# Asymptotic Approaches to Transition Modelling

**Stephen J. Cowley**
DAMTP
University of Cambridge
Silver Street
Cambridge CB3 9EW
UK

**Xuesong Wu**
Department of Mathematics
Imperial College
180 Queen's Gate
London SW7 2BZ
UK

## SUMMARY

The linear and nonlinear evolution of unstable disturbances in high-Reynolds-number flows is reviewed from the perspective of asymptotic theory. For non-parallel and/or unsteady flows, quasi-parallel and quasi-steady approximations can only be strictly justified by asymptotic expansions based on the smallness of the inverse Reynolds number. Further, such an asymptotic approach allows the inclusion of nonlinear effects in a self-consistent manner.

Attention is focussed primarily on three asymptotic regions: (i) the lower-branch Tollmien-Schlichting (TS) scaling for boundary layers, (ii) the upper-branch TS scaling for boundary layers, and (iii) the Rayleigh scaling for (decelerating) boundary layers, free shear layers, jets and wakes. For fixed-frequency disturbances in a decelerating boundary layer, these asymptotic regions occur at increasing distances from the leading edge. A disturbance propagating downstream from the leading edge will pass through each region in turn. The larger the initial disturbance, the further upstream nonlinear effects must be taken into account.

Weakly nonlinear theory is possible when the relative growth-rate of disturbances is small, e.g. near a neutral curve. Close to the lower branch, it is possible to take into account non-parallelism, wavetrain modulation (i.e. wavepackets), and three-dimensional effects such as those that lead to TS-wave/vortex interactions. A number of different models are described and critically assessed. Similar possibilities are examined on the upper-branch scaling, where an additional feature is the effect of nonlinear critical layers. Critical layers play a pre-eminent rôle on the Rayleigh scaling.

Physical effects explained include (i) the nonlinear saturation of two-dimensional disturbances in free shear layers and decelerating boundary layers, (ii) the explosive growth in amplitude of three-dimensional disturbances, and (iii) the generation of surprisingly large longitudinal vortices and spanwise-dependent mean flows.

## 1. INTRODUCTION

One of the parameters characterising a fluid flow is its Reynolds number. Low-Reynolds-number flows† are invariably laminar; the corresponding solutions of the Navier-Stokes (NS) equations are stable. At moderate or high Reynolds numbers, the NS equations admit solutions which describe laminar flow, but such solutions may be unstable to small perturbations; this is almost always the case when the Reynolds number is very large. When instability is present, the observed flow may be 'turbulent'. While it is difficult to give a precise definition of turbulence, Tennekes & Lumley (1972) list irregularity, 'diffusivity', vorticity fluctuations and 'dissipation' among its characteristics. 'Hydrodynamic stability theory' (i.e. the analysis of the stability of NS solutions) provides a possible starting-point for an understanding of turbulence. Moreover, such analysis may be directly relevant to those physical situations where a flow is observed to change from laminar to turbulent - as often happens on aircraft wings or turbine blades, for example. The process by which a laminar flow becomes turbulent is termed 'transition'; the reverse process is usually referred to as 'relaminarization'.

By its very nature, turbulence is a 'fully' nonlinear phenomenon, and hence a complete explanation of transition must also be nonlinear. In many cases, however, the initial variation from laminar flow caused by a disturbance can be described by a linear analysis. Furthermore, if in the transition process the linear-growth stage is much longer than the nonlinear stage, as often happens in the case of small initial disturbances, useful 'engineering' predictions of transition can be obtained using the $e^n$ method (e.g. Arnal, 1993). However, such an approach makes many empirical assumptions, and it answers no questions concerning the nonlinear dynamics.

† The term 'low Reynolds number' means $R \ll 1$. We note that in certain circles 'low Reynolds number' is sometimes used synonymously with 'laminar'.

Transition from laminar to turbulent flow has been a subject of intensive study for many decades and has been frequently reviewed. Recent reviews include those by Herbert (1988), Bayly & Orszag (1988), Kleiser & Zang (1992) and Huerre & Monkewitz (1990). The aim of the current review is to survey some of the methods by which nonlinear effects in transition can be taken into account *without* resort either to empirical modelling or to direct numerical solution of the NS equations. While such direct Navier–Stokes (DNS) simulations can often model specific flows, it is sometimes difficult to extract from them the underlying physical mechanisms causing transition. Analytically-based models, on the other hand, *do* highlight the underlying physics. In addition the influence of changes in disturbance amplitude, frequency, etc. are often more self-evident in such models.

The models we shall examine are systematic mathematical approximations based on asymptotic expansions. Terms in the governing equations will not be arbitrarily included or excluded to fit experimental data. This makes the underlying assumptions somewhat clearer than in some other models. On the other hand, since there are few empirical parameters, outstanding quantitative agreement with experiment is difficult to achieve – although not unattainable (e.g. Hultgren, 1992). Another advantage of the asymptotic approach is that once an effective model has been found, its application to different flows is not necessarily accompanied by a need to retune parameters.

### 1.1 The parallel-flow approximation

In order to illustrate the differences between, and relative merits of, *ad hoc* and systematic asymptotic approximations, let us consider the derivation of the well-known Orr–Sommerfeld (OS) equation. This equation is widely used to test the linear stability of both parallel flows (when it is exact) and almost parallel flows (when it is empirical). In particular, consider an incompressible unidirectional flow with a dimensional † velocity distribution

$$(\hat{U}_B(\hat{y}), 0, 0) \;, \tag{1.1}$$

where $(\hat{x}, \hat{y}, \hat{z})$ are Cartesian coordinates. Suppose that a typical lengthscale in the $\hat{y}$-direction is $\hat{\delta}$, and that a typical velocity is $\hat{U}_e$. We may define a Reynolds number, $R$, by

$$R = \frac{\hat{U}_e \hat{\delta}}{\hat{\nu}} \;, \tag{1.2}$$

† Dimensional quantities will be denoted by symbols with a caret over them.

where $\hat{\nu}$ is the kinematic viscosity of the fluid. As is well known, a velocity profile of the form of (1.1) is only an exact solution of the NS equations for certain very specific configurations, e.g. Couette flow, plane Poiseuille flow (PPF). However, at large Reynolds numbers,

$$R \gg 1 \;, \tag{1.3}$$

many boundary-layer and shear-layer flows have velocity profiles of the form of (1.1) *to leading order in an expansion in inverse powers of* $R$, i.e. such shear flows are 'quasi-parallel' in that limit. The stability of such flows is often studied by performing a perturbation analysis about the basic flow. To this end we write

$$\mathbf{U} = (U_B + \tilde{u}, \tilde{v}, \tilde{w}) \;, \tag{1.4}$$

where lengths, velocities and time have been nondimensionalised by $\hat{\delta}$, $\hat{U}_e$ and $\hat{\delta}\hat{U}_e^{-1}$ respectively, and variables with a tilde represent perturbation quantities. Next (1.4) is substituted into the NS equations, which are then linearised on the basis that:

($a$) (1.1) is an *exact* solution to the NS equations (the 'parallel-flow assumption'), and

($b$) $|\tilde{\mathbf{u}}| \ll 1$.

The resulting linear equations for $\tilde{\mathbf{u}}$ have coefficients that, according to the parallel flow assumption, are independent of $x$, $z$ and $t$. Hence solutions proportional to

$$E = \exp(\mathrm{i}\alpha x + \mathrm{i}\beta z - \mathrm{i}\omega t) \;, \tag{1.5}$$

can be sought. The final equation for $\tilde{v}$ is

$$(\mathrm{i}\alpha R)^{-1}\left(D^2 - \alpha^2 - \beta^2\right)^2 \tilde{v} + U_{Byy}\tilde{v} - (U_B - c)\left(D^2 - \alpha^2 - \beta^2\right)\tilde{v} = 0 \;, \tag{1.6}$$

where $c = \omega\alpha^{-1}$ and $D \equiv \frac{\mathrm{d}}{\mathrm{d}y}$. Equation (1.6) is the OS equation. Together with suitable boundary conditions it yields a dispersion relation

$$f(\alpha, \beta, \omega; R) = 0 \;. \tag{1.7}$$

A key point in the derivation of the OS equation has been assumption ($a$), which is *untrue* in general.

In effect the Reynolds number has been assumed to be *asymptotically large* for (1.1) to be a leading-order approximation, but then it has been assumed to be *finite* when substituting (1.4) into the NS equations. This is inconsistent (e.g. Smith, 1979$a$). To this extent the OS equation is an *ad hoc* approximation. Despite this rather shaky mathematical foundation linear stability results based on the OS equation can yield respectable agreement with experiment. To illustrate this consider solutions to (1.7) for which $\alpha$, $\beta$ and $\omega$ are all real. Under these conditions disturbances are pure travelling waves, i.e. they neither grow nor decay. If $\beta$ is known, then these solutions define a 'neutral curve' in the $R-\alpha$ plane; equivalently a neutral curve can be defined in the $R-\omega$ plane. Conventionally, the former/latter plane is used in the case of temporal/spatial stability analyses in which the wavelength/frequency is assumed to be real. The neutral curve for two-dimensional ($\beta = 0$) disturbances to the Blasius boundary-layer profile is plotted in figure 1; superimposed are experimental results. It can be seen that the agreement between the OS theory and the experiments is reasonably good at moderate Reynolds numbers, and improves at higher Reynolds numbers. This was to be expected since the quasi-parallel assumption is better justified at large Reynolds numbers; indeed it is then a valid leading-order approximation (see below).

The OS equation, in conjunction with the $e^n$ method, has proved to be a useful engineering tool. On strict mathematical grounds, however, a stability analysis of boundary-layer flows at *finite* Reynolds numbers requires solution of the full linearised NS equations; even the (linear) parabolised stability equations (PSE) are an *ad hoc* approximation. Thus it might be argued that the agreement between OS theory and experiment is fortuitous. Indeed, there are a number of other instances where the quasi-parallel approximation (or a close relative) has been used to much less effect. For example:

- Nonparallelism is a leading-order effect for Görtler instability in a boundary layer in the sense that the linear equations describing the instability have $x$-dependent coefficients (e.g. Hall 1983); this is so even if the Reynolds number of the boundary layer is large. At high Görtler numbers a quasi-parallel systematic approximation can be made (e.g. Hall, 1982), but this is *not* so at finite Görtler numbers. As in OS theory, an *ad hoc* quasi-parallel approximation can be tried. However, the agreement with experiment is not particularly good, and spurious results for the existence of very short wavelength Görtler vortices are easily obtained (see Hall (1990) and references therein).

- Quasi-parallel theory for cross-flow instability has had mixed success. For flow over a rotating disk, reasonable agreement with experiment has only been obtained by arbitrarily including and excluding certain terms in the generalised OS equation (e.g. see comments by Spalart, 1990).

- There is a close relation between the stability of unsteady shear layers and non-parallel shear layers. At finite Reynolds numbers the stability analysis of unsteady flows involves the solution of the linearised NS equations with time-dependent coefficients (e.g. von Kerczek & Davis 1974, Hall 1978). A quasi-steady approximation, in which the velocity profiles are frozen in time, is the equivalent of the quasi-parallel approximation for spatially-developing boundary layers. For Stokes layers, such an approach yields Reynolds numbers for instantaneous instability far below those at which Stokes layers are observed to become unstable for even part of their period (e.g. Cowley (1987) and references therein). The quasi-steady approximation *can* be mathematically justified, but again only at large Reynolds numbers.

We conclude that many linear theories of transition based on OS theory at finite Reynolds numbers cannot be justified rigorously. Of course, thirty or more years ago, such simplifications were often necessary in order to make progress; indeed a number of these approaches resulted in exceptionally useful tools for design and development. However, because of their *ad hoc* nature, it is often mathematically difficult to extend such theories into the nonlinear régime. For example, if certain terms have been arbitrarily omitted in the linear theory, when should they be reintroduced in the nonlinear theory? Such difficulties can be avoided by taking the Reynolds number (or other appropriate parameter) to be large throughout the analysis. Furthermore, such an asymptotic approach identifies the dominant terms in the NS equations, so enabling improved finite-Reynolds-number approximations to be proposed (e.g. Smith, Papageorgiou & Elliott 1984).

In the last twenty-five years there has been a rapid increase in the number of papers published that have adopted a high-Reynolds-number asymptotic approach. As such this review cannot be exhaustive. For the most part we will concentrate on incompressible flows. The extension to subsonic flows often just requires a rescaling based on Mach number. However, a number of the asymptotic scalings change significantly for transonic, supersonic and hypersonic flows, and there a number of new phenomenon (e.g. Smith & Bowles 1993, Smith 1989, Cowley & Hall 1990). Only passing references to such flows will be made. We also do not review asymptotic approaches to receptivity (e.g. see the review by Goldstein & Hultgren 1989), Görtler instability (e.g. see the review by Hall 1990), crossflow instability (e.g. Bassom & Gajjar 1988) or vortex breakdown (e.g. Leibovich & Stewartson 1983). We are also aware that we make inadequate reference to the Russian literature.

This review is organised as follows. In §2, we outline the scalings governing the linear TS instability in the lower- and upper- branch régimes. In §3, we survey some of the major ideas behind weakly nonlinear analysis, upon which much modern asymptotic theory, in one way or another, is based. In §4, we concentrate on nonlinear instability in the lower branch régime. The flow here is, in general, governed by the fully nonlinear triple-deck equations. However depending on the nature of the disturbance, different scalings can be derived, leading to different [weakly] nonlinear interactions. In §5 we turn to nonlinear instability in the upper-branch régime. Two distinct features which differ from the lower-branch régime are the rôle of nonlinear interactions within critical layers, and resonant-triad interactions. In §6, we summarise recent work on the nonlinear development of Rayleigh instability waves in shear layers. Much of this analysis involves nonlinear, nonequilibrium, viscous critical layer effects. While nonlinear interactions in different scaling régimes exhibit different features, the connections that exist are indicated.

## 2. ASYMPTOTIC LINEAR THEORY

Before the prevalence of modern computers, asymptotic methods were one of the principle means of solving the OS equation. To this end, uniform asymptotic approximations were derived on the assumption that $\alpha R \gg 1$ (for a review see Drazin & Reid, 1981). Since codes to solve differential equations numerically are now widespread, such approximations have diminished in importance as far as obtaining quantitatively accurate solutions to the OS equation. However, asymptotic methods still have an important rôle to play in understanding transition. In particular, for large Reynolds numbers they can be used to obtain consistent solutions to the full NS equations, rather than just the OS equation. In this way non-parallelism, unsteadiness, and nonlinearity can be incorporated in a mathematically consistent manner (e.g. Smith 1979*a*).

### 2.1 Plane Poiseuille flow

In order to illustrate the high-Reynolds-number asymptotic approach, we begin by considering PPF. This flow has the advantage that it is exactly parallel. Hence, the OS equation can be derived without approximation, and linear high-Reynolds-number asymptotic solutions can be sought from either the OS equation or the NS equations. The structure of the modes depends on whether they are 'nearer' the lower or upper branch of the neutral curve.[†] These different structures reflect the fact that there are different physical balances within the flow close to the upper and lower branches. Each of the branches can be said to represent a 'distinguished' asymptotic scaling. In many asymptotic analyses, a major aim/difficulty is to identify the appropriate distinguished scaling. As illustrated below, one approach to deriving such scalings is based on seeking 'maximal-interactions' between competing processes. Once derived, the asymptotic scaling can include rather exotic powers of the expansion parameter. However, the advantage of seeking a distinguished scaling is that a much wider range of parameter space can then be described by taking appropriate limits.

As an example of the derivation of a distinguished scaling, consider modes with a scaling near the lower branch of PPF. Since the Reynolds number is large, we begin by hypothesising that the modes have a three-zone asymptotic structure, i.e. a central inviscid core region, and two viscous layers of width $\Delta$ immediately adjacent to the walls (see figure 2*a*). Within the viscous layers we assume that there is a 'maximal-interaction' balance between the unsteady inertia ($\tilde{u}_t$), convective inertia ($U_B\tilde{u}_x$), pressure gradient ($\tilde{p}_x$) and viscous ($R^{-1}\tilde{u}_{yy}$) terms in the $x$-momentum equation. Since $U_B \sim \Delta$ in these layers, it follows that

$$\Delta \sim (\alpha R)^{-\frac{1}{3}} , \quad \omega \sim (\alpha^2 R^{-1})^{\frac{1}{3}} , \quad \tilde{p} \sim (\alpha R)^{-\frac{1}{3}}\tilde{u} . \tag{2.1a}$$

Since the two types of inertia term balance, these viscous wall layers are sometimes referred to as 'critical' layers, i.e. regions where the phase speed of the mode approximately equals the velocity of the undisturbed flow.

The flow in the viscous wall layers interacts with that in the core region through pressure variations. Hence we assume that the magnitude of the pressure variation is of the same order both in the core and the viscous layers (by analogy with classical boundary-layer theory there is almost no pressure variation across the viscous layers). Also, since the core is inviscid, the slip-velocity at the walls has the same magnitude as $\tilde{u}$ in the viscous layers. Normal modes, known as Tollmien–Schlichting (TS) waves, can be supported when there is a pressure variation *across* the core. This occurs when the $U_B\tilde{v}_x$ and $\tilde{p}_y$ terms in the $y$-momentum equation balance. Hence

$$\tilde{p} \sim \alpha\tilde{v} \sim \alpha^2\tilde{u} , \tag{2.1b}$$

[†] Note that the lower and upper branches are sometimes referred to as branches 1 and 2 respectively.

where the latter relation follows from the continuity equation. A comparison of the two expressions for $\tilde{p}$ in (2.1$a$) and (2.1$b$) yields

$$\alpha \sim R^{-\frac{1}{7}} \ . \tag{2.2}$$

Based on this three-layer scaling, † governing equations can be derived from the linearised NS equations and solved. Three-dimensional effects can be accounted for by assuming that $\beta \sim R^{-\frac{1}{7}}$. For a non-dimensionalisation based on the centreline velocity and half the channel width, the lower branch for two-dimensional modes is given by

$$\alpha_l \equiv \alpha R^{\frac{1}{7}} \approx 2.15 \tag{2.3}$$

(Lin 1946). For both temporal and spatial stability problems, the fastest-growing mode is two-dimensional and occurs for finite values of $\alpha_l$ and $\omega_l \equiv \omega R^{\frac{3}{7}}$.

Solutions of the dispersion relation $f_l(\alpha_l, \beta_l, \omega_l) = 0$ show that

- for the temporal problem, i.e. for $\alpha_l = O(1)$ and $\beta_l \equiv \beta R^{\frac{1}{7}} = O(1)$, the scaled growth-rate $\Im(\omega_l)$ is neither significantly smaller nor larger than the scaled frequency $\Re(\omega_l)$; similarly
- for the spatial problem, i.e. for $\omega_l = O(1)$ and $\beta_l = O(1)$, the scaled growth-rate $\Im(-\alpha_l)$ is similar in size to the scaled real wavenumber $\Re(\alpha_l)$.

However, near the neutral curve specified by (2.3), the growth-rate is much smaller than the frequency/real-wavenumber for the temporal/spatial case respectively. This is a key observation, since a relatively small growth-rate is a central assumption for much weakly nonlinear theory.

For this 'lower-branch' scaling there is another region of parameter space where the growth rate is relatively small and weakly nonlinear theory can be exploited; namely when $|\alpha_l| \gg 1$ and $|\omega_l| \gg 1$ (e.g. Smith & Burggraf, 1985). In this high-frequency/large-wavenumber limit the viscous wall regions in figure 2$a$ each split into three subregions (see figure 2$b$):

- a thin viscous region (the 'Stokes layer') of thickness $\omega_l^{-\frac{1}{2}}$ adjacent to the wall,
- an inviscid region of thickness $\omega_l \alpha_l^{-1}$ further from the wall, and
- a thin viscous critical layer of thickness $\alpha_l^{-\frac{1}{3}}$ sited within the inviscid region at the level where the phase speed of the wave equals the velocity of the undisturbed flow.

Detailed consideration of this asymptotic limit shows that to leading order the (real) phase speed of the modes is fixed by *inviscid* dynamics. However, the small non-zero growth-rate is a *viscous* effect and arises from a phase shift in the velocity due to the Stokes layer near the wall.

The upper branch of the linear neutral curve cannot be found from an analysis based on scaling (2.2). However, the high-frequency/large-wavenumber limit indicates how the analysis should be modified. In particular, in the high-frequency/large-wavenumber limit described above, the critical layer plays a passive rôle in determining the leading-order (linear) growth-rate; this is not so once the (scaled) frequency and wavenumber are sufficiently large. Specifically, when $(\alpha_l, \beta_l) = O(R^{\frac{4}{77}})$ and $\omega_l = O(R^{\frac{12}{77}})$, i.e. when

$$R^{\frac{1}{11}}(\alpha, \beta) = (\alpha_u, \beta_u) = O(1) \ , \ R^{\frac{3}{11}}\omega = \omega_u = O(1) \ ,$$

the curvature $U_{Byy}$ in the velocity profile leads to a velocity jump in $\tilde{u}$ across the critical layer that modifies the leading-order growth-rate (e.g. Lin 1955, Reid 1965). The upper branch neutral curve for two-dimensional modes is given by $\alpha_u \approx 1.79$ (Lin 1946).

Since $R^{-\frac{1}{11}} \gg R^{-\frac{1}{7}}$, the seven-layer upper-branch scaling covers a larger region of parameter space than the three-layer lower-branch scaling. This makes it an attractive scaling to study, especially since the relatively small growth-rates allow for an approach based on weakly nonlinear theory. On the other hand the fastest-growing modes occur on the lower-branch scaling.

