When is the weakly nonlinear evolution of a localized disturbance governed by the Ginzburg-Landau equation?

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The evolution of weakly nonlinear, modulated disturbances in marginally unstable systems has been the subject of intensive study. In a wide range of problems, it has been proposed that such disturbances are described by the (complex) Ginzburg-Landau equation, or in the case of two-dimensional modulations in certain systems, by the Davey-Stewartson equations with complex coefficients. In this paper we re-examine the evolution of an initially linear *localized* disturbance into the weakly nonlinear régime. We show that for one-dimensional modulations the weakly non-linear evolution is governed, initially at least, by the Landau equation, rather than the Ginzburg-Landau equation. For two-dimensional modulations, the evolution is governed by a reduced form of the Davey-Stewartson equations.

We apply our revised theory to Poiseuille-Couette flow. For both one- and two-dimensional modulations we find that a localized disturbance can always grow to order-one amplitude if its initial magnitude is sufficiently small (contrary to earlier predictions).

Keywords: marginally unstable system, amplitude evolution equation, focusing singularity, Davey-Stewartson equations, plane Poiseuille flow, Poiseuille-Couette flow

1. Introduction

(a) *Background*

The weakly nonlinear evolution of disturbances in marginally unstable systems has been the subject of intensive study since Stuart(1960) and Watson (1960) first demonstrated how amplitude equations can be derived to describe the growth of disturbances in slightly supercritical plane parallel flows, while Palm (1960) did likewise for a model of Bénard convection (see also Gor'kov 1957, Malkus & Veronis 1958). In these seminal papers of Stuart (1960) and Watson (1960), and in many subsequent analyses, attention was restricted to two-dimensional periodic disturbances, and the evolution was found to be governed by the Landau (1944) equation (that is, (2.15) below with $\hat{a}_2 = 0$). An extension to disturbances of slightly more general form was given by Newell & Whitehead (1969) and Segel (1969), who considered a slow spatial modulation of the amplitude of weakly nonlinear roll cells in Bénard convection. With a suitable choice of modulation length-scale, these authors showed that the development of a disturbance modulated in one direction can be described by the Ginzburg-Landau equation (Ginzburg & Landau 1950), i.e. (2.15) with real coefficients. The same equation, but with complex coefficients in general, was obtained by DiPrima, Eckhaus & Segel (1971), Newell & Whitehead (1971) and Stewartson & Stuart (1971) to describe weakly modulated disturbances in a variety of physical systems. The properties of the complex Ginzburg-Landau (CGL) equation were investigated by Hocking, Stewartson & Stuart (1972) and Hocking & Stewartson (1972), who found that depending on the values of the coefficients, solutions may or may not terminate in a localized finite-time singularity (which may take one of two different forms). More recent work has concentrated for the most part on bounded solutions, which may display chaotic behaviour (see for example Landman 1987 and references therein).

In this paper we shall examine the applicability of the CGL equation to a particular type of disturbance, namely one which starts off small enough for linear instability theory to be applicable, and is localized in one direction ('line disturbance') or both directions ('point disturbance'). This problem was first studied by Stewartson & Stuart (1971) who were concerned with the development of a two-dimensional line disturbance in plane Poiseuille flow (PPF); they claim that when weakly nonlinear effects first become significant, the disturbance amplitude is governed by the CGL equation. The coefficients are such that only bounded solutions are possible, which would seem to imply that two-dimensional disturbances cannot 'burst', that is, grow to order-one amplitudes, if they are initially small and localized. The analysis was extended to three-dimensional disturbances by Hocking *et al.* (1972) and Davey, Hocking & Stewartson (1974), who obtained a (non-integrable) version of the Davey-Stewartson equations (Davey & Stewartson 1974) with complex coefficients, namely (2.12); we shall refer to these as the DHSS equations. These equations reduce to the CGL equation in the case of disturbances modulated in one direction only. For such a 'skewed line disturbance' in PPF, the CGL equation admits a finite-time singularity only if the direction of modulation is inclined at more than $57.3°$ to the wave. Fully three-dimensional disturbances were considered by Hocking (1974), who showed that a localized finite-time singularity is possible for certain values of the coefficients, though not, apparently, for those appropriate to PPF. According to Hocking (1974), 'the present evidence supports the view that centred disturbances in slightly supercritical plane Poiseuille flow do not develop localized bursts of relatively large magnitude'. On the other hand, a corresponding analysis for the asymptotic suction boundary layer by Hocking (1975) led to the two-dimensional CGL equation, i.e. (2.12a) with $q = 0$. In this case, the coefficients are such as to admit a finite-time singularity, for both line and point disturbances. In other systems, such as Bénard convection, the amplitude equations for three-dimensional disturbances may take a different form: a comprehensive review of the subject can be found in Cross & Hohenberg (1993).

