mathbf{x}_n|\lambda\xi) \\ &= \lambda^{n\Delta_\phi}F(\lambda\mathbf{x}_1, \ldots, \lambda\mathbf{x}_n|\lambda\xi) \ . \end{aligned}$$

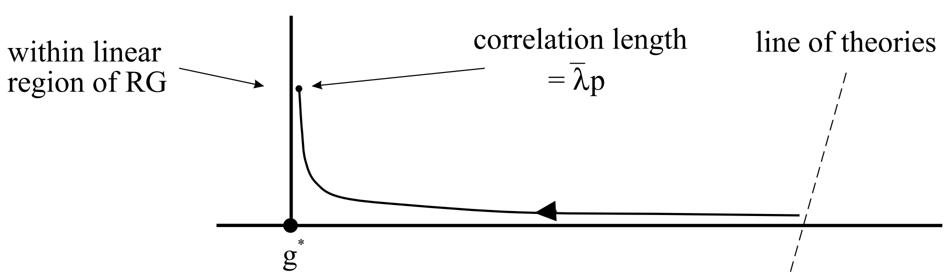
The last result follows since this further renormalization is applied to a theory near the fixed point (i.e. **only** the mass, $m = \xi^{-1}$, is not at its critical value) and so $\bar{A}$ takes the power-law form dictated by the linearized RG flow equations. Thus for $\lambda\xi = p, \quad p \gg 1$ and **fixed** we have

$$G(\mathbf{x}_1, \ldots, \mathbf{x}_n; \mathbf{g}|\xi) = \xi^{-n\Delta_\phi}E(\xi_1, \ldots, \xi_n) \ ,$$

where $\xi_i = \mathbf{x}_i/\xi$ and

$$\begin{aligned} E(\xi_1, \ldots, \xi_n) &= p^{n\Delta_\phi}F(p\xi_1, \ldots, p\xi_n|p) \ , \\ &= p^{n\Delta_\phi}\bar{A}^{\frac{n}{2}}(\bar{\lambda})G(\bar{\lambda}\mathbf{x}_1, \ldots, \bar{\lambda}\mathbf{x}_n; \mathbf{g}^*|\bar{\lambda}p) \ . \end{aligned}$$

This result is a verification of the **scaling hypothesis**. As can be seen, all results are ultimately expressed in terms of a theory with correlation length $\bar{\lambda}p$. In order that this theory lies in that neighbourhood of the fixed point in which the linearized RG equations apply we must have that $\bar{\lambda}p$ is sufficiently large. A diagram of the flow is shown below.



What matters is that the theory with correlation length $\bar{\lambda}p$ can be so chosen to lie in the region where the linear RG approximation is good. This region is a neighbourhood of the fixed point and as long as the flow line emanating from the **bare** theory passes through this region then the above analysis can be applied and the scaling behaviour derived is valid. Clearly, the range of values of temperature T (near T_c) for which this happens depends on how close the intersection of the trajectory of bare theories with the critical surface is to $\mathbf{g}^*$: if it is close to $\mathbf{g}^*$ then the scaling behaviour will be apparent over quite a wide range of temperatures near T_c, but the range will narrow as the distance of the intersection from $\mathbf{g}^*$ increases. This range of temperatures in which the scaling behaviour is observed and the critical exponents can be measured is called the **critical region**.

Now follows a few interesting consequences of scaling theory. Consider the generalized susceptibility

$$\Gamma_n = \int d\mathbf{x}_1 \dots d\mathbf{x}_{n-1} \; G(0, \mathbf{x}_1, \dots, \mathbf{x}_{n-1}; \mathbf{g}|\xi) \; .$$