### 2.2 The Blasius boundary layer

In the case of the Blasius boundary layer, asymptotic solutions of the OS equation similar to those described above can be sought when the Reynolds number is large. The lower-branch neutral curve for two-dimensional disturbances is given by

$$\alpha \sim 0.25 Re^{-\frac{1}{8}} \ , \tag{2.4}$$

where $Re = \hat{U}_e \hat{L} \hat{\nu}^{-1}$, $\hat{U}_e$ is the freestream velocity, $\hat{L}$ is the distance from the leading edge and, as is conventional, we have introduced the Reynolds number $Re$ based on distance from the leading edge. In order to facilitate comparison with the above results for PPF, the scale of the flow in the $\hat{y}$-direction may be taken to be $\hat{\delta} = Re^{-\frac{1}{2}} L$, in which case $R = Re^{\frac{1}{2}}$. ‡

Since the Blasius boundary layer develops over a

† In some sense this is the equivalent scaling for PPF to the famous 'triple-deck' scaling for boundary-layer flow (Stewartson 1969, Messiter 1970, Neiland 1969, Smith 1979$a$,$c$,$d$, Smith 1976$a$,$b$). See also the next subsection.

‡ The displacement thickness, $\hat{\delta}_1$, is sometimes used as the scale normal to the wall. It is then conventional to

non-dimensional length of order $R$, it follows that the $O(R^{\frac{1}{4}})$ wavelength of 'lower-branch' TS waves (see (2.4)) is much less than the lengthscale over which non-parallelism is noticeable. In other words, at large Reynolds numbers the quasi-parallel approximation is asymptotically valid to leading order.

For boundary-layer flows, non-parallel effects can be taken into account by means of an asymptotic solution based on the triple-deck structure illustrated in figure 3*a*. Smith (1979*a*) solved the linearised NS equations by seeking solutions of WKBJLG/multiple-scales form, e.g. in the 'lower deck', where $\hat{y} = O(Re^{-\frac{5}{8}}L)$, he wrote

$$\tilde{p} = Re^{-\frac{1}{4}}(\bar{p}_0(x) + Re^{-\frac{1}{8}}\bar{p}_1(x) + \ldots)E \ , \qquad (2.5a)$$

where

$$E = \exp\left(\mathrm{i}Re^{\frac{3}{8}}\int \alpha_l(x)\mathrm{d}x + \mathrm{i}\beta_l Z - \mathrm{i}\omega_l T\right) \ , \quad (2.5b)$$

and the independent coordinates have been non-dimensionalised according to

$$\hat{x} = \hat{L}x \ , \quad \hat{z} = \hat{L}Re^{-\frac{3}{8}}Z \ , \quad \hat{t} = Re^{-\frac{1}{4}}\hat{L}\hat{U}_e^{-1}T \ . \qquad (2.6)$$

In the analysis, the scaled frequency $\omega_l$ and spanwise wavenumber $\beta_l$ are taken as known, while $\alpha_l$ is expanded in powers of $Re^{-\frac{1}{8}}$ and $\log Re$. †

One of the major quantities of interest in such a calculation is the growth-rate; in the case of the pressure perturbation at the wall this is given by

$$\frac{\tilde{p}_x}{\tilde{p}} = \mathrm{i}Re^{\frac{3}{8}}\alpha_l(x) + \frac{\bar{p}_{0x}}{\bar{p}_0} + \ldots \ . \qquad (2.7)$$

In a non-parallel flow, the growth-rate depends on the quantity being measured (e.g. Bouthier 1973), i.e. while the leading-order term in (2.7) is universal, the second term clearly depends on the physical quantity in question. Further, we see that non-parallel effects enter the calculation at $O(Re^{-\frac{3}{8}})$ – as might be expected since this is the ratio of the TS wavelength to the development length of the boundary layer.

The lower-branch neutral curve for the Blasius boundary layer as calculated by Smith (1979*a*) is illustrated in figure 3*b*; also plotted is the neutral curve as calculated by the successive approximation method of Bouthier (1973) and Gaster (1974). The advantage of the latter method is that a 'full' neutral curve can be calculated; its disadvantage is that the Reynolds number again has to be assumed schizophrenic (i.e. finite at certain points in the analysis, but asymptotically large elsewhere).

As for the case of PPF, at large frequencies and wavenumbers the viscous wall region splits into three sub-layers (cf. figure 2*b*), and the TS waves described by the scaling (2.5) and (2.6) become almost neutral. The existence of a small parameter, i.e. the (relative) growth-rate, again makes this parameter régime amenable to asymptotic analysis. For example, Smith & Burggraf (1985) and Stewart & Smith (1987) have examined the linear stability of non-parallel flows such as separating boundary layers. In particular, Stewart & Smith (1987) find that for disturbances of given frequency the mode with the largest spatial growth-rate in a non-parallel flow may be three-dimensional.

The upper branch of the neutral curve again occurs for asymptotically large values of $\omega_l$, i.e. $\omega_l = O(Re^{\frac{3}{20}})$. As before, the curvature of the unperturbed velocity profile results in a velocity jump across the critical layer; it is this velocity jump which is responsible for the stabilisation. In a steady boundary layer above a stationary wall, the curvature of the unperturbed velocity profile at the wall is proportional to the pressure gradient (i.e. from the $\hat{x}$-momentum equation, $\hat{\mu}\hat{U}_{B\hat{y}\hat{y}} = \hat{p}_{\hat{x}}$ on $\hat{y} = 0$, where $\hat{\mu}$ is the fluid viscosity). Since there is no pressure gradient in the Blasius boundary layer, the curvature of the velocity profile *near* the wall is therefore very small. As a result the upper-branch scaling is different from that for a boundary layer in a favourable pressure gradient (see §2.3 and Bodonyi & Smith, 1981).

For the upper-branch scaling, (2.5*b*) is replaced by

$$E = \exp\left(\mathrm{i}Re^{\frac{9}{20}}\int \alpha_u(x)\mathrm{d}x + \mathrm{i}\beta_u Z - \mathrm{i}\omega_u T\right) \ , \qquad (2.8a)$$

where

$$\hat{x} = \hat{L}x \ , \quad \hat{z} = \hat{L}Re^{-\frac{9}{20}}Z \ , \quad \hat{t} = Re^{-\frac{2}{5}}\hat{L}\hat{U}_e^{-1}T \ . \qquad (2.8b)$$

Solutions can be obtained in each of the five regions illustrated in figure 4*a* in terms of asymptotic expansions in powers of $Re^{-\frac{1}{20}}$ and $\log Re$. Non-parallel effects are found to be of relative order $Re^{-\frac{3}{10}}\log Re$; this is much larger than the $O(Re^{-\frac{9}{20}})$ correction that might have been anticipated from the ratio of the TS wavelength to the development length of the boundary layer (Bodonyi & Smith 1981).

A comparison between the asymptotic theory, OS theory, and experiment is illustrated in figure 4*b* for the upper-branch neutral frequency of two-

define a Reynolds number

$$R_\delta = \hat{U}_e\hat{\delta}_1\hat{\nu}^{-1} \approx 1.72R \ .$$

† If the neutral frequency is sought then $\omega_l$ must also be expanded.

dimensional disturbances. The agreement between the raw four-term asymptotic expansion and experiment is distinctly worse than OS theory. Bodonyi & Smith (1981) observe that agreement can be improved either by an 'origin' shift of 300 in $R_\delta = 1.720R$, or by adding the leading-order *non-parallel* effect to a solution of the OS equation. Unfortunately, most (weakly) *nonlinear* theory is based on the *leading-order* (one-term) approximation which yields significant disagreement between *linear* theory and experiment. This does not bode well for quantitative comparisons between experiment and a leading-order nonlinear theory based on the upper-branch scaling.

However, Hultgren (1987) has shown that a surprisingly accurate prediction of the OS upper-branch neutral curve is possible by keeping the expansion for the lower-branch dispersion relation in what he refers to as its 'naturally occurring' form. In particular, a matched asymptotic solution for a lower-branch dispersion relation yields a solution

$$1 + l_1 F^{\frac{1}{6}} + l_2 F^{\frac{1}{3}} + O(F^{\frac{1}{2}}) = \frac{r}{1 + l_3 F^{\frac{1}{2}} \log(F^{\frac{1}{6}} L)} \ , \tag{2.9}$$

where the coefficients $l_j$, $L$ and $r$ are known in terms of the wavenumber and distance down the plate, and the frequency parameter,

$$F = \frac{\hat{\omega}\hat{\nu}}{\hat{U}_e^2} \ , \tag{2.10}$$

can be viewed as an inverse Reynolds number for a disturbance with given frequency. Solutions to (2.9) obtained *without* expanding the wavenumber, etc. in powers of $F^{\frac{1}{6}}$ (cf. Smith 1979*a*, Bodonyi & Smith 1981), yield quite good agreement with solutions to the OS equation for $F \le 10^{-5}$; this is so even for asymptotically large wavenumbers for which (2.9) is strictly not valid, e.g. at the upper branch.

For an important class of experiments, it is in fact more natural to work in terms of the frequency parameter, $F$, than the local Reynolds number, $Re$. For instance, suppose a perturbation of fixed (dimensional) frequency $\hat{\omega}$ is introduced into a Blasius boundary layer. As the disturbance propagates along the plate its Reynolds number varies with downstream distance. In particular:

- The lower-branch scaling corresponds to downstream distances for which

$$Re = O(F^{-\frac{4}{3}}) \ . \tag{2.11$a$}$$

  In this region TS disturbances have $O(F^{\frac{5}{6}}\hat{U}_e\hat{\nu}^{-1})$ wavenumbers. In addition it follows from (2.5*b*) that as the disturbance propagates through this region the *logarithm* of its amplitude experiences an integrated net growth of $O(F^{-\frac{1}{2}})$.

- Further downstream where

$$Re = O(F^{-\frac{5}{3}}) \ , \tag{2.11$b$}$$

  the disturbance enters the region described by the upper-branch scaling. Here TS disturbances have $O(F^{\frac{11}{12}}\hat{U}_e\hat{\nu}^{-1})$ wavenumbers, i.e. marginally *longer* wavelengths than for the lower-branch scaling. As a disturbance propagates through this region it follows from (2.8*a*), and the fact that $\Im(\alpha_u) = O(F^{\frac{1}{4}})$ (e.g. Reid 1965), that there is again an $O(F^{-\frac{1}{2}})$ net increase in logarithmic amplitude.

- However, the greatest net increase in amplitude occurs in the 'matching region', or 'intermediate region', between the upper and lower branches; specifically, the logarithmic amplitude experiences a net growth of $O(F^{-\frac{1}{2}} \log F)$ here (e.g. Goldstein & Hultgren 1989).

In the light of the above observations, arguments can be put forward for in-depth study of each of the asymptotic regions. For instance:

- From a theoretical standpoint, the upper-branch scaling is possibly a more attractive proposition than the lower-branch one, since from (2.11$a, b$), the former covers almost the entire unstable region (e.g. Goldstein & Durbin 1986). Further, in an experiment where the size of the input disturbance is slowly increased, nonlinear effects are likely to become important first where the amplitude of the disturbance is largest, i.e. near the upper branch. This scaling also has the advantage that a weakly nonlinear approach is possible, since the distance over which TS waves grow is much larger than the TS wavelength, i.e. the growth-rate is relatively small. However, as will be discussed below, the upper-branch scaling can involve very messy algebra when nonlinear effects are included.

- The lower-branch scaling also has attractions for theoretical study, not in the least because the agreement between linear theory and experiment is much better for the lower-branch neutral curve (see figure 3*b*). In addition the disturbance has its largest growth rate in this region.

- However, as an alternative to both these scalings, a number of studies have concentrated on the intermediate scaling, i.e.

$$F^{-\frac{4}{3}} \ll Re \ll F^{-\frac{5}{3}} \ . \tag{2.11$c$}$$

  It is in this 'fuzzy' intermediate region between the two distinguished scalings that the greatest net increase in amplitude occurs. This region can be viewed either as a far-downstream lower-branch scaling, or a far-upstream upper-branch scaling. While both approaches are asymptotically equivalent, most authors have approached the problem by taking the limit of the lower-branch scaling since the analysis is somewhat simpler from this direction, e.g. Smith & Burggraf (1985). † Further, Hultgren's (1987) observation on the agreement between lower-branch analysis and numerical solutions to the OS equations possibly adds credence to this approach. Another advantage of this intermediate scaling is that while the analysis is to a certain extent simpler than for the upper-branch scaling, the relative growth-rate is still small; hence a weakly nonlinear approach to finite-amplitude effects is possible.

### 2.3 Boundary layers with pressure gradients

At high Reynolds numbers the lower-branch triple-deck scaling of figure 3*a* also applies to steady boundary layers driven by a pressure gradient; the pressure gradient does not affect lower-branch TS waves at leading order. In fact this scaling describes the lower branch of most unbounded shear flows adjacent to a wall, including supersonic flows if three-dimensional disturbances are allowed (e.g. Smith 1989). An exception is when the wall shear is zero, i.e. for velocity profiles close to the onset of backflow (Goldstein *et al.* 1987, Elliott & Smith 1987). ‡

As indicated above, in a boundary layer the curvature of the streamwise velocity profile at the wall is proportional to the pressure gradient. As for the Blasius boundary layer, curvature plays a crucial rôle in stabilising/destabilising disturbances sufficiently far downstream. The larger curvature near the wall induced by the pressure gradient means that the 'upper-branch' scaling occurs for lower nondimensional frequencies than those specified by (2.8*b*): for an order-one pressure gradient, the scales of the wavenumbers and frequencies of such modes are

$$(\hat{\alpha}, \hat{\beta}) = O(Re^{-\frac{5}{12}}\hat{L}) \quad \text{and} \quad \hat{\omega} = O(Re^{\frac{1}{3}}\hat{U}_e\hat{L}^{-1}) \ , \tag{2.12}$$

respectively (Reid 1965).

If the pressure gradient is favourable, so that the flow at the edge of the boundary layer is accelerating, then $\hat{U}_{B\hat{y}\hat{y}} < 0$ near the wall and the velocity jump across the critical layer is stabilising; an upper-branch neutral frequency can then be identified. However, in an adverse pressure gradient, the velocity jump across the critical layer enhances the growth-rate of the disturbances with the result that *no* upper branch exists on this scale.

Whatever the sign of the pressure gradient, (2.12) specifies a 'distinguished' asymptotic scaling. As a shorthand, we will refer to this scaling as a 'viscous-layer/critical-layer balance' (VCB) scaling, since the contributions to the growth rate of linear TS waves from the viscous wall layer and the critical layer are comparable on this scaling. ⋆ The growth-rate of disturbances is relatively small for this VCB scaling. Hence, in principle, a weakly nonlinear analysis is possible.

In terms of the frequency parameter $F$, the VCB scaling corresponds to downstream distances where

$$Re = O(F^{-\frac{3}{2}}) \ ; \tag{2.13a}$$

TS disturbances have $O(F^{\frac{7}{8}}\hat{U}_e\hat{\nu}^{-1})$ wavenumbers on this scaling. Again we note from comparison with the lower-branch scaling, that the wavelength of fixed-frequency TS waves increases as they propagate downstream. Since $\Im(\alpha_u) = O(F^{\frac{1}{8}})$ at distances specified by (2.13*a*) (e.g. Reid 1965), the net growth in logarithmic amplitude on this scaling is again $O(F^{-\frac{1}{2}})$. In the case of a favourable pressure gradient, the total growth over the relatively 'long' upper-branch TS scaling is thus comparable with the total growth over the 'short' lower-branch scaling. The greatest net growth in logarithmic amplitude

† There is a slight problem with nomenclature here. Smith & Burggraf (1985) used a non-dimensionalisation based on distance from the leading edge. Hence for a disturbance with a fixed *dimensional* frequency, the scaled non-dimensional frequency increases downstream. This means that the intermediate region specified by (2.11*c*) corresponds to the 'high-frequency'/'large-wavenumber' region of the lower-branch parameter space studied by Smith & Burggraf (1985) and others. Further, this notation is even less intuitive if the wavenumber of a fixed-frequency disturbance is considered. In dimensional variables, the wavenumber of a fixed-frequency disturbance *decreases* as the disturbance propagates downstream out of the lower-branch region and into the intermediate region; however, with Smith & Burggraf's (1985) non-dimensionalisation, this region corresponds to the *large*-wavenumber limit!

‡ We prefer to use the term 'backflow' (or reversed flow) rather than 'separation' since it is only in two-dimensional steady flow adjacent to a non-moving boundary that the onset of backflow is synonymous with the separation of the boundary layer from the wall, e.g. Elliott *et al.* (1983).

⋆ An alternative name might be the 'quintuple-deck' scaling. However, in certain nonlinear calculations, more than five asymptotic layers are needed on this scaling, e.g. Wu (1993*a*).

is $O(F^{-\frac{1}{2}}\log F)$, and, as for the Blasius boundary layer, this occurs in the intermediate region between the lower-branch and upper-branch TS scalings.

For a flow in an order one adverse pressure gradient, the absence of a neutral curve on the VCB scaling (2.12) means that the amplitude (and wavelength) of linear fixed-frequency waves continues to grow downstream until

$$Re = O(F^{-2}) \ . \tag{2.13b}$$

Analysis shows that the modes then have $O(F\hat{U}_e\hat{\nu}^{-1})$ wavenumbers, or in terms of the local Reynolds number, $R = Re^{\frac{1}{2}}$,

$$(\hat{\alpha},\hat{\beta}) = O(R\hat{L}^{-1}) \quad \hat{\omega} = O(R\hat{U}_e\hat{L}^{-1}) \ . \tag{2.14}$$

(2.14) is just the classical Rayleigh-wave scaling, i.e. the mode wavelengths are comparable with the width of the boundary layer. The dynamics of the modes in this region are thus *inviscid* to leading order. The growth-rate of downstream propagating disturbances is larger in this Rayleigh region than in either the lower-branch or VCB regions. Moreover, since this region also 'covers' a much larger extent than both the lower-branch scaling (2.11*a*) and the VCB scaling (2.13*a*), the largest net growth in amplitude (specifically an $O(F^{-1})$ increase in logarithmic amplitude), occurs on this Rayleigh scale. However, the net growth in amplitude is bounded because sufficiently far downstream a neutral point is reached beyond which the waves decay. The structure of such decaying modes is a little surprising. In particular, in the direction normal to the boundary there are finite regions where viscous effects are important and where the modes have a very fine oscillatory structure (Foote & Lin, 1950).

In general, the spatial growth-rate for Rayleigh modes is *not* relatively small compared with the real wavenumber. However, the growth rate is small close to the neutral point, with the result that a weakly nonlinear analysis is possible if attention is focussed on disturbances with wavelengths/frequencies close to the neutral frequency (see §6). Similarly the intermediate region between the VCB scaling (2.13*a*) and the Rayleigh scaling (2.13*b*), i.e.

$$F^{-\frac{3}{2}} \ll Re \ll F^{-2} \ , \tag{2.15a}$$

is amenable to weakly nonlinear analysis because disturbances have relatively small growth-rates. For a scaling specified by (2.15) the analysis can be approached from two directions: either as a 'far-upstream' expansion of a Rayleigh scaling, or as a 'far-downstream' expansion of the VCB scaling. [†]

[†] In terms of a non-dimensionalisation based on boundary-layer thickness rather than the frequency of the disturbance, such an analysis is equivalent to a low-frequency expansion based on the Rayleigh scaling, or a high-frequency expansion based on the VCB scaling. We note, however, that there is not necessarily an unique low-frequency expansion based on the Rayleigh scaling. In the case of velocity profiles that have a local maximum or minimum there may exist additional low-frequency Rayleigh modes. Such modes have an asymptotic structure similar to that for a jet (e.g. Drazin & Howard 1962). When their frequencies are not excessively low, the modes have an inviscid structure. However, viscous forces modify the growth-rate of such modes for frequencies

$$\hat{\omega} = O(Re^{-\frac{3}{14}}\hat{U}_e\hat{\delta}^{-1}) = O(Re^{\frac{2}{7}}\hat{U}_e\hat{L}^{-1}) \ ,$$

and wavenumbers

$$(\hat{\alpha},\hat{\beta}) = O(Re^{-\frac{3}{14}}\hat{\delta}^{-1}) = O(Re^{\frac{2}{7}}\hat{L}^{-1}) \ .$$

Due to non-parallel effects no unique lower-branch neutral curve can be identified for these modes (e.g. Cowley, Hocking & Tutty 1985). While velocity profiles with such turning points are unusual in steady two-dimensional boundary-layer flow over stationary walls, they are relatively common in motion over downstream-moving walls, in three-dimensional boundary layers, and in unsteady flow.

However, details differ particularly when nonlinear effects are included. This is because the critical layer for the Rayleigh scalings is of 'non-equilibrium' (or 'unsteady') type, while the critical-layer is of viscous type for the VCB TS scaling. As a consequence there is an intermediate distinguished scaling

$$Re = O(F^{-\frac{14}{9}}) \ , \tag{2.15b}$$

where the critical layer is both of non-equilibrium and viscous type. This is a natural scaling to study especially as regard nonlinear effects since the disturbance modes might be viewed as *viscous* TS waves for $Re \ll F^{-\frac{14}{9}}$, but *inviscid* Rayleigh modes for $Re \gg F^{-\frac{14}{9}}$ (cf. Goldstein, Durbin & Leib 1987, Gajjar 1994).

