# On the nonlinear growth of two-dimensional Tollmien-Schlichting waves in a flat-plate boundary layer

By JUSTIN MOSTON, PHILIP A. STEWART AND STEPHEN J. COWLEY

DAMTP, University of Cambridge, Silver Street, Cambridge CB3 9EW, UK

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This paper studies the nonlinear development of two-dimensional Tollmien-Schlichting waves in an incompressible flat-plate boundary layer at asymptotically large values of the Reynolds number. Attention is restricted to the 'far-downstream lower-branch' régime where a multiple-scales analysis is possible. It is supposed that to leading-order the waves are inviscid and neutral, and governed by the [Davis-Acrivos-]Benjamin-Ono equation. This has a three-parameter family of periodic solutions, the large-amplitude (soliton) limit of which bears a qualitative resemblance to the 'spikes' observed in certain 'K-type' transition expermients. The variation of the parameters over slow length- and time-scales is controlled by a viscous sublayer. For the case of a purely temporal evolution, it is shown that a solution for this sublayer ceases to exist when the amplitude reaches a certain finite value. For a purely spatial evolution, it appears that an initially linear disturbance does not evolve to a fully nonlinear stage of the envisaged form. The implications of these results for the 'soliton' theory of spike formation are discussed.

## 1. Introduction

The development of Tollmien-Schlichting (TS) instability waves in a boundary layer and the consequent transition to turbulence was studied in a classic series of experiments by Schubauer & Klebanoff (1956), Klebanoff & Tidstrom (1959) and Klebanoff, Tidstrom & Sargent (1962). In these experiments, a two-dimensional TS input wave was observed to develop a spanwise-periodic variation in amplitude ('peak-valley splitting'), followed by the appearance of 'spikes' in the velocity profiles at the 'peaks' (regions of enhanced amplitude); this sequence of events is now known as 'K-type' transition. Early theoretical work attempted to explain the process of spike formation as a high-frequency secondary instability of the instantaneous inflectional velocity profiles which arise through amplification of the primary TS wave (e.g. Betchov 1960, Tani & Komoda 1962, Greenspan & Benney 1963, Landahl 1972). This interpretation was challenged by Kachanov & Levchenko (1984), who proposed that the spikes result from the purely deterministic generation of harmonics of the primary TS wave. A detailed experimental study by Borodulin & Kachanov (1988, 1990) indicated that high-frequency secondary instability does indeed take place near the wall, but it is not connected with the spikes which form higher in the boundary layer. Borodulin & Kachanov (1988) also noted that 'the spikes ... propagate steadily downstream within the boundary layer almost without change of their shape and amplitude', and claimed that it 'is highly probable that the behaviour of the spikes ... can be described within the framework of a theory of solitons', though the type of soliton was not identified.

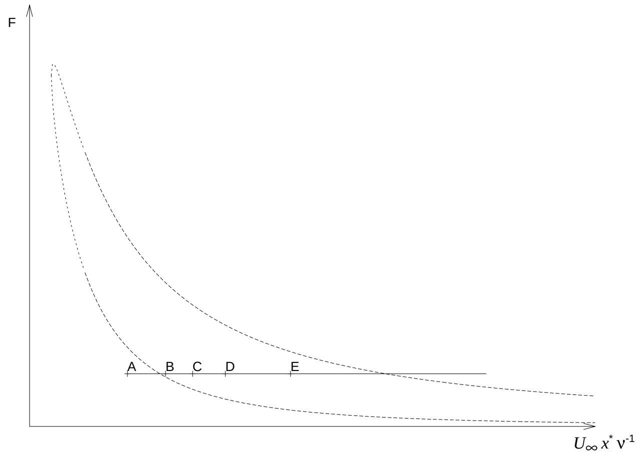


FIGURE 1. Schematic illustration (not to scale) of the different parameter régimes for an experiment where a fixed-frequency disturbance is introduced into a flat-plate boundary layer. The abscissa is proportional to downstream distance, and the ordinate to the frequency of the disturbance. The dashed lines represent the asymptotic ($U_\infty x^* \nu^{-1} \gg 1$) lower and upper branches of the linear neutral curve, while the dotted line represents the continuation of the neutral curve to finite Reynolds numbers provided by solutions to the (heuristic) Orr-Sommerfeld equation. A fixed-frequency disturbance introduced at point A moves along the solid horizontal line as it propagates downstream. Point B is asymptotically close to the lower-branch neutral curve, while point C is located in the main part of the 'lower-branch' régime, where the spatial growth-rate is comparable with the wavenumber. Point E is in the 'upper-branch' régime, and point D in the overlap region between the lower- and upper-branch scalings.

A mathematically self-consistent theory for the evolution of TS waves in spatially varying boundary layers can only be obtained for asymptotically large Reynolds numbers (e.g. Smith 1979, Zhuk & Ryzhov 1980, Bodonyi & Smith 1981). The precise evolution of the disturbances depends *inter alia* on where nonlinear effects become important. As an example, let us consider the flow of an incompressible fluid of kinematic viscosity $\nu$ at speed $U_\infty$ past an aligned flat plate. We suppose that a disturbance of dimensional frequency $\Omega$ is introduced in the vicinity of its lower-branch neutral point, with

$$F \equiv \Omega \nu U_\infty^{-2} \ll 1 \ . \tag{1.1}$$

If the disturbance is relatively large initially, it may become nonlinear while still in the 'lower-branch region', that is, at distances $x^*$ from the leading edge such that the local (chord-length) Reynolds number $U_\infty x^* \nu^{-1}$ is $O(F^{-\frac{4}{3}})$. A smaller disturbance might remain linear until the 'upper-branch region', where $U_\infty x^* \nu^{-1}$ is $O(F^{-\frac{5}{3}})$, while an even smaller disturbance might never become nonlinear, and simply decay after passing its upper-branch neutral point. These regions are indicated schematically in Figure 1.

Other than close to the lower-branch neutral curve (say at point B in Figure 1), the spatial growth-rate of the disturbance in the lower-branch region (say at point C) is comparable, in an asymptotic sense, with the real wavenumber of the disturbance (although in numerical terms the growth-rate may be relatively small). This makes an asymptotic description of nonlinear effects difficult if not impossible (e.g. Hall 1995). Analytical progress is much more feasible in the *upper-branch* scaling régime (at point E, say) since here the TS waves are neutral to leading order, and governed by inviscid dynamics (e.g. Bodonyi & Smith 1981). Amplitude growth, which is a viscous effect, takes place over a lengthscale/timescale long compared to the wavelength/period of the wave. This makes possible the use of *weakly nonlinear theory* to describe the evolution of small-amplitude TS waves. Study of the upper-branch régime would also seem to have practical relevance since in the experiments of Klebanoff *et al.* (1962) and others, nonlinear effects were observed to become significant near the upper branch rather than in the vicinity of the lower branch. Indeed, at asymptotically large Reynolds numbers the upper-branch scaling applies to almost the entire region of linear instability (Goldstein & Durbin 1986). On the other hand, Healey (1995) has noted that a naïve application of asymptotic theory does not predict the upper-branch neutral curve accurately at those (finite) Reynolds numbers that are expected to be of most interest (see also Bodonyi & Smith 1981, Cowley & Wu 1994).

As an alternative to studying the upper-branch régime, it is possible to consider the region of parameter space intermediate between the lower- and upper-branch scalings. This has conventionally been referred to as the 'high-frequency lower branch' (HFLB) régime. In the context of a fixed-frequency disturbance propagating downstream it is possibly better thought of as the 'far-downstream' limit of the lower-branch scaling, or equivalently the far-upstream limit of the upper-branch scaling (see Cowley & Wu 1994). In the HFLB régime, as in the upper-branch régime, the spatial (temporal) growth-rate is small compared with the wavenumber (frequency), thus making possible a weakly nonlinear approach (Smith & Burggraf 1985); indeed, as far as three-dimensional disturbances are concerned there would seem to be little difference, in general, between the HFLB and the upper branch proper (as shown for certain types of disturbance by Wu, Stewart & Cowley 1996, Jennings 1997). For two-dimensional disturbances, however, the onset of weakly nonlinear behaviour is quite different. In the upper-branch régime, nonlinearity acts initially by altering the critical-layer velocity jump which is produced by the curvature of the basic velocity profile there (e.g. Goldstein, Durbin & Leib 1987). In the HFLB régime, on the other hand, the critical layer is closer to the wall in a region where the basic-flow curvature is negligible; as a result the first effect of nonlinearity is felt outside the critical layer. We note in passing that there is, of course, an intermediate régime where the two effects are of equal importance. (For three-dimensional disturbances, there is an additional contribution to the critical-layer velocity jump which is curvature-independent, and this may be expected to dominate the dynamics in most cases except when the three-dimensionality is very weak.)

The weakly nonlinear development of a two-dimensional TS wave in the HFLB régime was investigated by Smith & Burggraf (1985), who found that nonlinearity initially affects only the phase of the wave, and not its amplitude. They proposed that the next distinct stage of development is fully nonlinear and is governed at leading order by the well-known [Davis-Acrivos-]Benjamin-Ono equation

$$\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x} + \frac{1}{\pi}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}\frac{A}{x-x'}\,dx' = 0 \tag{1.2}$$

(Davis & Acrivos 1967, Benjamin 1967, Ono 1975) — henceforth referred to as the DABO equation. Viscosity is assumed to be significant only in a passive sublayer adjacent to the plate. This viscous sublayer produces a higher-order correction to the inviscid solution, and forces a slow growth of the wave. Essentially the same asymptotic structure had earlier been proposed by Zhuk & Ryzhov (1982) (see also Balagondar 1981); a related HFLB régime for plane Poiseuille flow was studied by Stephanoff *et al.* (1983), and subsequently by Smith & Burggraf (1985) and Lam (1988).

A number of explicit solutions to the DABO equation are known. In particular, Benjamin (1967) found both a soliton solution, and a two-parameter family of periodic travelling waves; the latter is readily generalised, *via* a Galilean tranformation, to a three-parameter family. For the HFLB régime, the small-amplitude limit of the latter correspond to linear TS waves, while the infinite-amplitude limit gives the soliton solution. Rothmayer & Smith (1987) note the similarity between this soliton and the 'spikes' observed in K-type transition experiments (see also Ryzhov 1990). There would thus appear to be some theoretical underpinning for the 'spike-solition' conjecture made independently by Borodulin & Kachanov (1988). Further details of both the experimental and the theoretical work, and a review of other related literature, can be found in Kachanov, Ryzhov & Smith (1993), Kachanov (1994) and Ryzhov & Bogdanova-Ryzhova (1997).

On the other hand, the inviscid DABO description must be supplemented by the viscous sublayer, which is governed by the classical boundary-layer equations with a prescribed slip velocity. That a solution of this problem exists is not guaranteed *a priori* for DABO solutions of finite amplitude. Indeed, Smith & Burggraf (1985) point out that "the sublayer ... can develop a singularity in time for increased amplitudes ... and provide a 'burst' of vorticity penetrating the more outer flow zone". Rothmayer & Smith (1987) attempted to establish if and when the sublayer problem has a solution. Their results appear to indicate that an attached sublayer can be fitted under the inviscid DABO travelling-wave solution for a certain range of values of the parameters. However, Kachanov *et al.* (1993) comment that 'The choice of a one-parameter family of solutions restricts the application to an analysis, most likely, of alternative or bypass ways of laminar-turbulent transition rather than the properties of flashes-spikes in the periodic oscillations which can give rise to the K-regime'.

An aim of the present paper is to establish, for the full three-parameter family of periodic DABO travelling-waves, the existence or otherwise of a solution for the viscous sublayer flow. The slow evolution of the waves is determined by the solution in this sublayer. Our approach is similar to that of Lam (1988), who solved an analogous sublayer problem for the case of HFLB TS waves in plane Poiseuille flow. For the sake of definiteness, we restrict attention to the evolution of disturbances that start out as infinitesimal TS waves in the HFLB régime; all perturbation quantities are supposed to vanish far upstream and/or at large negative times. In particular, nonlinear interactions in the viscous Stokes layer adjacent to the wall generate a mean-flow distortion (i.e. a steady-streaming flow) which diffuses into the outer flow. Over the timescale of wave growth, the diffusion distance is relatively small; thus the mean-flow distortion is confined to a *diffusion layer* intermediate between the Stokes layer and the outer inviscid region. In this respect, our analysis differs from that of Smith & Burggraf (1985) and Rothmayer & Smith (1987) who apparently assume that a mean-flow distortion is (somehow) produced throughout the inviscid layer. this mean flow is not made clear. For the analogous channel-flow problem, Lam (1988) notes that in the Smith-Burggraf (1985) formulation 'both the mean pressure gradient and mass flux must change slowly in time in a specified manner. However, this is a rather artificial restriction and it is more natural to fix one or the other. This can be done by introducing another slow time scale† and a further asymptotic layer' (i.e. the diffusion layer).

This paper is organised as follows. In §2 we summarize the well-known high-Reynolds-number theory of lower-branch TS waves in a Blasius boundary layer. Then in §3 we set the scene for a multiple-scales analysis of such waves in the far-downstream/high-frequency limit of the lower-branch régime, where the growth-rate is relatively small. The weakly nonlinear phase of evolution is examined in §4. The analysis proceeds along the lines of that in Smith & Burggraf (1985), but modified by the inclusion of the diffu-

† In other words, the expansions (5.1) of wavenumber $\alpha$ and wavespeed $c$ proceed in powers of $\epsilon$, and not in powers of $\epsilon^2$ as implied by Smith & Burggraf (1985).

sion layer to accommodate the nonlinearly generated mean-flow distortion. This diffusion layer takes different forms for a spatio-temporal analysis and for a pure spatial analysis. The latter problem is discussed in Appendix A, where it is shown that the next stage of development is somewhat different from that proposed by Smith & Burggraf (1985) and envisaged in §3. In the remainder of the text, we restrict attention to the (spatio-)temporal case. In §5 we formulate the problem for the fully nonlinear stage of development, and show how the evolution of the wave amplitude is related to the solution of a periodic boundary-layer problem governing the behaviour of a viscous sublayer. Numerical solutions of this boundary-layer problem are presented in §6, and the evolution of a purely temporally growing TS wave is determined. It is shown that the sublayer solution ceases to exist when the wave amplitude reaches a certain value. In §7 some conclusions are drawn, and in particular an implication of the analysis for the 'soliton' theory of spike formation is discussed. Finally, in Appendix B, we consider the weakly nonlinear evolution of a shorter wavepacket for which linear dispersion is significant, and show how the amplitude equation obtained by Smith (1986a) alters when the diffusion layer is taken into account.

## 2. Formulation: the lower-branch régime

For the sake of definiteness the basic flow is taken to be an incompressible Blasius boundary layer over an aligned flat plate (although the analysis can also be applied to boundary layers with pressure gradients). Cartesian coordinates $[x^*, y^*]$ are chosen with $x^*$ along the plate and $y^*$ in the normal direction, and the corresponding velocity components are denoted by $[u^*, v^*]$. At distances from the leading edge for which $U_\infty x^*/\nu \gg 1$, where $U_\infty$ is the free-stream velocity and $\nu$ the kinematic viscosity of the fluid, the basic flow is given asymptotically by

$$u^* \sim U_\infty f_B'(\eta) \ , \quad v^* \sim \frac{1}{2}\left(\frac{U_\infty \nu}{x^*}\right)^{1/2} [\eta f_B'(\eta) - f_B(\eta)] \ , \quad \eta = y^* \left(\frac{U_\infty}{\nu x^*}\right)^{1/2} \ , \qquad (2.1)$$

where the Blasius function $f_B$ is the solution of

$$\frac{1}{2} f_B f_B'' + f_B''' = 0 \ , \quad f_B(0) = f_B'(0) = 0 \ , \quad f_B'(\eta) \to 1 \quad \text{as} \quad \eta \to \infty \ , \qquad (2.2)$$

and has the property

$$f_B'(\eta) \sim \hat{\lambda}_1 \eta + \hat{\lambda}_4 \eta^4 \quad \text{as} \quad \eta \to 0 \ , \quad \text{with} \quad \hat{\lambda}_1 = 0.332... \quad \text{and} \quad \hat{\lambda}_4 = -\hat{\lambda}_1^2/48 \ . \qquad (2.3)$$

We suppose that a disturbance of small amplitude and (dimensional) frequency $\Omega$ is introduced into the flow. The evolution of such a disturbance, in the linear stability régime, is conventionally analysed using so-called Orr-Sommerfeld theory, whereby the streamwise variation of the basic flow is neglected in calculating a local wavenumber and spatial growth-rate (the 'parallel-flow approximation'). While this approach gives results in good agreement with experiment (see for example Ross *et al.* 1970, Klingmann *et al.* 1993), it can only be justified in a strict asymptotic sense when the non-dimensional frequency parameter defined by (1.1) is small. As discussed in §1, if such a disturbance is introduced upstream of the region of instability (say at point A in Figure 1) it initially decays as it propagates downstream, and only starts to grow when it crosses the 'lower-branch' neutral curve at a distance from the leading edge for which

$$U_\infty x^*/\nu = O\left(F^{-4/3}\right) \ . \qquad (2.4)$$

It is convenient to define a reference length $L^*$ and a Reynolds number $Re$ by

$$U_\infty L^*/\nu = Re = F^{-4/3} \ . \tag{2.5}$$

In the vicinity of the lower-branch neutral point, the local wavelength is of order

$$L^* Re^{-3/8} \tag{2.6}$$

(e.g. Reid 1965), which is short compared to the $O(L^*)$ length-scale over which the basic flow varies (though long compared to the local boundary-layer thickness, which is of order $L^* Re^{-1/2}$). Orr-Sommerfeld theory thus provides a self-consistent approximation, at least to leading-order. Moreover, as pointed out by Smith (1979) and Zhuk & Ryzhov (1980), the scaling (2.6) implies that the disturbance structure is described by the (unsteady) 'triple-deck' equations (Stewartson 1969, Stewartson & Williams 1969, Neiland 1969, Messiter 1970, Schneider 1974, Brown & Daniels 1975, Ryzhov 1977, Ryzhov & Terent'ev 1977). Over most of the boundary layer (the 'main deck') the perturbation flow is effectively inviscid and quasi-steady, and described to leading order by the 'displacement' solution

$$u^* = U_\infty \left[ f_B'(\eta) + Re^{-1/8} \tilde{A}(\tilde{t}, \tilde{x}, x) f_B''(\eta) + \ldots \right] \ , \tag{2.7}$$

$$v^* = -U_\infty Re^{-1/4} \tilde{A}_{\tilde{x}}(\tilde{t}, \tilde{x}, x) f_B'(\eta) + \ldots \ , \tag{2.8}$$

$$p^* = \rho U_\infty^2 Re^{-1/4} \tilde{p}(\tilde{t}, \tilde{x}, x) + \ldots \ , \tag{2.9}$$

where

$$\tilde{t} = \Omega t^* \equiv Re^{1/4} U_\infty t^*/L^* \ , \quad x = x^*/L^* \ , \quad \tilde{x} = Re^{3/8} x \ , \quad \eta = (Re/x)^{1/2} y^*/L^* \ . \tag{2.10}$$

Viscous and unsteady effects are confined to a thin sublayer (the 'lower deck') where

$$\tilde{y} = Re^{5/8} y^*/L^* \tag{2.11}$$

is of order one. After the rescaling

$$u^* = U_\infty Re^{-1/8} \tilde{u}(\tilde{t}, \tilde{x}, x, \tilde{y}) + \ldots \ , \quad v^* = U_\infty Re^{-3/8} \tilde{v}(\tilde{t}, \tilde{x}, x, \tilde{y}) + \ldots \ , \tag{2.12}$$

the governing equations for the lower deck are

$$\tilde{u}_{\tilde{x}} + \tilde{v}_{\tilde{y}} = 0 \ , \quad \tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} + \tilde{v}\tilde{u}_{\tilde{y}} = -\tilde{p}_{\tilde{x}} + \tilde{u}_{\tilde{y}\tilde{y}} \ . \tag{2.13}$$

(The $Re^{-1/4}$ scaling for the amplitude of the pressure disturbance was made so as to give a fully nonlinear problem in this layer.) Equations (2.13) are to be solved subject to the no-slip condition

$$\tilde{u} = \tilde{v} = 0 \quad \text{on} \quad \tilde{y} = 0 \ ,$$

and the requirement of matching to the main deck

$$\tilde{u} \sim \lambda \left( \tilde{y} + \tilde{A} \right) \quad \text{as} \quad \tilde{y} \to \infty \ , \quad \lambda = \hat{\lambda}_1 x^{-1/2} \ . \tag{2.14}$$

Finally, it is apparent from (2.8) that there is a blowing velocity of order $U_\infty Re^{-1/4}$ at the edge of the boundary layer, indicating the need for an 'upper deck' outside the boundary layer whose width is comparable with the streamwise lengthscale, that is,

$$y^* = L^* Re^{-3/8} \tilde{Y} \ , \quad \text{with } \tilde{Y} \text{ of order one.} \tag{2.15}$$

To leading order the perturbation flow in this region is inviscid and irrotational, and the pressure

$$p^* = \rho U_\infty^2 Re^{-1/4} \tilde{P}(\tilde{t}, \tilde{x}, x, \tilde{Y}) + \ldots \tag{2.16}$$

satisfies

$$\tilde{P}_{\tilde{x}\tilde{x}} + \tilde{P}_{\tilde{Y}\tilde{Y}} = 0\ , \quad \tilde{P}_{\tilde{Y}}\Big|_{\tilde{Y}=0} = \tilde{A}_{\tilde{x}\tilde{x}}\ , \quad \tilde{P} \to 0 \quad \text{as} \quad \tilde{Y} \to \infty\ . \tag{2.17}$$

The (unknown) wall pressure

$$\tilde{p}(\tilde{t}, \tilde{x}, x) \equiv \tilde{P}(\tilde{t}, \tilde{x}, x, 0) \tag{2.18}$$

is thus related to the (unknown) displacement $\tilde{A}$ by

$$\tilde{p}(\tilde{t}, \tilde{x}, x) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{1}{\tilde{x} - \tilde{x}'} \frac{\partial \tilde{A}}{\partial \tilde{x}'}\, d\tilde{x}'\ , \tag{2.19}$$

where the bar indicates a Cauchy principal value. It may be noted that to leading order, the dependence on the 'slow' streamwise variable $x$ is purely parametric, and enters only through the wall-shear $\lambda$.