We have

$$\begin{aligned} \Gamma_n &\equiv \Gamma_n(\xi) \qquad \Rightarrow \\ \Gamma_n &= \frac{d\Gamma_{n-1}(\xi)}{dh} = \frac{d\xi}{dh}\frac{d\Gamma_{n-1}(\xi)}{d\xi} \; , \end{aligned}$$

where h is conjugate to ϕ.
From above

$$\begin{aligned} \Gamma_n(\xi) &= \int \; d^D\xi_1 \dots d^D\xi_{n-1} \; \xi^{(n-1)D - n\Delta_\phi} E(0, \xi_1, \dots, \xi_{n-1}) \\ &= \xi^{(n-1)D - n\Delta_\phi} \cdot C_n \; , \end{aligned}$$

where C_n is a constant given by

$$C_n = \int \; d^D\xi_1 \dots d^D\xi_{n-1} E(0, \xi_1, \dots, \xi_{n-1}) \; .$$

Then from above

$$\begin{aligned} &\Gamma_n = [(n-2)D - (n-1)\Delta_\phi] C_{n-1} \; \xi^{(n-2)D - (n-1)\Delta_\phi - 1} \; \frac{d\xi}{dh} \\ \Rightarrow \; &\frac{d\xi}{dh} \sim \xi^{D - \Delta_\phi - 1} \\ \Rightarrow \; &\xi \sim h^{-\nu_\phi} \qquad \text{where} \qquad \nu_\phi = \frac{1}{D - \Delta_\phi} \; . \end{aligned}$$

Thus the critical exponent associated with the **relevant** external field h conjugate to ϕ, is given in terms of the scaling dimension of ϕ, namely Δ_ϕ.

5.1 Scaling Relations

(1) From above the magnetic susceptibility satisfies the relation

$$\chi \sim \xi^{D - 2\Delta_\phi} \sim t^{-\nu(D - 2\Delta_\phi)} \; .$$

Since γ is **defined** by $\chi \sim t^{-\gamma}$ we have the relation

$$\nu D = \gamma + 2\nu\Delta_\phi \; .$$

(2) It follows from scaling theory that

$$M = <\phi> \sim \xi^{-\Delta_\phi} \sim t^{\nu\Delta_\phi} \ .$$

Since β is defined by $M \sim t^\beta$ we have that

$$\beta = \nu\Delta_\phi \ .$$

(3) β ($\equiv 1/T$) is conjugate to the energy operator $\epsilon(\mathbf{x})$ and hence

$$\xi \sim t^{-\nu} \quad \text{with} \quad \nu = \frac{1}{D - \Delta_\epsilon} \ .$$

Strictly speaking the only part of $\epsilon(\mathbf{x})$ that matters is the most singular, i.e., the component that has the lowest scaling dimension. In ϕ^4 theory this is the operator ϕ^2.

(4) The specific heat is

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_V \ ,$$

with $F = -T \log \ \mathcal{Z}$. Thus

$$\begin{aligned} C_V &= \beta^2 \frac{\partial^2}{\partial \beta^2} \log \ \mathcal{Z} \\ &= \beta^2 \int d\mathbf{x} \ < \epsilon(0)\epsilon(\mathbf{x}) > \qquad \Rightarrow \\ C_V &\sim \xi^{D-2\Delta_\epsilon} = t^{-\nu(D-2\Delta_\epsilon)} \ . \end{aligned}$$

Since α in defined by $C_V \sim t^{-\alpha}$ we find

$$\nu D = \alpha + 2\nu\Delta_\epsilon \ .$$

(5) From above $M \sim \xi^{-\Delta_\phi}$ and $\xi \sim h^{-\frac{1}{D-\Delta_\phi}}$. Since δ is defined by $M \sim h^{\frac{1}{\delta}}$ we find

$$\delta = \frac{D - \Delta_\phi}{\Delta_\phi} \ .$$

Combining these relations we derive the scaling relations

$$\begin{aligned} \alpha + 2\beta + \gamma &= 2 \\ \alpha + 2\beta\delta &= 2 + \gamma \ . \end{aligned}$$

Note the following

(i) Only two relevant or unstable directions occur in ϕ^4 field theory which correspond to h and T and are associated with the relevant operators ϕ and ϵ respectively.

(ii) All critical indices are controlled by Δ_ϕ and Δ_ϵ or equivalently by the dimensions assigned to h and T

$$[h] = D - \Delta_\phi \qquad \text{and} \qquad [T] = D - \Delta_\epsilon \ .$$

The rest is dimensional analysis. Hence in this case there are only two independent indices.