If there is a *mild* adverse pressure gradient, then the only unstable Rayleigh modes have wavelengths much longer than the local boundary-layer thickness. Since all such modes have relatively small growth-rates, a weakly nonlinear analysis is again feasible (e.g. Goldstein, Durbin & Leib 1987, Goldstein & Lee 1992). For even milder adverse pressure gradients, only (Blasius) upper-branch TS waves are unstable (i.e. there is no distinct inviscid, 'long-wavelength', Rayleigh scale). However, whenever the adverse pressure gradient is sufficiently strong that unstable waves exist downstream of the VCB scaling, the largest net growth in amplitude of fixed-frequency waves occurs downstream of this scaling. This is despite the fact that, for a range of mild adverse pressure gradients, the fastest growing mode can occur on the lower-branch scaling.

### 2.4 Free shear layers, jets and wakes

The high-Reynolds-number asymptotic theory described above for boundary layers can also be applied to free shear layers, jets and wakes. For all these flows the velocity profiles have inflection points, and are inviscidly unstable to Rayleigh modes of frequency $O(\hat{U}_e\hat{\delta}^{-1}R)$, where $\hat{U}_e$ is here a typical velocity, and $\hat{\delta}$ is the width over which there is a significant change in vorticity in the shear-layer/jet/wake. †

At least for a convectively unstable flow, non-parallelism can be accommodated by a straightforward multiple-scales approach since the wavelength of the Rayleigh modes is much shorter than the development length of the shear layer. To fix ideas, suppose that a fixed-frequency disturbance is introduced near the start of the shear layer (we will refer to this as the 'upstream' region). Assume also that the modes excited have a wavelength much greater than the thickness of the shear layer. As the disturbance propagates downstream, ‡ the ratio of wavelength to shear-layer thickness decreases as the local Reynolds number increases. At first this variation leads to an increase in growth-rate, but sufficiently far downstream the disturbance stabilises when, according to linear theory, the non-dimensional wavenumber of the disturbance equals that of the 'inflectional', or 'upper-branch', neutral mode (this part of the flow will be referred to as the 'downstream' region).

As we have emphasised already, weakly nonlinear theory is in general only feasible when the growth-rate of the instability wave is relatively small. This is clearly the case in the (downstream) region close to the neutral point (e.g. Goldstein & Leib 1988). However, unlike a boundary layer, the growth-rate in a subsonic shear layer is not relatively small for disturbances with wavelengths long compared to the shear-layer thickness, i.e. in the upstream region close to the source. As a consequence of this observation, most of the weakly nonlinear theory that has been developed has concentrated on describing the 'downstream' development of very small initial disturbances (although certain progress may be possible for disturbances which become nonlinear further upstream, through an extension of the long-wavelength analysis of Cowley, Tanveer, Baker & Page 1993). For sufficiently supersonic shear layers, matters are more tractable in the upstream region, since the growth-rate of long wavelength disturbances is small and a weakly nonlinear analysis is feasible (Miles 1958, Balsa & Goldstein 1991 (personal communication)).

† Since Rayleigh's theorem and Fjortoft's criterion are *sufficient* rather than *necessary* conditions for inviscid instability, the existence of an inflection point satisfying Fjortoft's criterion does not guarantee that the flow is unstable. While unbounded shear flows that tend to a *constant velocity* at infinity almost always admit unstable modes, this is not so in the case of parallel flows that tend to a *constant shear* at infinity (e.g. Smith & Bodonyi 1985, Tutty & Cowley 1986).

‡ Certain complications can arise if the flow is absolutely unstable, e.g. as can occur immediately behind a body. Huerre & Monkewitz (1990) review aspects of linear and weakly nonlinear absolute instability. Exactly how current ideas on this problem can be incorporated into a consistent high-Reynolds-number asymptotic framework, especially when nonlinear effects need to be included, still needs some clarification.

### 2.5 Unsteady boundary/shear layers

As explained in §1, for both parallel and non-parallel *unsteady* shear flows, the OS equation for normal modes can only be derived by making an *ad hoc* quasi-steady approximation. Moreover, this quasi-steady approximation often turns out to be less accurate than the quasi-parallel assumption. An alternative approach at high Reynolds numbers is to study the stability of unsteady shear flows using the analytical methods developed for steady almost parallel flows. For instance, the stability of high-Reynolds-number Stokes flow can be examined in a mathematically consistent manner by an expansion in inverse powers of the Reynolds number (Tromans 1977, Cowley 1987). Such a multiple-scales approach is possible since in general the timescale of the instability waves is asymptotically small compared to the timescale for the evolution of the shear layer.

## 3. WEAKLY NONLINEAR THEORY

If a flow is convectively unstable to exponentially growing travelling waves, then the question arises as to how the flow evolves when previously neglected nonlinear effects are included. If the spatial/temporal growth-rate is comparable with the wavenumber/frequency, then this question can only be answered, in general, by a fully numerical solution of the NS equations. However, when the growth-rate is relatively small, a weakly nonlinear theory is often possible.

At large Reynolds numbers, many scalings lead to weakly nonlinear theories. Some of these are extremely pertinent to experimental observations and place earlier *ad hoc* theories on a sound basis; others might be viewed as little more than displays of 'asymptotic muscle'! However, almost all turn out to be based on one or two basic concepts, which have their origin in the classical weakly nonlinear theory, even though the extension sometimes turns out to be rather elaborate mathematically.

In order to illustrate these basic concepts, we begin by briefly reviewing weakly nonlinear theory for PPF. This flow has the advantages ($a$) that non-parallel effects are absent and ($b$) that there is a well-defined *finite* critical Reynolds number, $R_c$, above which small disturbances may be amplified.

### 3.1 Uniform travelling waves in PPF

Since PPF is an exactly parallel flow, it is consistent to examine its stability from either a temporal or spatial approach; however, most of the nonlinear theory has taken the temporal approach. Weakly nonlinear calculations are possible when the timescale over which a (linear) wave grows is much longer than the period of the wave.

At the critical Reynolds number, $R = R_c$, there is a neutral mode with a (real) frequency $\hat{U}_e\hat{\delta}^{-1}\omega_c$ and a (real) wavenumber $\hat{\delta}^{-1}\alpha_c$. At slightly different wavenumbers $\alpha - \alpha_c = O(\epsilon)$, where $\epsilon = |R - R_c|^{\frac{1}{2}} \ll 1$, the growth-rates of the fastest growing linear disturbances are $O(\epsilon^2\hat{U}_e\hat{\delta}^{-1})$, i.e. much smaller than typical frequencies. This allows the introduction of a 'fast' timescale, $t = \hat{U}_e\hat{\delta}^{-1}\hat{t}$, and a slow timescale, $\tau = \epsilon^2\hat{U}_e\hat{\delta}^{-1}\hat{t}$. Nonlinear effects influence the growth of the disturbance through a cubic interaction. Specifically, suppose from (1.4) and (1.5) that the cross-stream velocity perturbation is given by

$$\tilde{v} = \bar{\epsilon}(A(\tau)E + c.c.) + \dots\ , \qquad (3.1a)$$

where

$$E = \exp(\mathrm{i}\alpha_c x - \mathrm{i}\omega_c t)\ , \quad \text{and} \quad 0 < \bar{\epsilon} \ll 1\ . \quad (3.1b, c)$$

Then as a result of the quadratic inertia terms in the NS equation, $O(\bar{\epsilon}^2)$ mean-flow and second-harmonic terms will be excited. The quadratic interaction of these terms with the original harmonic leads to an $O(\bar{\epsilon}^3)$ forced harmonic. If this forced harmonic is as large as the $O(\epsilon^2\bar{\epsilon})$ correction that arises because the Reynolds number differs from its critical value by an $O(\epsilon^2)$ amount, i.e. if $\bar{\epsilon} = O(\epsilon)$, then nonlinear effects cannot be neglected in estimating the growth of the wave. Stuart (1960) showed that this scaling leads to the 'Landau equation'

$$\frac{dA}{d\tau} = k_1 A + k_2|A|^2 A, \qquad (3.2)$$

where $k_1$ and $k_2$ are known complex constants, and the sign of $\Re(k_1)$ depends on the sign of $(R - R_c)$. For PPF, $k_{2r} > 0$ and $\mathrm{sgn}(k_{1r}) = \mathrm{sgn}(R - R_c)$, where $k_{1r} = \Re(k_1)$, $k_{1i} = \Im(k_1)$ etc.. The Landau equation describes the evolution of weakly nonlinear travelling waves.

- If $k_{1r}k_{2r} < 0$, there is a non-zero equilibrium amplitude corresponding to a nonlinear travelling wave.
  - *A subcritical bifurcation.* If $k_{2r} > 0$, the equilibrium solution is unstable and represents a 'threshold' amplitude; for PPF this corresponds to the case $R < R_c$. If the disturbance amplitude is smaller than this threshold, it decays, while larger disturbances grow to an 'infinite' (scaled) amplitude in a finite time. Of course, the unbounded growth in the latter case indicates that the asymptotic expansion procedure must ultimately fail and a new scaling must be tried.
  - *A supercritical bifurcation.* If $k_{2r} < 0$, the equilibrium amplitude is stable, and disturbances of any amplitude evolve to it.
- If $k_{1r}k_{2r} > 0$, there is no equilibrium amplitude.
  - If $k_{2r} > 0$, all non-zero disturbances become unbounded within a finite time; for PPF, this is the case when $R > R_c$.
  - If $k_{2r} < 0$, all disturbances decay to zero.

Note that instead of deriving the time-evolution equation (3.2), we could have sought nonlinear travelling-wave solutions, i.e. the equilibrium-amplitude solutions. This only leads to minor simplifications in the weakly nonlinear case, but has significant advantages for finite amplitudes. In particular, Herbert (1977) has pursued such an approach for two-dimensional travelling waves. The equilibrium amplitudes so found map out a nonlinear neutral surface in the parameter space of Reynolds number, wavenumber and amplitude (see figure 5).

Whether these travelling-wave solutions are stable to two-dimensional disturbances is a delicate question, and depends on whether the mass flux or the mean pressure gradient is fixed (e.g. Pugh & Saffman 1988). Small-amplitude travelling waves are certainly unstable to two-dimensional disturbances if the bifurcation is subcritical (see above). In addition, travelling waves that are stable to two-dimensional disturbances can be unstable to three-dimensional perturbations (Orszag & Patera 1980, 1981, Herbert 1983). However, whether such a linear stability analyses of travelling waves provides a (quantitative) description of transition in PPF is debatable since it is not clear how the unstable equilibria organise the long-time behaviour of the unstable motion. Nevertheless, in subcritical régimes the amplitude of the travelling wave *may* provide a useful indication of threshold amplitudes for instability.

### 3.2 Wavepackets in PPF: $R \approx R_c$

A restriction of the formulation leading to the Landau equation (3.2) is that it assumes a uniform wavetrain for all $\hat{x}$; this is physically unrealistic. Stewartson & Stuart (1971) and Davey, Hocking & Stewartson (1974) have shown how to generalise this formulation to wavetrains modulated in the streamwise direction, $\hat{x}$, and/or the spanwise direction, $\hat{z}$. They consider the region of parameter space close to

$$R = R_c\,, \quad \alpha = \alpha_c \quad \text{and} \quad \beta = 0\,. \tag{3.3}$$

From the dispersion relation (1.7), the neutral wave specified by (3.3) has a group velocity,

$$\mathbf{c_g} = \left(\frac{\partial \omega}{\partial \alpha}, \frac{\partial \omega}{\partial \beta}\right) = -\left(\frac{\partial f}{\partial \omega}\right)^{-1}\left(\frac{\partial f}{\partial \alpha}, \frac{\partial f}{\partial \beta}\right)\,, \tag{3.4}$$

that is *real* (in fact $\frac{\partial \omega}{\partial \beta} = 0$ by symmetry). This is important since it proves useful to transform into a frame moving with the group velocity. The appropriate scalings are

$$\epsilon^2 = |R - R_c|\,,\ t = \hat{U}_e \hat{\delta}^{-1}\hat{t}\,,\ \tau = \epsilon^2 \hat{U}_e \hat{\delta}^{-1}\hat{t}\,, \tag{3.5a-c}$$

$$x = \hat{\delta}^{-1}\hat{x}\,, \quad \xi = \epsilon\left(x - \frac{\partial \omega}{\partial \alpha}t\right)\,, \quad \eta = \epsilon\hat{\delta}^{-1}\hat{z}\,, \tag{3.5d-f}$$

where the amplitude $A$ in (3.1) is now a function of $\tau$, $\xi$ and $\eta$. The resulting governing equations are (after a suitable normalisation)

$$\frac{\partial A}{\partial \tau} - a\frac{\partial^2 A}{\partial \xi^2} - b\frac{\partial^2 A}{\partial \eta^2} = k_1 A + k_2 |A|^2 A + qAB\,, \tag{3.6a}$$

$$\frac{\partial^2 B}{\partial \xi^2} + \frac{\partial^2 B}{\partial \eta^2} = \frac{\partial^2 |A|^2}{\partial \eta^2}\,. \tag{3.6b}$$

The function $B(\tau, \xi, \eta)$ arises because of the need to account for a secular pressure term related to the mean-flow correction.

An initial condition for (3.6) needs to be specified. Stewartson & Stuart (1971) and Hocking, Stewartson & Stuart (1972) proposed to study the evolution of a small localised disturbance introduced at $\hat{t} = 0$. For $R > R_c$, this slowly grows as it propagates downstream. A (linear) steepest-descents analysis shows that for $\hat{\delta}\hat{U}_e^{-1} \ll \hat{t} \ll \epsilon^{-2}\hat{\delta}\hat{U}_e^{-1}$, the amplitude of the disturbance is largest in a small elliptical region that propagates downstream with the group velocity (see also Benjamin 1961, Gaster 1968*a*, *b*). It was proposed that appropriate initial conditions to (3.6) as $\tau \to 0$ might be

$$A \sim \frac{\Delta}{\tau^{\frac{1}{2}}} \exp\left(-\frac{\xi^2}{4a\tau}\right) \tag{3.7a}$$

for a line (two-dimensional) disturbance, and

$$A \sim \frac{\Delta}{\tau} \exp\left(-\frac{\xi^2}{4a\tau} - \frac{\eta^2}{4b\tau}\right) \tag{3.7b}$$

for a point (three-dimensional) disturbance. However, it is relatively straightforward to show that (3.7) do not provide uniformly valid initial conditions to (3.6) (Hocking, Stewartson, & Stuart 1972). Numerical calculations have been performed by instead applying (3.7) at some time $\tau_0 > 0$ (e.g. Hocking 1974).

For this modified initial condition, Davey *et al.* (1974) and Hocking & Stewartson (1971) considered wavepackets of two-dimensional waves that were modulated in one (possibly oblique) direction, i.e. $A \equiv A(\alpha\xi + \beta\eta, \tau)$. For wavepackets skewed at angles greater than 57.3° to the mean flow, they showed that a finite-time singularity develops for $R > R_c$, while for angles less than 57.3° the solution remains bounded. Hence modulation can suppress the unbounded growth of a uniform wavetrain.

Hocking & Stewartson (1971) refer to solutions that become unbounded in a finite time as 'bursts'. However such solutions are not *directly* related to the physical bursts commonly observed in transition. The latter have length and time scales much shorter than those of TS waves, while the modulational 'bursts' represented by solutions of (3.6) describe a 'focussing' of the wavetrain envelope from an initial $O(\hat{\delta}\epsilon^{-1})$ length to, for instance, a *still large* $O(|\log \epsilon|^{\frac{1}{2}}\hat{\delta})$ length (Hocking *et al.* 1972). The subsequent evolution of the flow, and its susceptibility to secondary instabilities, etc., have yet to be examined in detail.

Further, Hocking (1974) finds that, with the modified initial condition, a point-centred initial disturbance does *not* apparently 'burst' – that is, its amplitude remains bounded. This suggests that the system of equations (3.6) may not be the most pertinent to a description of transition in PPF. It also emphasises that the knowledge that a non-modulated disturbance 'bursts', should *not* be used to conclude that a localised initial disturbance 'bursts'. Since the theory is easier for the former, while experiments are often closer to the latter, this is a point not to be forgotten – no matter how tempting it might be to do so!

While the results for the modified initial condition are interesting, the question remains as to how nonlinearity modifies the evolution of an initially localised linear disturbance. As discussed above, this is a physical situation that can in principle be realised experimentally. Analysis suggests that because of a necessary logarithmic shift in the time origin, the Landau equation (3.2), rather than the system (3.6), is recovered when the flow becomes nonlinear. Hence the spatial modulation plays a passive rôle. Further, it follows immediately (see §3.1) that the scaled disturbance $A$ becomes unbounded at a finite time due to nonlinear effects. Hence an initially localised linear disturbance in PPF eventually 'bursts'.

### 3.3 Extensions

Hocking (1975) has extended this weakly nonlinear analysis to the (exactly-parallel) asymptotic suction profile. The most significant change in equations (3.6) is that $q = 0$, because the absence of an upper rigid boundary means that an induced pressure gradient cannot be supported. For the initial conditions (3.7) applied at $\tau = \tau_0 > 0$, Hocking (1975) finds that both line- and point-centred initial disturbances can develop into localised modulational 'bursts' in a finite time.

The analysis leading to (3.6) is for modulated two-dimensional waves. The carrier wave cannot be three-dimensional, since when $R = R_c$ there is no neutral three-dimensional wave. However, a weakly nonlinear analysis can in principle be performed close to any point on a neutral curve; as indicated above, weakly nonlinear two-dimensional travelling-wave solutions have been obtained in this way. It is relatively straightforward to extend this analysis to the time-evolution of modulated wavetrains of two-dimensional waves, etc.. However, for $R \neq R_c$, the group velocity, $\mathbf{c_g} = \nabla_\alpha$, is in general not real; hence the scaling (3.5$e$) for $\xi$ has to be replaced by

$$\xi = \epsilon^2 x. \qquad (3.8a)$$

For example, in the case of the asymptotic suction profile, (3.6) becomes

$$\frac{\partial A}{\partial \tau} - d\frac{\partial A}{\partial \xi} - b\frac{\partial^2 A}{\partial \eta^2} = k_1 A + k_2 |A|^2 A \ , \qquad (3.8b)$$

where $d$ is complex. Care must be taken in the formulation of suitable initial conditions for disturbances that start as linear localised wavepackets. This is because (3.4) is derived by expanding about a neutral mode, and there exist modes with neighbouring wavenumbers that have larger growth-rates. Since a linear *localised* disturbance includes some of the faster-growing modes, it is necessary to ensure that these modes do not grow to dominate the solution before the slow timescale defined by (3.5$c$) comes into play. Of course, there is also the question of how such disturbances can be experimentally realised.

Likewise, the evolution of *three*-dimensional oblique almost-neutral modes could be studied. Suppose that $R_c^\beta$ is the 'critical' Reynolds number above which unstable TS waves with spanwise wavenumber $\beta$ can be found. Then weakly nonlinear expansions, based on either a single neutral carrier wave, or a pair, are in principle possible for $|R - R_c^\beta| \ll 1$. This problem does not appear to have been tackled at finite Reynolds numbers, either for uniform wavetrains or for modulated wavepackets (but see Hall & Smith (1990), Smith & Blennerhassett (1992), Smith & Bowles (1992) for aspects of the high-Reynolds-number limit). In a situation where disturbances with a broad band of wavenumbers are excited, there is a good reason for not doing this, namely that the fastest-growing disturbance is likely to dominate. In a carefully controlled experiment, however, or possibly in flow through a grooved channel, near-neutral waves might be preferentially excited with larger amplitudes and so cause transition. We will return to this point later, particularly in relation to non-parallel flows and wave/vortex interactions.

### 3.4 Summary of finite-Reynolds-number results

At finite Reynolds numbers we have seen that:

($a$) Weakly nonlinear theory can be used to describe both two- and three-dimensional, small-amplitude, equilibrium travelling waves (although only two-dimensional solutions seem to have been sought). It is possible to test the stability of these solutions ($a$) to disturbances of the same wavelength using (3.2), and ($b$) to disturbances of slightly different wavelengths by means of a 'sideband' instability analysis using (3.6). If stable, the flow *may* evolve to these waves; if unstable, the equilibrium wave amplitude predicts a threshold amplitude for nonlinear instability.

($b$) Weakly nonlinear theory can also be used to describe the nonlinear evolution of both uniform wavetrains (see (3.2)) and wavepackets, i.e. wavetrains modulated in the streamwise and/or spanwise directions (see (3.6) and (3.8$b$)). Unbounded growth ('bursts') can occur for both types of disturbances. 'Bursting' indicates the development of large-amplitude waves and, in the case of wavepackets, the focussing of the disturbance on the slow modulational lengthscale. This type of 'bursting' does *not* describe the development of sub-wavelength turbulent bursts, although it may be a precursor of the latter.

($c$) The method can be applied at any point on the neutral curve, although the governing equations for streamwise-modulated wavetrains differ according to whether the group velocity is real or complex.

(*d*) For disturbances with a broad band of wavenumbers, the theory is most applicable for $R \approx R_c$ since weakly nonlinear theory can then describe the fastest-growing mode. However, in experiments where (almost neutral) two- or three-dimensional disturbances with a preferred wave number are excited, the theory can also be applied for $R > R_c$.