The stability of the undisturbed solution

$$(\tilde{u}, \tilde{v}, \tilde{p}, \tilde{A}) = (\lambda \tilde{y}, 0, 0, 0) \tag{2.20}$$

to a small disturbance may be analysed by substituting

$$(\tilde{u} - \lambda\tilde{y}, \tilde{v}, \tilde{p}, \tilde{A}) = \Delta\, [\mathcal{U}(\tilde{y}), \mathcal{V}(\tilde{y}), \mathcal{P}(\tilde{y}), \mathcal{A}(\tilde{y})]\, \mathrm{e}^{\mathrm{i}\tilde{\alpha}\tilde{x} - \mathrm{i}\tilde{\omega}\tilde{t}} + c.c.\ , \tag{2.21}$$

in (2.13)–(2.19), where $\Delta$ is an infinitesimal parameter and *c.c.* denotes complex conjugate. This leads to the dispersion relation (e.g. Smith 1979, Zhuk & Ryzhov 1980)

$$\mathrm{Ai}'(\zeta) = \mathrm{e}^{\mathrm{i}\pi/6} \left(\frac{\tilde{\alpha}}{\lambda^{5/4}}\right)^{4/3} \int_{\zeta}^{\infty} \mathrm{Ai}(\zeta')\, d\zeta'\ , \quad \zeta = -\mathrm{e}^{\mathrm{i}\pi/6} \left(\frac{\tilde{\alpha}}{\lambda^{5/4}}\right)^{-2/3} \left(\frac{\tilde{\omega}}{\lambda^{3/2}}\right)\ , \tag{2.22}$$

where Ai is the Airy function (Abramowitz & Stegun 1964) and $-3\pi/2 < \arg\tilde{\alpha} < \pi/2$. In fact, (2.22) merely reproduces a familiar result from Orr-Sommerfeld theory (see Reid 1965 and references therein). For a disturbance of fixed frequency $\tilde{\omega}$, (2.22) may be solved numerically to determine the local value of (complex) wavenumber $\tilde{\alpha}$ as a function of the local wall-shear $\lambda$. There are an infinite number of such solutions (cf. Ryzhov & Zhuk 1980), the least stable being the 'Tollmien-Schlichting' mode for which it is found that

$$\mathrm{Im}(\tilde{\alpha}) \gtrless 0 \quad \text{for} \quad \tilde{\omega}\lambda^{-3/2} \lessgtr S_0 \equiv 2.297...\ , \tag{2.23}$$

where Im denotes imaginary part. The other modes have $\mathrm{Im}(\tilde{\alpha}) > 0$ for all $\lambda$, implying spatial decay everywhere. It follows from (2.14) and (2.23) that

$$x = \left(S_0^{4/3} \hat{\lambda}_1^2\right) \tilde{\omega}^{-4/3} \tag{2.24}$$

specifies (to leading order) the lower-branch neutral point for a disturbance of (scaled) frequency $\tilde{\omega}$; upstream of this point, the disturbance decays, while downstream it grows.

Far downstream of the lower-branch neutral point, that is, at large values of $x$, the wall-shear is small, i.e. $\lambda \ll 1$, and the Tollmien-Schlichting branch of the dispersion relation has $\zeta \gg 1$†; the Airy functions in (2.22) may accordingly be replaced by their asymptotic expansions for large argument to give

$$\tilde{\alpha} \sim \tilde{\omega}^{1/2}\lambda^{1/2} - \tfrac{1}{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\tilde{\omega}^{-1/2}\lambda^2 + \dots\ . \tag{2.25}$$

The first term is real, and an imaginary part only enters at second order. This implies the existence of two length-scales: a 'fast' wavelength scale of order $\tilde{\omega}^{-1/2}\lambda^{-1/2}$, over which

† The other (decaying) modes have $\zeta$ of order one in this limit.

the disturbance is quasi-neutral, and a slower $O(\tilde{\omega}^{1/2}\lambda^{-2})$ scale which characterises the amplitude growth. As observed by Smith & Burggraf (1985), the asymptotic form (2.25) is also obtained in the 'high-frequency' limit $\tilde{\omega} \gg 1$ at fixed $\lambda$, that is, by concentrating on a fixed position on the plate (i.e. a fixed Reynolds number) and allowing the input frequency to be large compared to the frequency of the local lower-branch neutral mode. For such a 'high-frequency' disturbance, the first term of (2.25) indicates that the wavenumber is large compared to that of the local neutral mode. This 'short-wavelength' interpretation may be slightly misleading, however, since a fixed-frequency disturbance propagating downstream has wavelength *increasing* like $\lambda^{-1/2}$, albeit more slowly than the boundary-layer thickness, which grows like $\lambda^{-1}$ (see Cowley & Wu 1994).

We end this section by remarking that the result (2.25) is not valid for arbitrarily small $\lambda$ (or arbitrarily large $\tilde{\omega}$); rather, when $\lambda = O(F^{-1/6})$ (fixed frequency) or $\tilde{\omega} = O(Re^{3/20})$ (fixed Reynolds number), the second term of (2.25) is modified by the negative curvature of the Blasius profile and this ultimately causes the disturbance to stabilise at the 'upper-branch' neutral position. For a fixed-frequency disturbance, this corresponds to distances from the leading-edge such that

$$U_\infty x^*/\nu = O(F^{-5/3}) \ , \tag{2.26}$$

(e.g. Reid 1965). In what follows, however, we shall restrict attention to the intermediate régime where the dispersion relation (2.25) is applicable.

## 3. Far-downstream lower-branch: multiple-scales formulation for a nonlinear travelling wave

The relative smallness of the linear growth rate in the far-downstream ($\lambda \ll 1$) or high-frequency ($\tilde{\omega} \gg 1$) triple-deck makes possible the use of standard *weakly nonlinear* theory to describe the onset of nonlinearity. In particular, Smith & Burggraf (1985) show that nonlinear effects first become significant when the amplitude of the pressure $\tilde{p}$ rises to order $\lambda^{-3/2}$. They find, however, that at this stage nonlinearity affects only the phase of the wave, while the amplitude continues to grow exponentially, at the same rate as in linear theory. Smith & Burggraf (1985) propose that the next distinct phase of evolution will be fully nonlinear, with pressure of order $\lambda^{-3}\tilde{\omega}$, and evolving on the same length- and/or time-scales as before. In §4 below, we repeat their weakly nonlinear analysis, but with the inclusion of the diffusion layer. Our amplitude equation differs from that given by Smith & Burggraf (1985) but at least in the case of temporal evolution, their aforementioned conclusions still hold good. We therefore proceed directly to formulate the problem for the fully nonlinear stage of development; results for the earlier weakly nonlinear régime are readily recovered (§4). It should be emphasised at the outset, however, that at this stage the presentation is purely formal, and that solutions which match to initially small-amplitude waves will be presented only for the case of purely temporal evolution. Indeed, for the purely spatial case the analysis in Appendix A appears to indicate that a nonlinear régime of the form below does not follow on from the weakly nonlinear phase. Only preliminary results have been found as yet for the fully nonlinear evolution of a disturbance modulated in both space and time, and these are not reported here.

We thus suppose, in the first instance, that the fully nonlinear problem may be characterised by the two length- and/or time-scales identified in the linear régime. 'Slow' variables $X$ and $T$ are accordingly defined by

$$\tilde{x} = \lambda^{-2}\tilde{\omega}^{1/2}X \ , \quad \tilde{t} = \lambda^{-3/2}T \ , \tag{3.1$a,b$}$$

while the 'fast' length and time scales are of order $\lambda^{-1/2}\tilde{\omega}^{-1/2}$ and $\tilde{\omega}^{-1}$ respectively, that is, faster by a factor

$$\epsilon^2 \equiv \lambda^{3/2}\tilde{\omega}^{-1} \ . \tag{3.2}$$

Further we assume that the disturbance takes the form of a slowly modulated travelling wave, with 'fast' dependence occurring only through the phase variable

$$\epsilon^{-2}\theta(X,T) \equiv \sigma \ . \tag{3.3}$$

The derivatives thus transform as

$$\frac{\partial}{\partial \tilde{x}} \to \lambda^{1/2}\tilde{\omega}^{1/2}\left[\alpha\frac{\partial}{\partial\sigma} + \epsilon^2\frac{\partial}{\partial X}\right] \ , \quad \frac{\partial}{\partial \tilde{t}} \to \tilde{\omega}\left[-\alpha c\frac{\partial}{\partial\sigma} + \epsilon^2\frac{\partial}{\partial T}\right] \ , \tag{3.4}$$

where the local wavenumber $\alpha(X,T)$ and wavespeed $c(X,T)$ are defined by

$$\alpha = \theta_X \ , \quad \alpha c = -\theta_T \ , \tag{3.5}$$

and satisfy the compatibility condition

$$\alpha_T + (\alpha c)_X = 0 \ . \tag{3.6}$$

In addition to (3.4), we make the substitutions

$$\tilde{y} = \tilde{\omega}^{1/2}\lambda^{-3/2}\hat{y} \ , \quad \tilde{u} = \tilde{\omega}^{1/2}\lambda^{-1/2}\hat{u} \ , \quad \tilde{v} = \tilde{\omega}^{3/2}\lambda^{-3/2}\hat{v} \ , \tag{3.7}$$

$$\tilde{p} = \tilde{\omega}\lambda^{-1}p \quad \text{and} \quad \tilde{A} = \tilde{\omega}^{1/2}\lambda^{-3/2}A, \tag{3.8}$$

in (2.13)–(2.14) to obtain the rescaled lower-deck problem

$$\alpha\hat{u}_\sigma + \epsilon^2\hat{u}_X + \hat{v}_{\hat{y}} = 0 \ , \tag{3.9}$$

$$\alpha(\hat{u}-c)\hat{u}_\sigma + \epsilon^2\left(\hat{u}_T + \hat{u}\hat{u}_X\right) + \hat{v}\hat{u}_{\hat{y}} = -\alpha p_\sigma - \epsilon^2 p_X + \epsilon^4\hat{u}_{\hat{y}\hat{y}} \ , \tag{3.10}$$

$$\hat{u} = 0 \ , \quad \hat{v} = 0 \quad \text{on} \quad \hat{y} = 0 \ , \quad \hat{u} \sim \hat{y} + A \quad \text{as} \quad \hat{y} \to \infty \ . \tag{3.11}$$

According to Zhuk & Ryzhov (1982) and Smith & Burggraf (1985), the 'far-downstream' lower deck described by (3.9)–(3.11) with $\epsilon \ll 1$ subdivides into two asymptotic layers. The outermost region, which will be referred to as the *Tollmien layer*, has $\hat{y}$ of order one and is quasi-inviscid. The leading-order solution in this region does not satisfy the no-slip condition at the wall, and hence the viscous term must be reinstated in a *Stokes layer* adjacent to the wall, where $\hat{y}$ is of order $\epsilon^2$ (from the unsteady-viscous balance $-\alpha c\hat{u}_\sigma \sim \epsilon^4\hat{u}_{\hat{y}\hat{y}}$). However, the Stokes layer gives rise to a $\sigma$-independent *steady-streaming* at its outer edge, and it is to be expected that this will diffuse into a region where $\epsilon^2\hat{u}_T \sim \epsilon^4\hat{u}_{\hat{y}\hat{y}}$. Thus, as indicated by Lam (1988) for the analogous channel-flow problem, and by Bowles, Caporn & Timoshin (1998) in the boundary-layer context†, a *diffusion layer* with $\hat{y}$ of order $\epsilon$ should also be included, and the original lower deck subdivides into the *three-layer* structure illustrated in figure 2. (This argument requires modification in the case of a purely spatial analysis, with $\partial/\partial T = 0$: see Appendix A.)

Finally, we consider the rescaling of the pressure-displacement interaction law (2.19). For a modulated wave described by the scales (3.4), we split the upper-deck pressure

† The need for a diffusion layer was also noted in Stewart & Smith 1992 (where it is termed the 'buffer zone'), Wu, Stewart & Cowley 1996 ('wall-buffer layer') and Jennings 1997 (ditto), as well as in the related upper-branch/Rayleigh-wave analyses of Mankbadi, Wu & Lee 1993 ('wall layer'), Wu 1993 (ditto) and Smith, Brown & Brown 1993 ('thicker wall layer'). These studies are all concerned with three-dimensional disturbances, however, for which this diffusion layer is found to play a purely passive rôle.

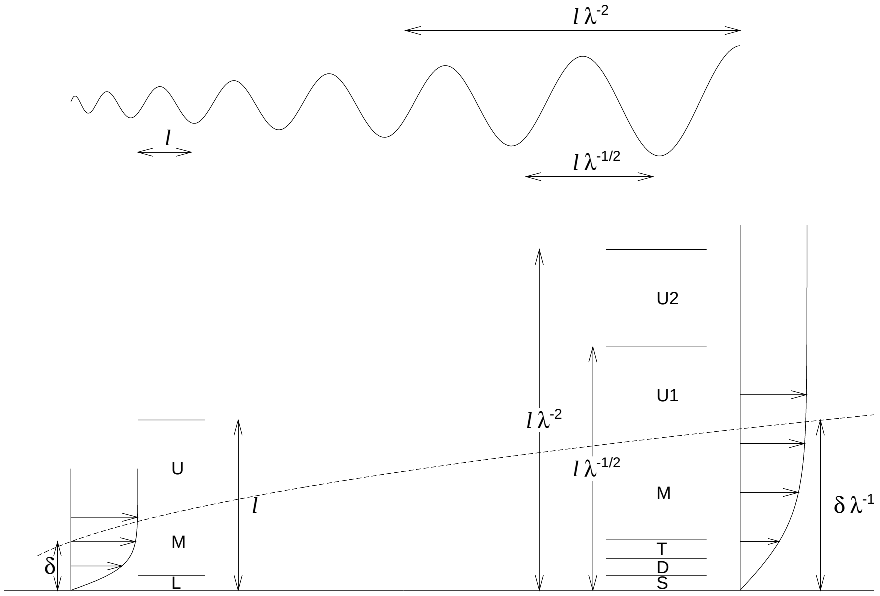


FIGURE 2. Schematic diagram (not to scale) of the evolution of a fixed-frequency disturbance in the lower-branch régime, and the emergence of a six-layer structure in the far-downstream limit. The left of the picture corresponds to the lower branch proper (e.g. point C in Figure 1); here the disturbance is characterised by a single horizontal length-scale $l = O(\delta Re^{1/8})$ where $\delta$ is the local boundary-layer thickeness, and a three-layer (triple-deck) structure with upper deck (U) of width $O(l)$, main deck (M) of width $O(\delta)$ and lower deck (L) of width $O(\delta Re^{-1/8})$. In the far-downstream limit ($\lambda \ll 1$), corresponding to points such as D in Figure 1, the oscillation wavelength increases downstream like $l\lambda^{-1/2}$, while the amplitude evolves on a slower scale $l\lambda^{-2}$; the upper deck accordingly subdivides into two regions U1 and U2 whose widths are comparable with the wavelength and evolution lengthscale respectively. The lower deck subdivides into a viscous Stokes layer (S), an inviscid Tollmien layer (T), and an intermediate diffusion layer (D), as described in the text.

$\tilde{P}$ into its mean ($\sigma$-independent) and fluctuating parts, and anticipate that the latter will tend to zero over a $\tilde{Y}$-scale of order $\lambda^{-1/2}\tilde{\omega}^{-1/2}$, comparable with the wavelength, while the former will tend to zero over a slower $O(\lambda^{-2}\tilde{\omega}^{1/2})$ scale comparable with the modulation lengthscale (cf. Hocking 1975, Balagondar, Maslowe & Melkonian 1987). Accordingly we write

$$\tilde{P} = \lambda^{-1}\tilde{\omega}\left[P^{(f)}(\sigma, T, X, Y) + P^{(m)}(T, X, \bar{Y})\right] , \tag{3.12}$$

where

$$Y = \lambda^{1/2}\tilde{\omega}^{1/2}\tilde{Y} , \quad \bar{Y} = \lambda^2\tilde{\omega}^{-1/2}\tilde{Y} = \epsilon^2 Y , \quad \text{and} \quad \int_0^{2\pi} P^{(f)} d\sigma = 0 . \tag{3.13}$$

On substituting (3.12) into (2.17)–(2.18) and separating mean and fluctuating components, we obtain

$$\alpha^2 P^{(f)}_{\sigma\sigma} + P^{(f)}_{YY} = -2\epsilon^2\alpha P^{(f)}_{X\sigma} - \epsilon^2\alpha_X P^{(f)}_{\sigma} - \epsilon^4 P^{(f)}_{XX} , \quad P^{(f)} \to 0 \quad \text{as} \quad Y \to \infty , \tag{3.14}$$

$$P^{(f)}_Y(\sigma, T, X, 0) = \alpha^2 A_{\sigma\sigma} + 2\epsilon^2\alpha A_{X\sigma} + \epsilon^2\alpha_X A_\sigma + \epsilon^4 \left[A - \langle A\rangle\right]_{XX} \tag{3.15}$$

for the fluctuating part, and

$$P^{(m)}_{XX} + P^{(m)}_{\bar{Y}\bar{Y}} = 0 \;, \quad P^{(m)} \to 0 \quad \text{as} \quad \bar{Y} \to \infty \;, \tag{3.16}$$

$$P^{(m)}_{\bar{Y}}(T, X, 0) = \epsilon^2 \langle A \rangle_{XX} \tag{3.17}$$

for the mean part, where $<>$ denotes the average over a period in $\sigma$:

$$\langle A \rangle (T, X) = \frac{1}{2\pi} \int_0^{2\pi} A(\sigma, T, X) d\sigma \;. \tag{3.18}$$

The rescaled wall pressure, namely

$$p = P^{(f)}(\sigma, T, X, 0) + P^{(m)}(T, X, 0) \tag{3.19}$$

is thus given by

$$p = \frac{1}{\pi} \left( \alpha \frac{\partial}{\partial \sigma} + \epsilon^2 \frac{\partial}{\partial X} \right) \fint_{-\infty}^{\infty} \frac{A(\sigma', T, X)}{\sigma - \sigma'} d\sigma' + \frac{\epsilon^2}{\pi} \frac{\partial}{\partial X} \fint_{-\infty}^{\infty} \frac{\langle A \rangle (T, X')}{X - X'} dX' + O(\epsilon^4) \;. \tag{3.20}$$

With the subdivision of the upper deck implied by (3.13), the far-downstream lower-branch régime is chararcterised by a six-layer structure in all, as shown in Figure 2 (except for a purely temporal evolution, in which case there is no need for the outermost region U2 which is associated with the slow spatial evolution scale).

## 4. Weakly nonlinear (spatio-)temporal evolution

From standard weakly nonlinear theory (see for example Stuart 1960), we expect that as a result of cubic interactions, nonlinearity first comes into play when the (scaled) disturbance quantities appearing in (3.9)–(3.11) have amplitude of order $\epsilon$. According to Smith & Burggraf (1985), the evolution of the disturbance at this stage is governed by the Landau equation, with a purely imaginary coefficient for the nonlinear term. We now indicate how their analysis should be modified to incorporate the diffusion layer, and obtain a different amplitude equation governing the weakly nonlinear stage of development, namely (4.47)–(4.48) below.