5.2 Renormalized Green Functions

This is a very large topic since the ideas underlying the concept of renormalization and their implementation are central to the modern theory of analyzing both quantum field theory and quantum and classical statistical mechanics. In this course we have time only to introduce the main ideas using the critical theory as the example, and to link these ideas with the notions which have gone before.

We work with a scalar field theories which lie in the critical surface ($\xi = \infty \quad \Rightarrow \quad m^2 = 0$). Of course, the approach can be extended to study near critical theories and non-leading corrections to the predicted leading scaling behaviour.

Define $\Gamma_n(\mathbf{p}, \mathbf{g}_0, \Lambda)$ to be the 1PI truncated Green function. We will consider the effect of the "thinning" procedure but will **not** rescale the observable quantities. In other words we shall use the fact that the "thinning" procedure produces a new theory with the same low-energy predictions as the original one. The effect of "thinning" is encoded directly in the relation

$$\Gamma_n(\mathbf{p}, \mathbf{g}_0, \Lambda) = Z^{-\frac{n}{2}}\left(\frac{\Lambda}{\mu}\right)\Gamma_n(\mathbf{p}, \mathbf{g}(\mu), \mu) \ .$$

μ is the new cut-off after thinning and the $\mathbf{g}(\mu)$ are dimensionless couplings with $\mathbf{g}(\Lambda) = \mathbf{g}_0$.

The LHS is independent of μ and hence

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_i(\mathbf{g})\frac{\partial}{\partial g_i} - \gamma^{\frac{n}{2}}(\mathbf{g})\right)\Gamma_n(\mathbf{p}, \mathbf{g}(\mu), \mu) = 0 \ ,$$

where

$$\begin{aligned}
\gamma(\mathbf{g}) &= \mu\frac{\partial}{\partial\mu}\log\ Z = \left(-\Lambda\frac{\partial}{\partial\Lambda} + \beta_i(\mathbf{g})\frac{\partial}{\partial g_i}\right)\log\ Z \\
\beta_i(\mathbf{g}) &= \mu\frac{\partial}{\partial\mu}g_i(\mu) \ .
\end{aligned}$$

This equation satisfied by Γ_n is the **renormalization group equation**.

Note that

$$\lim_{\Lambda\to\infty}\Gamma_n(\mathbf{p}, \mathbf{g}(\mu), \mu) = \lim_{\Lambda\to\infty} Z^{\frac{n}{2}}\left(\frac{\Lambda}{\mu}\right)\Gamma_n(\mathbf{p}, \mathbf{g}_0, \Lambda)$$

is **finite**. $\Gamma_n(\mathbf{p}, \mathbf{g}(\mu), \mu)$ is the **renormalized** Green function.

Consider ϕ^4 field theory.

(i) $\gamma(g)$

$$\begin{aligned}
\frac{d}{dp^2}\Gamma_2(\mathbf{p}, g_0, \Lambda)_{\mathbf{p}^2=\mu^2} &= Z^{-1}\left(\frac{\Lambda}{\mu}\right)\frac{d}{dp^2}\Gamma_2(\mathbf{p}, g(\mu), \mu)_{\mathbf{p}^2=\mu^2} \\
&= Z^{-1}\left(\frac{\Lambda}{\mu}\right)\ f(g(\mu)) \ .
\end{aligned}$$

$f(g)$ is a dimensionless function of g and can be calculated in perturbation theory

$$f(g) = 1 + ag^2 + \ldots$$

There are no **divergences** because $|\mathbf{p}| = \mu$, which is the cut-off. We also have (calculating in the limit $\epsilon \to 0$)

$$\frac{d}{dp^2}\Gamma_2(\mathbf{p}, g_0, \Lambda)_{\mathbf{p}^2=\mu^2} = 1 + g_0^2 \log\left(\frac{\Lambda}{\mu}\right) + \ldots$$