(*e*) Little quantitative agreement between theory and experiment has yet been presented for PPF (but see Smith & Bowles 1992), or for boundary layers with suction – the two best-known exactly-parallel unstable profiles.

In summary, the key assumption (or restriction) in weakly nonlinear theory is that the growth rate is small. As such, the leading-order dispersion relation and the eigenfunction can be determined by a linear analysis while nonlinear effects are included at higher order. By taking advantage of this, the nonlinear development of uniform or modulated wavetrains in exactly parallel flows can be described. For nonparallel flows, a high-Reynolds-number asymptotic approach is crucial if nonlinearity and nonparallelism are to be included in a self-consistent and systematic manner. As will be seen, the modern asymptotic approach has both revealed new features in transition, and provided deeper insight into the physical mechanisms. However, many (if not all) of these theories are based on the assumption of small growth rate, and as such should be regarded as being weakly nonlinear; this is so despite the fact that sometimes nonlinear partial differential equations have to be solved.

### 3.5 High-Reynolds-number weakly nonlinear theory

Before 1979, attention concentrated on the parameter régime where $R \approx R_c$, since in that régime the *fastest-growing modes* can be described by an asymptotic theory; this is crucial if disturbances with a broad band of wavenumbers/frequencies are excited. However, there are important problems where there exists a preferred frequency (e.g. arising from engine noise or vibration of the structure) and/or a preferred lengthscale (e.g. corresponding to the size of particles impinging on the flow). Furthermore, in a spatially-dependent or unsteady flow, disturbances with preferred frequencies/lengthscales may be stable in one part of the flow régime but unstable in another.

For much of the rest of this review we will concentrate on examples of transition which are caused by disturbances with a preferred frequency. An oft-repeated experiment modelling this type of transition is that of Schubauer & Skramstad (1947), on the evolution of TS waves in a Blasius boundary layer. By means of a vibrating ribbon, waves of fixed frequency and known amplitude are introduced into the flow. Suppose that the forcing occurs upstream of the lower branch (see figure 6, where the abscissa measures the distance downstream). The waves initially decay as they propagate downstream, until they reach the lower branch; beyond this, there is a region of amplitude growth. When the upper branch is crossed, decay once again sets in, *provided* that nonlinear effects are still not significant. Due to the region of exponential growth, the initial amplitude at the point of forcing is usually required to be exponentially small (in some sense) if the flow is to remain linear everywhere. If the amplitude of the initial disturbance is suitably adjusted, nonlinear effects can be made to come into play as the waves reach the upper branch. If the forcing amplitude is increased further, nonlinearity becomes important at positions upstream of the upper branch, and for sufficiently large input disturbances, nonlinear effects may develop near the lower branch.

The analysis describing the evolution of the disturbance differs depending on where the flow first becomes nonlinear. For the Blasius boundary layer, there are three major regions to be considered – these are illustrated schematically in figure 7*a*. For definiteness, suppose that a fixed-frequency disturbance is introduced at the lower branch neutral point. As discussed above, for very small initial amplitudes the disturbance can remain linear at all points. For slightly larger disturbances, the initial amplitude can be adjusted so that the perturbation becomes nonlinear at an $O(F^{-\frac{5}{3}}\hat{U}_e\hat{\nu}^{-1})$ distance downstream, i.e. in the region described by the VCB scaling. Aspects of this case are discussed in §5 on the assumption that the flow does not become nonlinear *very* close to the upper branch. In the latter special case, the very small growth rate of the disturbance can mean that non-parallel effects, *inter alia*, play an enhanced rôle, with the result that the asymptotic analysis is modified.

If the initial amplitude is larger, then the disturbance can become nonlinear in the triple-deck region, i.e. an $O(F^{-\frac{4}{3}}\hat{U}_e\hat{\nu}^{-1})$ distance from the leading edge. In general, since the growth-rate is not small here, a numerical solution of the fully nonlinear equations (see (4.1)) is required (e.g. Duck 1985, 1990, Cowley *et al.* 1990). However, if the initial amplitude is sufficiently large, then the disturbance can become nonlinear close to the lower branch. A weakly nonlinear analysis is then possible; this case is discussed in §4. Further, as discussed there, if the initial disturbance is large enough, then the flow can become nonlinear sufficiently close to the neutral curve for non-parallel effects to be important.

For an accelerating boundary layer, there are again three major regions to be considered where the disturbance can first become nonlinear (see figure 7*b*). These correspond to the three regions for the Blasius boundary layer. As explained in §2.3, the major difference is that the VCB scaling occurs at an $O(F^{-\frac{3}{2}}\hat{U}_e\hat{\nu}^{-1})$ distance downstream of the leading edge. Much of the nonlinear analysis described in §5 remains valid after an appropriate change of scaling.

For a decelerating boundary layer the disturbance can first become nonlinear in one of *at least* seven distinguished regions. The three upstream regions closest to the lower branch are the same as for an accelerating boundary layer (see figure 7*c*). However, there is no upper-branch neutral point on the VCB scaling. The fourth distinguished region occurs at an $O(F^{-\frac{14}{9}}\hat{U}_e\hat{\nu}^{-1})$ distance downstream (see (2.15*b*)). While on the VCB scaling the critical layer is primarily viscous, in this region the dynamics in the critical layer involves both 'unsteady' and viscous forces (e.g. Goldstein, Durbin & Leib, 1987). Since nonlinearities first come into play within the critical layers, this change in the dynamics of the critical layer affects the amplitude-evolution equation. Since this scaling and the VCB scaling differ only by a factor of $F^{-\frac{1}{18}}$, before any comparison with experiment, it is probably best to form a composite expansion of the two scalings (Dr M.E. Goldstein 1992 (personal communication)). If the disturbance becomes nonlinear for $F^{-\frac{14}{9}} \ll Re \ll F^{-2}$, then the weakly nonlinear analysis is primarily inviscid.

If the initial disturbance amplitude is even smaller, so that the flow becomes nonlinear in the fifth distinguished region (defined to be an $O(F^{-2}\hat{U}_e\hat{\nu}^{-1})$ distance downstream), then a fully nonlinear analysis is in general necessary since the growth-rate of the linear waves is not relatively small. However, for suitable initial amplitudes the flow perturbation can become nonlinear in the vicinity of the upper branch (which we take to be at $Re = Re_c$); a weakly nonlinear analysis is then possible. † There are two distinguished regions close to the upper branch. The first is an $O(F^{-\frac{5}{3}}\hat{U}_e\hat{\nu}^{-1})$ distance upstream of the upper branch. The analysis here is not dissimilar to that for the fourth region since both unsteady and viscous effects are significant in the critical layer (e.g. Goldstein & Hultgren 1988). The limits $F^{-\frac{5}{3}} \ll |Re - Re_c| \ll 1$ and $|Re - Re_c| \ll F^{-\frac{5}{3}}$ correspond to predominantly inviscid and viscous limits respectively (e.g. Goldstein & Leib 1988). However, the latter limiting solution is not valid arbitrarily close to the upper branch. For $|Re-Re_c| = O(F^{-\frac{3}{2}})$ there is a seventh distinguished region. Here the growth-rate of the disturbance is so small that nonparallel effects cannot be neglected at leading order (e.g. Shukhman 1989, Smith, Brown & Brown 1993).

The above classification is not comprehensive, particularly if two or more disturbance modes are excited and/or subsequent nonlinear stages are considered. However, the different scalings serve to emphasise the different physical mechanisms that can influence transition. Further, while the multitude of scalings makes comparison with experiment less than straightforward, some spectacular successes have been achieved by use of composite expansions.

### 3.6 Types of disturbance

In addition to varying the position where the perturbed flow becomes nonlinear, the evolution of a number of different types of disturbance can be considered. For example the 'carrier' TS/Rayleigh wave can be

- ($a$) a two-dimensional mode,
- ($b$) a single oblique mode,
- ($c$) a pair of oblique modes, or
- ($d$) a combination of the above,

while the wavetrain may be

- ($i$) uniform, or
- ($ii$) be modulated so as to form a wavepacket that, for instance, is localised in time and/or space,

Further, the disturbance may

- ($\alpha$) evolve from an initially linear state (as in §3.5), or
- ($\beta$) be introduced as a weakly nonlinear perturbation.

It is also possible to consider disturbances excited

- ($A$) at a point,
- ($B$) along a line, or
- ($C$) over some broader region.

A number of these possibilities will be reviewed below.

† The fact that the initial amplitude has to be tuned so that the flow becomes nonlinear in the *vicinity* of the upper branch may at first sight suggest that the analysis has limited validity. However, in an analogous problem for a shear layer (see §6.2), Hultgren (1992) has shown that the weakly nonlinear analysis has a surprisingly large range of validity.

## 4. ASYMPTOTIC NONLINEAR THEORY – TS LOWER BRANCH

In this section we will assume that a disturbance is introduced into the boundary layer in such a way that the flow becomes nonlinear very close to the lower branch. For definiteness we start by assuming that the disturbance is introduced in a region where its evolution can be described by linear dynamics. We then briefly review a situation where 'nonlinear' disturbances are introduced in the neighbourhood of the lower branch - an example of bypass transition.

### 4.1 Growth of a two-dimensional carrier wave

If the carrier wave is two-dimensional, and the disturbance initially linear, then an analysis similar to that in §3 is expected to hold. Smith (1979*b*) and Hall & Smith (1982) have derived the Landau equation for an attached boundary layer (e.g. the Blasius boundary layer) and Poiseuille flow respectively. In addition Hall & Smith (1982) allow for a wavelike forcing at the flow boundary. Their analysis is based on a nonlinear extension of the triple-deck scalings discussed in §2. For boundary-layer flow, the scaled governing equations are

$$U_T + UU_X + VU_Y + WU_Z = -P_X + U_{YY}\,, \tag{4.1a}$$
$$W_T + UW_X + VW_Y + WW_Z = -P_Z + W_{YY}\,, \tag{4.1b}$$
$$P_Y = 0\,,\quad U_X + V_Y + W_Z = 0\,, \tag{4.1c,d}$$

with boundary conditions

$$U = V = W = 0 \quad \text{on} \quad Y = 0\,, \tag{4.1e}$$
$$U \to \Lambda(x)(Y + A)\,,\ W \to 0 \quad \text{as} \quad Y \to \infty\,, \tag{4.1f}$$
$$U \to \Lambda(x)Y \quad \text{as} \quad X \to -\infty\,, \tag{4.1g}$$

and interaction relation

$$P = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{A_{\xi\xi}}{(X-\xi)^2+(z-\zeta)^2}\,d\zeta d\xi\ , \tag{4.1h}$$

where for future convenience we have included a spanwise dependence with the same scale as the triple-deck TS wavelength. The slow variable $x$ (which is here non-dimensionalised by the distance from the leading edge), represents the slow downstream evolution of the boundary layer. Its effect is felt through the slowly-varying wall shear $\Lambda(x)$.

For a given disturbance frequency, there will be a location, say $x = x_0$, where the TS wave is neutral. The weakly nonlinear analysis assumes that the wave remains linear as it propagates downstream until it reaches a position $x = x_0 + \epsilon^2 x_1$, where $0 < \epsilon \ll 1$ and $x_1 = O(1)$ (this downstream displacement corresponds to the $O(\epsilon^2)$ variation from the critical Reynolds number in (3.5*a*)). At this point the (scaled) spatial growth-rate of the wave is $O(\epsilon^2)$, while the (scaled) wavenumber is $O(1)$. As in §3, if the wave has a (scaled) amplitude of $O(\epsilon)$, a nonlinear interaction ensues. The governing equation for the amplitude is then the spatial Landau equation

$$\frac{d\bar{P}}{d\bar{X}} = k_3 x_1 \bar{P} + k_2|\bar{P}|^2\bar{P}\,, \tag{4.2a}$$

where $\bar{X} = \epsilon^2 X$ and $\bar{P}$ is the scaled amplitude of the leading-order pressure perturbation.

Note that if a disturbance is introduced at a fixed value of $x$, then $\epsilon$ can be *deduced* in terms of the initial amplitude by the condition that the disturbance should have an $O(\epsilon)$ amplitude at a distance $O(\epsilon^2)$ downstream of neutral. Observe also for future reference that there are four lengthscales involved in this calculation:

(1) the very 'slow' downstream boundary-layer evolution scale, $x = O(1)$;

(2) the distance downstream from the lower branch at which nonlinear effects develop, i.e. $x - x_0 = O(\epsilon^2)$;

(3) the 'slowish' lengthscale over which the TS waves grow, i.e. $\bar{X} = O(1)$ or $X = O(\epsilon^{-2})$ or $x - x_0 = O\left(\epsilon^{-2}Re^{-3/8}\right)$;

(4) the 'fast' wavelength of the TS wave, i.e. $X = O(1)$ or $x - x_0 = O(Re^{-3/8})$.

Hall & Smith (1984) observed that non-parallel effects enter the analysis at leading order if the second and third scales are identical, i.e. if

$$\epsilon = O(Re^{-3/32})\ . \tag{4.2b}$$

In terms of figure 7, this corresponds to nonlinear effects becoming important within an $O(F^{-\frac{13}{12}})$ distance of the lower neutral point. For such a scaling the Landau equation (4.2*a*) is slightly modified because $\bar{X}$ and $x_1$ are now the same variable. The solution to the linear equation is then proportional to $\exp(\frac{1}{2}k_3\bar{X}^2)$; that this is correct can be easily checked by taking the lower bound of the integral in (2.5*b*) to be $x_1$, expanding in powers of $(x - x_1)$, and using the fact that $\Im(\alpha_\ell(x_1)) = 0$.

For boundary-layer flow, Smith (1979*b*) and Hall & Smith (1982, 1984) show that solutions to (4.2*a*) tend to a slowly-varying amplitude downstream, i.e. the bifurcation is supercritical. This 'equilibrium' amplitude is stable to two-dimensional disturbances of the same frequency, but can be unstable to three-dimensional disturbances of a different frequency. *Inter alia*, Hall & Smith (1984) † show that

† *Inter alia*, Hall & Smith (1984) derive equations which describe the nonlinear interaction of two or more almost neutral nodes. Note that a 'diffusion layer' of

such instabilities can force the disturbance amplitude to become singular a finite, i.e. $\bar{X} = O(1)$, distance downstream. As a result of this singular growth, the flow reverts to being described by the fully nonlinear system of equations (4.1).

While such analyses yield interesting results, they are equivalent to a 'uniform-waveform' calculation (cf. §3.1). No detailed consideration is given to how the wavetrain is established, e.g. by including some modulation in time. In addition, while the rapid growth in disturbance amplitude identified by Hall & Smith (1984) in their multi-mode analysis is consistent with transition, their weakly nonlinear analysis is based on the assumption that the disturbance consists of multi-frequency, *almost-neutral* three-dimensional waves. This may be possible in very carefully controlled experiments, but in most flows where the disturbance has a significant multi-frequency component, it is likely that at least some of the (linear) modes will have $O(1)$ growth-rates – in which case there will be other stronger instabilities.

Smith & Walton (1989) have extended the two-dimensional analysis to include a slow spanwise modulation over distances $\bar{Z} \equiv \epsilon Z = O(1)$. For the distinguished limit when non-parallel effects are important, i.e. $\epsilon = O(R^{-3/32})$, the scaled amplitude equation is

$$\frac{\partial \bar{P}}{\partial \bar{X}} = k_3 \bar{X} \bar{P} + k_4 \frac{\partial^2 \bar{P}}{\partial \bar{Z}^2} + k_2 |\bar{P}|^2 \bar{P} \ . \tag{4.3}$$

This equation has the same form as the finite-Reynolds-number amplitude equation (3.8*b*) but with the inclusion of the linear non-parallel effects, and the exclusion of the modulation in time. For appropriate parameter values, Smith & Walton (1989) show that solutions to this equation can end in a finite-distance singularity due to a focussing of the spanwise modulation. The subsequent evolution of the disturbance is described by the full nonlinear equations (4.1). A time-modulated wavepacket corresponding to a finite period of excitation does not seem to have been studied.

### 4.2 A 'weak' TS-wave/vortex interaction

Smith & Walton (1989) show that the large-$\bar{X}$, two-dimensional, quasi-equilibrium solution to (4.3) is unstable to three-dimensional disturbances, particularly those with large spanwise wavenumbers.

width $O(\epsilon^{-\frac{2}{3}})$ seems to have been omitted in their analysis. This layer is needed in order to eliminate a 'steady-streaming' spanwise mean flow induced far from the boundary. Since this velocity is independent of the spanwise coordinate it is presumably passive (cf. Smith & Walton 1989).

However, *if* such disturbances can be suppressed then the two-dimensional equilibrium solution can lose stability to another type of interaction further downstream. In particular Smith & Walton (1989) show that if $\epsilon = O(R^{-3/40})$ then this new 'TS-wave/vortex' interaction is described by the normalised amplitude equation

$$(k_3 \bar{X} + k_5 \Lambda_0) \bar{P} + k_4 \frac{\partial^2 \bar{P}}{\partial \bar{Z}^2} + k_2 \bar{P} |\bar{P}|^2 = 0 \ , \tag{4.4a}$$

where the vortex skin-friction factor, $\Lambda_0$, is given by‡

$$\Lambda_0 = -\frac{\partial^2}{\partial \bar{Z}^2} \int_{-\infty}^{\bar{X}} (\bar{X} - \xi)^{-\frac{1}{3}} \, |\bar{P}|^2(\xi, \bar{Z}) \, d\xi \ . \tag{4.4b}$$

These vortex equations arise because the quadratic interaction between the two-dimensional TS carrier wave and the spanwise TS wave induced by the modulation, drives a surprisingly large $\bar{X}$-independent spanwise velocity – indeed this grows logarithmically as $Y \to \infty$. The growth is resolved in a *'diffusion layer'* (sometimes referred to as a 'buffer layer') of thickness $Y = O(\epsilon^{-1})$ far from the wall. Such diffusion layers are common in situations where relatively large mean flows are forced as a result of nonlinear interactions in critical layers (e.g. Brown & Stewartson 1978, Haynes & Cowley 1986). These mean flows are generated over the same lengthscale/timescale as that on which the nonlinear effects evolve. In the problem under discussion nonlinear effects develop over a distance $X = O(\epsilon^3)$. Far from the wall the magnitude of the mean flow is $O(Y)$, and hence from a balance between the convective-inertia and viscous terms it follows that the spanwise perturbation will have diffused a distance $Y = O(\epsilon^{-1})$ from the wall. The streamwise and spanwise lengthscales are then chosen (after accounting for logarithmic terms) so that this induced 'vortex' feeds back, via the skin-friction factor $\Lambda_0$, to interact with the evolving (nonlinear) TS wave.

Smith & Walton (1989) show

(*a*) that the two-dimensional quasi-equilibrium solution is unstable to three-dimensional perturbations, and

(*b*) that three-dimensional solutions to (4.4) may either terminate in a finite-distance singularity, or decay downstream.

Again, these solutions have features in common with transition, but 'no quantitative comparison with transition experiments and computations have been attempted yet' (Smith & Walton 1989).

‡ The original version of this analysis was corrected in outline by Smith & Blennerhassett (1992).

### 4.3 A 'medium' TS-wave/vortex interaction

Having recognised the powerful influence that an induced longitudinal vortex can have on the evolution of a wave, Smith & Walton (1989) also considered disturbances with a spanwise lengthscale comparable with the wavelength of the two-dimensional TS carrier wave. The flow is again assumed to evolve over a distance $O(\epsilon^{-3})$, where the amplitude of the wave for $Y = O(1)$ is $O(\epsilon(\log\epsilon)^{-1/2})$. The analogous equation to (4.4$a$) yields a dispersion relation which has a nonlinear dependence on $\bar{P}$ through an unknown skin-friction factor. The equations governing the vortex in the diffusion layer where $Y = O(\epsilon^{-1})$ are the full classical three-dimensional, zero-pressure-gradient, boundary-layer equations, with a spanwise forcing at the boundary as a result of matching to the $Y = O(1)$ region.

A key point in the analysis is that the *leading-order* wavenumber must be allowed to vary over the slow $O(\epsilon^{-3})$ evolution scale, whereas in previous 'almost-neutral' stability analyses the carrier wave wavenumber/frequency only varies slightly. As the flow evolves downstream the linear three-dimensional TS waves force the vortex flow. This in turn modifies the wall shear which governs the TS wavelength (i.e. assuming the frequency of the disturbance is known). The TS waves *must* remain neutral for the theory to be applicable. This is achieved by a feedback loop in which the amplitude of the TS wave automatically adjusts itself so that the forced vortex leads to a value of wall shear consistent with a neutral wave. This condition might be viewed as a nonlinear secularity condition for fixing the amplitude of a three-dimensional, weakly nonlinear, TS wave.

### 4.4 A 'strong' TS-wave/vortex interaction

The above analysis has been extended by Hall & Smith (1991) to the case when $\epsilon = O(Re^{-1/8})$; the downstream evolution lengthscale then merges with the non-parallel evolution scale of the Blasius boundary layer. The TS-wave/vortex interaction is then sufficiently strong to produce a complete alteration of the Blasius boundary-layer velocity profile.

The governing equations for both the above TS-wave/vortex interactions are 'strongly' nonlinear. Numerical solutions have been obtained by Hall & Smith (1991) (see also Walton & Smith 1992). The initial conditions for these calculations are rather arbitrary, and in general require the spanwise velocity to satisfy a compatibility constraint. A slight question-mark remains as to whether such initial conditions can be realised experimentally. However, as noted by Smith & Walton (1989), an alternative initial condition can be constructed by considering the secondary instability of a two-dimensional TS wave to three-dimensional perturbations. As indicated in §4.4 another possible initial condition arises from the evolution of a pair of oblique waves (Hall & Smith 1989, Blackaby 1993).