In this section, we shall follow the presentation and notation of Smith & Burggraf (1985) as closely as possible, and expand

$$p = \epsilon \left[ p_{11}(T, X) \mathrm{e}^{\mathrm{i}\sigma} + c.c. \right] + \epsilon^2 \left[ p_{22} \mathrm{e}^{2\mathrm{i}\sigma} + c.c. + p_{2M} \right] + \epsilon^3 \left[ p_{31} \mathrm{e}^{\mathrm{i}\sigma} + \ldots \right] + \ldots \;, \tag{4.1}$$

$$A = \epsilon \left[ A_{11}(T, X) \mathrm{e}^{\mathrm{i}\sigma} + c.c. \right] + \epsilon^2 \left[ A_{22} \mathrm{e}^{2\mathrm{i}\sigma} + c.c. + A_{2M} \right] + \epsilon^3 \left[ A_{31} \mathrm{e}^{\mathrm{i}\sigma} + \ldots \right] + \ldots \;, \tag{4.2}$$

where $p_{11}$, $A_{11}$, etc. are complex, and

$$\sigma = \epsilon^{-2} \left( \alpha X - T \right) \;, \tag{4.3}$$

with $\alpha$ a constant. From the pressure-displacement interaction law (3.20) we obtain

$$p_{11} = \alpha A_{11} \;, \quad p_{22} = 2\alpha A_{22} \;, \quad p_{2M} = 0 \;, \quad p_{31} = \alpha A_{31} - \mathrm{i} \frac{\partial A_{11}}{\partial X} \;. \tag{4.4$a,b,c,d$}$$

(In fact, there would seem to be no loss of generality in setting $A_{31}$, and all higher-order terms in $\mathrm{e}^{\mathrm{i}\sigma}$ in the $A$-expansion, to zero.)

We now consider each of the three lower-deck sublayers identified in the previous section.

(*a*) In the *Stokes layer*

$$\hat{y} = \epsilon^2 \bar{y} \ , \tag{4.5}$$

and

$$\begin{aligned} \hat{u} &= \epsilon \left[\bar{u}_{11}(T,X,\bar{y})\mathrm{e}^{\mathrm{i}\sigma} + c.c.\right] + \epsilon^2 \left[\bar{u}_{22}\mathrm{e}^{2\mathrm{i}\sigma} + c.c. + \bar{y} + \bar{u}_{2M}\right] \\ &\quad + \epsilon^3 \left[\bar{u}_{31}\mathrm{e}^{\mathrm{i}\sigma} + \bar{u}_{33}\mathrm{e}^{3\mathrm{i}\sigma} + c.c.\right] + \dots \ , \end{aligned} \tag{4.6}$$

$$\begin{aligned} \hat{v} &= \epsilon^3 \left[\bar{v}_{11}(T,X,\bar{y})\mathrm{e}^{\mathrm{i}\sigma} + c.c.\right] + \epsilon^4 \left[\bar{v}_{22}\mathrm{e}^{2\mathrm{i}\sigma} + c.c.\right] + \epsilon^5 \left[\bar{v}_{31}\mathrm{e}^{\mathrm{i}\sigma} + \bar{v}_{33}\mathrm{e}^{3\mathrm{i}\sigma} + c.c.\right] \\ &\quad + \epsilon^6 \left[\bar{v}_{4M} + \dots\right] + \dots \ . \end{aligned} \tag{4.7}$$

Substitution into the continuity and momentum equations gives, exactly as in Smith & Burggraf (1985),

$$\mathrm{i}\alpha\bar{u}_{11} + \frac{\partial \bar{v}_{11}}{\partial \bar{y}} = 0 \ , \quad -\mathrm{i}\bar{u}_{11} = -\mathrm{i}\alpha p_{11} + \frac{\partial^2 \bar{u}_{11}}{\partial \bar{y}^2} \ , \tag{4.8}$$

$$\frac{\partial \bar{u}_{2M}}{\partial X} + \frac{\partial \bar{v}_{4M}}{\partial \bar{y}} = 0 \ , \quad \bar{v}_{11}^{*}\frac{\partial \bar{u}_{11}}{\partial \bar{y}} + c.c. = \frac{\partial^2 \bar{u}_{2M}}{\partial \bar{y}^2} \ , \tag{4.9}$$

and so on, which are to be solved subject to the no-slip condition

$$\bar{u}_{11} = \bar{v}_{11} = \bar{u}_{2M} = \bar{v}_{4M} = \dots = 0 \quad \text{on} \quad \bar{y} = 0 \ . \tag{4.10}$$

It follows that

$$\bar{u}_{11} = \alpha p_{11}\left[1 - \mathrm{e}^{m\bar{y}}\right] \ , \quad \bar{v}_{11} = -\mathrm{i}\alpha^2 p_{11}\left[\bar{y} - m^{-1}\mathrm{e}^{m\bar{y}} + m^{-1}\right] \ , \tag{4.11}$$

where $m = \mathrm{e}^{3\pi\mathrm{i}/4}$, and

$$\begin{aligned} \bar{u}_{2M} = \alpha^3 |p_{11}|^2 \Big[ & 3 + \left(2 - \sqrt{2}\bar{y}\right)\mathrm{e}^{-\bar{y}/\sqrt{2}}\sin\left(\bar{y}/\sqrt{2}\right) \\ & - \left(4 + \sqrt{2}\bar{y}\right)\mathrm{e}^{-\bar{y}/\sqrt{2}}\cos\left(\bar{y}/\sqrt{2}\right) + \mathrm{e}^{-\sqrt{2}\bar{y}}\Big] \ ; \end{aligned} \tag{4.12}$$

no further terms are required for the present purpose.

(*b*) The *diffusion layer* has

$$\hat{y} = \epsilon y \ . \tag{4.13}$$

The appropriate expansions are

$$\begin{aligned} \hat{u} &= \epsilon \left[u_{11}(T,X,y)\mathrm{e}^{\mathrm{i}\sigma} + c.c. + y\right] + \epsilon^2 \left[u_{21}\mathrm{e}^{\mathrm{i}\sigma} + u_{22}\mathrm{e}^{2\mathrm{i}\sigma} + c.c. + u_{2M}\right] \\ &\quad + \epsilon^3 \left[u_{31}\mathrm{e}^{\mathrm{i}\sigma} + u_{32}\mathrm{e}^{2\mathrm{i}\sigma} + u_{33}\mathrm{e}^{3\mathrm{i}\sigma} + c.c. + u_{3M}\right] + \dots \ , \\ \hat{v} &= \epsilon^2 \left[v_{11}(T,X,y)\mathrm{e}^{\mathrm{i}\sigma} + c.c.\right] + \epsilon^3 \left[v_{21}\mathrm{e}^{\mathrm{i}\sigma} + v_{22}\mathrm{e}^{2\mathrm{i}\sigma} + c.c.\right] \\ &\quad + \epsilon^4 \left[v_{31}\mathrm{e}^{\mathrm{i}\sigma} + v_{32}\mathrm{e}^{2\mathrm{i}\sigma} + v_{33}\mathrm{e}^{3\mathrm{i}\sigma} + c.c.\right] + \epsilon^5 \left[v_{5M} + \dots\right] + \dots \ . \end{aligned} \tag{4.14}$$

The continuity and momentum equations take the form

$$\mathrm{i}\alpha u_{11} + \frac{\partial v_{11}}{\partial y} = 0 \ , \quad -\mathrm{i}u_{11} = -\mathrm{i}\alpha p_{11} \ , \tag{4.15}$$

$$\mathrm{i}\alpha u_{21} + \frac{\partial v_{21}}{\partial y} = 0 \ , \quad -\mathrm{i}u_{21} + \mathrm{i}\alpha y u_{11} + v_{11} = 0 \ , \tag{4.16}$$

$$2\mathrm{i}\alpha u_{22} + \frac{\partial v_{22}}{\partial y} = 0 \ , \quad -2\mathrm{i}u_{22} + \mathrm{i}\alpha u_{11}^2 + v_{11}\frac{\partial u_{11}}{\partial y} = -2\mathrm{i}\alpha p_{22} \ , \tag{4.17}$$

$$\mathrm{i}\alpha u_{31} + \frac{\partial u_{11}}{\partial X} + \frac{\partial v_{31}}{\partial y} = 0 \ , \tag{4.18}$$

$$-\mathrm{i}u_{31}+\frac{\partial u_{11}}{\partial T}+\mathrm{i}\alpha y u_{21}+v_{21}+\mathrm{i}\alpha u_{22}u_{11}^{*}+\mathrm{i}\alpha u_{2M}u_{11}+v_{11}^{*}\frac{\partial u_{22}}{\partial y}+v_{22}\frac{\partial u_{11}^{*}}{\partial y}+v_{11}\frac{\partial u_{2M}}{\partial y}$$
$$=-\mathrm{i}\alpha p_{31}-\frac{\partial p_{11}}{\partial X}+\frac{\partial^2 u_{11}}{\partial y^2}\ , \tag{4.19}$$

which may be solved subject to matching with the Stokes layer to give

$$u_{11}=\alpha p_{11}\ ,\quad v_{11}=-\mathrm{i}\alpha^2 p_{11}y\ ,\quad u_{21}=0\ ,\quad v_{21}=\mathrm{e}^{3\pi\mathrm{i}/4}\alpha^2 p_{11}\ , \tag{4.20}$$

$$u_{22}=\alpha\left(p_{22}+\tfrac{1}{2}\alpha^2 p_{11}^2\right)\ ,\quad v_{22}=-2\mathrm{i}\alpha^2\left(p_{22}+\tfrac{1}{2}\alpha^2 p_{11}^2\right)y\ , \tag{4.21}$$

$$u_{31}=\alpha p_{31}-\mathrm{i}\left(\alpha\frac{\partial p_{11}}{\partial T}+\frac{\partial p_{11}}{\partial X}\right)+\alpha^2 p_{11}\left(\mathrm{e}^{\mathrm{i}\pi/4}+u_{2M}-y\frac{\partial u_{2M}}{\partial y}\right)$$
$$+\alpha^2 p_{11}^{*}\left(p_{22}+\tfrac{1}{2}\alpha^2 p_{11}^2\right)\ , \tag{4.22}$$

and so on. The leading-order mean-flow distortion is found to satisfy

$$\frac{\partial u_{2M}}{\partial T}-\frac{\partial^2 u_{2M}}{\partial y^2}=-\Big[u_{11}^{*}\left(\mathrm{i}\alpha u_{31}\right)+u_{21}^{*}\left(\mathrm{i}\alpha u_{21}\right)+u_{31}^{*}\left(\mathrm{i}\alpha u_{31}\right)+u_{22}^{*}\left(2\mathrm{i}\alpha u_{22}\right)$$
$$+u_{11}^{*}\frac{\partial u_{11}}{\partial X}+v_{11}^{*}\frac{\partial u_{31}}{\partial y}+v_{21}^{*}\frac{\partial u_{21}}{\partial y}+v_{31}^{*}\frac{\partial u_{11}}{\partial y}+v_{22}^{*}\frac{\partial u_{22}}{\partial y}+c.c.\Big]$$
$$=-\alpha^2\frac{\partial|p_{11}^2|}{\partial X}\ , \tag{4.23}$$

with

$$u_{2M}|_{y=0}=3\alpha^3|p_{11}|^2 \tag{4.24}$$

in order to match to the Stokes layer, and $u_{2M}$ bounded as $y\to\infty$. For definiteness, we take the initial condition for a (spatio-)temporally modulated disturbance to be

$$u_{2M}\to 0\quad \text{as}\quad T\to-\infty\ , \tag{4.25}$$

in which case

$$u_{2M}=A_{2M}+\int_0^\infty \frac{y\mathrm{e}^{-y^2/4s}}{(4\pi s^3)^{1/2}}f(T-s,X)ds\ , \tag{4.26}$$

with

$$A_{2M}=-\alpha^2\frac{\partial}{\partial X}\int_{-\infty}^{T}|p_{11}|^2(T',X)dT'\ ,\quad f(T,X)=3\alpha^3|p_{11}|^2-A_{2M}\ . \tag{4.27}$$

($c$) The *Tollmien layer* has $\hat{y}$ of order one. The velocity components expand in the form

$$\hat{u}=\hat{y}+\epsilon\left[\hat{u}_{11}(T,X,\hat{y})\mathrm{e}^{\mathrm{i}\sigma}+c.c.\right]+\epsilon^2\left[\hat{u}_{22}\mathrm{e}^{2\mathrm{i}\sigma}+c.c.+\hat{u}_{2M}\right]$$
$$+\epsilon^3\left[\hat{u}_{31}\mathrm{e}^{\mathrm{i}\sigma}+\hat{u}_{33}\mathrm{e}^{3\mathrm{i}\sigma}+c.c\right]+\dots\ , \tag{4.28}$$

$$\hat{v}=\epsilon\left[\hat{v}_{11}(T,X,\hat{y})\mathrm{e}^{\mathrm{i}\sigma}+c.c.\right]+\epsilon^2\left[\hat{v}_{22}\mathrm{e}^{2\mathrm{i}\sigma}+c.c.\right]$$
$$+\epsilon^3\left[\hat{v}_{31}\mathrm{e}^{\mathrm{i}\sigma}+\hat{v}_{33}\mathrm{e}^{3\mathrm{i}\sigma}+c.c\right]+\epsilon^4\left[\hat{v}_{4M}+\dots\right]+\dots\ . \tag{4.29}$$

Substitution into the continuity and momentum equations gives

$$\mathrm{i}\alpha\hat{u}_{11}+\frac{\partial\hat{v}_{11}}{\partial\hat{y}}=0\ ,\quad \mathrm{i}\left(\alpha\hat{y}-1\right)\hat{u}_{11}+\hat{v}_{11}=-\mathrm{i}\alpha p_{11}\ , \tag{4.30}$$

$$2\mathrm{i}\alpha\hat{u}_{22}+\frac{\partial\hat{v}_{22}}{\partial\hat{y}}=0\ ,\quad 2\mathrm{i}\left(\alpha\hat{y}-1\right)\hat{u}_{22}+\hat{v}_{22}+\mathrm{i}\alpha\hat{u}_{11}^2+\hat{v}_{11}\frac{\partial\hat{u}_{11}}{\partial\hat{y}}=-2\mathrm{i}\alpha p_{22}\ , \tag{4.31}$$

$$\frac{\partial \hat{u}_{11}}{\partial X} + \mathrm{i}\alpha\hat{u}_{31} + \frac{\partial \hat{v}_{31}}{\partial \hat{y}} = 0 \ , \tag{4.32}$$

$$\frac{\partial \hat{u}_{11}}{\partial T} + \hat{y}\frac{\partial \hat{u}_{11}}{\partial X} + \mathrm{i}\left(\alpha\hat{y}-1\right)\hat{u}_{31} + \hat{v}_{31} + \mathrm{i}\alpha\hat{u}_{11}^{*}\hat{u}_{22} + \hat{v}_{11}^{*}\frac{\partial \hat{u}_{22}}{\partial \hat{y}} + \hat{v}_{22}\frac{\partial \hat{u}_{11}^{*}}{\partial \hat{y}}$$
$$+\mathrm{i}\alpha\hat{u}_{11}\hat{u}_{2M} + \hat{v}_{11}\frac{\partial \hat{u}_{2M}}{\partial \hat{y}} = -\mathrm{i}\alpha p_{31} - \frac{\partial p_{11}}{\partial X} \ , \tag{4.33}$$

from which we have

$$\frac{\partial \hat{u}_{11}}{\partial \hat{y}} = 0 \ , \quad \frac{\partial \hat{u}_{22}}{\partial \hat{y}} = 0 \ , \quad \frac{\partial \hat{u}_{31}}{\partial \hat{y}} = \frac{\mathrm{i}\hat{v}_{11}}{(\alpha\hat{y}-1)}\frac{\partial^2 \hat{u}_{2M}}{\partial \hat{y}^2} \quad \text{for} \quad \hat{y} \neq \alpha^{-1} \ . \tag{4.34}$$

The mean-flow distortions $\hat{u}_{2M}$, $\hat{v}_{4M}$ are found to satisfy

$$\frac{\partial \hat{u}_{2M}}{\partial X} + \frac{\partial \hat{v}_{4M}}{\partial \hat{y}} = 0 \ , \tag{4.35}$$

$$\frac{\partial \hat{u}_{2M}}{\partial T} + \hat{y}\frac{\partial \hat{u}_{2M}}{\partial X} + \hat{v}_{4M} = -\Big[\hat{u}_{11}^{*}\left(\mathrm{i}\alpha\hat{u}_{31}\right) + \hat{u}_{31}^{*}\left(\mathrm{i}\alpha\hat{u}_{31}\right) + \hat{u}_{22}^{*}\left(2\mathrm{i}\alpha\hat{u}_{22}\right)$$
$$+\hat{u}_{11}^{*}\frac{\partial \hat{u}_{11}}{\partial X} + \hat{v}_{11}^{*}\frac{\partial \hat{u}_{31}}{\partial \hat{y}} + \hat{v}_{31}^{*}\frac{\partial \hat{u}_{11}}{\partial \hat{y}} + \hat{v}_{22}^{*}\frac{\partial \hat{u}_{22}}{\partial \hat{y}} + c.c. + \frac{\partial p_{2M}}{\partial X}\Big] \ , \tag{4.36}$$

from which

$$\left(\frac{\partial}{\partial T} + \hat{y}\frac{\partial}{\partial X}\right)\frac{\partial \hat{u}_{2M}}{\partial \hat{y}} = 0 \quad \text{for} \quad \hat{y} \neq \alpha^{-1} \ . \tag{4.37}$$

On the assumption that all perturbation quantities go to zero as $T \to -\infty$ and/or $X \to -\infty$, it follows that

$$\frac{\partial \hat{u}_{2M}}{\partial \hat{y}} = 0 \quad \text{for} \quad \hat{y} \neq \alpha^{-1} \ . \tag{4.38}$$

We thus have that in $\hat{y} > \alpha^{-1}$

$$\hat{u}_{11} = A_{11} \ , \quad \hat{u}_{22} = A_{22} \ , \quad \hat{u}_{2M} = A_{2M} \ , \quad \hat{u}_{31} = A_{31} \ , \tag{4.39}$$

$$\hat{v}_{11} = -\mathrm{i}\alpha p_{11} - \mathrm{i}\left(\alpha\hat{y}-1\right)A_{11} \ , \quad \hat{v}_{22} = -2\mathrm{i}\alpha p_{22} - 2\mathrm{i}\left(\alpha\hat{y}-1\right)A_{22} - \mathrm{i}\alpha A_{11}^{2} \ , \tag{4.40}$$

$$\hat{v}_{31} = -\mathrm{i}\alpha p_{31} - \frac{\partial p_{11}}{\partial X} - \mathrm{i}\left(\alpha\hat{y}-1\right)A_{31} - \frac{\partial A_{11}}{\partial T} + \hat{y}\frac{\partial A_{11}}{\partial X} - \mathrm{i}\alpha A_{11}A_{2M} - \mathrm{i}\alpha A_{11}^{*}A_{22} \ , \tag{4.41}$$

$$\hat{v}_{4M} = -\frac{\partial |A_{11}|^2}{\partial X} - \frac{\partial A_{2M}}{\partial T} - \hat{y}\frac{\partial A_{2M}}{\partial X} \ , \tag{4.42}$$

and so on. The same solutions apply in $y < \alpha^{-1}$ but with $A_{11}$ replaced by $A_{11} - J_{11}$, etc. , where the 'velocity jumps' $J_{11}$, etc. are determined by the solution in the viscous *critical layer* where $\hat{y} - \alpha^{-1} = O(\epsilon^{4/3})$. An analysis of this layer (which we do not present) shows that the required velocity jumps are in fact zero, and thus the solutions (4.39)–(4.42) apply in $\hat{y} < \alpha^{-1}$ also.