The corrections introduced by keeping ϵ finite in this calculation give rise to corrections to γ which are higher order in ϵ. Then from above we get

$$\begin{aligned} Z\left(\frac{\Lambda}{\mu}\right) &= \frac{1 + ag^2(\mu) + \ldots}{1 + \alpha g_0^2 \log\left(\frac{\Lambda}{\mu}\right) + \ldots} \\ \Rightarrow \gamma &\equiv \mu\frac{\partial}{\partial\mu}\log\ Z \\ &= \alpha g_0^2 \ + 2ag(\mu)\beta(g(\mu)) \ldots \end{aligned}$$

From the renormalization group equation γ is a **finite** (i.e., Λ independent) function of $g(\mu)$. This follows because both $\beta(g)$ and $\Gamma_n(\mathbf{p}, g(\mu), \mu)$ are independent of Λ and the RG equation cannot balance unless γ is independent of Λ too.

Since $g_0 = g(\mu) \ + O(g^2(\mu))$ we can make the substitution for g_0 in terms of $g(\mu)$ and find

$$\gamma(g) = \alpha g^2 \ + 2ag\beta(g) \ldots$$

(ii) $\beta(g)$

$$\Gamma_4(\mathbf{p}, g_0, \Lambda)_{\mathbf{p}^2=\mu^2} = Z^{-2}\left(\frac{\Lambda}{\mu}\right)\Gamma_4(\mathbf{p}, g(\mu), \mu)_{\mathbf{p}^2=\mu^2} \ .$$

But

$$\Gamma_4(\mathbf{p}, g(\mu), \mu)_{\mathbf{p}^2=\mu^2} = \mu^\epsilon(g(\mu) + bg^2(\mu) + \ldots) \ ,$$

again a **finite** function: b is an ϵ-independent number. Note that Γ_4 has **engineering** or **ordinary** dimension ϵ: $[\Gamma_4] = \epsilon$. These dimensions are carried by the μ^ϵ factor.

We also have that

$$\Gamma_4(\mathbf{p}, g_0, \Lambda)_{\mathbf{p}^2=\mu^2} = \Lambda^\epsilon g_0 - \frac{\beta_0}{\epsilon}\Lambda^{2\epsilon}\mu^{-\epsilon}g_0^2 + \ldots$$

To see how the form of the second term on the RHS arises we note that it comes from an integral of the form

$$\text{Const.} \cdot \int_\mu^\Lambda \frac{d^{4-\epsilon}}{q^4} \ .$$

Ignore the μ-independent terms and terms that are not leading non-trivial order in ϵ.

[We note in passing that the above equations show to lowest order that $g(\mu) = \left(\frac{\mu}{\Lambda}\right)^\epsilon g_0$. In the calculation for γ we worked in the $\epsilon \to 0$ limit since we only required the leading order results in that case. This is why we were able to use $g_0 = g(\mu)$ in that calculation. If we want to work to higher orders in ϵ we must be more careful.]

Then

$$\mu^\epsilon(g(\mu) + bg^2(\mu)) \approx \Lambda^\epsilon g_0 - \frac{\beta_0}{\epsilon}\Lambda^{2\epsilon}\mu^{-\epsilon}g_0^2 \ ,$$

(Note that $Z = 1$ to the order in which we are working.) Now apply the operator $\mu\frac{\partial}{\partial\mu}$ with $\beta = \mu\frac{\partial g(\mu)}{\partial\mu}$. We get

$$\begin{aligned} \epsilon(g + bg^2) \ + \ (\beta + 2bg\beta) \ &\approx \ \beta_0\left(\frac{\Lambda}{\mu}\right)^{2\epsilon} g_0^2 \\ &= \ \beta_0 g^2 \ + \ldots, \end{aligned}$$

where the expression for g_0 in terms of g has been used in the last step. Hence

$$\beta \approx \frac{-\epsilon g \ + \ (\beta_0 - \epsilon b)g^2}{1 \ + \ 2bg} \ .$$

Keeping only non-trivial orders in ϵ, (β_0 is $O(1)$) gives

$$\beta = -\epsilon g \ + \ \beta_0 g^2 \ + \ldots$$

We have already computed β_0 before when calculating the critical exponent ν (note that when comparing these two calculations we must remember that $d\log\mu = -db$. This accounts for the sign difference between the two otherwise identical versions of the flow equation). For $\mu \ll \Lambda$ we find $g(\mu) = g^*$ where

$$\beta(g^*) = 0 \qquad \Rightarrow \qquad g^* = \frac{\epsilon}{\beta_0} \ ,$$

and we found before that $g^* = \frac{16\pi^2}{3}\epsilon$.