The calculations of Hall & Smith (1991) and Walton & Smith (1992) suggest that TS-wave/vortex interactions of this kind terminate in singularities a finite distance downstream (see also Smith & Walton, 1989). Smith & Walton (1989) and Walton & Smith (1992) tentatively suggest that the singularity formation is related to the creation of lambda vortices. However, in an extension of work by Hall & Horseman (1991), Hall (1992, personal communication) has shown that before the singularity develops, the three-dimensional distorted boundary-layer flow induced by the TS-wave/vortex interaction is itself unstable to a 'secondary' Rayleigh instability. Hall proposes that it is this instability which is responsible for the so-called 'spike' formation (see also Greenspan & Benney 1963).

Hall & Smith (1991) show how the ideas underlying the TS-wave/vortex interaction can be extended to other flows supporting almost neutral waves, e.g. Rayleigh-wave/vortex interactions (see also Bennett, Hall & Smith 1991). However, as noted by Smith *et al.* (1993), there are difficulties in establishing such Rayleigh-wave/vortex interactions from realistic upstream boundary conditions (see also Wu, Lee & Cowley 1993, Brown, Brown, Smith & Timoshin 1993). †

### 4.5 A pair of oblique modes

Other TS-wave/vortex interactions can be sought at locations close to the lower-branch neutral point (i.e. in region I of figure 7). For instance, Hall & Smith (1989), Smith & Blennerhassett (1992) and Blackaby 1993 consider the interaction of two oblique three-dimensional TS waves with an induced streamwise vortex.

If the TS waves are weakly three-dimensional then the induced vortex is passive and the amplitude evolution equation is just (4.3). However, if the TS waves are sufficiently oblique, the induced vortex

† This point is possibly not surprising once it is realised that the almost-neutral Rayleigh (or TS) carrier modes are in fact weakly nonlinear, quasi-equilibrium, three-dimensional travelling waves which slowly evolve downstream – cf. equilibrium solutions to (4.2$a$) when $x_1$ is viewed as a slowly varying parameter. Experience suggests that such quasi-equilibrium travelling waves are highly likely to be unstable to three-dimensional disturbances at asymptotically large Reynolds numbers.

can affect the nonlinear evolution of the disturbance. If $\beta$ is the spanwise wavenumber based on the triple-deck scaling, then the distinguished limit where the two types of nonlinearity balance is given by †

$$\beta = O(\epsilon^{\frac{2}{3}}(\log\epsilon)^{-\frac{1}{2}}) \ , \tag{4.5}$$

where $\epsilon$ is as in §4.1 Nonparallel effects can be allowed for by scaling $\epsilon$ as in (4.2*b*). For stronger three-dimensionality than (4.5) the nonlinearity forced by the streamwise vortex dominates the cubic Landau nonlinearity. If $\beta = O(1)$, the scaled amplitude equation for two oblique waves of equal amplitude is (Blackaby 1993)

$$\frac{\partial \bar{P}}{\partial \bar{X}} = K_3 \bar{X}\bar{P} + K_5 \bar{P}\int_0^{\bar{X}} (\bar{X}-\xi)^{-\frac{1}{3}} |\bar{P}(\xi)|^2 d\xi \ , \tag{4.6}$$

where it has been assumed that the disturbance is introduced at $X = 0$. The same equation, but with different coefficients, is obtained if compressibility effects are included. For incompressible flow, Blackaby argues that if the obliqueness angle, $\theta$, of the TS waves satisfies $32.21° < \theta < 45°$, then their amplitude blows-up in a finite distance. The subsequent evolution of the waves is then described by the fully nonlinear triple-deck equations (4.1) (Hall & Smith 1989). For other values of $\theta$ the disturbance amplitude grows algebraically as $\bar{X} \to \infty$. Hall & Smith (1989) argue that the disturbance subsequently evolves to a TS-wave/vortex interaction in which the entire boundary layer is altered (Hall & Smith (1991) and §4.4). As far as transition to turbulence is concerned (i.e. the development of short[er] scale motions), the existence of a finite-distance singularity is potentially the most relevant result (see also §6.4). However, such singularities only develop for a relatively narrow range of obliqueness angles. Further, Blackaby (1993) shows that range of angles is dependent on Mach number, and disappears for Mach numbers above $\sqrt{2}$.

If the initial disturbance is sufficiently small so that nonlinear effects only become important in a region where non-parallel effects are negligible to leading order, i.e. if $Re^{-3/32} \ll \epsilon \ll 1$, then instead of algebraic growth, nonlinear effects cause the disturbance amplitudes to *decay* exponentially (cf. Smith & Blennerhassett 1992, Wu *et al.* 1993). The disturbance only reverts to algebraic growth even further downstream as a result of non-parallel effects (cf. Smith *et al.* 1993)

### 4.6 'Far-downstream' lower-branch analyses

As indicated in §2, if TS waves propagate to a position far downstream from the lower branch (as measured on the lower-branch scaling), then the growth-rate of TS waves is relatively small. This means that if a flow becomes nonlinear in this region, its evolution can in general be described by a weakly nonlinear analysis. Solutions for the weakly nonlinear evolution of two-dimensional disturbances in this so-called 'high-frequency' limit have been obtained by Smith & Burggraf (1985) and Smith (1986*a*). For two-dimensional disturbances of larger amplitude, the inviscid dynamics are apparently described by the well-known Benjamin-Ono equation. Theories have been proposed (e.g. Rothmayer & Smith 1987, Kachanov, Ryzhov & Smith 1993) suggesting that soliton solutions of this equation may be associated with the 'spikes' in the velocity profiles observed in many transition experiments. However, this idea cannot be fully evaluated until the theory is extended to three-dimensional disturbances. In addition, the problem of incorporating viscous effects into the analysis appears to be unresolved at present.

The nonlinear development of weakly three-dimensional wavetrains has been considered by Stewart & Smith (1992) for boundary layers, and Smith & Bowles (1992) for channel flow. These authors make comparisons with the experiments of Klebanoff & Tidstrom (1959) and Nishioka, Asai & Iida (1980) respectively; their 'wave-vortex' interaction theory apparently captures the main features observed. Stewart & Smith (1987) and Doorly & Smith (1992) have looked at various fully three-dimensional linear problems. In addition, Smith & Stewart (1987) observe that a resonant-triad interaction is a possibility in the HFLB régime. Solutions of their equations apparently yield good agreement with the experiments by Kachanov & Levchenko (1984) on subharmonic transition in boundary layers.

However, rather than reviewing this work in detail, we will proceed to consider analyses performed with the upper-branch scaling. The 'far-upstream' (or 'low-frequency') limit of such analyses should match with the above 'far-downstream' lower-branch solutions.

## 5. ASYMPTOTIC NONLINEAR THEORY – TS UPPER BRANCH

### 5.1 Two-dimensional instability

One of the major strands of work based on the upper-branch scaling has been the derivation of two-dimensional travelling-wave equilibrium solutions for flows in favourable (or zero) pressure gradients. As the amplitude of a linear TS wave is increased, non-

† University of Cambridge, Mathematical Tripos Part III (1993), Paper 33, Question 3.

linear effects first come into play within the thin critical layer situated in the buffer region adjacent to the wall (see figure 4*a*). The possibility that nonlinearity could play a crucial rôle in critical layers was realised by Lin (1957), and then exploited by Benney & Bergeron (1969) and Davis (1969). The latter authors concentrated on *almost*-inviscid critical layers, while Haberman (1972) and Brown & Stewartson (1978) incorporated viscous effects into nonlinear critical layers (and corrected earlier ideas). As a result of these studies, the wavespeed of possible travelling waves could be identified, but the amplitude of the wave was not fixed. Smith & Bodonyi (1982*a*) explained how this could be done for a boundary layer in a favourable pressure gradient. Like Haberman (1972), they considered a scaling in which viscous and convective-inertia effects were comparable within the critical layer.They showed how the amplitude-dependent 'phase jump' across the critical layer (essentially one Fourier component of the critical-layer velocity jump) could balance the phase jump arising from the Stokes layer adjacent to the wall, and so fix the amplitude. Bodonyi, Smith & Gajjar (1983) and Smith, Doorly & Rothmayer (1990) extended this analysis to larger-amplitude modes in which the critical layer is situated either in the middle of the classical boundary layer, or at the outer edge of the boundary layer, respectively.

For PPF and (non-decelerating) boundary layers, it was shown by Reutov (1982) and Gajjar & Smith (1985), respectively, that upper-branch travelling-wave solutions are unstable – i.e. the bifurcation is subcritical. Hence the amplitudes of the travelling waves may provide an indication of nonlinear threshold amplitudes for two-dimensional instability (subject to the condition that the diffusion layers mentioned below are passive). Goldstein & Durbin (1986) also conclude that nonlinear critical-layer effects completely 'eliminate' the upper branch predicted by linear theory; this is in agreement with numerical results of Bayliss *et al.* (1985).

We note that Gajjar & Smith (1985) and Goldstein & Durbin (1986) base their analysis on the assumption that an induced mean-flow perturbation has diffused *completely across* the developing boundary layer. If the TS wave has grown from an initially linear disturbance, this assumption is incorrect. Instead, it is necessary to introduce extra 'diffusion layers' which sandwich the critical layer (e.g. Brown & Stewartson 1978). For two-dimensional disturbances, the diffusion layers are expected to be passive, (e.g. Churilov & Shukhman 1987, Goldstein & Hultgren 1988); however in three dimensions this may not be the case, since a wave/vortex interaction can arise in these layers (e.g. Wu 1993*a*).

It is not immediately clear what the rôle of such diffusion layers is in equilibrium, travelling-wave analyses for developing boundary layers, e.g. Bodonyi *et al.* (1983) and Smith *et al.* (1990). † In a developing boundary layer, the equilibrium travelling waves must have evolved from some upstream initial condition. Unless this initial condition is chosen so that the mean-flow change is introduced across the whole boundary layer, then diffusion layers will be needed. However, even with such carefully chosen initial conditions, it is likely that diffusion layers will develop as the non-parallelism of the mean flow comes into play. Hence there seems a need to confirm that any such diffusion layers are passive.

For boundary layers in adverse pressure gradients, similar nonlinear critical-layer analyses can be performed. While no equilibrium travelling-wave solutions can be found, the growth of perturbations can be followed from the linear through to the strongly nonlinear stage, e.g. Gajjar & Smith (1985). In the vicinity of the 'upper-branch' scaling, the growth remains exponential even after the critical layer becomes nonlinear. However, if the disturbance becomes nonlinear far downstream of the VCB scaling, a simple extension of Gajjar & Smith's (1985) analysis shows that the initial exponential growth of the linear wave gives way to algebraic growth, before reverting to a smaller exponential growth once the amplitude is large enough. ‡

We note that the distance of the critical layer from the wall can be amplitude-dependent. If an almost-inviscid nonlinear critical layer moves *too* rapidly through the ambient vorticity then there may not be time for the vorticity within the 'cats-eyes' of the critical layer, i.e. the region of closed streamlines, to equilibrate to a uniform vorticity. Examples of such non-uniform vorticity régimes in *forced* problems were found by Cowley (1981, 1985) and

† In an *exactly* parallel flow, e.g. Hagen-Poiseuille flow (Smith & Bodonyi 1982*b*), the diffusion layers need not be present since in principle there is always sufficient 'time' for the mean flow perturbation to have diffused completely across the flow.

‡ Churilov & Shukhman (1987) and Goldstein & Hultgren (1988) find algebraic growth in closely related problems concerned with the growth of two-dimensional instabilities in shear layers. An important difference between shear layers and boundary layers is the existence of the Stokes layer adjacent to the wall in the latter case. When the effects of this Stokes layer are felt at leading order, the instability wave grows exponentially.

Haynes & Cowley (1986), while Gajjar & Smith (1985) proposed a scenario for a wavepacket of modulated upper-branch TS waves that could lead to such critical layers. †

If in a flow with an adverse pressure gradient, the disturbance first becomes nonlinear in the fourth region of figure 7*c*, i.e. where $Re = F^{-\frac{14}{9}}$, then the initial nonlinear evolution of the TS waves is governed by a *non-equilibrium*, nonlinear, viscous critical-layer equation (e.g. Gajjar 1994). Again nonlinear effects result in a transformation from exponential to algebraic growth. Even further downstream, i.e. effectively in the Rayleigh régime, viscous effects are initially negligible in the critical layer. In this case Goldstein, Durbin & Leib (1987) show that the growth-rate can oscillate about zero before viscous effects reassert themselves and lead to algebraic growth.

### 5.2 Spatial modulation of a planar carrier wave

Imperfections are invariably present in, say, a vibrating-ribbon experiment. As a result, a supposedly two-dimensional wavetrain may have an amplitude that is weakly modulated in the spanwise direction. Models for such flows when the disturbance first becomes nonlinear near the lower branch have been discussed in §4 (e.g. equation (4.3)). Suppose instead that the initial amplitude of the disturbance is smaller, so that nonlinear effects become important in the upper-branch (VCB) régime. As an example we discuss a boundary layer with a pressure gradient, but with a simple modification of scaling, the analysis also applies to the Blasius boundary layer, or to the 'high-frequency lower-branch' régime. For our typical lengthscale (see §1.1) we use the boundary-layer thickness and we write $F = \sigma^8$. The leading-order pressure perturbation is assumed to be of the form

$$\epsilon A(\bar{X}, \bar{Z}) \exp \mathrm{i}\sigma(\bar{\alpha}x - \sigma\bar{\omega}t) + c.c.\ , \qquad (5.1a)$$

where $\bar{X}, \bar{Z}$ are 'slow' spatial scales, $\bar{\alpha}$ and $\bar{\omega}$ are of order one, and $\epsilon \ll 1$.

If the three-dimensionality is extremely weak, then the evolution of the disturbance will still be described by the two-dimensional analysis of the previous section. However, if a typical spanwise lengthscale of the warping is $O(\sigma^{-\frac{23}{12}})$, then the solution to the strongly nonlinear critical layer is believed to be influenced by a diffusion layer that surrounds it. This problem has yet to be studied in detail.

If the warping is more rapid than $O(\sigma^{-\frac{23}{12}})$ then the critical layer ceases to be strongly nonlinear. Wu, Stewart & Cowley (1994) identify a distinguished scaling which accommodates the effects of linear growth, linear spanwise dispersion and weak nonlinearity at the same order, viz.

$$\bar{X} = \sigma^2 x\ ,\ \bar{Z} = \sigma^{3/2} z\ ,\ \epsilon = \sigma^{15/4}\ . \qquad (5.1b)$$

The dominant nonlinear effect arises in the critical layer and surrounding diffusion layer, and can be regarded as a cubic 'wave-vortex' interaction between the fundamental mode and an induced spanwise-dependent mean flow. The amplitude equation takes the form

$$\frac{\partial A}{\partial \bar{X}} - \mathrm{i}\frac{\partial^2 A}{\partial \bar{Z}^2} = A + \mathrm{i}A\int_0^\infty \zeta^{-\frac{1}{2}} \left(A(\bar{X}-\zeta, \bar{Z}) A^*_{\bar{Z}}(\bar{X}-\zeta, \bar{Z})\right)_{\bar{Z}} d\zeta\ . \qquad (5.2)$$

It is straightforward to include a temporal modulation in this equation.

Equation (5.2) has exact solutions describing pure plane waves. A *linear* secondary-stability analysis shows that they are susceptible to a 'sideband' instability of 'exponential-of-exponential form'. Solutions of the full nonlinear equation can terminate in a finite-distance singularity associated with a shortening of both spanwise and streamwise scales. The subsequent evolution of such solutions has not been studied in detail. However an analysis of scaling changes indicates that in the subsequent phase of development, the slow streamwise variation will enter at leading order in the critical layer equations, so that the critical layer becomes of non-equilibrium viscous type – a situation similar to that studied by Wu (1993*b*); see §6.6.

As an alternative to considering the warping of a two-dimensional mode, equation (5.2) can be used to study the nonlinear evolution of, say, a pair of slightly oblique TS waves. If the three-dimensionality is much stronger than that specified by (5.1), then the governing equations simplify. In particular, the energy of the waves can be shown to grow exponentially in a frame moving downstream with the group velocity.

In the next section we consider a case where the pair of oblique TS waves have comparable spanwise and streamwise wavelengths. However, first we note that there is an intermediate scaling with spanwise lengthscale of $O(\sigma^{-13/12})$. On this scaling there are

† While studying the spatial stability of shear layers, Goldstein & Hultgren (1988) and Hultgren (1992) find quasi-equilibrium nonlinear critical layers which have non-uniform vorticity within almost-inviscid 'cat-eyes'. However, the source of this non-uniformity is vorticity swept downstream rather than a variation in the ambient arising from the sideways movement of the critical layer.

two competing forms of wave/vortex interaction. As a result the $\zeta^{-\frac{1}{2}}$ kernel function in (5.2) needs to be replaced by $(\zeta^{-\frac{1}{2}} + \kappa)$, where the magnitude of $\kappa$ is proportional to the obliqueness (cf. Timoshin & Smith 1993). However, the precise form of the kernel function seems to have no significant effect on the qualitative form of solutions, which again display an exponential increase in both amplitude and phase.

### 5.3 Resonant-triad interactions

Rather than considering a general pair of oblique modes (which can be viewed as a special case), we study the interaction of a pair of oblique modes with a two-dimensional mode. Such interactions are of interest because experiments have shown that subharmonic resonance is a possible mechanism for transition to turbulence in boundary layers (e.g. Raetz 1959, Craik 1971, Kachanov *et al.* 1977, 1984). Resonance occurs when for *any* three waves

$$\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0 \quad \text{and} \quad \omega_1 + \omega_2 + \omega_3 = 0 \ , \quad (5.3)$$

where the $\mathbf{k}_j$ and $\omega_j$ are the wavenumbers and frequencies of the waves respectively. The three waves need not all be of the same type, although in this section we only consider TS waves.

The resonance condition (5.3) cannot be satisfied by three almost-neutral *lower*-branch TS waves. However, Smith & Stewart (1987) show that this condition *can* be satisfied by three TS waves in the 'high-frequency' limit of the lower-branch régime. In other words, for disturbances of given frequency, resonance can take place sufficiently far downstream of the lower branch. Resonance can likewise take place in the upper-branch (VCB) régime (where the asymptotic structure is very similar). We focus on the VCB scaling since subharmonic resonance is experimentally observed to occur near, and to continue downstream of, the upper branch neutral point.

A resonant-triad interaction in the upper-branch régime has been studied by Mankbadi (1991) and Mankbadi, Wu & Lee (1993) for the Blasius boundary layer, and by Wu (1993a) for boundary layers with non-zero pressure gradients (see also §6.5). We take the latter case as an example.

In the notation of §5.2, the leading-order pressure perturbation is assumed to be of the form

$$\epsilon B(\bar{X}) e^{i\sigma(\bar{\alpha}x - \sigma\bar{\omega}t)} + \delta A_+(\bar{X}) e^{i\sigma(\bar{\alpha}x/2 + \bar{\beta}z - \sigma\bar{\omega}t/2)} + \delta A_-(\bar{X}) e^{i\sigma(\bar{\alpha}x/2 - \bar{\beta}z - \sigma\bar{\omega}t/2)} + c.c. \ , \quad (5.4)$$

where $\bar{\alpha}$, $\bar{\beta}$, and $\bar{\omega}$ are all of order one and $\bar{X} = \sigma^2 x$. As shown by Smith & Stewart (1987), the resonance condition (5.3) is satisfied when

$$\tan^{-1} \bar{\beta}/\bar{\alpha} = 60^\circ .$$

Since upper-branch TS modes have an inviscid structure at leading order, this resonance condition is the same as for long-wavelength Rayleigh modes. The three waves have (nearly) equal phase speeds, and so share the same critical layer.

The maximal interaction scaling to ensure that the oblique waves are both influenced by the two-dimensional wave, and interact nonlinearly among themselves (cf. §5.2), is given by

$$\delta \sim \sigma^{13/3} \ , \quad \epsilon \sim \sigma^5 \ .$$

For these scalings, quadratic interactions between the two-dimensional wave and one or other of the oblique modes within the critical layer lead to a parametric resonance. Moreover, while the quadratic interaction between the oblique waves does not affect the development of the two-dimensional mode, it does drive a mean-flow distortion which becomes unbounded towards the edge of the critical layer. In order to match the mean-flow distortion in a self-consistent manner, a diffusion layer must be introduced. In this layer the slow-spatial-growth term balances the viscous diffusion term. The interaction between an oblique mode and the mean-flow distortion produces a cubic nonlinear term in the amplitude equations. This interaction is non-local because the diffusion layer is 'unsteady'. The final amplitude equations are found to be

$$\begin{aligned} \frac{dA_+}{d\bar{X}} &= aA_+ + qA_-^* B \\ &\quad + hA_- \int_0^{+\infty} A_-^*(\bar{X}-\zeta) A_+(\bar{X}-\zeta) d\zeta \ , \\ \frac{dA_-}{d\bar{X}} &= aA_- + qA_+^* B \\ &\quad + hA_+ \int_0^{+\infty} A_+^*(\bar{X}-\zeta) A_-(\bar{X}-\zeta) d\zeta \ , \\ \frac{dB}{d\bar{X}} &= bB \ , \end{aligned} \quad (5.5)$$

where $a$, $b$, $q$ and $h$ are constants. The analysis can readily be extended to allow for temporal and/or spanwise modulation (Jennings, Stewart & Wu 1994).