We may now match the Tollmien-layer solutions to those in the diffusion layer, which requires in particular that

$$\hat{v}_{11} \to 0 \ , \quad \hat{v}_{22} \to 0 \ , \quad \hat{v}_{31} \to \mathrm{e}^{3\pi\mathrm{i}/4}\alpha^2 p_{11} \quad \text{as} \quad \hat{y} \to 0 \ . \tag{4.43}$$

From the first of these, we have that

$$A_{11} = \alpha p_{11} \ , \tag{4.44}$$

which combines with the pressure-displacement relation (4.4a) to give

$$\alpha = 1 \ . \tag{4.45}$$

The second of (4.43) in conjunction with (4.4b) gives

$$A_{22} = -\tfrac{1}{2}A_{11}^2 \ , \quad p_{22} = -A_{11}^2 \ . \tag{4.46}$$

Finally, by combining the last of (4.43) with (4.4d), and making use of (4.27) and (4.44)–(4.46), we obtain the amplitude-evolution equation

$$\frac{\partial A_{11}}{\partial T} + 2\frac{\partial A_{11}}{\partial X} = \mathrm{e}^{-\mathrm{i}\pi/4}A_{11} + \mathrm{i}A_{11}B \ , \tag{4.47}$$

where $B$ is a real quantity specified by

$$B = \frac{1}{2}|A_{11}|^2 + \frac{\partial}{\partial X}\int_{-\infty}^{T} |A_{11}|^2(T', X)dT' \ . \tag{4.48}$$

Two comments are in order. First, it is apparent that the foregoing analysis is not valid for a pure spatial evolution, with $\partial/\partial T = 0$, since then the integral in (4.48) does not exist. The modifications required to handle this case are outlined in Appendix A. Second, the amplitude equation (4.47)–(4.48) differs from that found by Smith & Burggraf (1985) who do not include the diffusion layer, but instead assume that the mean-flow correction $\hat{u}_{2M}$ is equal to the Stokes-layer steady-streaming value, $3|A_{11}|^2$, throughout the Tollmien layer. In consequence, their amplitude equation (for temporal and/or spatial modulation) takes the form (4.47) but with $B = -\frac{5}{2}|A_{11}|^2$. Bowles *et al.* (1998) do take account of the diffusion layer, and obtain (for a purely temporal evolution) the amplitude equation (4.47)–(4.48). It may be noted, however, that provided $B$ is real, its precise form does not affect the (temporal or spatio-temporal) evolution of the amplitude $|A_{11}|$, which satisfies

$$\frac{\partial |A_{11}|^2}{\partial T} + 2\frac{\partial |A_{11}|^2}{\partial X} = \sqrt{2}|A_{11}|^2 \ , \tag{4.49}$$

and thus grows exactly as in linear theory. On substituting for $\partial|A_{11}|^2/\partial X$ in the last term of (4.48), we may rewrite the amplitude equation in the form

$$\frac{\partial A_{11}}{\partial T} + 2\frac{\partial A_{11}}{\partial X} = \mathrm{e}^{-\mathrm{i}\pi/4}A_{11} + \frac{\mathrm{i}A_{11}}{\sqrt{2}}\int_{-\infty}^{T} |A_{11}|^2(T', X)dT' \ , \tag{4.50}$$

from which we obtain

$$\left(\frac{\partial}{\partial T} + 2\frac{\partial}{\partial X}\right)\arg A_{11} = \frac{1}{\sqrt{2}}\left[-1 + \int_{-\infty}^{T} |A_{11}|^2(T', X)dT'\right] \ . \tag{4.51}$$

The general solutions of (4.49) and (4.51) are

$$|A_{11}|^2 = \mathrm{e}^{\sqrt{2}T}h(\xi) \ , \quad \arg A_{11} = \phi(\xi) - \frac{T}{\sqrt{2}} + \frac{\mathrm{e}^{\sqrt{2}T}}{2\sqrt{2}}\int_0^{\infty} \mathrm{e}^{-\xi'/\sqrt{2}}h\,(\xi + \xi')\,d\xi' \ , \tag{4.52}$$

where $h$ and $\phi$ are arbitrary functions of the 'group-velocity' co-ordinate

$$\xi = X - 2T \ . \tag{4.53}$$

It follows that in a frame of reference moving with the group velocity ($\xi$ fixed), the amplitude is ultimately growing like $\mathrm{e}^{T/\sqrt{2}}$, and the phase like $\mathrm{e}^{\sqrt{2}T}$, without change of spatial scale. We might expect, therefore, that the next stage of development occurs when $A_{11}$ rises to $O(\epsilon^{-1/2})$, that is when the disturbance amplitude reaches $O(\epsilon^{1/2})$, and quintic nonlinearity comes into play. This régime is described by the variables

$$T = \frac{1}{\sqrt{2}} \ln \epsilon^{-1} + \tilde{T} \ , \quad \tilde{\sigma} = \epsilon^{-2} (X - T) + \frac{\epsilon^{-1}}{2\sqrt{2}} \int_0^\infty \mathrm{e}^{-\xi'/\sqrt{2}} h(\xi + \xi')\, d\xi' - \tfrac{1}{2} \ln \epsilon^{-1} \quad (4.54)$$

and expansions

$$A = \epsilon^{1/2} \left[ \tilde{A}_{11}(\tilde{T}, \xi) \mathrm{e}^{\mathrm{i}\tilde{\sigma}} + c.c. \right] + \dots \ , \quad \text{etc.} \quad (4.55)$$

Once again, however, the analysis leads to an amplitude equation of the form

$$\frac{\partial \tilde{A}_{11}}{\partial \tilde{T}} = \mathrm{e}^{-\mathrm{i}\pi/4} \tilde{A}_{11} + \mathrm{i} \tilde{A}_{11} \tilde{B} \ , \quad (4.56)$$

with $\tilde{B}$ a *real* quantity. It follows that the amplitude $|\tilde{A}_{11}|$ is still unaffected by nonlinearity, viz.

$$|\tilde{A}_{11}|^2 = \mathrm{e}^{\sqrt{2}\tilde{T}} h(\xi) \ . \quad (4.57)$$

The same behaviour is found at amplitudes of order $\epsilon^{1/3}$, $\epsilon^{1/4}$, etc. when higher-order nonlinearities enter the reckoning: at each stage nonlinearity affects only the phase, and the evolution length- and time-scales are unchanged. We conclude, as did Smith & Burggraf (1985) for their amplitude equation, that the next significant stage of development is fully nonlinear, and described by the length- and time-scales identified in §3 above.

## 5. Fully nonlinear stage: derivation of the evolution equations

The fully nonlinear phase of evolution is described by (3.9)–(3.11) and (3.20) with all variables expanded in powers of $\epsilon$, thus

$$(p, A, \alpha, c) = (p_0, A_0, \alpha_0, c_0) + \epsilon(p_1, A_1, \alpha_1, c_1) + \epsilon^2(p_2, A_2, \alpha_2, c_2) + \dots , \quad (5.1)$$

where the $p_i$ and $A_i$ are functions of $\sigma$, $T$ and $X$, and the $\alpha_i$ and $c_i$ are functions of $T$ and $X$. From the pressure-displacement interaction relation (3.20), we have that

$$p_0 = \frac{1}{\pi} \frac{\partial}{\partial \sigma} \fint_{-\infty}^{\infty} \frac{\alpha_0 A_0(\sigma', T, X)}{\sigma - \sigma'}\, d\sigma' \ , \quad p_1 = \frac{1}{\pi} \frac{\partial}{\partial \sigma} \fint_{-\infty}^{\infty} \frac{\alpha_0 A_1 + \alpha_1 A_0}{\sigma - \sigma'}\, d\sigma', \quad (5.2a,b)$$

$$p_2 = \frac{1}{\pi} \frac{\partial}{\partial \sigma} \fint_{-\infty}^{\infty} \frac{\alpha_0 A_2 + \alpha_1 A_1 + \alpha_2 A_0}{\sigma - \sigma'}\, d\sigma' + \frac{1}{\pi} \frac{\partial}{\partial X} \left[ \fint_{-\infty}^{\infty} \frac{A_0 d\sigma'}{\sigma - \sigma'} + \fint_{-\infty}^{\infty} \frac{\langle A_0 \rangle\, dX'}{X - X'} \right] . \quad (5.2c)$$

Each of the three lower-deck sublayers will now be considered in turn.

### 5.1. **Tollmien Layer**

The Tollmien layer has $\hat{y}$ of order one. Substitution of the expansions

$$(\hat{u}, \hat{v}) = (\hat{u}_0, \hat{v}_0) + \epsilon(\hat{u}_1, \hat{v}_1) + \epsilon^2(\hat{u}_2, \hat{v}_2) + \dots$$

into the continuity and momentum equations (3.9)–(3.10) gives for the first three orders

$$\alpha_0 \hat{u}_{0\sigma} + \hat{v}_{0\hat{y}} = 0 \ , \quad \left[ \tfrac{1}{2} \alpha_0 (\hat{u}_0 - c_0)^2 \right]_\sigma + \hat{v}_0 \hat{u}_{0\hat{y}} = -\alpha_0 p_{0\sigma}; \quad (5.3)$$

$$\alpha_0 \hat{u}_{1\sigma} + \alpha_1 \hat{u}_{0\sigma} + \hat{v}_{1\hat{y}} = 0 \ , \quad (5.4)$$

$$\left[ \alpha_0 (\hat{u}_0 - c_0)(\hat{u}_1 - c_1) + \tfrac{1}{2} \alpha_1 (\hat{u}_0 - c_0)^2 \right]_\sigma + \hat{v}_0 \hat{u}_{1\hat{y}} + \hat{v}_1 \hat{u}_{0\hat{y}} = -\alpha_0 p_{1\sigma} - \alpha_1 p_{0\sigma} \ ; \quad (5.5)$$

$$\alpha_0 \hat{u}_{2\sigma} + \alpha_1 \hat{u}_{1\sigma} + \alpha_2 \hat{u}_{0\sigma} + \hat{u}_{0X} + \hat{v}_{2\hat{y}} = 0 \ , \quad (5.6)$$

$$\left[\alpha_0(\hat{u}_0-c_0)(\hat{u}_2-c_2)+\tfrac{1}{2}\alpha_0(\hat{u}_1-c_1)^2+\alpha_1(\hat{u}_0-c_0)(\hat{u}_1-c_1)+\tfrac{1}{2}\alpha_2(\hat{u}_0-c_0)^2\right]_\sigma$$
$$+\hat{u}_{0T}+\hat{u}_0\hat{u}_{0X}+\hat{v}_0\hat{u}_{2\hat{y}}+\hat{v}_1\hat{u}_{1\hat{y}}+\hat{v}_2\hat{u}_{0\hat{y}}=-\alpha_0 p_{2\sigma}-\alpha_1 p_{1\sigma}-\alpha_2 p_{0\sigma}-p_{0X}\ . \tag{5.7}$$

The outer boundary condition requires

$$\hat{u}_0\sim\hat{y}+A_0\ ,\quad \hat{u}_1\to A_1\ ,\quad \hat{u}_2\to A_2\quad \text{as}\quad \hat{y}\to\infty\ , \tag{5.8}$$

and we expect that

$$\hat{v}_0\to 0\quad \text{as}\quad \hat{y}\to 0 \tag{5.9}$$

in order to match to the diffusion layer.

The problem (5.3)–(5.9) has the simple solution (Smith & Burggraf 1985, Lam 1988)

$$\hat{u}_0=\hat{y}+A_0,\qquad \hat{v}_0=-\hat{y}\alpha_0 A_{0\sigma},\qquad p_0=R_0-\tfrac{1}{2}(A_0-c_0)^2+\tfrac{1}{2}c_0^2\ ; \tag{5.10}$$

$$\hat{u}_1=A_1,\qquad \hat{v}_1=\hat{v}_{1b}-\hat{y}(\alpha_0 A_{1\sigma}+\alpha_1 A_{0\sigma})\ , \tag{5.11}$$

$$p_1=R_1-(A_0-c_0)(A_1-c_1)+c_0c_1-\frac{1}{\alpha_0}\int_0^\sigma \hat{v}_{1b}(\sigma',X,T)\,d\sigma'\ ; \tag{5.12}$$

$$\hat{u}_2=A_2,\qquad \hat{v}_2=\hat{v}_{2b}-\hat{y}(\alpha_0 A_{2\sigma}+\alpha_1 A_{1\sigma}+\alpha_2 A_{0\sigma}+A_{0X})\ , \tag{5.13}$$

$$\begin{aligned}p_2=R_2&-(A_0-c_0)(A_2-c_2)-\tfrac{1}{2}(A_1-c_1)^2+c_0c_2+\tfrac{1}{2}c_1^2\\&-\frac{1}{\alpha_0}\int_0^\sigma\left[A_{0T}+(c_0A_0)_X+R_{0X}+\hat{v}_{2b}-\frac{\alpha_1}{\alpha_0}\hat{v}_{1b}\right]d\sigma'\ .\end{aligned} \tag{5.14}$$

Here the $R_i(X,T)$ are at present arbitrary functions of integration, as are $\hat{v}_{1b}(\sigma,X,T)$, $\hat{v}_{2b}(\sigma,X,T)$. While we do not claim that this solution is unique, at least for a temporal evolution it can be matched at earlier times to the weakly nonlinear solution of the previous section; indeed it provides the unique continuation of that solution as long as viscous effects remain negligible and closed streamlines do not develop in a frame moving with the wave (see below).

The leading-order pressure $p_0$ can be eliminated between (5.10) and (5.2$a$) to give

$$B_0B_{0\sigma}+\frac{\alpha_0}{\pi}\frac{\partial^2}{\partial\sigma^2}\,⨍_{-\infty}^{\infty}\frac{B_0}{\sigma-\sigma'}\,d\sigma'=0\ ,\quad B_0=A_0-c_0\ , \tag{5.15}$$

which may be recognised as the travelling-wave reduction of the DABO equation (1.2). For our problem, we seek solutions which are $2\pi$-periodic and which may without loss of generality be taken to be even in $\sigma$. It was shown by Benjamin (1967) that (for any $\alpha_0$ and $c_0$) there is a one-parameter family of such solutions, namely

$$B_0=\alpha_0\left[\frac{1}{(1-b_0^2)^{1/2}}-\frac{2(1-b_0^2)^{1/2}}{1-b_0\cos\sigma}\right]\ , \tag{5.16}$$

where the arbitrary parameter $b_0$ is a measure of wave amplitude and must be allowed to depend on the slow scales $X$ and $T$ (as do $\alpha_0$ and $c_0$, we recall). The solution (5.16) is illustrated in Figure 3 for various representative amplitudes. For $b_0$ small

$$B_0\sim-\alpha_0\left[1+2b_0\cos\sigma\right]\ , \tag{5.17}$$

while as $b_0\to 1^-$, with $\sigma$ of order $(1-b_0)^{1/2}$,

$$B_0\sim\frac{\alpha_0}{2^{1/2}(1-b_0)^{1/2}}\left[1-\frac{8(1-b_0)}{\sigma^2+2(1-b_0)}\right]\ , \tag{5.18}$$

which is the soliton solution of the DABO equation. It may also be noted that $B_0$ is

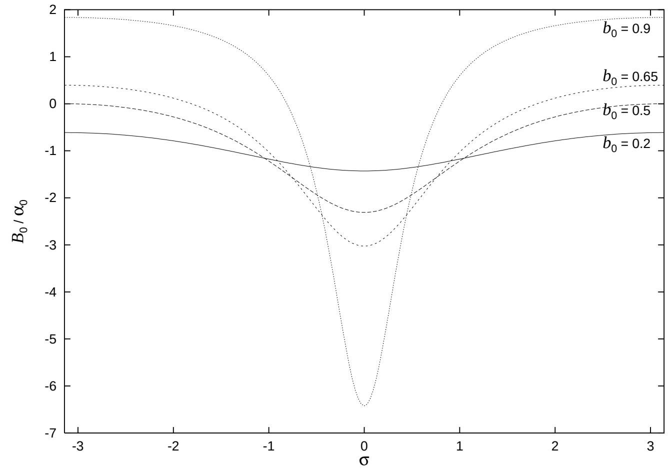


FIGURE 3. The periodic DABO solution (5.16) for various values of the amplitude $b_0$.

negative for all $\sigma$ if $|b_0| < 0.5$, but takes both positive and negative values if $|b_0| > 0.5$. In the latter case, this implies the presence of a region of closed streamlines in a frame of reference moving with the wavespeed, and the validity of the solution (5.10)–(5.14) is not immediately clear.

Proceeding to the next two orders, we may eliminate $p_1$ and $p_2$ between the inviscid-layer solutions (5.12), (5.14) and the interaction equations (5.2) to obtain

$$[B_0(A_i - c_i)]_\sigma + \frac{\alpha_0}{\pi}\frac{\partial^2}{\partial\sigma^2}\int_{-\infty}^{\infty}\frac{A_i}{\sigma-\sigma'}\,d\sigma' = F_i \quad \text{for} \quad i = 1, 2\ ,$$

where the right-hand sides are given by

$$F_1 = \frac{1}{\alpha_0}\hat{v}_{1b} - \frac{\alpha_1}{\pi}\frac{\partial^2}{\partial\sigma^2}\int_{-\infty}^{\infty}\frac{A_0}{\sigma-\sigma'}\,d\sigma'\ , \tag{5.19}$$

$$\begin{aligned} F_2 = &-\frac{1}{\alpha_0}\left[A_{0T} + (c_0A_0)_X + R_{0X} + \hat{v}_{2b} - \frac{\alpha_1}{\alpha_0}\hat{v}_{1b}\right] - \frac{1}{\pi}\frac{\partial^2}{\partial X\partial\sigma}\int_{-\infty}^{\infty}\frac{A_0(\sigma',T,X)}{\sigma-\sigma'}\,d\sigma' \\ &-\frac{1}{\pi}\frac{\partial^2}{\partial\sigma^2}\int_{-\infty}^{\infty}\frac{\alpha_1A_1+\alpha_2A_0}{\sigma-\sigma'}\,d\sigma' - [\tfrac{1}{2}(A_1-c_1)^2]_\sigma \qquad (5.20) \\ = &-\frac{1}{\alpha_0}\left[A_{0T} + 2\,(c_0A_0)_X - A_0A_{0X} + 2R_{0X} + \hat{v}_{2b} - \frac{\alpha_1}{\alpha_0}\hat{v}_{1b}\right] \\ &+\frac{\alpha_{0X}}{\alpha_0^2}\left[R_0 - \tfrac{1}{2}A_0^2 + c_0A_0\right] - \frac{1}{\pi}\frac{\partial^2}{\partial\sigma^2}\int_{-\infty}^{\infty}\frac{\alpha_1A_1+\alpha_2A_0}{\sigma-\sigma'}\,d\sigma' - [\tfrac{1}{2}(A_1-c_1)^2]_\sigma\ , \end{aligned}$$

the last following from (5.2$a$) and (5.10). In order that the equations for $A_1$ and $A_2$ have a solution, secularity conditions must be satisfied by their right-hand sides, namely that the inner products

$$<\phi F_i> \equiv \frac{1}{2\pi}\int_0^{2\pi}\phi F_i\,d\sigma\ ,$$

must vanish for any $2\pi$-periodic solution $\phi$ of the adjoint equation

$$B_0\phi_\sigma + \frac{\alpha_0}{\pi}\int_{-\infty}^{\infty}\frac{\phi_{ss}}{\sigma-s}\,ds = 0\ . \tag{5.21}$$

It is clear that equation (5.21) is satisfied by $\phi = 1$ and $\phi = A_0$; no other independent solution could be found. The inner products of the $A_1$-equation with $\phi = 1$ and $\phi = A_0$ impose the requirements that

$$\langle \hat{v}_{1b} \rangle = 0 \quad \text{and} \quad \langle A_0 \hat{v}_{1b} \rangle = 0 \ , \tag{5.22}$$

respectively. It will be evident from the diffusion-layer solution that these are indeed satisfied, and moreover that $\hat{v}_{1b}$ is an odd function of $\sigma$. It follows that $F_1$ is also odd in $\sigma$, and consequently $A_1$ may be taken to be an even function of $\sigma$. (It is possible to add an arbitrary multiple of $\partial B_0/\partial\sigma$ to $A_1$, but this odd term may be eliminated by an $O(\epsilon)$ shift in the origin of $\sigma$.)