A consequence of the two calculations above is that for $\mu \ll \Lambda$ we have

$$\frac{d}{d\log\mu} Z\left(\frac{\Lambda}{\mu}\right) = \gamma(g(\mu)) = \gamma(g^*) \ .$$

Or, writing $\lambda = \frac{\mu}{\Lambda}$

$$Z = Z_0\lambda^{\gamma(g^*)} \ .$$

But from the scaling analysis near the fixed point

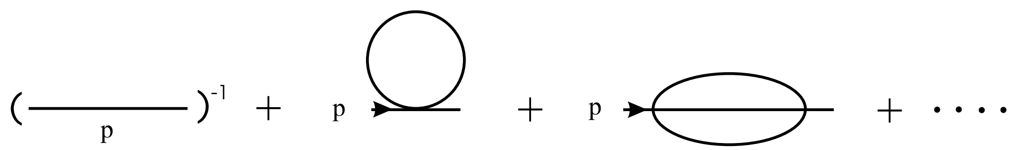
$$Z \sim \lambda^{2\eta_\phi} \ .$$

Thus

$$\begin{aligned} \eta_\phi &= \frac{1}{2}\gamma(g^*) \\ \Delta_\phi &= \left(\frac{D}{2} - 1\right) \ + \ \frac{1}{2}\gamma(g^*) \ . \end{aligned}$$

From (i) above $\eta_\phi = \frac{1}{2}\alpha g^{*2}$ since $\beta(g^*) = 0$.

We now indicate how to calculate α. Γ_2 is given by the graphs



$$\mathbf{p}^2 \qquad + \qquad C\Lambda^2 g_0^2 \qquad + \qquad \left(g_0^2 D\Lambda^2 + \alpha\mathbf{p}^2\right)\log\frac{\Lambda}{\mu} \qquad + \ldots$$

The calculation of α is somewhat involved and will not be reproduced here. From Raymond's book "A Modern Primer on Field Theory" we find

$$\begin{aligned} \alpha &= \frac{1}{6(16\pi^2)^2} \\ \Rightarrow & \\ \eta_\phi &= \frac{1}{2}\,\frac{1}{6(16\pi^2)^2}\,\left(\frac{16\pi^2}{3}\right)\epsilon^2 \\ &= \frac{1}{108}\epsilon^2 \ . \end{aligned}$$

What happened to all the couplings corresponding to the **irrelevant** operators (i.e., those with dimension $\Delta > D$)? A calculation of the β functions for these operators in the same way as in (ii) above shows that the fixed point for these couplings is at the **origin**. This is expected because

(i) the operators corresponding to these couplings do not give rise to extra IR divergences: $D > D_c$ for these operators. This implies that Landau's approach works and the critical exponents receive no anomalous contributions from these operators.

(ii) mean field theory works for operators whose associated couplings have their fixed point at the **origin**.

$$\eta_\phi = \frac{1}{2}\gamma(\mathbf{g}^*) \ ,$$

but only g^*, the ϕ^4 coupling, is non-zero.

(iii) The corollary is that only the **renormalizable** operators in the usual sense, i.e. $\Delta < D$ are **relevant** since only **their** couplings flow to IR stable fixed points which are **not** at the origin. For those operators with $\Delta = D$ the fixed point is at the origin **but** we find log, rather than power-law, corrections to mean field theory. These operators constitute a special case but an important one since, for example, **scaling violations** in Quantum Chromodynamics are exactly of this kind.