Numerical solutions of these equations indicate that if the oblique modes initially have small amplitudes, they first experience a rapid 'exponential-of-exponential' growth caused by parametric resonance (Goldstein & Lee 1992, Wu 1993*a*). † In a subsequent stage, the cubic interactions of the oblique

† If/when the small growth-rate of the two-dimensional mode is much less than that of the three-dimensional modes, then the secondary stability of the two-dimensional mode to three-dimensional disturbances can be tested using Floquet theory (e.g. Herbert 1988).

modes inhibit the growth and lead to a wavelength shortening. On the other hand, if the initial amplitudes of the oblique modes are sufficiently large, the parametric resonance can be completely bypassed and the amplitude growth is merely exponential. Numerical solutions also suggest that oblique modes with unequal initial amplitudes evolve to an equal-amplitude state.

Similar results to the above have been obtained by Mankbadi *et al.* (1993) for the upper-branch régime of the Blasius boundary layer, and by Jennings *et al.* (1994) for the 'high-frequency lower-branch' régime . Mankbadi (1993) presents comparisons between theory and experiment. However, in order to obtain good agreement, he has to include additional higher-order terms without performing a full systematic asymptotic analysis.

Recently, Goldstein (1994) observed that if the initial amplitude of the subharmonic mode is sufficiently small, the 'exponential-of-exponential' growth of the subharominc induced by the parametric resonance can lead it to evolve on a faster spatial scale. As a result, the viscous critical layer becomes of nonequilibrium type (see also Khokhlov 1991, 1993). Goldstein (private communication) also points out that at the streamwise locations where experimental measurements were made, the critical layer has actually entered the nonequilibrium régime. Therefore, this régime must be considered in order to perform an appropriate comparison with experiment.

## 6. RAYLEIGH WAVES IN SHEAR FLOWS

So far we have discussed asymptotic approaches to transition when fixed-frequency disturbances initially become nonlinear either very close to the lower branch, or in the upper-branch régime, i.e. regions I and III in figure 7. In this section we will concentrate on disturbances that become nonlinear near a Rayleigh wave neutral point, i.e. regions VI and VII in figure 7*c*. Although we do not discuss region IV in figure 7*c* explicitly, we note that many of the properties of weakly nonlinear disturbances in this region are similar to those in region VI. This is because non-equilibrium, viscous critical layers play an important rôle in both regions. There are also certain common properties between region III and region VII since a viscous critical layer surrounded by a diffusion layer is a key element of both regions. However, the amplitude equations in region VII have a slightly more general form; additionally they include non-parallel effects.

Rather than concentrating on a decelerating boundary layer, we consider a general two-dimensional spatially developing, and/or temporally varying, shear layer that has an inflectional velocity profile. Such a shear layer usually has a vertical velocity component of magnitude of order $R^{-1}$, where $R$ is the Reynolds number based on shear-layer width. As indicated in §1.1, a quasi-parallel linear instability analysis can only be justified when $R \gg 1$. Such an analysis shows that shear layers with inflectional profiles (e.g. a free shear layer, a decelerating boundary layer, a Stokes Layer) can often support inviscid Rayleigh instability waves (§2.4). For shear layers with a monotonic or symmetric velocity profile, e.g. the free mixing layer or wake, the instability is 'associated' with an inflection point. However for shear layers with non-monotonic profiles, other modes may exist (Foote & Lin 1950).

In general Rayleigh modes have order-one growth-rates, so that a weakly nonlinear analysis is not possible. However, as in §3.5 suppose that a small disturbance of given frequency is introduced upstream. Because of the viscous spreading of the shear layer, the (spatial) growth-rate varies, and eventually goes to zero at some location downstream, say $x_0$. As the disturbance approaches this neutral point, critical layers emerge in the structure of the modes. Owing to the singular nature of the eigenfunction close to the critical layer, the vorticity disturbance generally has a larger magnitude within the critical layer than elsewhere. As a result that nonlinear effects are most significant in the critical layer – a fact first highlighted by Lin (1957).

Early weakly nonlinear studies exploiting the weak growth near the neutral curve include those of Robinson (1974), Huerre (1980) and Huerre & Scott (1980). In these studies, the critical layer was assumed to be viscous and in equilibrium. Goldstein & Leib (1988) criticised this work on the grounds that the growth-rates were so small that either the nonlinear stage could not be matched onto the linear one upstream, or the quasi-parallel assumption could not be justified unless artificial body forces were applied (see also Huerre & Scott 1980). In particular, following Stewartson & Stuart (1971), Goldstein & Leib (1988) proposed that for a theory to be relevant, the nonlinear stage should be a natural continuation of an earlier linear development. They therefore chose scalings such that the slow spatial growth could be retained at leading order in the critical-layer equations. Goldstein & Leib (1988) refer to such criti-

When the three-dimensional modes have growth-rates comparable with, or even smaller than, that of the two-dimensional mode, the multiple-scales approach seems necessary (Smith 1986*b*, Goldstein & Lee 1992).

cal layers as *non-equilibrium* critical layers, although they are also referred to as an *unsteady* critical layers, e.g. Dickinson 1970.

In this section we concentrate on cases where non-equilibrium viscous critical layers play a central rôle (i.e. region VI in figure 7*c*). However, we will indicate certain of the changes that occur if disturbances become nonlinear in the region where the critical layer is viscous and non-parallel effects are important at leading order (i.e. region VII). We consider in turn the evolution of (*a*) a 'regular' two-dimensional mode, (*b*) either a 'singular' two-dimensional mode or a single oblique mode, (*c*) a pair of oblique modes, (*d*) a resonant triad, and (*e*) a modulated two-dimensional mode. First we provide an example of how a nonlinear scaling can be derived.

### 6.1 An example scaling

As appropriate for a weakly nonlinear analysis in the vicinity of a neutral point we write

$$x = x_0 + \mu x_1 \ , \quad \omega = \omega_0 + \mu\omega_1 \ , \qquad (6.1a, b)$$

where $x$ is the slow variable describing the viscous spreading of the unperturbed shear flow, $\mu \ll 1$, $x_1 = O(1)$ and $\omega$ is the local Strouhal number. Since $\omega_0$ is real the local growth rate is $O(\mu)$. The variables

$$X = Rx \ , \quad \bar{x} = \mu X \qquad (6.1c)$$

are introduced to describe the carrier wave and the slow streamwise development of the amplitude.

The flow is described by a double-layered structure: (*i*) an outer region occupying the main part of shear layer where the flow is basically linear and inviscid, and (*ii*) a critical layer centred at $y = y_c$, say, where nonlinear interactions are important. If the disturbance becomes nonlinear in region VI the critical is layer is required to be of viscous, nonequilibrium type. From a balance of terms in the $x$-momentum equation it follows that the thickness of the critical layer is of order $\mu = O(R^{-\frac{1}{3}})$.

The precise effect of nonlinearity depends on the type of disturbance introduced and the nature of the singularity associated with the critical layer[s]. Nevertheless, scalings for a number of different cases can be derived in a unified manner. Assume that as a critical layer is approached, the vorticity of the disturbance $\tilde{u}_y$ can be expressed as

$$\tilde{u}_y \sim O\left(\frac{\epsilon}{(y - y_c)^k}\right) \sim O\left(\frac{\epsilon}{\mu^k}\right) \ , \qquad (6.2a)$$

where $\epsilon$ is the amplitude of the disturbance in the outer region and

$k = 0$ for two-dimensional disturbances with a 'regular' critical layer, i.e. $\bar{U}_{yy}(y_c) = 0$,

$k = 1$ for two-dimensional disturbances with a 'singular' critical layer, i.e. $\bar{U}_{yy}(y_c) \neq 0$,

$k = 2$ for three-dimensional disturbances.

By means of a scaling argument it is possible to analyse how the nonlinear interactions inside the critical layers generate higher harmonics and a mean-flow distortion. In particular, the cubic interaction drives a fundamental of $O(\epsilon^3\mu^{-k-3})$. For this to affect the evolution of the disturbance, it must balance the $O(\epsilon\mu)$ linear, 'slow-growth', correction in the outer region (Hickernell 1984, Goldstein & Choi 1989):

$$O(\epsilon^3\mu^{-k-3}) \sim O(\epsilon\mu) \ , \quad \text{i.e.} \quad \mu = O(\epsilon^{\frac{2}{k+4}}) \ . \qquad (6.2b, c)$$

Hence in region VI the critical amplitude for nonlinear effects to be felt by the disturbance is

$$\epsilon = (\lambda R)^{-\frac{k+4}{6}} \ , \qquad (6.2d)$$

where $\lambda$ is a (generalised) Haberman parameter.

### 6.2 Two-dimensional 'regular' modes

The development of nonlinear effects for two-dimensional modes in an incompressible free shear layer has been studied by Goldstein & Leib (1988), Goldstein & Hultgren (1988) and Hultgren (1992). For a free shear layer, the critical layer is located at the inflection point in the velocity profile; hence $k = 0$ in (6.2*b, c*) (Goldstein & Leib 1988, Goldstein & Hultgren 1988).

After an appropriate rescaling, the critical-layer dynamics in this region is governed by

$$\left(\frac{\partial}{\partial\bar{x}} + Y\frac{\partial}{\partial X} - \Re(\mathrm{i}Ae^{\mathrm{i}X})\frac{\partial}{\partial Y} - \bar{\lambda}\frac{\partial^2}{\partial Y^2}\right)Q = \Re\left(\mathrm{e}^{\mathrm{i}X}(\mathrm{i}A + \frac{1}{2}\bar{U}\frac{dA}{d\bar{x}})\right) \ , \qquad (6.3a)$$

where $A$ is the amplitude function, $Q$ is the disturbance vorticity within the critical layer, $\bar{U}$ is a measure of the mean speed of the critical layer, $\bar{\lambda}$ is proportional to $\lambda$, $Y$ is a critical layer coordinate. From matching with the outer layer, and an upstream linear mode, it follows that

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\int_0^{2\pi} Q\mathrm{e}^{-\mathrm{i}X}dXdY = \mathrm{i}\chi\frac{dA}{d\bar{x}} + JA \ , \qquad (6.3b)$$

and

$$A \to \mathrm{e}^{\kappa\bar{x}} \quad \text{as} \quad \bar{x} \to -\infty \ , \qquad (6.3c)$$

respectively, where $\chi$, $J$ and $\kappa$ are constants depending on the shear-layer velocity profile.

The system (6.3) uniquely determines $A$ and $Q$. Similar equations were derived by Churilov & Shukhman (1987) when studying the evolution of a free Rayleigh mode on a $\beta$ plane. This system also governs the development of a forced Rossby wave (Warn & Warn (1976, 1978), Stewartson (1978)). †

Numerical solutions to (6.3) have been obtained by Goldstein & Leib (1988) in the inviscid, $\lambda = 0$, case. Their results show that nonlinearity can cause the instability to saturate well upstream of the linear neutral stability point. The vorticity rolls up under nonlinear effects, and small-scale eddies are generated downstream (Warn & Warn 1978). The 'large-time' solution does not seem to tend to a well-defined limit (Stewartson 1978). However, when viscosity is included, Goldstein & Hultgren (1988) found that sufficiently far downstream viscous effects cause the vorticity distribution to diffuse into a much simpler pattern. Far downstream the instability wave grows algebraically and the critical layer evolves into a quasi–equilibrium state. This is similar to that of Benney & Bergeron (1969), but the detailed flow is different in that the vorticity within the cat's-eye boundary is *non–uniform*. Of course there are the ubiquitous diffusion layers sandwiching the critical layer. As this quasi–equilibrium state propagates downstream, Goldstein & Hultgren (1988) show how to incorporate non-parallel effects. They demonstrate that on passing the neutral point of *linear* waves, the *nonlinear* quasi-equilibrium states decay!

In both Goldstein & Leib (1988) and Goldstein & Hultgren (1988), the basic profile was chosen to be '$\tanh y$' so that the eigenfunction could be solved analytically. Strictly speaking, for complete self-consistency the basic flow should be obtained by solving the steady two-dimensional boundary-layer equations. Hultgren (1992) adopted this approach using an experimentally measured velocity distribution as an initial profile. Based on this more realistic profile and using composite expansions, Hultgren (1992) obtained theoretical predictions for the development of the wave that are in *quantitative* agreement with experiment (see figure 8).

Using a similar approach, Leib & Goldstein (1989) have investigated the nonlinear interaction between the marginally unstable sinuous and varicose modes in the 'Bickley jet'. Both modes are regular since the critical levels are located at the symmetric inflection points. The varicose mode is the subharmonic of the sinuous mode, and thus they form a subharmonic resonance. The critical layers are strongly nonlinear, and the equations governing the evolution of the amplitudes and the distribution of the vorticity of each mode are coupled.

In fact, 'strongly' nonlinear critical layers are not limited to flows with regular normal modes. As indicated at the end of §5.1, a strongly nonlinear critical-layer structure also describes the vorticity roll-up for long-wavelength modes in boundary layers with (weak) adverse pressure gradients (Goldstein, Durbin & Leib, 1987). In addition, Goldstein & Wundrow (1990) found that the nonlinear evolution of the so-called 'acoustic' mode in hypersonic boundary layers is governed by a nonlinear critical-layer vorticity equation coupled with a nonlinear energy equation.

### 6.3 Two-dimensional 'singular' modes

Not all flows of interest have regular critical layers (e.g. Stokes layers). In the case of a two-dimensional flow where a mode's critical layer[s] is/are not located at inflection points, the outer solution for the linear vorticity perturbation has a pole singular at the critical layer[s]. On setting $k = 1$ and $\mu = \epsilon^{\frac{2}{5}}$ in (6.2), we recover Hickernell's (1984) scaling $\epsilon = R^{-\frac{5}{6}}$. It follows that the mode needs a smaller amplitude for nonlinearity to influence the disturbance. ‡

For this scaling Hickernell (1984) showed that, as in classical weakly nonlinear theory (e.g. Stuart 1960), nonlinearity enters through inhomogeneous terms in the governing equations. Specifically, within the critical layer a sequence of *linear* partial differential equations of the form

$$\left(\frac{\partial}{\partial \bar{x}} + Y\frac{\partial}{\partial X} - \lambda\frac{\partial^2}{\partial Y^2}\right)\Phi = F(X, \bar{x}, Y) \qquad (6.4)$$

need to be solved (cf. (6.3*a*)). The evolution equation for the amplitude then follows from matching the critical-layer solutions to those in the outer layer. In particular, as a result of retaining the slow growth term at leading order in the critical-layer equations, the amplitude equation is of integro-differential type (cf. (4.4), (4.6), (5.2), (5.5)). Hickernell (1984) assumed that a disturbance was introduced at a specific upstream location, say at $\bar{x}$. † An alternative initial condition is to follow Stewartson & Stuart (1971) and assume that as $\bar{x} \to -\infty$, the nonlinear stage matches onto the earlier linear one. The amplitude

† While equation (6.3*a*) is strongly nonlinear, we believe that the theory as a whole should be regarded as being weakly nonlinear in the sense stated at the end of §3.4.

‡ Of course this statement on relative *local* amplitudes ignores any upstream region of linear exponential growth.

† Actually, Hickernell (1984) considered a temporal stability problem, and so introduced his disturbance at a specific *time*.

equation in this case is

$$\frac{dA}{d\bar{x}} = g_0 x_1 A + \int_0^{+\infty} d\xi \int_0^{+\infty} d\eta \sum_j g_j K_j(\xi,\eta) \times A(\bar{x} - \xi) A(\bar{x} - \xi - \eta) A^*(\bar{x} - 2\xi - \eta), \quad (6.5a)$$

where the sum is over all critical layers, $g_0$ and $g_j$ are constants, and

$$K_j(\xi,\eta) = \xi^2 e^{-\frac{1}{3}\lambda\beta_j(2\xi^3+3\xi^2\eta)}, \quad \beta_j = \alpha^2 \bar{U}_y^2 . \quad (6.5b)$$

Numerical investigations of this equation by Goldstein & Leib (1989) and others show that solutions can either blow up at a finite distance downstream, or evolve into an equilibrium state. The outcome depends on the relative size of the disturbance, the Reynolds number, and the $\Re(g_j)$. The rapid increase in amplitude associated with a finite-distance singularity indicates the development of a shorter-scale flow, consistent with observations of transition. We note, however, that this result is analogous to a finite-time singularity of a *uniform* wavetrain described by the Landau equation (3.2). Since real experiments have finite spanwise extent, and are performed over a finite period of time, the effects of modulation on this result need to be accounted for (§6.6).

Hickernell's (1984) equation, and generalisations of it, have also been derived by Goldstein & Leib (1989) and Leib (1991) when studying the evolution of a *single* oblique mode in a compressible shear layer. Although the neutral eigensolutions for the streamwise and spanwise velocity exhibit a pole type of singularity in such a shear layer, these singularities can be eliminated by an appropriate Squire transformation (cf. Bodonyi & Smith 1982*b*). ‡ Goldstein & Leib (1989) and Leib (1991) show that the simple pole in the *temperature* fluctuation (plus the logarithmic branch point of the leading–order solutions for the pressure and the vertical velocity, if the critical level does not coincide with a generalised inflexion point), lead to a critical-layer nonlinearity similar to that of Hickernell (1984).

Shukhman (1991) showed that equation (6.5) also describes the nonlinear evolution of unstable spiral density waves on a rotating compressible shear layer. Wu & Cowley (1994) have used this approach to describe the two-dimensional weakly nonlinear temporal instability of Stokes layers. They propose that the explosive growth associated with a finite-time singularity may be related to the rapid growth of high-frequency disturbances observed in experiments.

### 6.4 A pair of oblique modes

In two-dimensional incompressible flows, the planar mode usually has the largest linear growth rate. Thus the *early* stages of transition are often observed to be two-dimensional. Sufficiently far downstream, however, three-dimensionality can no longer be ignored. For example, if nonlinear effects modify the growth of a two-dimensional instability through a strongly nonlinear critical layer (see §6.2), then the initial exponential growth of the two-dimensional mode is reduced to an algebraic growth (e.g. Churilov & Shukhman 1987, Goldstein & Hultgren 1988). Wu *et al.* (1993) observe that this may allow a exponentially growing three-dimensional instability to overtake the two-dimensional mode, for instance through secondary instability (cf. Kelly & Maslowe 1970, Killworth & Mcintyre 1985, and Haynes 1985). ⋆

In addition, there are supersonic compressible shear flows where, if the Mach number is not too large, the most rapidly growing mode is three-dimensional. Even in flows such as Stokes layers where the fastest-growing mode is two-dimensional, transition caused by instability waves growing from background noise is often is observed to be three-dimensional throughout.

For these and other reasons, three-dimensional disturbances have been studied. The case of a single oblique mode in a (compressible) shear layer has been mentioned in §6.3. However, since there is no preferred spanwise direction, it is likely that if one oblique mode is excited, so is its 'twin' propagating at an equal but opposite angle to the mean-flow direction. Furthermore, the nonlinear interaction of

‡ A key point here is that Goldstein & Leib (1989) and Leib (1991) assume that the oblique mode grows in its direction of propagation. As part of a study on the nonlinear spatial evolution of helical disturbances on an axial jet, Churilov & Shukhman (1994) have shown that the governing amplitude equation changes if, instead, the oblique mode grows in the direction of the underlying shear-layer flow. Although the scaling is still given by (6.2*d*) with $k = 1$, the interactions within the critical layer differ from those given in §6.1, and arise from a need to satisfy the continuity equation on the modulational lengthscale. In the very viscous limit Churilov & Shukhman (1994) obtain

$$\frac{dA}{d\bar{X}} = G_0 x_1 A + G_1 A \frac{d}{d\bar{X}} \int_0^{\infty} \bar{\zeta}^{-\frac{1}{2}} |A(\bar{X} - \zeta)|^2 d\zeta ,$$

where the $G_i$ are constants, as their scaled amplitude equation. Depending on the sign of $\Re(G_1)$, solutions to this equation either terminate in a finite distance singularity, or grow algebraically as $\bar{X} \to \infty$.

⋆ Goldstein & Lee (1993) note that this observation has recently been placed on a firmer analytical footing.

these oblique modes generates a longitudinal vortex. The presence of such longitudinal vortices is one of the characteristics of transitional flows.

The nonlinear spatial evolution of a pair of oblique waves in a free shear layer has been studied by Goldstein & Choi (1989). They showed that because of the double-pole vorticity singularity at the critical layer, nonlinear effects must be included when $\epsilon = R^{-1}$ (put $k = 2$ in (6.2$d$)). This is a smaller amplitude than needed for nonlinear effects to influence a single oblique mode. This analysis was extended by Wu *et al.* (1993) to include viscous effects and to allow critical layers to be situated away from an inflection point of the velocity profile. Their analysis applies to any 'nearly parallel' two-dimensional shear layer which is inviscidly unstable, although by way of example they apply the theory to Stokes layers.