The inner product of the $A_2$ equation with $\phi = 1$ leads to

$$\frac{\partial}{\partial T}\langle A_0 \rangle + \frac{\partial}{\partial X}\left[c_0 \langle A_0 \rangle\right] + \frac{\partial R_0}{\partial X} + \langle \hat{v}_{2b} \rangle = 0 \ , \tag{5.23}$$

and it will turn out that $\langle \hat{v}_{2b} \rangle = 0$. Further, it follows from (5.2) and (5.10) that

$$0 = \langle p_0 \rangle = -\tfrac{1}{2}\left\langle A_0^2 \right\rangle + c_0 \langle A_0 \rangle + R_0 \ , \tag{5.24}$$

and elimination of $R_0$ from (5.23) and (5.24) yields

$$\frac{\partial}{\partial T}\langle A_0 \rangle + \tfrac{1}{2}\frac{\partial}{\partial X}\left\langle A_0^2 \right\rangle = 0 \ . \tag{5.25}$$

Finally, the condition that $\langle A_0 F_2 \rangle = 0$ gives

$$\tfrac{1}{2}\frac{\partial D}{\partial T} + \left[c_0 + \tfrac{1}{2}M\right]\left(\frac{\partial D}{\partial X} - \frac{D}{\alpha_0}\frac{\partial \alpha_0}{\partial X}\right) + 2D\frac{\partial c_0}{\partial X} - \tfrac{1}{3}\frac{\partial S}{\partial X} + \frac{S}{2\alpha_0}\frac{\partial \alpha_0}{\partial X} = -\langle A_0 \hat{v}_{2b} \rangle \ , \tag{5.26}$$

where we write

$$M = \langle A_0 \rangle \ , \quad D = \left\langle A_0^2 \right\rangle - \langle A_0 \rangle^2 \ , \quad S = \left\langle A_0^3 \right\rangle - \langle A_0 \rangle^3 \ . \tag{5.27}$$

The right-hand side of (5.26) is determined by the solutions in the diffusion layer and Stokes layer, and in fact turns out to be a local function of $\alpha_0$, $c_0$ and $b_0$. With the substitutions

$$M = c_0 - \alpha_0 - e_0 \ , \quad D = 4\alpha_0 e_0 \ , \quad S = 12\alpha_0 e_0 \left(c_0 - \alpha_0\right) \ , \tag{5.28}$$

where

$$e_0 = \alpha_0 \left[\frac{1}{\left(1 - b_0^2\right)^{1/2}} - 1\right] \ , \tag{5.29}$$

(3.6), (5.25) and (5.26) provide three simultaneous equations for the evolution of the leading-order wavenumber $\alpha_0$, wavespeed $c_0$ and amplitude $b_0$ (or equivalently $e_0$), namely

$$\frac{\partial}{\partial T}\begin{pmatrix} \alpha_0 \\ c_0 \\ e_0 \end{pmatrix} + \begin{pmatrix} c_0 & \alpha_0 & 0 \\ \alpha_0 & c_0 & 0 \\ e_0 & e_0 & \alpha_0 + c_0 + e_0 \end{pmatrix} \frac{\partial}{\partial X}\begin{pmatrix} \alpha_0 \\ c_0 \\ e_0 \end{pmatrix} = \begin{pmatrix} 0 \\ -F \\ F \end{pmatrix} \tag{5.30}$$

where

$$F = -\frac{1}{2\alpha_0}\langle A_0 \hat{v}_{2b} \rangle \ . \tag{5.31}$$

The 'inviscid' form of this system, without the right-hand side of (5.26) or (5.30), has been obtained by Dobrokhotov & Krichever (1991), who also identify Riemann invariants

$$R_1 = c_0 + \alpha_0 \ , \quad R_2 = c_0 - \alpha_0 \ , \quad R_3 = c_0 + \frac{\alpha_0}{\left(1 - b_0^2\right)^{1/2}} \ . \tag{5.32}$$

These evolve according to

$$\frac{\partial R_1}{\partial T} + R_1 \frac{\partial R_1}{\partial X} = \frac{\partial R_2}{\partial T} + R_2 \frac{\partial R_2}{\partial X} = -F \ , \quad \frac{\partial R_3}{\partial T} + R_3 \frac{\partial R_3}{\partial X} = 0 \ . \qquad (5.33)$$

For purely spatial evolution, with $\partial/\partial T = 0$, it is apparent from (5.25) that matching to an upstream linear solution is not possible, since if $A_0 \to 0$ as $X \to -\infty$ then $A_0 \equiv 0$. In Appendix A, we examine the small-amplitude stage for the purely spatial case, and show that it is not expected to evolve to a fully nonlinear régime of the form assumed here. Rather, the mean-flow in the diffusion layer goes fully nonlinear and develops a Goldstein-type singularity while the wave-amplitude is still small. On the other hand, in the case of a purely temporal evolution, with $\partial/\partial X = 0$, the kinematic equation (3.6) states merely that $\alpha_0$ is constant (in fact we could take $\alpha_0 = 1$ without loss of generality). Further, (5.25) together with the requirement that $A_0 \to 0$ as $T \to -\infty$ gives

$$M \equiv \langle A_0 \rangle = 0 \ . \qquad (5.34)$$

It then follows from the first of (5.28) that

$$c_0 = \alpha_0 - e_0 \equiv \alpha_0 \left[ 2 - \frac{1}{(1 - b_0^2)^{1/2}} \right] \ , \qquad (5.35)$$

and hence

$$A_0 \equiv c_0 + B_0 = \alpha_0 \left[ 2 - \frac{2(1 - b_0^2)^{1/2}}{1 - b_0 \cos\sigma} \right] \ . \qquad (5.36)$$

Finally, on substituting (5.28) into (5.26) we obtain a single equation for the evolution of the amplitude $b_0$, namely

$$2\alpha_0 \frac{de_0}{dT} \equiv \frac{\alpha_0^2}{(1 - b_0^2)^{3/2}} \frac{d\left(b_0^2\right)}{dT} = -\left\langle A_0 \hat{v}_{2b} \right\rangle \ . \qquad (5.37)$$

The blowing velocity $\hat{v}_{2b} = \hat{v}_2|_{\hat{y}=0}$ will now be found by considering in turn the diffusion layer and the Stokes layer, thus enabling the right-hand side of (5.26) or (5.37) to be calculated.

## 5.2. **Diffusion Layer**

The diffusion-layer problem is obtained by substituting the rescaling

$$\hat{y} = \epsilon y \ , \quad (\hat{u}, \hat{v}) = (u, \epsilon v)$$

in (3.9) and (3.10) to obtain continuity and momentum equations,

$$\alpha u_\sigma + \epsilon^2 u_X + v_y = 0 \ , \qquad (5.38)$$

$$\alpha(u - c)u_\sigma + \epsilon^2(u_T + u u_X) + v u_y = -\alpha p_\sigma - \epsilon^2 p_X + \epsilon^2 u_{yy} \ . \qquad (5.39)$$

It is convenient to make a von Mises transformation, that is, to define a modified stream-function $\psi(\sigma, T, X, y)$ by

$$u = -\psi_y + c \ , \quad v = \bar{v}_\infty + \alpha\psi_\sigma + \epsilon^2(\psi_X - y c_X) \ , \quad \psi = 0 \quad \text{on} \quad y = 0 \ ,$$

where

$$\bar{v}_\infty \equiv v|_{y=0} = \epsilon \bar{v}_{0\infty} + \epsilon^2 \bar{v}_{1\infty} + \ldots \qquad (5.40)$$

is the blowing velocity of the Stokes layer, and to use $(\sigma, T, X, \psi)$ as independent variables. In this formulation,

$$y = \int_0^\psi \frac{1}{c - u} \, d\psi' \ , \quad v = \bar{v}_\infty + (u - c)\left(\alpha \frac{\partial y}{\partial \sigma} + \epsilon^2 \frac{\partial y}{\partial X}\right) - \epsilon^2 y c_X \ , \qquad (5.41)$$

and the momentum equation is transformed to

$$\frac{\partial}{\partial\sigma}\left[\tfrac{1}{2}\alpha(u-c)^2\right]+\epsilon^2\left[\frac{\partial u}{\partial T}+u\frac{\partial u}{\partial X}\right]+\epsilon^2\frac{\partial}{\partial\psi}\left[\tfrac{1}{2}(u-c)^2\right]\left[\frac{\partial y}{\partial T}+\frac{\partial}{\partial X}(cy)\right]$$
$$-\bar{v}_\infty\frac{\partial}{\partial\psi}\left[\tfrac{1}{2}(u-c)^2\right]=-\left(\alpha\frac{\partial}{\partial\sigma}+\epsilon^2\frac{\partial}{\partial X}\right)p+\epsilon^2(u-c)\frac{\partial^2}{\partial\psi^2}\left[\tfrac{1}{2}(u-c)^2\right]\ . \quad (5.42)$$

For this transformation to be valid, it is necessary to have $(c-u)$ everywhere positive; since

$$u\to c+B_0+O(\epsilon)\quad \text{as}\quad y\to\infty \quad (5.43)$$

this requires $B_0<0$ for all $\sigma$, or in other words $|b_0|<0.5$ (as illustrated in Figure 3). The question of whether solutions can be found with $|b_0|>0.5$ (in which case a region of closed streamlines is present in a frame of reference moving at the wavespeed $c$) will not be considered here.†

The expansions

$$(u,v,y,p)=(u_0,v_0,y_0,p_0)+\epsilon(u_1,v_1,y_1,p_1)+\epsilon^2(u_2,v_2,y_2,p_2)+\cdots\ , \quad (5.44)$$

may now be substituted into (5.41)–(5.42), with the $p_i$ given by (5.10), (5.12) and (5.14), and for the first two orders the solutions are found to be

$$\tfrac{1}{2}(u_0-c_0)^2=g_0(\psi,X,T)+R_0+\tfrac{1}{2}c_0^2-p_0\equiv g_0(\psi,X,T)+\tfrac{1}{2}B_0^2\ , \quad (5.45)$$

$$(u_0-c_0)(u_1-c_1)=-g_1(\psi,X,T)-p_1+\frac{g_{0\psi}}{\alpha_0}\int_0^\sigma \bar{v}_{0\infty}\,d\sigma'\ , \quad (5.46)$$

$$y_0=\int_0^\psi\frac{1}{c_0-u_0}\,d\psi'\ ,\quad y_1=-\int_0^\psi\frac{c_1-u_1}{(c_0-u_0)^2}\,d\psi'\ , \quad (5.47)$$

where $g_0$ and $g_1$ are undetermined at this order. Matching to the Tollmien layer requires

$$g_0\to 0\ ,\quad g_{1\psi}\to 1\quad \text{as}\quad \psi\to\infty \quad (5.48a,b)$$

and

$$u_0=c_0-\left[2g_0(T,X,\psi)+B_0^2\right]^{\frac{1}{2}}\ , \quad (5.49)$$

where the positive square root is taken.

The $O(\epsilon^2)$ problem takes the form

$$\alpha_0\tfrac{\partial}{\partial\sigma}\left[(u_0-c_0)(u_2-c_2)+\tfrac{1}{2}(u_1-c_1)^2\right]+\alpha_1\tfrac{\partial}{\partial\sigma}\left[(u_0-c_0)(u_1-c_1)\right]$$
$$+\alpha_2\tfrac{\partial}{\partial\sigma}\left[\tfrac{1}{2}(u_0-c_0)^2\right]+\tfrac{\partial u_0}{\partial T}+u_0\tfrac{\partial u_0}{\partial X}+\tfrac{\partial}{\partial\psi}\left[\tfrac{1}{2}(u_0-c_0)^2\right]\left[\tfrac{\partial y_0}{\partial T}+\tfrac{\partial}{\partial X}(c_0y_0)\right]$$
$$-\bar{v}_{0\infty}\tfrac{\partial}{\partial\psi}\left[(u_0-c_0)(u_1-c_1)\right]-\bar{v}_{1\infty}\tfrac{\partial}{\partial\psi}\left[\tfrac{1}{2}(u_0-c_0)^2\right]$$
$$=-\tfrac{\partial}{\partial\sigma}\left[\alpha_0p_2+\alpha_1p_1+\alpha_2p_0\right]-\tfrac{\partial p_0}{\partial X}+(u_0-c_0)\tfrac{\partial^2}{\partial\psi^2}\left[\tfrac{1}{2}(u_0-c_0)^2\right]\ . \quad (5.50)$$

Averaging this equation over a period in $\sigma$, recalling that $\langle p_0\rangle=0$ and anticipating also that

$$\langle\bar{v}_{0\infty}\rangle=0\ ,\quad \langle\bar{v}_{1\infty}\rangle=0 \quad (5.51)$$

† While we impose the restriction that the flow is unidirectional in the frame of reference moving at the wavespeed $c$, the flow near the wall can be either positive or negative in the wall frame. To see this first recall that $\hat{u}_0=A_0$ at the bottom of the Tollmien layer. Then note for, say, the purely temporal case, that from (5.36) $A_0<0$ and $A_0>0$ when $\sigma=0$ and $\sigma=\pi$ respectively.

(which follow from (5.70) below), we obtain a (nonlinear) evolution equation for $g_0$, namely

$$I_\psi \frac{\partial g_0}{\partial T} + (c_0 I_\psi - 1)\frac{\partial g_0}{\partial X} - \left[I_T + (c_0 I)_X\right]\frac{\partial g_0}{\partial \psi} - (I_\psi)^{-1}\frac{\partial^2 g_0}{\partial \psi^2} = J \ , \tag{5.52}$$

where

$$I = \int_0^\psi \left\langle \left(2g_0 + B_0^2\right)^{-1/2} \right\rangle d\psi' \ , \tag{5.53}$$

and

$$J = c_{0T} + c_0 c_{0X} - \left\langle \left(2g_0 + B_0^2\right)^{1/2} \right\rangle c_{0X} - \left\langle \frac{B_0\left(B_{0T} + c_0 B_{0X}\right)}{\left(2g_0 + B_0^2\right)^{1/2}} \right\rangle + \langle B_0 B_{0X} \rangle \ . \tag{5.54}$$

In order to solve (5.52) it is necessary to impose the value of

$$g_{0b} \equiv g_0(T, X, 0) \ , \tag{5.55}$$

which is in fact determined by the Stokes-layer solution (see below), and a suitable initial condition, which in the purely temporal case may be taken to be simply

$$g_0 \to 0 \quad \text{as} \quad T \to -\infty \ . \tag{5.56}$$

The diffusion-layer blowing velocity may be evaluated in primitive variables as

$$\hat{v}_{1b} + \epsilon \hat{v}_{2b} + \ldots = \lim_{y\to\infty}\left\{ v + y\left(\alpha\frac{\partial}{\partial\sigma} + \epsilon^2\frac{\partial}{\partial X}\right)u \right\} \ , \tag{5.57}$$

which in the von Mises formulation is equivalent to

$$\bar{v}_\infty + \lim_{\psi\to\infty}\left\{ \frac{\partial}{\partial\sigma}\left[\alpha(u-c)y\right] + \alpha y\frac{\partial y}{\partial\sigma}\frac{\partial}{\partial\psi}\left[\tfrac{1}{2}(u-c)^2\right] \right\} + O(\epsilon^2) \ . \tag{5.58}$$

We thus obtain

$$\hat{v}_{1b} = \lim_{\psi\to\infty}\frac{\partial}{\partial\sigma}\left[\alpha_0 B_0 y_0\right] \ , \tag{5.59}$$

$$\hat{v}_{2b} = \bar{v}_{0\infty} + \lim_{\psi\to\infty}\frac{\partial}{\partial\sigma}\left[\alpha_0 B_0 y_1 + \alpha_1 B_0 y_0 - \alpha_0 B_0^{-1}(g_1 + p_1)y_0 - \tfrac{1}{2}\alpha_0 y_0^2\right] \ . \tag{5.60}$$

which together with (5.51) justify our earlier assertions that

$$<\hat{v}_{1b}> = 0 \ , \quad <\hat{v}_{2b}> = 0 \ . \tag{5.61}$$

As noted in the previous section, there is no loss of generality in taking $A_0$ and $A_1$ (and hence $B_0$ and $p_1$) to be even in $\sigma$, in which case the same is true of $y_0$, while

$$\begin{aligned} y_1 &= \frac{1}{\alpha_0}\int_0^\psi \left[\frac{g_{0\psi'}}{(u_0 - c_0)^3}\,d\psi'\right]\left[\int_0^\sigma \bar{v}_{0\infty}\,d\sigma'\right] + \text{terms even in } \sigma \\ &= \frac{1}{\alpha_0}\left[\left(2g_0 + B_0^2\right)^{-1/2} - \left(2g_{0b} + B_0^2\right)^{-1/2}\right]\left[\int_0^\sigma \bar{v}_{0\infty}\,d\sigma'\right] + \ldots \ . \end{aligned} \tag{5.62}$$

It follows that $\hat{v}_{1b}$ is odd in $\sigma$, which justifies the second of (5.22), while

$$\hat{v}_{2b} = -\frac{\partial}{\partial\sigma}\left[\frac{B_0}{\left(2g_{0b} + B_0^2\right)^{1/2}}\int_0^\sigma \bar{v}_{0\infty}d\sigma'\right] + \text{terms odd in } \sigma \ . \tag{5.63}$$

Only the part of $\hat{v}_{2b}$ even in $\sigma$ contributes to the right-hand side of the amplitude equation (5.26). After integration by parts, this can be expressed in the form

$$-\langle A_0\hat{v}_{2b}\rangle = \frac{1}{2\pi}\int_0^{2\pi}\left(2g_{0b}+B_0^2\right)^{1/2}\bar{v}_{0\infty}d\sigma \equiv \frac{1}{2\pi}\int_0^{2\pi}\left(c_0-\bar{u}_e\right)\bar{v}_{0\infty}d\sigma\ , \qquad (5.64)$$

where

$$\bar{u}_e(\sigma,X,T) \equiv u_0|_{y=0} = c_0 - \left(2g_{0b}+B_0^2\right)^{\frac{1}{2}} \qquad (5.65)$$

is the leading-order slip velocity of the Stokes layer.

A solution of equation (5.52) with boundary conditions (5.48$a$), (5.55) and (5.56) will be presented only for the the case of purely temporal evolution (see Figure 7 below), although preliminary calculations suggest that a solution can be found for a more general spatio-temporal evolution, at least for sufficiently small values of the amplitude $b_0$. (The foregoing analysis is only valid in any case for $|b_0| < 0.5$). We note, however, that provided a diffusion-layer solution exists, its precise form has no bearing on the leading-order evolution of the disturbance. This is because the quantities $\bar{v}_{0\infty}$ and $g_{0b}$ appearing in (5.64) and (5.65) are determined as functions of $\alpha_0$, $c_0$ and $b_0$ purely by the Stokes-layer solution, as we now show.

### 5.3. **Stokes Layer**

The Stokes layer is described by the rescaling

$$\hat{y} = \epsilon^2\bar{y}\ ,\quad \hat{u} = \bar{u}_0 + \epsilon\bar{u}_1 + \dots\ ,\quad \hat{v} = \epsilon^2\bar{v}_0 + \epsilon^3\bar{v}_1 + \dots\ , \qquad (5.66)$$

which gives at leading order

$$\alpha_0\bar{u}_{0\sigma} + \bar{v}_{0\bar{y}} = 0\ ,\quad \alpha_0(\bar{u}_0-c_0)\bar{u}_{0\sigma} + \bar{v}_0\bar{u}_{0\bar{y}} = -\alpha_0 p_{0\sigma} + \bar{u}_{0\bar{y}\bar{y}}\ , \qquad (5.67)$$

$$\bar{u}_0 = \bar{v}_0 = 0 \quad \text{on} \quad \bar{y}=0\ ,\quad \bar{u}_0 \to \bar{u}_e(\sigma,X,T) \quad \text{as} \quad \bar{y}\to\infty, \qquad (5.68)$$

with $\bar{u}_e$ specified by (5.65), and $-p_{0\sigma} = \bar{u}_e\bar{u}_{e\sigma}$. At next order, we have *inter alia* that

$$\alpha_0\bar{u}_{1\sigma} + \alpha_1\bar{u}_{0\sigma} + \bar{v}_{1\bar{y}} = 0\ ,\quad v_1 = 0 \quad \text{on} \quad \bar{y}=0\ . \qquad (5.69)$$

It follows that

$$<\bar{v}_0> \equiv 0\ ,\quad <\bar{v}_1> \equiv 0\ , \qquad (5.70)$$

which justifies the earlier assumptions (5.51).

The leading-order problem (5.67)–(5.68) may be simplified by writing

$$\sigma = -x\ ,\quad \bar{y} = \frac{z}{\sqrt{\alpha_0 c_0}}\ ,\quad \bar{u}_0 - c_0 = -c_0 U(x,z)\ ,\quad \bar{v}_0 = \sqrt{\alpha_0 c_0}V(x,z) \qquad (5.71)$$

(the dependence on the slow variables $T$ and $X$ is purely parametric and has been suppressed). The problem then reduces to

$$U_x + V_z = 0\ ,\quad UU_x + VU_z = U_e(x)U_e'(x) + U_{zz}\ , \qquad (5.72)$$

$$U = 1\ ,\quad V = 0 \quad \text{on} \quad z=0\ ,\quad U \to U_e(x) \quad \text{as} \quad z\to\infty\ , \qquad (5.73)$$

where

$$U_e(x) \equiv \left[-2w + c_0^{-2}B_0^2(x)\right]^{1/2}\ ,\quad w = -g_{0b}/c_0^2\ , \qquad (5.74)$$

and

$$c_0^{-1}B_0(x) = \gamma_0\left[\frac{1}{(1-b_0^2)^{1/2}} - \frac{2(1-b_0^2)^{1/2}}{1-b_0\cos x}\right]\ ,\quad \gamma_0 = \frac{\alpha_0}{c_0} \qquad (5.75)$$

We note that $c_0^{-1}B_0(x)$ is $2\pi$-periodic in $x$ and depends on only two parameters, namely $b_0$ and $\gamma_0$. The constant $w$ is an eigenvalue uniquely determined by the requirement that (5.72)–(5.75) has a solution which is $2\pi$-periodic in $x$, as we shall show in the next section. Thus the Stokes-layer solution provides both the boundary condition (5.55) required to solve for $g_0$ in the diffusion layer, and the forcing (5.64) on the right-hand side of the amplitude-evolution equation (5.26).