The amplitude evolution equation still takes the form of (6.5$a$), but the kernel functions $K_j(\xi,\eta|\lambda)$ are different, and in the viscous case are algebraically messy. In the inviscid limit, $\lambda = 0$, Goldstein & Choi (1989) obtain

$$K_j(\xi,\eta|\lambda) = K(\xi,\eta) = (2\xi^3 + \xi^2\eta) - 2\sin^2\theta(2\xi^3 - \xi\eta^2) - 4\sin^4\theta(\xi^2\eta + \xi\eta^2)\ . \quad (6.6)$$

Goldstein & Choi (1989) show that solutions of the inviscid equation always terminate in a finite-distance singularity (or finite-time singularity in the corresponding temporal problem). As usual this can be interpreted as indicating an evolution towards shorter scales, usually the fully nonlinear Euler stage. Numerical solutions of the viscous problem show that for a range of obliqueness angles a singularity always forms, while for other angles the solution *decays* exponentially downstream if viscous effects are sufficiently large (i.e. if the flow becomes nonlinear sufficiently close to the neutral point). Indeed for large viscosity, and after a suitable rescaling, the amplitude equation simplifies to

$$\frac{dA}{d\bar{x}} = g_0 x_1 A + \bar{g}A\int_0^{+\infty} |A(\bar{x}-\eta)|^2 d\eta\ . \quad (6.7)$$

This equation has also been derived by Smith & Blennerhassett (1992) in a TS-wave/vortex study of the evolution of almost-neutral, *lower-branch*, oblique waves in PPF. Solutions to (6.7) develop a finite-distance singularity if $\Re(\bar{g}) > 0$, decay exponentially far downstream (but leave a mean flow growing algebraically) if $\Re(\bar{g}) < 0$, and grow exponentially if $\Re(\bar{g}) = 0$. The last case always corresponds to waves propagating at 45° to the freestream. Depending on the flow in question, a singularity forms, or the solutions decays, for angles greater than or less than 45° respectively — or vice versa.

The dynamics underlying (6.7) are a Rayleigh-wave/vortex interaction as a result of the development of a diffusion layer that sandwiches the critical layer. If viscous effects are sufficiently large, i.e. if nonlinear effects become important in region VII (or equivalent) of figure 7$c$, then the $\bar{x}$ and $x_1$ variables in (6.7) become identical (cf. §4.1). This scaling has been considered by Smith *et al.* (1993) in some detail in the context of a wall shear layers. In that case an extra linear term proportional to $A$ arises as an result of an $O(R^{-\frac{1}{2}})$ correction to the growth-rate from a Stokes layer adjacent to the wall (see also Cowley, 1987).

For the case where the shear layer is about to stabilise (cf. figure 7$c$), Smith *et al.* (1993) show that solutions to their equation in general either hit a finite-distance singularity or decay (although in one special case a Hall & Smith (1991) Rayleigh-wave/vortex interaction is generated). However, the analysis also applies close to a neutral point if a shear flow is becoming unstable to Rayleigh waves. Then, if the initial disturbances is large enough, solutions either hit a finite-distance singularity or evolve to a periodic solution.

Since Smith *et al.* (1993) arrived at their equation by considering a limiting problem of a wave/vortex interaction, while Wu *et al.* (1993) started from a non-equilibrium critical-layer approach, there is clearly a mathematical link between the two approaches. Further, Wu *et al.* (1993) suggest that because their solutions either decay or terminate in a singularity, it is unlikely that an initially linear wave will evolve to a Rayleigh-wave/vortex interaction of the type described by Hall & Smith (1991).

### 6.5 Resonant-triad interactions

A resonant triad of Rayleigh instability waves was first studied in the context of non-equilibrium critical layers by Goldstein & Lee (1992) for long wavelength modes (in fact for the inviscid limit of region IV in figure 7$c$). The case of modes with $O(1)$ wavelengths (i.e. inviscid limit of region VI) was studied by Wu (1992). In both situations, the amplitude, $\epsilon$, of the subharmonic waves was assumed to be $O(\delta/\mu)$, where $\delta$ is the magnitude of the planar wave at the fundamental frequency ($\mu$ satisfies (6.2$c$) with $k = 2$, and while $\lambda \ll 1$ in (6.2$d$) so that the flow is primarily inviscid). Essentially the same amplitude equations are obtained by both Goldstein &

Lee (1992) and Wu (1992), namely

$$\frac{dA}{d\bar{x}} = g_{10}x_1 A + g_{11}\int_0^\infty K_{11}A^*(\bar{x}-2\xi)B(\bar{x}-\xi)d\xi + g_{12}\int_0^\infty d\xi\int_0^\infty d\eta K_{12}\times A(\bar{x}-\xi)A(\bar{x}-\xi-\eta)A^*(\bar{x}-2\xi-\eta)\ , \quad (6.8a)$$

and

$$\begin{aligned}\frac{dB}{d\bar{x}} &= g_{20}x_1 B \\ &+ g_{21}\int_0^\infty d\eta\int_0^\infty d\zeta K_{21}\times B(\bar{x}-\zeta)A(\bar{x}-\zeta-\eta)A^*(\bar{x}-3\zeta-\eta) \\ &+ g_{22}\int_0^\infty d\eta\int_0^\infty d\zeta K_{22}\times B(\bar{x}-\zeta-\eta)A(\bar{x}-\zeta)A^*(\bar{x}-3\zeta-2\eta) \\ &+ g_{23}\int_0^\infty d\xi\int_0^\infty d\eta\int_0^\infty d\zeta K_{23}A(\bar{x}-\eta)\times A(\bar{x}-\eta-\zeta)A(\bar{x}-\eta-\zeta-\xi)A^*(\bar{x}-3\eta-2\zeta-\xi)\ ,\end{aligned} \quad (6.8b)$$

where $A$ and $B$ are the amplitudes of the oblique modes and the two-dimensional mode respectively, and the two oblique modes have been assumed to have equal amplitudes (for the sake of simplicity). The kernel functions are polynomials in $\xi$, $\eta$ and $\zeta$. For a triad of modes with $O(1)$ wavelengths, the coefficients are usually complex numbers, while for long-wavelength modes they are pure imaginary. Whatever their value, numerical solutions always seem to develop a singularity within a finite distance (or time).

The above results are obtained by assuming that the Reynolds number is sufficiently large so that viscosity can be ignored. Given the rather surprising rôle played by viscosity in the case of pair of oblique modes, it would be interesting to examine whether the singularity still forms when viscosity is included. Goldstein & Lee (private communication) are tackling this problem by solving the critical-layer equations numerically, while Wu is taking an analytical approach.

As mentioned above, in order to derive the fully coupled equations (6.8$a,b$), it is assumed that $\epsilon = O(\delta/\mu)$, i.e. the three-dimensional modes are required to have a much larger magnitude than the two-dimensional mode. If $\epsilon \ll \delta/\mu$, then the cubic term in (6.8$a$), and all the nonlinear terms in (6.8$b$), can be dropped from from the equations. Since the oblique modes have no feedback effect on the planar mode, this is referred to as parametric resonance. Goldstein & Lee (1992) show that the oblique waves can experience a *exponential-of-exponential* growth, while the planar waves evolve exponentially. Depending on the initial magnitude of the oblique modes, there are several possibilities for the subsequent development. If the oblique modes are not too small initially, their magnitude will quickly overtake that of the planar mode, and the evolution soon enters the fully coupled stage described by (6.8). However, if the initial magnitudes of the oblique modes are sufficiently small (exponentially small in some sense), Wundrow, Hultgren & Goldstein (1994) observe that the planar wave can go nonlinear before the oblique waves can produce a feedback effect. For shear layers with a *regular* critical layer, the evolution of the planar modes is then governed by (6.3), while the oblique modes evolve over a much faster (inviscid) 'time' scale in this stage. The continuing increase of the amplitude of the oblique modes eventually leads to a feedback on the plane mode so that ultimately all waves evolve on the short (inviscid) scale, and become fully coupled. This final stage is largely described by (6.8$a,b$) except that the linear terms are dropped. In addition a different upstream condition is imposed which is given by the asymptotic behaviour of the previous stage. However, if the critical layer is *singular*, then following the parametric resonance stage the development of the planar mode is governed by (6.5$a$). Because the solution of this equation can develop a singularity or equilibrate, the final stage may be different from that of Wundrow *et al.* (1994).

### 6.6 Spatial modulation of a planar carrier wave

In the studies summarised so far in this section, the disturbances have been assumed to be two-dimensional or to have a sinusoidal dependence on the spanwise variable. As in earlier weakly nonlinear studies this restriction can be relaxed by allowing for modulated wavetrains. For example, Wu (1993$b$) has studied the spanwise modulated version of the amplitude equation (6.5$a$) for modes with singular critical layers. He shows that for weak spanwise variations over $O(\epsilon^{-\frac{1}{5}})$ distances, nonlinear interactions within the critical layer affect the spanwise distribution of the vorticity, as well as the streamwise development, in region VI (i.e. at $O(\epsilon^{2/5}R)$ distances upstream of the neutral curve). The modulation equation is

$$\frac{\partial A}{\partial \bar{x}} - p\frac{\partial^2 A}{\partial \bar{Z}^2} = g_0 x_1 A + \int_0^\infty d\xi\int_0^\infty d\eta K(\xi,\eta|\lambda)\times \Big(g_1\xi^2 A(\bar{x}-\xi)A(\bar{x}-\xi-\eta)A^*(\bar{x}-2\xi-\eta) + h\xi A(\bar{Z},\bar{x}-\xi)A(\bar{Z},\bar{x}-\xi-\eta)A^*_{\bar{Z}\bar{Z}}(\bar{Z},\bar{x}-2\xi-\eta) + h\eta A(\bar{Z},\bar{x}-\xi)[A(\bar{Z},\bar{x}-\xi-\eta)A^*_{\bar{Z}}(\bar{Z},\bar{x}-2\xi-\eta)]_{\bar{Z}}$$

$$+ h\xi[A(\bar{Z},\bar{x}-\xi)A(\bar{Z},\bar{x}-\xi-\eta)A^*_{\bar{Z}}(\bar{Z},\bar{x}-2\xi-\eta)]_{\bar{Z}}\Big) , \qquad (6.9a)$$

where $\bar{Z} = \epsilon^{\frac{1}{5}} z$, $K(\xi,\eta)$ is defined by (6.5b), and $p$, $g_0$, $g_1$ and $h$ are constants.

Numerical solutions show that a disturbance centred at a spanwise position can propagate laterally to form concentrated, quasi-periodic streamwise vortices. This qualitatively captures phenomena observed in free shear layer experiments. The focussing of the vorticity appears to be associated with a localised singularity at a finite distance downstream.

In the viscous limit the scaled amplitude equation becomes

$$\frac{\partial A}{\partial \bar{x}} - p\frac{\partial^2 A}{\partial \bar{Z}^2} = g_0 x_1 A + hA\int_0^\infty \eta^{-\frac{1}{2}}[A(\bar{x}-\eta)A^*_{\bar{Z}}(\bar{x}-\eta)]_{\bar{Z}} d\eta \ . \qquad (6.9b)$$

As in the case of the oblique modes, the equation obtained in the viscous limit is non-local. Further, we note that the amplitude equation governing the modulation of TS waves in the upper-branch régime, i.e. equation (5.2), is a special case of (6.9*b*).

As for oblique waves, if the disturbance becomes nonlinear very close to the neutral point (i.e. region VII), then non-parallel effects need to be included by identifying $x_1$ with $\bar{x}$ in (6.9*b*) (and, in the case of a wall shear layer, by including an extra linear term in $A$ to allow for the influence of the Stokes layer on the wall). This régime has been studied by Timoshin & Smith (1993), who also note there is an intermediate scaling between this slow-spanwise-modulation scaling, and the oblique wave scaling of §6.4 (cf. §5.2).

## 7. CONCLUDING REMARKS

In this paper we have attempted to review some of the more recent high-Reynolds-number asymptotic approaches to understanding transition in incompressible/subsonic shear flows. By necessity we have had to be selective in our choice of material because of the fact that in the high-Reynolds-number limit there are an abundance of possible scalings. We have attempted to explain how some of these scalings relate to each other in the hope that this will make the choice of which theory to compare with which experiment slightly easier. However, in preparing the review we were struck by disparity between the large number of papers based on an asymptotic approach, and the small number of papers that obtain good quantitative (or even qualitative) agreement with experiment. To a certain extent this is a consequence of our choice of subject material, i.e. [weakly] nonlinear models of transition caused by the growth of TS/Rayleigh waves. We note that in related topics (for which they are already recent reviews) a number of favourable comparisons have been obtained between high-Reynolds-number theories and experiment; e.g. for the linear receptivity problem see Goldstein & Hultgren (1987), while for Görtler vortex flows see Hall (1983), Hall & Seddougui (1989), Denier, Hall & Seddougui (1991) and Hall & Horseman (1991).

There is of course agreement between the lower branch neutral curve for the Blasius boundary layer and experiment (Smith 1979*a*). Hultgren (1992) has also obtained good quantitative agreement between theory and experiment by imaginative use of composite expansions. Further, Smith & Stewart (1987), Stewart & Smith (1992) and Smith & Bowles (1992) all note quantitative agreement with various weakly nonlinear theories and experiment. However, in each of these three theories an *assumption* is apparently made that the nonlinear critical layer plays a passive rôle, whereas in other theories (e.g. Wu *et al.* 1994) critical layers are known to play an active part in the dynamics. Verification that the critical layer in the aforementioned theories was indeed passive would be reassuring.

Further, if improved quantitative agreement between theory and experiment is to be obtained, then more attention seems to be required concerning initial conditions to the nonlinear amplitude equations, etc. For instance, it appears that careful attention in this area may revise our view of whether slightly supercritical PPF ‘bursts’. There also appears to be merit in greater use of composite expansions.

On the plus side, one of the great strengths of asymptotic theory has been in identifying mechanisms. They key rôle of the critical layer has been emphasised in many of the papers quoted. The work of Hall, Smith and their colleagues has also highlighted the powerful influence that wave/vortex (or wave/mean-flow) interactions can have on high-Reynolds-number flows. Further, the occurrence of finite-distance/finite-time singularities is also suggestive of the development of small scales. The possibilities of singularities in the fully nonlinear triple-deck equations (e.g. Smith 1988), and the potential of Van Dommelen singularities (Van Dommelen & Shen 1980) in the classical unsteady boundary layer equations to explain sub-layer bursting (e.g. Smith & Burggraf 1985, Hoyle, Smith & Walker 1991), are worthy of special mention.

## ACKNOWLEDGEMENTS

The authors are grateful to Dr S.T. Lam, M.J. Jennings, J. Moston and P.A. Stewart for many useful discussions and much practical help. They also thank Dr A. Bassom and Dr B. Singer for their comments on an earlier version of the manuscript.

## REFERENCES

Arnal D. 1993. Boundary layer transition: predictions based on linear theory. *Progress in Transition Modelling: AGARD FDP/VKI Special Course, Report 793.*

Bassom A.P. & Gajjar J.S.B. 1988. Non-stationary cross-flow vortices in three-dimensional boundary-layer flows. *Proc. R. Soc. Lond. A*, **417**, 179–212.

Bassom A.P. & Hall P. 1989. On the generation of mean flows by the interaction of Görtler vortices and Tollmien–Schlichting waves in curved channel flows. *Stud. Appl. Math.*, **81**, 185-219.

Bassom A.P. & Hall P. 1992. On the interaction of Görtler vortices and Tollmien–Schlichting waves in boundary layer flows. In *Instability, Transition and Turbulence*, proceedings of the 1991 workshop on transition and turbulence, NASA Langley, VA, USA (editors M.Y. Hussaini, A. Kumar and C.L. Streett), Springer Verlag, New York.

Bayliss A., Maestrello L., Parikh P. & Turkel E. 1985. *AIAA Paper* 85–0565.

Bayly B.J., Orszag S.A. 1988. Instability mechanisms in shear-flow transition. *Ann. Rev. Fluid Mech.* **20**, 359-391.

Benjamin T.B. 1961. The development of three-dimensional disturbances in an unstable film of liquid flowing down an inclined plane. *J. Fluid Mech.* **10**, 401–419.

Bennett J., Hall P. & Smith F.T. 1991. The strong nonlinear interaction of Tollmien–Schlichting waves and Taylor–Görtler vortices in curved channel flow. *J. Fluid Mech.* **223**, 475–495.

Benney D.J. & Bergeron R.F.,Jr. 1969. A new class of nonlinear waves in parallel flows. *Stud. Appl. Math.* **48**, 181–204.

Blackaby N.D. 1993. Tollmien-Schlichting/vortex interactions in compressible boundary layer flows. *ICASE Report 93-55* and submitted to *IMA J. Appl. Math.*

Bodonyi R.J.& Smith F.T. 1981. The upper branch stability of the Blasius boundary layer, including non-parallel flow effects. *Proc. R. Soc. Lond.* A**375**, 65–92.

Bodonyi R.J., Smith F.T. & Gajjar J. 1983. Amplitude-dependent stability of boundary-layer flow with a strongly non-linear critical layer. *IMA J. Appl. Math.* **30**, 1–19.

Bouthier M. 1973. Stabilité linéaire des écoulements presque parallèles. La couche limite de Blasius. *J. de Mécanique* **12**, 75–95.

Brown P.G., Brown S.N., Smith F.T. & Timoshin S.N. 1993. On the starting process of strongly nonlinear vortex Rayleigh-wave interactions. *Mathematika* **40**, 7-29.

Brown S.N. & Stewartson K. 1978. The evolution of the critical layer of a Rossby wave. Part II. *Geophys. Astrophys. Fluid Dyn.* **10**, 1–24.

Churilov S.M. & Shukhman I.G. 1987. The nonlinear development of disturbances in a zonal shear flow. *Geophys. Astrophys. Fluid Dyn.* **38**, 145–175.

Churilov S.M. & Shukhman I.G. 1994. Nonlinear spatial evolution of helical disturbances on an axial jet. Submitted to *J. Fluid Mech.*

Cowley S.J. 1981. *High Reynolds number flows through distorted channels and flexible tubes.* Ph.D Thesis, University of Cambridge.

Cowley S.J. 1985. Pulsatile flow through distorted channels: low-Strouhal-number and translating-critical-layer effects. *Q. J. Mech. Appl. Maths.* **38**, 589–619.

Cowley S.J. 1987. High frequency Rayleigh instability of Stokes layers. In *Stability of time dependent and spatially varying flows* (ed. D.L. Dwoyer & M.Y. Hussaini), Springer-Verlag.

Cowley S.J., Duck P.W., Tutty O.R. & Van Dommelen L.L 1990. Genesis of three-dimensional separation. *Workshop on Analytical Methods in Unsteady Separation*, Ohio State University, January 1990.

Cowley S.J. & Hall P. 1990. On the instability of hypersonic flow past a wedge. *J. Fluid Mech.* **214**, 17–42.

Cowley S.J., Hocking L.M. & Tutty O.R. 1985. The stability of solutions of the classical unsteady boundary-layer equation. *Phys. Fluids* **28**, 441–443.

Cowley S.J., Tanveer S., Baker G.R. & Page M.A. 1993. An asymptotic description of the formation of a curvature singularity in a vortex sheet. In Preparation.

Craik A.D.D. 1971. Non-linear resonant instability in boundary layers. *J. Fluid Mech.* **50**, 393–413.

Dickinson R.E. 1970. Development of a Rossby wave critical level. *J. Atmos. Sci.* **27**, 627–633.

Davey A., Hocking L.M. & Stewartson K. 1974. On the nonlinear evolution of three-dimensional disturbances in plane Poiseuille flow. *J. Fluid Mech.* **63**, 529–536.

Davis R.E. 1969. On the high Reynolds number flow over a wavy boundary. *J. Fluid Mech.* **30**, 337–346.

Denier J.P., Hall P. & Seddougui S.O. 1991. On the receptivity problem for Görtler vortices: vortex motions induced by wall roughness. *Phil. Trans. R. Soc. Lond.* A**335**, 51-85.

Doorly D.J. & Smith F.T. 1992. Initial-value problems for spot disturbances in incompressible or compressible boundary layers. *J. Eng. Mech.* **26**, 87–106.

Drazin P.G. & Howard L.N. 1962. The instability to long waves of unbounded parallel inviscid flow. *J. Fluid Mech.* **14**, 257–283.

Drazin P.G. & Reid W.H. 1981. *Hydrodynamic Stability.* Cambridge Univ. Press.

Duck P.W. 1985. Laminar-flow over unsteady humps - the formation of waves. *J. Fluid Mech.* **160**, 465-498.

Duck P.W. 1990. Triple-deck flow over unsteady surface disturbances - the three-dimensional development of Tollmien-Schlichting waves. *Comput. Flu.* **18**, 1-34.

Elliott J.W., Cowley S.J. & Smith F.T. 1983. Breakdown of boundary layers: (i) on moving surfaces; (ii) in semi-similar unsteady flow; (iii) in fully unsteady flow. *Geophys. Astrophys. Fluid Dyn.* **25**, 77–138.

Elliott J.W. & Smith F.T. 1987 Dynamic stall due to unsteady marginal separation. *J. Fluid Mech.* **179**, 489-512.

Foote J.R. & Lin C.C. 1950. Some recent investigations in the theory of hydrodynamic stability. *Quart. Appl. Math.* **8**, 265–280.

Freymuth P. 1966. On transition in a separated linear boundary layer. *J. Fluid Mech.* **25**, 683–704.

Gajjar J. 1994. Nonlinear evolution of the first mode supersonic oblique waves in compressible boundary layers. Part I: Heated/cooled walls. To appear in *IMA J. Appl. Math.*.

Gajjar J. & Smith F.T. 1985. On the global instability of free disturbances with a time-dependent nonlinear viscous critical layer. *J. Fluid Mech.* **157**, 53–77.

Gaster M. 1968*a*. Growth of disturbances in both space and time. *Phys. Fluids* **11**, 723–727.

Gaster M. 1968*b*. The development of three–dimensional wave packets in a boundary layer. *J. Fluid Mech.* **32**, 173–184.

Gaster M. 1974. On the effects of boundary–layer growth on flow stability *J. Fluid Mech.* **66**, 465–480.

Goldstein M.E. & Choi S.-W. 1989. Nonlinear evolution of interacting oblique waves on two-dimensional shear layers. *J. Fluid Mech.* **207**, 97-120. Corrigendum, 1990. *J. Fluid Mech.* **216**, 659–663.

Goldstein M.E. & Durbin P.A. 1986. Nonlinear critical layers eliminate the upper branch of spatially growing Tollmien-Schlichting waves. *Phys. Fluids* **29**, 2344–2345.

Goldstein M.E., Durbin P.A. & Leib S.J. 1987. Roll–up of vorticity in adverse–pressure–gradient boundary–layers. *J. Fluid Mech.* **183**, 325–342.

Goldstein M.E. & Hultgren L.S. 1987. A note on the generation of Tollmien-Schlichting waves by sudden surface-curvature change. *J. Fluid Mech.* **181**, 519–525.

Goldstein M.E. & Hultgren L.S. 1988. Nonlinear spatial evolution of an externally excited instability wave in a free shear layer. *J. Fluid Mech.* **197**, 295–330.

Goldstein M.E. & Hultgren L.S. 1989. Boundary-layer receptivity to long-wave free-stream distrurbances. *Ann. Rev. Fluid Mech.* **21**, 137–166.

Goldstein M.E. & Lee S.S. 1992. Fully coupled resonant-triad interaction in an adverse-pressure-gradient boundary layer. *J. Fluid Mech.* **245**, 523–551.

Goldstein M.E. & Lee S.S. 1993. Oblique instability waves in nearly parallel shear flows. In *'Nonlinear waves and weak turbulence'*, Birkhauser, Boston.

Goldstein M.E. & Leib S.J. 1988. Nonlinear roll-up of externally excited free shear layers. *J. Fluid Mech.* **191**, 481–515.

Goldstein M.E. & Leib S.J. 1989. Nonlinear evolution of oblique waves on compressible shear layers. *J. Fluid Mech.* **207**, 73–96.

Goldstein M.E. & Wundrow D.W. 1990. Spatial evolution of nonlinear acoustic mode instabilities on hypersonic boundary layers. *J. Fluid Mech.* **219**, 585–607.

Greenspan H.P. & Benney D.J. 1963. On shear-layer instability, breakdown and transition. *J. Fluid Mech.* **15**, 133–153.

Haberman R. 1972. Critical layers in parallel flows. *Stud. Appl. Math.* **51**, 139–161.

Hall P. 1978. The linear stability of flat Stokes layers. *Proc. R. Soc. Lond.* A**359**, 151–166.

Hall P. 1982. Taylor-Görtler vortices in fully-developed or boundary-layer flows – linear-theory *J. Fluid Mech.* **124**, 475–494.

Hall P. 1983. The linear development of Görtler vortices in growing boundary layers. *J. Fluid Mech.* **130**, 41–58.

Hall P. 1990. Görtler vortices in growing boundary layers: the leading edge receptivity problem, linear growth and the nonlinear breakdown stage. *Mathematika* **37**, 151–189.

Hall P. & Horseman N.J. 1991. The linear inviscid secondary instability of longitudinal vortex structures in boundary layers. *J. Fluid Mech.* **232**, 357–375.

Hall P. & Seddougui S.O. 1989. On the onset of three-dimensionality and time-dependencd in Görtler vortices. *J. Fluid Mech.* **204**, 405-420.

Hall P. & Smith F.T. 1982. A suggested mechanism for nonlinear wall roughness effects on high Reynolds number flow stability. *Stud. Appl. Math.* **66**, 241–265.

Hall P. & Smith F.T. 1984. On the effects of nonparallelism, three-dimensionality, and mode interaction in nonlinear boundary-layer stability. *Stud. Appl. Math.* **70**, 91–120.

Hall P, & Smith F.T. 1988 The nonlinear-interaction of Tollmien-Schlichting waves and taylor-gortler vortices in curved channel flows. *Proc. R. Soc. Lond.* A**417**, 255-282.

Hall P. & Smith F.T. 1989. Nonlinear Tollmien-Schlichting vortex interaction in boundary-layers. *Euro. J. Mech. B-Fluids* **8**, 179-205. *For corrections see* Smith F.T. & Blennerhassett P. (1992)

Hall P. & Smith F.T. 1990. Near planar TS waves and longitudinal vortices in channel flow: nonlinear interaction and focussing. In *Instability and transition II* (ed. M.Y. Hussaini & R.G. Voigt), Springer-Verlag, New York. *For corrections see* Smith F.T. & Blennerhassett P. (1992)

Hall P. & Smith F.T. 1991. On strongly nonlinear vortex/wave interactions in boundary-layer transition. *J. Fluid Mech.* **227**, 641–666.

Haynes P.H. 1985. Nonlinear instability of a Rossby-wave critical layer. *J. Fluid Mech.* **161**, 493-511.

Haynes P.H. & Cowley S.J. 1986. The evolution of an unsteady translating nonlinear Rossby-wave critical layer. *Geophys. Astrophys. Fluid Dyn.* **35**, 1–55.

Herbert T. 1977. Finite amplitude stability of plane parallel flows. *Laminar-Turbulent Transition: Proc. AGARD Conf. No. 224, Lyngby.*

Herbert T. 1983. Secondary instability of plane channel flow to subharmonic three-dimensional disturbances. *Phys. Fluids* **26**, 871–874.

Herbert T. 1988 Secondary instability of boundary layers. *Ann. Rev. Fluid Mech.* **20**, 487–526.

Hickernell F.J. 1984. Time-dependent critical layers in shear flows on the beta-plane. *J. Fluid Mech.* **142**, 431–449.

Hocking L.M. 1974. On the structure of instability bursts in three-dimensional parallel flow. *Mathematika* **21**, 134–146.

Hocking L.M. 1975. Non-linear instability of the asymptotic suction velocity profile. *Quart. J. Mech. Appl. Maths.* **28**, 341–353.

Hocking L.M. & Stewartson K. 1971. On the non-linear response of a marginally unstable plane parallel flow to a three-dimensional disturbance. *Mathematika* **18**, 219–239.

Hocking L.M., Stewartson K. & Stuart J.T. 1972. A non-linear instability burst in plane parallel flow. *J. Fluid Mech.* **51**, 705–735.

Hoyle JM, Smith F.T., Walker J.D.A. 1991 On sublayer eruption and vortex formation. *Comp. Phys. Comm.* **65**, 151-157.

Huerre P. 1980. The nonlinear stability of a free shear layer in the viscous critical layer regime. *Phil. Trans. R. Soc. Lond.* A**293**, 643–675.

Huerre P. & Monkewitz P.A. 1990. Local and global instabilities in spatially developing flows. *Ann. Rev. Fluid Mech.* **22**, 473–537.

Huerre P. & Scott J.F. 1980. Effects of critical layer structure on the nonlinear evolution of waves in free shear layers. *Proc. R. Soc. Lond.* A**371**, 509–524.

Hultgren L.S. 1987. Higher eigenmodes in the Blasius boundary-layer stabilty problem. *Phys. Fluids* **30**, 2947–2951.

Hultgren L.S. 1992. Nonlinear spatial equilibration of an externally excited instability wave in a free shear layer. *J. Fluid Mech.* **236**, 635–664.

Jennings M.J., Stewart P.A. & Wu X. 1993. In preparation.

Kachanov Yu.S., Kozlov V.V. & Levchenko V.Ya. 1977. Nonlinear development of a wave in a boundary layer. *Izv. Akad. Nauk SSSR, Mekh. Zhidk i Gaza* **3**, 49–53.

Kachanov Yu.S. & Levchenko V.Ya. 1984. The resonant interactions of disturbances at laminar-turbulent transition in a boundary layer. *J. Fluid Mech.* **138**, 209–247.

Kachanov Yu.S., Ryzhov O.S. & Smith F.T. 1993. Formation of solitons in transitional boundary-layers - theory and experiment. *J. Fluid Mech.* **251**, 273 -297

Kelly R.E. & Maslowe S.A. 1970. The nonlinear critical layer in a slightly stratified shear flow. *Stud. Appl. Math.* **49**, 301-326.

Khokhlov A.P. 1991. The temporal evolution of Tollmien-Schlichting waves near nonlinear resonance. Preprint.

Khokhlov A.P. 1993 Evolution of resonant triads in a boundary layer. Submitted to *J. Fluid Mech.*

Killworth P.D. & McIntyre M.E. 1985. Do Rossby-wave critical layers absorb, reflect or over-reflect? *J. Fluid Mech.* **161**, 449–492.

Klebanoff P.S. & Tidstrom K.D. 1959. Evaluation of amplified waves leading to transition in a boundary layer with zero pressure gradient. *NASA TN D-195.*

Kleiser L., & Zang T.A. 1991. Numerical simulation of transition in wall-bounded shear flows. *Annu. Rev. Fluid Mech.* **23**, 495-537.

Leib S.J. 1991. Nonlinear evolution of subsonic and supersonic disturbances on a compressible free shear-layer. *J. Fluid Mech* **224**, 551–578.

Leib S.J. & Goldstein M.E. 1989. Nonlinear interaction between the sinuous and varicose instability modes in a plane wake. *Phys. Fluids* **A1**, 513–521.

Leibovich S. & Stewartson K. 1983. A sufficient condition for the instability of columnar vortices. *J. Fluid Mech.* **126**, 335–356.

Lin C.C. 1946. On the stability of two-dimensional parallel flows. Part 3 — Stability in a viscous fluid. *Quart. Appl. Math.* **3**, 277-301.

Lin C.C. 1955. *The Theory of Hydrodynamic Stability.* Cambridge Univ. Press.

Lin C.C. 1957. On instability of laminar flow and transition to turbulence. In *IUTAM symposium Freiberg/BR.*, 144.

Mankbadi R.R. 1991. Resonant triad in boundary-layer stability. *NASA TM* 105208.

Mankbadi R.R. 1993. A critical-layer analysis of the resonant triad in boundary-layer transition. Part II. Composite solution and comparison with observations. Submitted to *J. Fluid Mech.*.

Mankbadi R.R., Wu X. & Lee S.S. 1993. A critical-layer analysis of the resonant triad in Blasius boundary layer: nonlinear interactions. *J. Fluid Mech.* **256**, 85-106.

Messiter A.F. 1970. Boundary-layer flow near the trailing edge of a flat plate. *SIAM J. Appl. Math.* **18**, 241–257.

Miles J.W. 1958. On the disturbed motion of a plane vortex sheet. *J. Fluid Mech.* **4**, 538–552.

Neiland V.Ya. 1969. Towards a theory of separation of the laminar boundary layer in a supersonic stream. *Izv. Akad. Nauk SSSR, Mekh. Zhidk. i Gaza* **4**, 33–35.

Nishioka M., Asai M. & Iida S. 1980. An experimental investigation of secondary instability. In *Laminar-turbulent transtion* (ed. R. Eppler & H. Fasel), Springer-Verlag, Berlin.

Orszag S.A. & Patera A.T. 1980. Subcritical transition to turbulence in plane channel flows. *Phys. Rev. Lett.* **45**, 989–993.

Orszag S.A. & Patera A.T. 1981. Subcritical transition to turbulence in planar shear flows. In *'Transition and Turbulence'*, 127–146. Academic, New York.

Pugh J.D. & Saffman P.G. 1988. Two-dimensional superharmonic stability of finite-amplitude waves in plane Poiseuille flow. *J. Fluid Mech.* **194**, 295–307.

Raetz G.S. 1959. A new theory of the cause of transition in fluid flows. *Norair Rep.* NOR-59-383, Hawthorne, California.

Reid W.H. 1965. In *Basic Developments in Fluid Mechanics* (ed. M. Holt), Academic, New York.

Reutov V.P. 1982. Nonstationary critical layer and nonlinear stage of instability in a plane Poiseuille flow. *Zh. Prikl. & Tekh. Fiz* **4**, 43–54.

Robinson J.L. 1974. The inviscid nonlinear instability of parallel shear flows. *J. Fluid Mech.* **63**, 723–752.

Rothmayer A.P. & Smith F.T. 1987 Strongly nonlinear wavepackets in boundary layers. *Trans. ASME.* 67.

Schubauer G.B. & Skramstad H.K. 1947. Linear boundary-layer oscillations and transition on a flat plate. *J. Res. Nat. Bur. Stand.* **38**, 251–292.

Shukhman I.G. 1989. Nonlinear stability of a weakly supercritical mixing layer in a rotating fluid. *J. Fluid Mech.* **200**, 425-450.

Shukhman I.G. 1991. Nonlinear evolution of spiral density waves generated by the instability of the shear-layer in a rotating compressible fluid. *J. Fluid Mech.* **233**, 587–612.

Smith F.T. 1976*a*. Flow through constricted or dilated pipes and channels: Part 1. *Quart. J. Mech. Appl. Maths.* **29**, 343–364.

Smith F.T. 1976*b*. Flow through constricted or dilated pipes and channels: Part 2. *Quart. J. Mech. Appl. Maths.* **29**, 365–376.

Smith F.T. 1979*a*. On the non-parallel flow stability of the Blasius boundary layer. *Proc. R. Soc. Lond.* A**366**, 91–109.

Smith F.T. 1979*b*. Nonlinear stability of boundary layers for disturbances of various sizes. *Proc. R. Soc. Lond.* A**368**, 573–589. *See also* Corrections to 'Nonlinear stability of boundary layers for disturbances of various sizes'. *Proc. R. Soc. Lond.* A**371**, 439–440.

Smith F.T. 1979*c*. Instability of flow through pipes of general cross-section: Part 1. *Mathematika* **26**, 187-210.

Smith F.T. 1979*d*. Instability of flow through pipes of general cross-section: Part 2. *Mathematika* **26**, 211-223.

Smith F.T. 1986*a*. Two-dimensional disturbance travel, growth and spreading in boundary layers. *J. Fluid Mech.* **169**, 353–377.

Smith F.T. 1986*b*. The strong nonlinear growth of three-dimensional disturbances in boundary layers. *Utd. Tech. Res. Cent., E. Hartford, Conn., Rep. UTRC-86-10*

Smith F.T. 1988. Finite-time break-up can occur in any unsteady interacting boundary layer. *Mathematika* **35**, 256-273

Smith F.T. 1989. On the first-mode instability in subsonic, supersonic or hypersonic boundary layers. *J. Fluid Mech.* **198**, 127–153.

Smith F.T. & Blennerhassett P. 1992. Nonlinear interaction of oblique three-dimensional Tollmien-Schlichting waves and longitudinal vortices, in channel flows and boundary layers. *Proc. R. Soc. Lond.* A**436**, 585–602.

Smith F.T. & Bodonyi R.J. 1982*a*. Nonlinear critical layers and their development in streaming-flow stability. *J. Fluid Mech.* **118**, 165–185.

Smith F.T. & Bodonyi R.J. 1982*b*. Amplitude-dependent neutral modes in the Hagen-Poiseuille flow through a circular pipe. *Proc. R. Soc. Lond.* A**384**, 463–489.

Smith F.T. & Bodonyi R.J. 1985. On short-scale inviscid instabilities in flow past surface-mounted obstacles and other non-parallel motions. *Aero. J.* **89**, 205-212.

Smith F.T. & Bowles R.I. 1992. Transition theory and experimental comparisons on (i) amplification into streets and (ii) a strongly nonlinear break-up criterion. *Proc. R. Soc. Lond.* A**439**, 163-175.

Smith F.T. & Bowles R.I. 1993. On boundary-layer transition in transonic flow. *J. Eng. Math.* **27** 309-342.

Smith F.T., Brown S.N. & Brown P.G. 1993. Initiation of three-dimensional nonlinear transition paths from an infectional profile. *Eur. J. Mech. B* **12**, 447-473.

Smith F.T. & Burggraf O.R. 1985. On the development of large-sized short-scaled disturbances in boundary layers. *Proc. R. Soc. Lond.* A**399**, 25–55.

Smith F.T., Doorly D.J. & Rothmayer A.P. 1990. On displacement-thickness, wall-layer and mid-flow scales in turbulent boundary layers, and slugs of vorticity in channel and pipe flows. *Proc. R. Soc. Lond.* A**428**, 255–281.

Smith F.T., Papageorgiou D. & Elliott J.W. 1984. An alternative approach to linear and nonlinear stability calculations at finite Reynolds numbers. *J. Fluid Mech.* **146**, 313–330.

Smith F.T. & Stewart P.A. 1987. The resonant-triad nonlinear interaction in boundary-layer transition. *J. Fluid Mech.* **179**, 227–252.

Smith F.T. & Walton A.G. 1989. Nonlinear interaction of near-planar TS waves and longitudinal vortices in boundary-layer transition. *Mathematika* **36**, 262–289. *For corrections see* Smith F.T. & Blennerhassett P. (1992)

Spalart P. 1990. Direct numerical study of crossflow instability. In *Laminar-turbulent transition: IUTAM Symposium, Toulouse/France 1989* (ed. D. Arnal & R. Michel), Springer-Verlag, Berlin.

Stewart P.A. & Smith F.T. 1987. Three-dimensional instabilities in steady and unsteady non-parallel boundary layers, including effects of Tollmien-Schlichting disturbances and cross flow. *Proc. R. Soc. Lond.* A**409**, 229–248.

Stewart P.A. & Smith F.T. 1992. Three-dimensional nonlinear blow-up from a nearly planar initial disturbance, in boundary-layer transition: theory and experimental comparisons. *J. Fluid Mech.* **244**, 79–100.

Stewartson K. 1969. On the flow near the trailing edge of a flat plate. Part II. *Mathematika* **16**, 106–121.

Stewartson K. 1978. The evolution of the critical layer of a Rossby wave. *Geophys. Astrophys. Fluid Dyn.* **9**, 185–200.

Stewartson K. & Stuart J.T. 1971. A non-linear instability theory for a wave system in plane Poiseuille flow. *J. Fluid Mech.* **48**, 529–545.

Stuart J.T. 1960. On the non-linear mechanics of wave disturbances in stable and unstable parallel flows. Part

1. The basic behaviour in plane Poiseuille flow. *J. Fluid Mech.* **9**, 353–370.

Tennekes H. & Lumley J.L. 1972. *A first course in turbulence.* The MIT Press, Cambridge, Mass.

Timoshin S.N. & Smith F.T. 1993. On the nonlinear vortex-Rayleigh wave interaction in a boundary-layer flow. In preparation. Presented at the International Workshop on Advances in Analytical Methods in Aerodynamics, 12-14 July 1993, Miedzyzdroje, Poland.

Tromans P. 1977. Ph.D. Thesis, University of Cambridge.

Tutty O.R. & Cowley S.J. 1986. On the stability and the numerical solution of the unsteady interactive boundary-layer equation. *J. Fluid Mech.* **168**, 431–456.

Van Dommelen L.L. & Shen S.F. 1980 The spontaneous generation of the singularity in a separating laminar boundary layer. *J. Comp. Phys.* **38**, 125-140.

von Kerczek C. & Davis S.H. 1974. Linear stability theory of oscillatory Stokes layers. *J. Fluid Mech.* **62**, 753–773.

Walton A.G. & Smith F.T. 1992. Properties of strongly nonlinear vortex/ Tollmien-Schlichting-wave interactions. *J. Fluid Mech.* **244**, 649–676.

Warn T. & Warn H. 1976. On the development of a Rossby wave critical level. *J. Atmos. Sci.* **33**, 2021–2024.

Warn T. & Warn H. 1978. The evolution of a nonlinear critical level. *Stud. Appl. Math.* **59**, 37–71.

Wu X. 1992. The nonlinear evolution of high-frequency resonant-triad waves in an oscillatory Stokes-layer at high Reynolds number. *J. Fluid Mech.* **245**, 553–597.

Wu X. 1993*a*. On critical-layer and diffusion-layer nonlinearity in the three-dimensional stage of boundary-layer transition. *Proc. R. Soc. Lond.* A**443**, 95-106.

Wu X. 1993*b*. Nonlinear temporal-spatial modulation of near-planar Rayleigh waves in shear flows: formation of streamwise vortices. *J. Fluid Mech.* **256**, 685-719.

Wu X. & Cowley S.J. 1994. The nonlinear evolution of high-frequency disturbances in the Stokes layer at high Reynolds number. Submitted to *Quart. J. Mech. Appl. Maths.*. *See also* Wu X. 1991. *Nonlinear instability of Stokes layers.* Ph.D. Thesis, University of London.

Wu X., Lee S.S. & Cowley S.J. 1993. On the nonlinear three-dimensional instability of Stokes layers and other shear layers to pairs of oblique waves. *J. Fluid Mech.* **253**, 681-721.

Wu X., Stewart P.A. & Cowley S.J. 1994 On the weakly nonlinear development of Tollmien-Schlichting wavetrains in boundary layers. Submitted to *J. Fluid Mech.*

Wundrow D.W. 1993. Nonlinear spatial evolution of inviscid instabilities on hypersonic boundary layers. To appear in *J. Fluid Mech.*.

Wundrow D.W., Hultgren L.S. & Goldstein M.E. 1993 Interaction of oblique instability waves with a nonlinear plane wave. To appear in *J. Fluid Mech.*