## 6. Numerical solution

The problem (5.72)–(5.74) has the form of the classical boundary-layer equations with a moving lower boundary and a periodic slip velocity. Similar problems involving periodic boundary layers have been studied in a number of contexts, such as flow past a rotating and translating cylinder (Glauert 1957, Moore 1957, Nikolayev 1982, Lam 1988), driven cavity flows with closed streamlines (Wood 1956, Kuwahara & Imai 1969, Riley 1981, Chipman & Duck 1993), periodic oscillatory flows at large values of the streaming Reynolds number (Riley 1978), flows over wavy surfaces (Bordner 1978) and rotating-annulus flows with bottom topography (Page 1982). Our numerical method is an adaptation of a code written by Lam (1988) which she used to study both the flow past a rotating and translating cylinder, and periodic boundary layers associated with two-dimensional TS waves in plane Poiseuille flow.

### 6.1. The periodic boundary-layer problem

The Stokes-layer problem (5.72)–(5.73) was rewritten in von Mises variables as

$$UU_x = U_eU_e' + U(UU_\psi)_\psi \ , \quad U = 1 \quad \text{on} \quad \psi = 0 \ , \tag{6.1$a$,$b$}$$

$$U \to U_e(x) \quad \text{as} \quad \psi \to \infty \ . \tag{6.2}$$

The problem was then solved in a finite computational domain, with the far-field boundary condition (6.2) replaced by

$$U = U_e(x) \quad \text{on} \quad \psi = \psi_\infty \ . \tag{6.3}$$

As noted by Riley (1981), for given values of the parameters $b_0$ and $\gamma_0$, periodic solutions can be found for a range of values of the parameter $w$ when this finite boundary condition is used. However, only one of these satisfies the true far-field boundary condition in the limit as $\psi_\infty \to \infty$. In order to identify this solution we follow Riley (1981) and examine the behaviour towards the edge of the boundary layer ($\psi \gg 1$) , where

$$U(x,\psi) = U_e(x) + \tilde{U}(x,\psi) \quad \text{with} \quad |\tilde{U}| \ll 1 \ .$$

On substituting into the governing equation (6.1) and retaining only terms linear in $\tilde{U}$, we find that the resulting equation for $\tilde{U}$ has a separable solution that is periodic in $x$ (cf. Riley 1981), namely

$$U_e\tilde{U} = D_0 + E_0\psi + \sum_{n=1}^{\infty}\left[D_n \exp\left(\frac{\mathrm{i}n}{\langle U_e\rangle}\int_0^x U_e\,dx' - \mathrm{e}^{\mathrm{i}\pi/4}\left(\frac{n}{\langle U_e\rangle}\right)^{\frac{1}{2}}\psi\right) + c.c.\right] \ . \tag{6.4}$$

Replacing the outer boundary condition (6.2b) by (6.3) leads for most values of $w$ to a solution with non-zero values of $D_0$ and $E_0$ (although $D_0 + E_0\psi_\infty \approx 0$ for $\psi_\infty \gg 1$). In order to obtain the unique value of $w$ which gives the correct exponentially decaying behaviour for $\tilde{U}$ at the edge of the boundary layer it is necessary to apply an additional condition which forces $E_0$, and hence $D_0$, to zero as $\psi_\infty \to \infty$, for example

$$< U_e \tilde{U}_\psi > \equiv \frac{1}{2\pi} \int_0^{2\pi} U_e \tilde{U}_\psi dx = 0 \quad \text{at} \quad \psi = \psi_\infty \; . \tag{6.5}$$

The method we adopted to solve for $w(b_0, \gamma_0)$ was as follows. First, for given values of $b_0$ and $\gamma_0$ a guess was made for $w$. For these values of $b_0$, $\gamma_0$ and $w$, a second-order-accurate Crank-Nicolson scheme was used to integrate (6.1) subject to (6.2) through repeated periods of $2\pi$ until the solution settled down to a state that was periodic to within a specified tolerance. We expect the rate of convergence to be exponentially fast, as may be seen by noting that once the solution is close to a periodic state, the linearised equation for the perturbation has coefficients which are $2\pi$-periodic in $x$. On the basis of Floquet theory (cf. von Kzercek & Davis 1974), this suggests that the velocity perturbation has the form

$$\mathrm{e}^{-\Lambda x} \tilde{\mathcal{U}}(x, \psi) + c.c., \tag{6.6}$$

where $\tilde{\mathcal{U}}$ is a $2\pi$-periodic function of $x$ and $\Lambda$ is a complex constant; the relevant solutions have $\mathrm{Re}(\Lambda) > 0$. Once a periodic solution had been obtained, the value of $E_0$ in (6.4) was estimated by evaluating $< U_e \tilde{U}_\psi >$ at $\psi = \psi_\infty$. An improved approximation for the eigenvalue $w$ could then be found by secant iteration with the aim of forcing $E_0$ to zero. The procedure was repeated until $|E_0|$ was less than some specified tolerance.

For each new value of the parameters $b_0$ and $\gamma_0$, a velocity profile was needed at the starting position $x_0$ (say $x_0 = 0$). It was desirable that this starting profile should satisfy the boundary condition at $\psi = 0$ and model the correct oscillatory exponential decay for large $\psi$. For small values of $b_0$, a velocity profile loosely based on Glauert's (1957) 'small-amplitude' solution was used, namely

$$U = U_e(x_0) + [1 - U_e(x_0)] \frac{\cos\left(\theta + 2^{-1/2}\psi\right)}{\cos\theta \exp\left(2^{-1/2}\psi\right)} \; , \quad \text{where} \quad \theta = \tfrac{1}{2} \sin\left[\frac{U_e(x_0) - 1}{2 - U_e(x_0)}\right] \; . \tag{6.7}$$

However, for larger amplitudes the velocity profile at $x_0$ for a new values of $b_0$ and $\gamma_0$ was obtained by linear interpolation from the velocity profiles at $x_0$ previously calculated for two nearby pairs of values of $b_0$ and $\gamma_0$. An interpolated update for the velocity profile was necessary in order to avoid introducing thin viscous shear layers at $\psi_\infty$, since if introduced these took many periods to diffuse out of the flow. A similar interpolation was generally used for obtaining the initial velocity profile when $w$ was updated for a given value of $b_0$ and $\gamma_0$ (especially for larger values of $b_0$).

For a given value of $\gamma_0$, a solution of the sublayer problem can only be found for $b_0$ less than some critical value $b_{0c}(\gamma_0)$, indicated in figure 4. As $b_0$ approaches such a critical value from below, the minimum velocity in the boundary layer decreases until for $b_0 = b_{0c}$ a stagnation point seems to develop within the boundary layer, at $x = x_c$ and $\psi = \psi_c$ say. In order to help resolve the velocity profile for values of $b_0$ when this stagnation point is close to developing, a co-ordinate transformation $(x, \psi) \to (\chi(x), \phi(\psi))$ specified implicitly by

$$x = \chi - \left[1 - \left(\hat{b}_{0c} - b_0\right)^{1/2}\right] \sin\chi \; ,$$

$$\psi = \phi - \beta(1 - \hat{\varepsilon}) \left[\tanh\left(\frac{\phi - \hat{\phi}_c}{\beta}\right) + \tanh\left(\frac{\hat{\phi}_c}{\beta}\right)\right]$$

was introduced, where $\hat{b}_{0c}$ and $\hat{\phi}_c$ are approximations to $b_{0c}$ and $\phi_c \equiv \phi(\psi_c)$, and $\hat{\varepsilon}$ and $\beta$ are constants (typically $\hat{\varepsilon} = 0.005$, $\beta = 0.656$). It was not found necessary to apply a

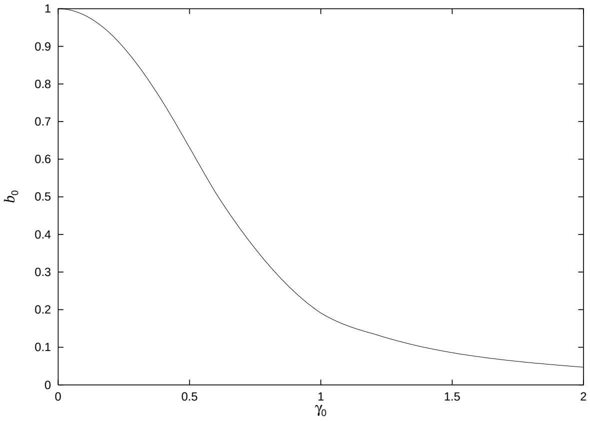


FIGURE 4. The critical value $b_{0c}(\gamma_0)$ above which a solution of the Stokes-layer problem cannot be found.

shift in $\chi$ since $|x_c| \ll 1$. The values of these stretching constants were fixed by trial and error so as to give an enhanced grid resolution near the position of minimum velocity (an equi-spaced grid in $\chi$ and $\phi$ was used).

### 6.2. Temporal evolution of a DABO travelling wave

Once the solution of the Stokes-layer problem has been obtained, the forcing on the right-hand side of (5.26) or (5.37) may be calculated as

$$-<A_0\hat{v}_{2b}>=\alpha_0^{1/2}c_0^{3/2}S\left[b_0,\gamma_0\right] \ , \quad S\left[b_0,\gamma_0\right]=\frac{1}{2\pi}\int_0^{2\pi}U_eV_\infty dx \ , \tag{6.8}$$

where $V_\infty$ is the blowing velocity defined as

$$V_\infty=\lim_{z\to\infty}(V+z\frac{\mathrm{d}U_e}{\mathrm{d}x})\equiv\int_0^\infty\left(\frac{U_e}{U}\right)_x d\psi \ . \tag{6.9}$$

As indicated earlier, for simplicity we restrict attention to the case of a purely temporal evolution. This is characterised by a single parameter, namely the amplitude $b_0$, since from (5.35)

$$\frac{\alpha_0}{c_0}=\gamma_0(b_0)=\frac{\left(1-b_0^2\right)^{1/2}}{2\left(1-b_0^2\right)^{1/2}-1} \ , \tag{6.10}$$

and from (5.37) and (6.8), we have the evolution equation

$$\frac{1}{(1-b_0^2)^{3/2}}\frac{d\left(b_0^2\right)}{dT}=\left[\gamma_0(b_0)\right]^{-3/2}S\left[b_0,\gamma_0(b_0)\right] \ . \tag{6.11}$$

In figure 5($a$) the eigenvalue $w\left(b_0,\gamma_0(b_0)\right)$ is plotted as a function of $b_0$. A solution for $w$ could not be found for $b_0 > b_{0s} \approx 0.191$. A reason for this may be surmised from figure 5($b$) and ($c$) which are plots against $b_0$ of the square of $U_{min} \equiv \min_{x,\psi} U$ and the reciprocal of $V_{max} \equiv \max_x V_\infty$, respectively. These figures suggest that as $b_0 \to b_{0s}$, a stagnation point is developing within the boundary layer, and the blowing velocity is becoming unbounded. Similar behaviour is observed in other periodic boundary-layer flows over downstream-moving walls when the adverse pressure gradient becomes too large over a sustained section of the wall (e.g. Lam 1988). The behaviour of the blowing velocity in particular suggests that a 'marginal separation' singularity will be present in the solution for $b_0 = b_{0s}$.

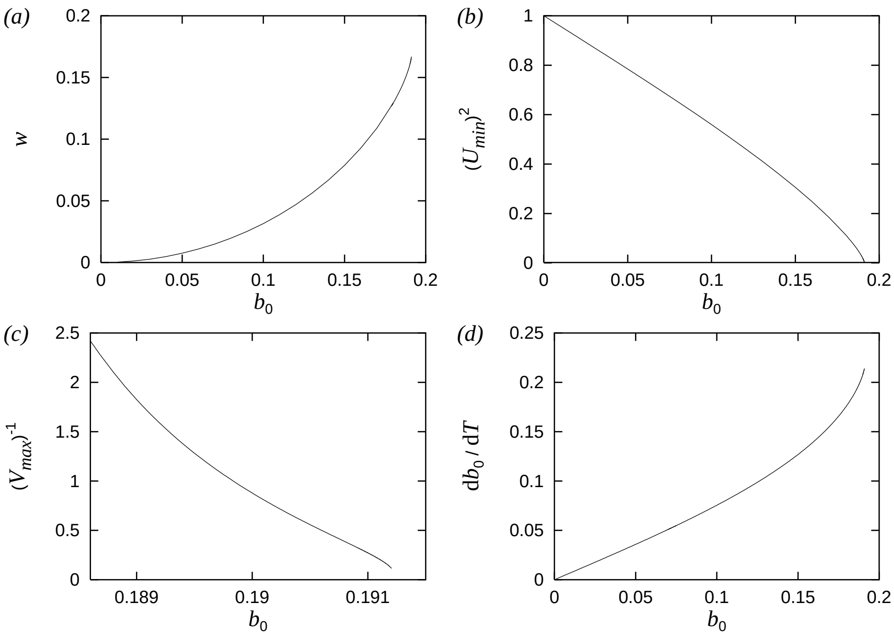


FIGURE 5. ($a$) The eigenvalue $w$, ($b$) the square of the minimum streamwise velocity $U_{min} \equiv \min_{x,\psi} U$, ($c$) the reciprocal of the maximum blowing velocity $V_{max} \equiv \max_x V_\infty$, and ($d$) the rate-of-change of amplitude $\mathrm{d}b_0/\mathrm{d}T$, as functions of the amplitude $b_0$, for a purely temporal evolution.

We have not been able to determine numerically the precise form of the singularity. Its analytic structure will depend on whether or not the position of the singularity, $x_s$, is located at the maximum point in the external pressure distribution, i.e. whether or not $x_s = 0$ (Prof. S.N. Brown, private communication 1987, Sychev 1987, Negoda & Sychev 1987, Timoshin 1996). For the time being we note that more detailed calculations for other external velocities $U_e$ by S.J.C. and Prof. S.I. Chernyshenko (private communication) suggest that $x_s \neq 0$. If this is the case here, then the singularity for $b = b_{0s}$ will have the form proposed by Timoshin (1996) for marginal separation in non-periodic boundary-layer flow over a downstream-moving wall.

Figure 5($d$) is a plot against $b_0$ of the rate-of-change of amplitude $\mathrm{d}b_0/\mathrm{d}T$ calculated from (5.37) and (5.71). Since $\mathrm{d}b_0/\mathrm{d}T$ is positive, $b_0$ will be a monotonically increasing function of time. We have solved for $b_0(T)$ for the initial condition

$$b_0 \sim \mathrm{e}^{T/\sqrt{2}} \quad \text{as} \quad T \to -\infty \;, \tag{6.12}$$

using a fourth-order Runge-Kutta integrator and interpolants of the graph in figure 5($d$). The result is plotted in figure 6($a$). This shows that when the singularity develops in the Stokes layer, at $T = T_s \approx -2.47$, both the amplitude $b_0$ and its rate-of-change $\mathrm{d}b_0/\mathrm{d}T$ are finite. Moreover, we have verified (numerically) that the diffusion-layer problem (5.52)–(5.56) has a solution for $g_0(\psi, T)$ right up to the singularity time; this is shown in figure 7. In order to highlight the reason why no solution can be found for $T > T_s$ (i.e. $b_0 > b_{0s}$), figure 6($b$) shows $c_0 - \max_{\sigma,\bar{y}} \bar{u}_0$ as a function of $T$. We conclude that for $T < T_s$ the TS wave is propagating downstream faster than the fluid in the Stokes layer beneath it. However, as $T \to T_s$ a stagnation point develops in the frame moving with the instantaneous velocity of the TS wave. For times larger than this we anticipate that

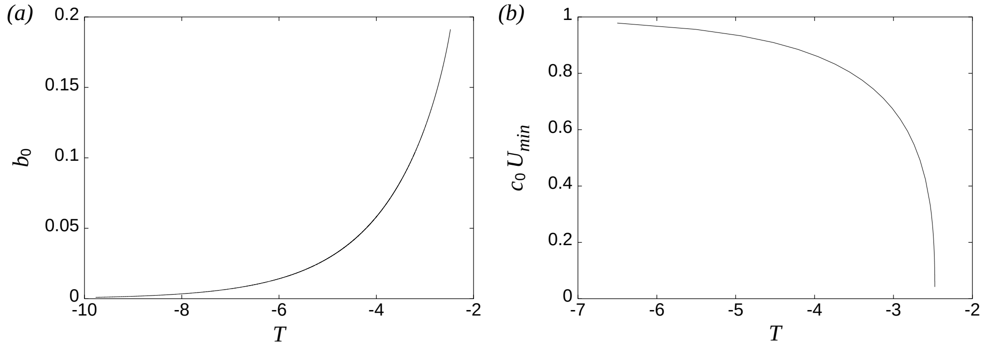


FIGURE 6. The temporal evolution of ($a$) amplitude $b_0$, and ($b$) $c_0U_{min} \equiv (c_0 - \max_{\sigma,\bar{y}} \bar{u}_0)$, as specified by (6.10)–(6.12).

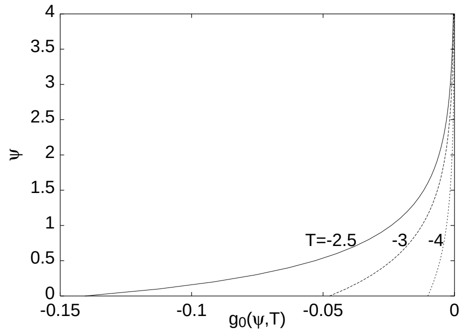


FIGURE 7. The solution of (5.52)–(5.56) for the temporal evolution specified by (6.10)–(6.12).

there would be fluid moving both faster and slower than the TS wave. However, by this time boundary-layer ‘separation’ will have occurred (albeit ‘separation’ because the fluid is moving too fast relative to the TS wave), and a new asymptotic régime will have been entered.

We do not pursue the details of this new régime here, other than to note that when the separation is small, i.e. marginal, and if $x_s \neq 0$, then the flow in the Stokes layer will be as given by Timoshin (1996).

## 7. Conclusions

This paper has examined the evolution of a certain class of two-dimensional TS waves in a boundary layer. Following earlier work by Zhuk & Ryzhov (1982) and Smith & Burggraf (1985), we have concentrated on the ‘far-downstream lower-branch’ régime of a high-Reynolds-number Blasius boundary layer. In this régime linear TS waves are asymptotically inviscid and neutral to leading order and their slow growth in amplitude is a relatively slow viscous effect. If this is still the case in the subsequent nonlinear stage of development, the form of such quasi-periodic TS waves will then be governed to leading order by the DABO equation. This equation is known to have a three-parameter family of periodic solutions (characterised by wavenumber $\alpha_0$, wavespeed $c_0$, and amplitude parameter $b_0$), which asymptote in the large-amplitude limit ($b_0 \to 1^-$) to a soliton solution. As Rothmayer & Smith (1987) and Kachanov *et al.* (1993) have noted, this soliton bears a striking resemblance to the ‘spikes’ observed in certain ‘K-type’ transition experiments. Moreover, a quantitative comparison has been made by Kachanov *et al.* (1993) between the vibrating-ribbon experiments of Borodulin & Kachanov (1988) and periodic DABO solutions with $\gamma_0 \equiv \alpha_0/c_0 = 1$ and a range of values of $b_0$ (which was used as an adjustable parameter). The agreement appears reasonably good for $b_0$ up to about 0.65 (discrepancies for larger $b_0$ were attributed to three-dimensional effects).

At higher order, the slow evolution of the solution parameters is determined by a system of equations involving the behaviour of a periodic viscous Stokes layer governed by nonlinear classical boundary-layer equations. A careful formulation of the problem shows that nonlinear interactions within the Stokes layer generate a mean-flow distortion (steady streaming) which diffuses outwards from the wall. Over the slow timescale of growth of the TS waves, this mean flow diffuses through a distance that is large compared to the Stokes-layer thickness, but small compared to the width of the outer inviscid region (Tollmien layer). Thus our formulation differs in technical respects from that of Smith & Burggraf (1985) who do not include this diffusion layer, but implicitly assume that a uniform mean-flow distortion is somehow produced throughout the Tollmien layer. In the case of temporal or spatio-temporal evolution, the diffusion layer would seem to play a formally passive rôle† in the sense that an explicit solution for this layer is not required in order to determine the evolution of the wave parameters, although it needs to be verified that a solution exists (which is certainly the case for a purely temporal evolution). Nevertheless the inclusion of the diffusion layer does indirectly influence the evolution of the waves by altering the outer boundary condition for the Stokes layer. This will depend, too, on the particular initial condition chosen for the diffusion layer. The solutions obtained in this paper assume that the mean-flow correction goes to zero at asymptotically large negative times (and/or far upstream), as would be expected if the evolution starts as a linear instability.

This is not the end of the story, however, since as observed by previous authors (e.g. Smith & Burggraf 1985, Lam 1988) there is no guarantee that a Stokes-layer solution will continue to exist for indefinitely large disturbances. Indeed, our calculations confirm that unless the parameter $\gamma_0 \equiv \alpha_0/c_0$ goes to zero, the Stokes layer develops a 'marginal-separation' singularity at a certain finite value of wave amplitude ($b_0 < 1$), in which case the quasi-inviscid DABO description ceases to be strictly valid before the asymptotic soliton form in attained. We note that the parameter values $\gamma_0 = 1$, $b_0 = 0.65$ used in the experimental comparisons of Kachanov *et al.* (1993) are well outside the range for which, in our choice of formulation, an attached Stokes-layer solution could be found (see Figure 4).‡ It is not clear, therefore, how a solution with these parameters would develop from an initially linear HFLB TS wave (and in most controlled experiments the initial disturbance is in the linear régime). Admittedly, there seems no reason why Kachanov *et al.* (1993) could not have tuned their data with smaller values of $\gamma_0$, for which larger-amplitude solutions with an unseparated Stokes layer are available. In a complete theoretical description, however, the parameters $\gamma_0$ and $b_0$ cannot be assigned arbitrarily, but should be obtained from a solution of the evolution equations.

As a model problem, we have studied the temporal evolution of a spatially periodic disturbance that starts life at large negative times as an infinitesimal TS mode. While this condition might be difficult to realise experimentally, it represents a particularly simple initial-value problem since the evolution is governed by a single first-order ordinary

† As shown in Appendix A, the purely spatial case is a different matter and the diffusion layer plays a determining rôle, albeit for smaller disturbance amplitudes.

‡ We emphasise again that while the Stokes-layer solutions are required to be unidirectional in the frame of reference moving at the wavespeed, all the solutions presented here have regions of 'reversed-flow' in the wall frame.

differential equation (6.11), the right-hand side of which is determined by the solution to the classical boundary-layer equations in the Stokes layer. It turns out that the marginal separation singularity is encountered at a rather modest amplitude, $b_0 \approx 0.19$. At this stage the wave profile has not attained a shape resembling the 'spiky' soliton form; indeed it is still roughly sinusoidal in appearance (see Figure 3). We conclude that an infinitesimal TS wave may not evolve temporally into a wave with an approximate soliton form because of the failure of a solution to exist in the Stokes layer.

More generally, our formulation allows us to consider the spatio-temporal evolution of a localized wavepacket. In this case the slow evolution of the three wave parameters (wavenumber, wavespeed and amplitude) is governed by a nonlinear hyperbolic system of equations, namely (3.6), (5.25) and (5.26), and the possibility of 'shock' formation represents another way in which the asymptotic description may break down. Preliminary numerical solutions of these equations indicate that for sufficiently short packets, a shock does indeed form on the trailing edge of the packet, whereas for longer packets breakdown occurs through a Stokes-layer singularity. Such rear-facing shocks were in fact predicted by Rothmayer & Smith (1987) on the basis of a model equation; a connection was suggested with experiments of Amini & Lespinard (1982) on 'incipient spots', although these were certainly three-dimensional.

Finally, in Appendix A we consider a time-periodic disturbance evolving in space, as is presumably appropriate to vibrating-ribbon experiments such as those of Borodulin & Kachanov (1988). In this case, the inclusion of the diffusion layer has more far-reaching consequences; the mean-flow correction in this layer goes fully nonlinear and then develops a classical Goldstein singularity while the wave-amplitude (in the Tollmien layer) is still small. This suggests that the DABO formulation may not apply at all to the nonlinear régime.

It must be emphasised that our results assume that the solution starts out as a linear TS wave (or wavepacket) in the HFLB régime. This implies that $\gamma_0 = 1$ initially, and in fact for all the solutions found to date, $\gamma_0$ differs little from unity in the subsequent evolution, with the result that the wave amplitude $b_0$ cannot grow beyond about 0.2 before some form of breakdown occurs. Nevertheless, the development of DABO travelling-wave solutions with smaller values of $\gamma_0$ (and consequently a larger critical amplitude $b_{0c}$) is not ruled out for other types of initial condition, such as those appropriate to bypass transition. For example, numerical solutions of the full nonlinear triple-deck by Ryzhov & Savenkov (1989) and Ryzhov & Timofeev (1995) suggest the emergence, possibly through a bypass mechanism, of wavepackets with soliton-like structures (although quantitative agreement with DABO travelling-wave solutions remains to be confirmed). In this context, we note an alternative proposal by Li, Walker, Bowles & Smith (1998), that a TS wave which goes nonlinear in the lower-branch régime proper may encounter a finite-time singularity (Smith 1988) leading to a new stage of development governed by an 'extended' Benjamin-Ono equation. Further numerical work would seem desirable to clarify the relationship between these, and possibly other, scenarios.

The motivation for this work came from the PhD thesis of Dr S. T. Lam, who kindly provided the basis of the code used to solve the periodic boundary-layer problem of §6.1. It has also benefitted from discussions with Prof. S. N. Brown, Prof. S. I. Chernyshenko, Dr M. O. de Souza and Dr S. N. Timoshin, and from the helpful comments of the referees, in particular Dr O. S. Ryzhov.

## Appendix A. Pure spatial evolution: weakly nonlinear stage

As remarked in the text, the amplitude equation (4.48) is not valid for a purely spatial evolution, with $\partial/\partial T = 0$. To understand the modification required in this case, let us consider the limit of weak dependence on $T$, and set

$$A_{11}(T,X) = \tilde{A}_{11}(\tau, X) \ , \quad \tau = \mu T \ , \quad \mu \ll 1 \ . \tag{A 1}$$

We have then that

$$\int_{-\infty}^{T} |A_{11}|^2(T',X)dT' = \mu^{-1} \int_{-\infty}^{\tau} |\tilde{A}_{11}|^2(\tau',X)d\tau' \ , \tag{A 2}$$

implying that the disturbance amplitude should be scaled down by a factor $\mu^{1/2}$. Further, it is apparent from (4.23) that the diffusion-layer thickness must shrink by the same factor. A new régime comes into play when $\mu = O(\epsilon^{4/3})$, since then a $y\partial u_{2M}/\partial X$-term must be included in equation (4.23). We shall not pursue the details of this transitional scaling, however, but proceed directly to the purely spatial case.

### A.1. **Stage I**

For a weakly nonlinear disturbance whose amplitude is a function of $X$ only, we expect the appropriate expansions to be

$$p = \epsilon^{4/3}\left[p_{11}(X)\mathrm{e}^{\mathrm{i}\sigma} + c.c.\right] + \epsilon^2 p_{2M} + \epsilon^{8/3}\left[p_{22}\mathrm{e}^{2\mathrm{i}\sigma} + c.c.\right] + \epsilon^{10/3}\left[p_{31}\mathrm{e}^{\mathrm{i}\sigma} + \ldots\right] + \ldots \ , \tag{A 3}$$

$$A = \epsilon^{4/3}\left[A_{11}(X)\mathrm{e}^{\mathrm{i}\sigma} + c.c.\right] + \epsilon^2 A_{2M} + \epsilon^{8/3}\left[A_{22}\mathrm{e}^{2\mathrm{i}\sigma} + c.c.\right] + \epsilon^{10/3}\left[A_{31}\mathrm{e}^{\mathrm{i}\sigma} + \ldots\right] + \ldots \ . \tag{A 4}$$

The results (4.4) are unaltered, however (and we are free to set $A_{31}$, etc. to zero). The Tollmien-layer velocity components expand as

$$\begin{aligned} \hat{u} = \hat{y} &+ \epsilon^{4/3}\left[\hat{u}_{11}(X,\hat{y})\mathrm{e}^{\mathrm{i}\sigma} + c.c.\right] + \epsilon^2 \hat{u}_{2M} + \epsilon^{8/3}\left[\hat{u}_{22}\mathrm{e}^{2\mathrm{i}\sigma} + c.c.\right] \\ &+\epsilon^{10/3}\left[\hat{u}_{31}\mathrm{e}^{\mathrm{i}\sigma} + c.c.\right] + \ldots \ , \end{aligned} \tag{A 5}$$

$$\begin{aligned} \hat{v} = \epsilon^{4/3}&\left[\hat{v}_{11}(X,\hat{y})\mathrm{e}^{\mathrm{i}\sigma} + c.c.\right] + \epsilon^{8/3}\left[\hat{v}_{22}\mathrm{e}^{2\mathrm{i}\sigma} + c.c.\right] + \epsilon^{10/3}\left[\hat{v}_{31}\mathrm{e}^{\mathrm{i}\sigma} + c.c.\right] \\ &+\epsilon^4\left[\hat{v}_{4M} + \ldots\right] + \ldots \ , \end{aligned} \tag{A 6}$$

where

$$\sigma = \epsilon^{-2}\left(\alpha X - T\right) \ , \tag{A 7}$$

and again the previous results (4.39)–(4.41) apply except that terms involving $\partial/\partial T$ should be dropped, as should the term $-\mathrm{i}\alpha A_{11}^* A_{22}$ in (4.41). Likewise, in the Stokes layer the appropriate expansions are

$$\hat{u} = \epsilon^{4/3}\left[\bar{u}_{11}(X,\bar{y})\mathrm{e}^{\mathrm{i}\sigma} + c.c.\right] + \epsilon^2\bar{y} + \epsilon^{8/3}\left[\bar{u}_{2M} + \ldots\right] + \ldots \ , \tag{A 8}$$

$$\hat{v} = \epsilon^{10/3}\left[\bar{v}_{11}(X,\bar{y})\mathrm{e}^{\mathrm{i}\sigma} + c.c.\right] + \ldots \ , \tag{A 9}$$

with the same results as before for the leading-order streamwise and normal velocity, namely (4.11), as well as (4.12) for $\bar{u}_{2M}$. The significant new feature concerns the diffusion layer, which is now described by the rescaling

$$\hat{y} = \epsilon^{2/3}\eta \ , \tag{A 10}$$

and expansions

$$\hat{u} = \epsilon^{2/3}\eta + \epsilon^{4/3}\left[u_{11}(X,\eta)\mathrm{e}^{\mathrm{i}\sigma} + c.c\right] + \epsilon^2 u_{2M} + \ldots \ , \tag{A 11}$$

$$\hat{v} = \epsilon^2\left[v_{11}(X,\eta)\mathrm{e}^{\mathrm{i}\sigma} + c.c\right] + \ldots + \epsilon^{14/3}v_{4M} + \ldots \ . \tag{A 12}$$

Again the results (4.20) apply, but now the mean-flow distortion is obtained from

$$\frac{\partial u_{2M}}{\partial X} + \frac{\partial v_{4M}}{\partial \eta} = 0 \ , \quad \eta \frac{\partial u_{2M}}{\partial X} + v_{4M} = -\frac{\partial |A_{11}^2|}{\partial X} + \frac{\partial^2 u_{2M}}{\partial \eta^2} \ , \tag{A 13}$$

with boundary conditions

$$u_{2M}|_{y=0} = v_{4M}|_{y=0} = 0 \ , \tag{A 14}$$

together with the requirement that $u_{2M}$ is bounded as $\eta \to \infty$, and the initial condition

$$u_{2M} \to 0 \quad \text{as} \quad X \to -\infty \ . \tag{A 15}$$

It follows from (A 13)–(A 15) that

$$A_{2M} \equiv \lim_{\eta \to \infty} u_{2M} = -\frac{\partial}{\partial X} \int_0^\infty (3s)^{-2/3} \, |A_{11}|^2 (X - s) \, ds \ . \tag{A 16}$$

Matching between the Tollmien layer and diffusion layer again imposes the requirements (4.43), which here give (4.44)–(4.46) together with the amplitude-evolution equation

$$\frac{\partial A_{11}}{\partial X} = \gamma \, (1 - \mathrm{i}) \, A_{11} - \frac{\mathrm{i}}{2} A_{11} A_{2M} \ , \quad \gamma = \frac{1}{2\sqrt{2}} \ . \tag{A 17}$$

As before, the nonlinear term affects only the phase, while the amplitude continues to grow exponentially as for a linear mode. Specifically,

$$|A_{11}| = a_0 \mathrm{e}^{\gamma X} \ , \quad \arg A_{11} = \phi_0 - \gamma X + C \mathrm{e}^{2\gamma X} \ , \quad \text{where} \quad C = \frac{\left(-\frac{2}{3}\right)! a_0^2}{2^{5/6} 3^{2/3}} \ , \tag{A 18}$$

and $a_0$ and $\phi_0$ are arbitrary. However, the subsequent phase of evolution differs from the (spatio-)temporal case since the mean flow in the diffusion layer goes nonlinear when $u_{2M}$ rises to order $\epsilon^{-4/3}$. We anticipate, in view of (A 13), that this will happen when $a_0 \mathrm{e}^{\gamma X}$, and hence $A_{11}$, is of order $\epsilon^{-2/3}$.

### A.2. **Stage II**

The new régime, which comes into play at order-one values of

$$\bar{X} \equiv X - \frac{2}{3\gamma} \ln \epsilon^{-1} + \frac{\ln a_0}{\gamma} \ , \tag{A 19}$$

may be expected, in view of (A 18), to take the WKBJ form

$$\begin{aligned} p = &\Big[\epsilon^{2/3} \bar{p}_{11}(\bar{X}) + \epsilon^{4/3} \bar{p}_{21}(\bar{X}) + \epsilon^2 \bar{p}_{31}(\bar{X}) + \epsilon^{8/3} \bar{p}_{41}(\bar{X}) + \ldots\Big] E \\ &+ \Big[\epsilon^{4/3} p_{22}(\bar{X}) + \epsilon^2 p_{32}(\bar{X}) + \ldots\Big] E^2 + c.c. + \epsilon^{8/3} \bar{p}_{4M}(\bar{X}) + \ldots \ , \end{aligned} \tag{A 20}$$

$$\begin{aligned} A = &\, \epsilon^{2/3} \bar{A}_{11}(\bar{X}) E + \Big[\epsilon^{4/3} \bar{A}_{22}(\bar{X}) + \epsilon^2 \bar{A}_{32}(\bar{X}) + \ldots\Big] E^2 + c.c. \\ &+ \epsilon^{2/3} \bar{A}_{1M}(\bar{X}) + \epsilon^{4/3} \bar{A}_{2M}(\bar{X}) + \epsilon^2 \bar{A}_{3M}(\bar{X}) + \ldots \ , \end{aligned} \tag{A 21}$$

where

$$E = \mathrm{e}^{\mathrm{i}(x-t) + \mathrm{i}\epsilon^{-4/3} \Theta_1(\bar{X}) + \mathrm{i}\epsilon^{-2/3} \Theta_2(\bar{X}) + \frac{2}{3} \mathrm{i} \ln \epsilon - \mathrm{i} \ln a_0} \ . \tag{A 22}$$

As before, the pressure-displacement interaction law yields

$$\bar{p}_{11} = \bar{A}_{11} \ , \quad \bar{p}_{21} = \Theta_1' \bar{A}_{11} \ , \quad \bar{p}_{31} = \Theta_2' \bar{A}_{11} \ , \quad \bar{p}_{41} = -\mathrm{i} \frac{\partial \bar{A}_{11}}{\partial \bar{X}} \ , \tag{A 23}$$

$$\bar{p}_{22} = 2 \bar{A}_{22} \ , \quad \bar{p}_{32} = 2 \bar{A}_{32} + 2 \Theta_1' \bar{A}_{22} \ , \tag{A 24}$$

$$\bar{p}_{4M} = \frac{1}{\pi}\frac{\partial}{\partial \bar{X}} \int_{-\infty}^{\infty} \frac{\bar{A}_{1M}}{\bar{X}-\bar{X}'}\, d\bar{X}' \ , \quad \text{and so on.} \tag{A 25}$$

In the Tollmien layer

$$\hat{u} = \hat{y} + A \ , \tag{A 26}$$

$$\hat{v} = \left[\epsilon^{2/3}\hat{v}_{11}(\bar{X},\hat{y}) + \ldots\right] E + + \left[\epsilon^{4/3}\hat{v}_{22} + \ldots\right] E^2 + c.c. + \epsilon^{8/3}\hat{v}_{4M} + \ldots \ , \tag{A 27}$$

with

$$\hat{v}_{11} = -\mathrm{i}\bar{A}_{11}\hat{y} \ , \quad \hat{v}_{21} = -\mathrm{i}\left[\bar{p}_{21} + \bar{A}_{11}\bar{A}_{1M}\right] - \mathrm{i}\Theta_1' \left[\bar{p}_{11} + \hat{y}\bar{A}_{11}\right] \ , \tag{A 28}$$

$$\hat{v}_{31} = -\mathrm{i}\left[\bar{p}_{31} + \bar{A}_{11}\bar{A}_{2M} + \bar{A}_{11}^*\bar{A}_{22}\right] - \mathrm{i}\Theta_1'\left[\bar{p}_{21} + \bar{A}_{11}\bar{A}_{1M}\right] - \mathrm{i}\Theta_2'\left[\bar{p}_{11} + \hat{y}\bar{A}_{11}\right] \ , \tag{A 29}$$

$$\begin{aligned}\hat{v}_{41} = &-\mathrm{i}\left[\bar{p}_{41} + \bar{A}_{11}\bar{A}_{3M} + \bar{A}_{11}^*\bar{A}_{32}\right] - \mathrm{i}\Theta_1'\left[\bar{p}_{31} + \bar{A}_{11}\bar{A}_{2M} + \bar{A}_{11}^*\bar{A}_{22}\right] \\ &-\mathrm{i}\Theta_2'\left[\bar{p}_{21} + \bar{A}_{11}\bar{A}_{1M}\right] - \frac{\partial}{\partial \bar{X}}\left[p_{11} + \hat{y}A_{11}\right] \ ,\end{aligned} \tag{A 30}$$

$$\hat{v}_{22} = -2\mathrm{i}\left[\bar{p}_{22} + (\hat{y}-1)\bar{A}_{22} + \frac{1}{2}\bar{A}_{11}^2\right] \tag{A 31}$$

$$\hat{v}_{32} = -2\mathrm{i}\left[\bar{p}_{32} + (\hat{y}-1)\bar{A}_{32}\right] - 2\mathrm{i}\Theta_1'\left[\bar{p}_{22} + \hat{y}\bar{A}_{22} + \frac{1}{2}\bar{A}_{11}^2\right] \ , \tag{A 32}$$

The Stokes layer velocity components expand as

$$\hat{u} = \epsilon^{2/3}\left[\bar{u}_{11}(\bar{X},\bar{y})E + c.c.\right] + \epsilon^{4/3}\left[\bar{u}_{22}E^2 + \bar{u}_{21}E + c.c. + \bar{u}_{2M}\right] + \ldots \ , \tag{A 33}$$

$$\hat{v} = \epsilon^{8/3}\left[\bar{v}_{11}(\bar{X},\bar{y})E + c.c.\right] + \epsilon^{10/3}\left[\bar{v}_{22}E^2 + \bar{v}_{21}E + c.c.\right] + \ldots \ , \tag{A 34}$$

with solutions (4.11)–(4.12) as before. The main difference comes in the diffusion layer, $\hat{y} = \epsilon^{2/3}\eta$, where the mean-flow correction is now comparable with the basic flow. The velocity expansions in this layer are

$$\hat{u} = \epsilon^{2/3}\left[u_{11}(\bar{X},\eta)E + c.c. + u_{1M}\right] + \epsilon^{4/3}\left[u_{22}E^2 + u_{21}E + c.c. + u_{2M}\right] + \ldots \ , \tag{A 35}$$

$$\hat{v} = \epsilon^{4/3}\left[v_{11}(\bar{X},\eta)E + c.c.\right] + \epsilon^{6/3}\left[v_{22}E^2 + v_{21}E + c.c.\right] + \ldots \ ; \tag{A 36}$$

it is found that

$$u_{11} = \bar{A}_{11} \ , \quad v_{11} = -\mathrm{i}\bar{A}_{11}\eta \ , \quad u_{22} = \bar{A}_{22} \ , \quad v_{22} = -2\mathrm{i}\bar{A}_{22}\eta \ , \tag{A 37}$$

$$u_{21} = \bar{A}_{11}F_{1M} \ , \quad v_{21} = -\mathrm{i}\Theta_1'\bar{A}_{11}\eta - \mathrm{i}\bar{A}_{11}\int_0^\eta F_{1M}\, d\eta' \ , \tag{A 38}$$

$$u_{31} = \bar{A}_{11}\left[F_{2M} + u_{1M}F_{1M} - \frac{\partial u_{1M}}{\partial \eta}\int_0^\eta F_{1M}\, d\eta'\right] \ , \tag{A 39}$$

$$\begin{aligned}v_{31} = &\,\mathrm{e}^{3\pi\mathrm{i}/4}\bar{A}_{11} - \mathrm{i}\Theta_2'\bar{A}_{11}\eta - \mathrm{i}\bar{A}_{11}\int_0^\eta F_{2M}\, d\eta' - \mathrm{i}\Theta_1'\bar{A}_{11}\int_0^\eta F_{1M}\, d\eta' \\ &-\mathrm{i}\bar{A}_{11}\int_0^\eta\left[u_{1M}F_{1M} - \frac{\partial u_{1M}}{\partial \eta'}\int_0^{\eta'} F_{1M}\, d\eta''\right] d\eta' \ ,\end{aligned} \tag{A 40}$$

where

$$F_{1M} = u_{1M} - \bar{A}_{1M} - \eta\frac{\partial u_{1M}}{\partial \eta} \ , \quad F_{2M} = u_{2M} - \bar{A}_{2M} - \eta\frac{\partial u_{2M}}{\partial \eta} \ . \tag{A 41}$$

The leading-order mean flow is governed by the equations

$$\frac{\partial u_{1M}}{\partial \bar{X}} + \frac{\partial v_{4M}}{\partial \eta} = 0 \ , \quad u_{1M}\frac{\partial u_{1M}}{\partial \bar{X}} + v_{4M}\frac{\partial u_{1M}}{\partial \eta} = -\frac{\partial |\bar{A}_{11}|^2}{\partial \bar{X}} + \frac{\partial^2 u_{1M}}{\partial \eta^2} \ , \tag{A 42}$$

together with the boundary conditions

$$u_{1M} = v_{4M} = 0 \text{ on } \eta = 0 \ , \quad \frac{\partial u_{1M}}{\partial \eta} \to 1 \text{ as } \eta \to \infty \ , \tag{A 43}$$

and upstream condition

$$u_{1M} \sim \eta \quad \text{as} \quad \bar{X} \to -\infty \ . \tag{A 44}$$

Matching between the Tollmien layer and diffusion layer, and making use of (A 23)–(A 24), we obtain

$$\Theta_1' = -\frac{1}{2}\bar{A}_{1M} \ , \quad \Theta_2' = -\frac{1}{2}\bar{A}_{2M} - \frac{1}{8}|\bar{A}_{11}^2| - \frac{1}{16}\bar{A}_{1M}^2 + \int_0^\infty \left(u_{1M} - \bar{A}_{1M}\right) d\eta \ , \tag{A 45}$$

and

$$\frac{\partial \bar{A}_{11}}{\partial \bar{X}} = \gamma\,(1-\mathrm{i})\,\bar{A}_{11} + \mathrm{i}\bar{A}_{11}\bar{B} \ , \quad \text{where} \quad \gamma = \frac{1}{2\sqrt{2}} \tag{A 46}$$

and $\bar{B}$ is a real quantity. It follows that

$$|\bar{A}_{11}|^2 = \mathrm{e}^{\bar{X}/\sqrt{2}} \ . \tag{A 47}$$

In order to determine the evolution of the leading-order phase $\Theta_1$ from (A 45), it is necessary to obtain

$$\bar{A}_{1M} = \lim_{\eta\to\infty} (u_{1M} - \eta) \tag{A 48}$$

from a numerical solution of the mean-flow problem (A 42)–(A 44) and (A 47).

This mean-flow solution is shown in figure 8. As might be anticipated for a boundary-layer flow in an adverse pressure gradient, the solution is seen to terminate in a Goldstein-type singularity at a finite distance downstream, $\bar{X} = \bar{X}_s$ say, with wall-shear $\partial u_{1M}/\partial\eta(\eta = 0)$ going to zero like $(\bar{X}_s - \bar{X})^{1/2}$, and displacement gradient $\partial \bar{A}_{1M}/\partial \bar{X}$ going to infinity like $(\bar{X}_s - \bar{X})^{-1/2}$.

As the singularity is approached, the mean pressure (A 25) is growing in importance. It might be supposed that at some point this mean pressure makes itself felt in equation (A 42), thus giving an interactive boundary-layer problem for the subsequent evolution of the mean flow over a shortened lengthscale. We shall not pursue this new problem, however, other than to note two possible scenarios.

The first scenario is that it is possible to solve the rescaled problem and any subsequent rescaled problems, and to obtain a downstream solution well past the position of the singularity without altering the solution already found upstream of the singularity.

The second scenario, and possibly the more likely, is analogous to steady classical-boundary-layer flow past a bluff body, where it is known that the Goldstein singularity is not 'removable' (Stewartson 1970). This is the case in which one of the rescaled problems downstream of the singularity does not have a solution, and the entire formulation is invalid — with the conclusion that a correct description (if, indeed, a solution exists) requires an order-one change to the upstream solution.

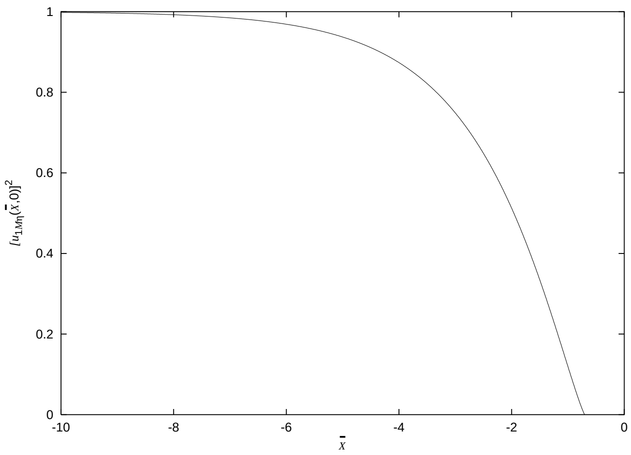


FIGURE 8. The square of the mean wall-shear, $\left[u_{1M\eta}(\bar{X},0)\right]^2$. A Goldstein-type singularity is apparently forming as the wall-shear falls to zero at a finite value of $\bar{X}$.

## Appendix B. Weakly nonlinear evolution of a dispersing wavepacket

As illustrated in §4, it is often convenient to study the weakly nonlinear evolution of a wavepacket using a frame of reference moving at the [linear] group velocity of the carrier wave. In the fully nonlinear régime studied in \$ 5, the carrier waves are nonlinear and the 'group velocities' are amplitude-dependent (and hence non-unique). As a result it is not possible to simplify the analysis by moving into the frame of the group velocity. However for smaller-amplitude waves the group velocity of the carrier wave is constant (to leading order). By transforming into a frame moving with this group velocity it is then possible to consider the evolution of a wavepacket which is modulated over a shorter spatial scale than (3.1*a*), and for which linear dispersion plays a significant rôle over the slow evolution timescale (3.1*b*). Here we outline a derivation of the amplitude equation for this problem.

In order to identify the appropriate scaling, we rewrite (4.50) in a frame of reference moving at the group velocity by means of the substitution

$$A_{11}(T,X) = \Delta\tilde{A}_{11}(T,\tilde{\xi}) \ , \quad \tilde{\xi} = \mu^{-1}(X-2T) \ , \tag{B 1}$$

which gives

$$\frac{\partial\tilde{A}_{11}}{\partial T} = \mathrm{e}^{-\mathrm{i}\pi/4}\tilde{A}_{11} + \frac{\mathrm{i}\mu\Delta^2\tilde{A}_{11}}{2\sqrt{2}}\int_0^\infty |\tilde{A}_{11}|^2\left(T-\tfrac{1}{2}\mu\tilde{\xi}',\tilde{\xi}+\tilde{\xi}'\right)d\tilde{\xi}' \ . \tag{B 2}$$

It follows that for a 'short' wavepacket, with $\mu \ll 1$, it is appropriate to scale the amplitude with $\Delta = \mu^{-1/2}$ to preserve the nonlinear term. As shown by Smith (1986a), a new régime comes into play when $\mu = O(\epsilon)$, and dispersive effects enter the reckoning. Accordingly we introduce the variables

$$\sigma = \epsilon^{-2}\,(X-T) \ , \quad \xi = \epsilon^{-1}\,(X-2T) \tag{B 3}$$

and expand

$$\begin{aligned} p &= \epsilon^{1/2}\left[p_{11}(T,\xi)\mathrm{e}^{\mathrm{i}\sigma} + c.c.\right] + \epsilon\left[p_{22}\mathrm{e}^{2\mathrm{i}\sigma} + c.c. + p_{2M}\right] + \epsilon^{3/2}\left[p_{31}\mathrm{e}^{\mathrm{i}\sigma} + p_{33}\mathrm{e}^{3\mathrm{i}\sigma} + c.c.\right] \\ &\quad + \epsilon^2\left[p_{42}\mathrm{e}^{2\mathrm{i}\sigma} + p_{44}\mathrm{e}^{4\mathrm{i}\sigma} + c.c. + p_{4M}\right] + \epsilon^{5/2}\left[p_{51}\mathrm{e}^{\mathrm{i}\sigma} + \ldots\right] + \ldots \ , \qquad \text{(B 4)} \\ A &= \epsilon^{1/2}\left[A_{11}(T,\xi)\mathrm{e}^{\mathrm{i}\sigma} + c.c.\right] + \epsilon\left[A_{22}\mathrm{e}^{2\mathrm{i}\sigma} + c.c. + A_{2M}\right] \end{aligned}$$

$$+\epsilon^{3/2}\left[A_{31}\mathrm{e}^{\mathrm{i}\sigma}+A_{33}\mathrm{e}^{3\mathrm{i}\sigma}+c.c.\right]+\epsilon^{2}\left[A_{42}\mathrm{e}^{2\mathrm{i}\sigma}+A_{44}\mathrm{e}^{4\mathrm{i}\sigma}+c.c.+A_{4M}\right]$$
$$+\epsilon^{5/2}\left[A_{51}\mathrm{e}^{\mathrm{i}\sigma}+\ldots\right]+\ldots\ ; \tag{B 5}$$

without loss of generality we may take

$$A_{31}=A_{51}=\ldots=0\ . \tag{B 6}$$

For simplicity, we restrict attention to localised wavepackets, for which

$$A_{11}\to 0 \quad \text{as} \quad \xi\to\pm\infty \tag{B 7}$$

(as opposed to $\xi$-periodic wavetrains, say, which can in fact be handled by a slight modification of the following analysis).

The pressure-displacement interaction law gives

$$p_{11}=A_{11}\ ,\quad p_{22}=2A_{22}\ ,\quad p_{2M}=0\ ,\quad p_{31}=-\mathrm{i}\frac{\partial A_{11}}{\partial\xi}\ ,\quad p_{33}=3A_{33}\ , \tag{B 8}$$

$$p_{42}=2A_{42}-\mathrm{i}\frac{\partial A_{22}}{\partial\xi}\ ,\quad p_{4M}=\frac{1}{\pi}\frac{\partial}{\partial\xi}\int_{-\infty}^{\infty}\frac{A_{2M}}{\xi-\xi'}\,d\xi'\ ,\quad p_{51}=0\ . \tag{B 9}$$

In the Tollmien layer (where $\hat{y}$ is of order one) we may take

$$\hat{u}=\hat{y}+A\ , \tag{B 10}$$

and

$$\hat{v}=\epsilon^{1/2}\left[\hat{v}_{11}(T,\xi,\hat{y})\mathrm{e}^{\mathrm{i}\sigma}+c.c.\right]+\epsilon\left[\hat{v}_{22}\mathrm{e}^{2\mathrm{i}\sigma}+c.c.\right]+\epsilon^{3/2}\left[\hat{v}_{31}\mathrm{e}^{\mathrm{i}\sigma}+\hat{v}_{33}\mathrm{e}^{3\mathrm{i}\sigma}+c.c.\right]$$
$$+\epsilon^{2}\left[\hat{v}_{42}\mathrm{e}^{2\mathrm{i}\sigma}+\hat{v}_{44}\mathrm{e}^{4\mathrm{i}\sigma}+c.c.+\hat{v}_{4M}\right]+\epsilon^{5/2}\left[\hat{v}_{51}\mathrm{e}^{\mathrm{i}\sigma}+\ldots\right]+\epsilon^{3}\left[\hat{v}_{6M}+\ldots\right]+\ldots\ , \tag{B 11}$$

where

$$\hat{v}_{11}=-\mathrm{i}A_{11}\hat{y}\ ,\quad \hat{v}_{22}=-2\mathrm{i}A_{22}\left(\hat{y}+1\right)-\mathrm{i}A_{11}^{2}\ ,\quad \hat{v}_{31}=-\frac{\partial A_{11}}{\partial\xi}\hat{y}-\mathrm{i}G_{1}\ , \tag{B 12}$$

$$\hat{v}_{33}=-3\mathrm{i}A_{33}\left(\hat{y}+2\right)-3\mathrm{i}A_{11}A_{22}\ , \tag{B 13}$$

$$\hat{v}_{42}=-2\mathrm{i}A_{42}\left(\hat{y}+1\right)-\frac{\partial A_{22}}{\partial\xi}\left(\hat{y}+2\right)-2\mathrm{i}\alpha\left[A_{22}A_{2M}+A_{11}^{*}A_{33}\right]-A_{11}\frac{\partial A_{11}}{\partial\xi}\ , \tag{B 14}$$

$$\hat{v}_{4M}=-\frac{\partial}{\partial\xi}\left[A_{2M}\left(\hat{y}-2\right)+\left|A_{11}\right|^{2}\right]\ ,\quad \hat{v}_{51}=\mathrm{i}\frac{\partial^{2}A_{11}}{\partial\xi^{2}}-\frac{\partial A_{11}}{\partial T}-\frac{\partial G_{1}}{\partial\xi}-\mathrm{i}G_{2}\ , \tag{B 15}$$

$$\hat{v}_{6M}=-\frac{1}{\pi}\frac{\partial^{2}}{\partial\xi^{2}}\int_{-\infty}^{\infty}\frac{A_{2M}}{\xi-\xi'}\,d\xi'-\frac{\partial A_{2M}}{\partial T}-\frac{\partial}{\partial\xi}\left[A_{4M}\left(\hat{y}-2\right)+\tfrac{1}{2}A_{2M}^{2}+\left|A_{22}\right|^{2}\right]\ , \tag{B 16}$$

and we have defined

$$G_{1}=A_{11}A_{2M}+A_{11}^{*}A_{22}\ ,\quad G_{2}=A_{11}A_{4M}+A_{11}^{*}A_{42}+A_{22}^{*}A_{33}\ . \tag{B 17}$$

We omit the analysis of the Stokes layer and diffusion layer (which in this case has $\hat{y}$ of order $\epsilon^{3/2}$), and merely note that it imposes the requirements

$$\hat{v}_{22}\to 0\ ,\quad \hat{v}_{33}\to 0 \quad \text{and} \quad \hat{v}_{42}\to 0 \quad \text{as} \quad \hat{y}\to 0\ , \tag{B 18}$$

from which we obtain

$$A_{22}=-\tfrac{1}{2}A_{11}^{2}\ ,\quad A_{33}=\frac{1}{4}A_{11}^{3}\ ,\quad A_{42}=-\frac{\mathrm{i}}{4}\frac{\partial\left(A_{11}^{2}\right)}{\partial\xi}+\tfrac{1}{2}A_{11}G_{1}\ . \tag{B 19}$$

It is further required that

$$\hat{v}_{4M} \to 0 \; , \quad \hat{v}_{6M} \to 0 \quad \text{as} \quad \hat{y} \to 0 \; , \tag{B 20}$$

and we expect that all perturbation quantites go to zero ahead of the wavepacket, i.e. as $\xi \to +\infty$, in which case

$$A_{2M} = \tfrac{1}{2}|A_{11}|^2 \; , \tag{B 21}$$

$$A_{4M} = \frac{3}{16}|A_{11}|^4 + \frac{1}{2\pi}\frac{\partial}{\partial \xi}\,⨍_{-\infty}^{\infty} \frac{A_{2M}}{\xi - \xi'}\,d\xi' - \frac{1}{2}\int_{\xi}^{\infty} \frac{\partial A_{2M}}{\partial T}d\xi' \; , \tag{B 22}$$

Finally, we have that

$$\hat{v}_{31} \to 0 \; , \quad v_{51} \to \mathrm{e}^{3\pi\mathrm{i}/4}A_{11} \quad \text{as} \quad \hat{y} \to 0 \; . \tag{B 23}$$

The first of (B 23) is automatically satisfied since $G_1 = 0$; the second gives the amplitude-evolution equation

$$\frac{\partial A_{11}}{\partial T} - \mathrm{i}\frac{\partial^2 A_{11}}{\partial \xi^2} = \mathrm{e}^{-\mathrm{i}\pi/4}A_{11} - \tfrac{1}{2}|A_{11}|^2\frac{\partial A_{11}}{\partial \xi} + \mathrm{i}A_{11}\tilde{B} \; , \tag{B 24}$$

where

$$\tilde{B} = -\frac{1}{16}|A_{11}|^4 - \frac{1}{4\pi}\frac{\partial}{\partial \xi}\,⨍_{-\infty}^{\infty} \frac{|A_{11}|^2}{\xi - \xi'}\,d\xi' + \frac{1}{4}\int_{\xi}^{\infty} \frac{\partial |A_{11}^2|}{\partial T}\,d\xi' \; . \tag{B 25}$$

Since $\tilde{B}$ is a real quantity, it follows from (B 24) that

$$\frac{\partial}{\partial T}\left(|A_{11}|^2\right) = \frac{\partial}{\partial \xi}\left(\mathrm{i}A_{11}^*\frac{\partial A_{11}}{\partial \xi} - \mathrm{i}A_{11}\frac{\partial A_{11}^*}{\partial \xi} - \frac{1}{4}|A_{11}|^4\right) + \sqrt{2}|A_{11}|^2 \; , \tag{B 26}$$

and hence

$$\int_{-\infty}^{\infty} |A_{11}|^2\,d\xi = H_0\mathrm{e}^{\sqrt{2}T} \; , \tag{B 27}$$

where $H_0$ is a constant; the evolution of this 'energy' integral is thus unaffected by nonlinearity (as in Smith 1986a). On making use of (B 26), we may rewrite the amplitude equation (B 24)–(B 25) in the form

$$\begin{aligned} \frac{\partial A_{11}}{\partial T} - \mathrm{i}\frac{\partial^2 A_{11}}{\partial \xi^2} &= \mathrm{e}^{-\mathrm{i}\pi/4}A_{11} - \frac{1}{4}A_{11}\frac{\partial |A_{11}^2|}{\partial \xi} \\ &+\mathrm{i}A_{11}\left[\frac{1}{2\sqrt{2}}\int_{\xi}^{\infty} |A_{11}|^2\,d\xi' - \frac{1}{4\pi}\frac{\partial}{\partial \xi}\,⨍_{-\infty}^{\infty} \frac{|A_{11}|^2}{\xi - \xi'}\,d\xi'\right] \; . \end{aligned} \tag{B 28}$$

We shall not study further properties of the amplitude equation here, and merely note that it differs from that of Smith (1986a) for the same reason that (4.47)–(4.48) differs from that of Smith & Burggraf (1985), namely the absence of a diffusion layer in the analysis of Smith (1986a). Instead, he takes $A_{2M} = 3|A_{11}|^2$, with the consequence that $G_1 \equiv \frac{5}{2}|A_{11}|^2 A_{11}$, and $v_{31}$ cannot satisfy the boundary condition (B 23). It is thus necessary for his amplitude expansion to proceed in powers of $\epsilon$ rather than $\epsilon^{1/2}$ as here, leading to an amplitude equation of Ginzburg-Landau type. In our analysis, the (fortuitous) vanishing of $G_1$ gives rise to an amplitude equation involving higher-order nonlinearities (as in Johnson 1977, Dysthe 1979 for example). For a more general interactive boundary-layer problem, however, there is no reason to expect that $G_1$ will vanish. In a wall jet, for example, for which the pressure-displacement interaction law (2.19) is replaced by $\tilde{p} = -\tilde{A}_{\tilde{x}\tilde{x}}$ (Smith & Duck 1977, Ryzhov 1982), we find that $G_1 = \frac{1}{6}|A_{11}|^2 A_{11}$. It follows that a weakly nonlinear wavepacket of the kind under consideration here is described by an expansion of the form

$$A = \epsilon A_{11}(T,\xi)\mathrm{e}^{\mathrm{i}\sigma} + c.c. + O\left(\epsilon^2\right) \ , \tag{B 29}$$

where

$$\sigma = \epsilon^{-2}\left(X - T\right) \ , \quad \xi = \epsilon^{-1}\left(X - 3T\right) \ , \tag{B 30}$$

and $A_{11}$ satisfies the Ginzburg-Landau equation

$$\frac{\partial A_{11}}{\partial T} - 3\mathrm{i}\frac{\partial^2 A_{11}}{\partial \xi^2} = \mathrm{e}^{-\mathrm{i}\pi/4}A_{11} - \tfrac{1}{6}\mathrm{i}|A_{11}|^2 A_{11} \ . \tag{B 31}$$

Solutions of this equation are discussed by Smith (1986*a*,*b*) and Smith, Stewart & Bowles (1994).

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