The principal aim of this paper is to show that when nonlinear effects first become important in the evolution of an initially localized linear disturbance, the amplitude equations have a simpler form than previously recognised. In the case of a line disturbance, the amplitude equation is the Landau equation, rather than the Ginzburg-Landau equation: the linear 'dispersive' term is missing from the amplitude equation even though the disturbance is spatially modulated. For a point disturbance in PPF the amplitude is governed by a reduced form of the DHSS equations, without the dispersive terms. It may be noted that dispersion is indeed important during the initial linear phase of evolution, but by the time nonlinearity comes into play, the disturbance has spread to a sufficient extent that the dispersive terms are asymptotically small compared to the linear growth term. Moreover, the smaller the initial amplitude, the more the disturbance will have spread before the weakly nonlinear phase is reached, and the smaller the dispersive terms will then be.

For a 'subcritical' system such as PPF, solutions of both the Landau equation and reduced DHSS equations are found to blow up in a finite time. This does not necessarily mean that the disturbance 'bursts' since it turns out that the dispersion terms are not uniformly negligible as the singularity is approached. Nevertheless, if these terms are sufficiently small at the onset of nonlinearity, the disturbance will attain order-one amplitude before dispersion can come into play. A consequence is that in PPF, *any* localized (line or point) disturbance can 'burst' *if its initial magnitude is sufficiently small.*

(*b*) *Outline of paper*

We believe that these observations can apply to a wide range of marginally unstable situations. However, since the evolution of initially small localized distubances was first considered by Stewartson & Stuart (1971) for the particular case of PPF, we concentrate on this example. We also extend the analysis to plane Poiseuille-Couette flow (PPCF), and to a three-dimensional version of this flow.

At the start of §2 we briefly outline the approach of Stewartson & Stuart (1971) and Davey *et al.* (1974), and indicate their distinguished scaling for the modulation length. We then explain why we believe that a small logarithmic change to this scaling is needed if the first effects of weak nonlinearity are to be accounted for correctly.

In §3 we give the modifications necessary for a line disturbance. We show that in the subcritical case, the solution terminates in a finite-time singularity on this scaling. It turns out that if the initial amplitude is sufficiently small, this singularity describes the evolution up to 'bursting', that is, when the disturbance attains order-one amplitude (and any weakly nonlinear theory becomes invalid). For slightly larger initial amplitudes, however, the linear dispersive term is reintroduced before this happens. The next stage of evolution is then described by a (complex) Ginzburg-Landau equation, but without a linear growth term. In §4–§5, the analysis is extended to point disturbances. Again it is found that for a subcritical system the solution terminates in a finite-time singularity, which may or may not be valid right up to 'bursting'.

The implications of this revised analysis for the well-known results for PPF are discussed in §7. First, however, in §6 the amplitude equations are derived for a model three-dimensional shear flow, namely, three-dimensional Poiseuille-Couette flow. After a suitable rotation of axes, these amplitude equations can be reduced to those for a two-dimensional basic flow (albeit with a slightly different initial condition); there seems no reason to expect any significantly new qualitative features for a three-dimensional basic state.

2. Formulation

(a) The large-time linear solution for a small localized disturbance

For definiteness the analysis will be presented for the specific example of plane Poiseuille-Couette flow (PPCF), following as closely as possible the notation of Stewartson & Stuart (1971), Hocking *et al.* (1972) and Davey *et al.* (1974). We thus consider the flow of an incompressible fluid of density ρ and kinematic viscosity ν between two infinite parallel plates a distance $2h$ apart, at a Reynolds number $R = U_0 h/\nu$ where U_0 is the centreline speed of the undisturbed flow. A set of Cartesian axes $Oxyz$ is chosen with origin O on the centreline, and $h[x, y, z]$ measuring distances in the streamwise (flow) direction, in the spanwise direction, and in the direction perpendicular to the plates, respectively. The components of fluid velocity are written as $U_0[u, v, w]$, while ht/U_0 measures time and $\rho U_0^2 P$ pressure. If the upper and lower walls move with velocity $\pm U_1$ in the x-direction, the undisturbed flow takes the form

$$u = \bar{U}(z) \equiv 1 - z^2 + u_{\mathrm{w}} z \ , \quad v = w = 0 \ , \quad \text{where } u_{\mathrm{w}} = U_1/U_0. \tag{2.1}$$

For sufficiently small wall velocities, with $|u_{\mathrm{w}}| < u_{\mathrm{w}}^* \approx 0.3519$, there is a critical Reynolds number, R_{cr}, above which there exist exponentially growing linear eigenmodes of the form

$$[\tilde{u}, \tilde{v}, \tilde{w}] \equiv \left[u - \bar{U}(z), v, w\right] = \Delta \left[\tilde{U}(z), \tilde{V}(z), \tilde{W}(z)\right] \mathrm{e}^{\mathrm{i}\alpha(x-ct)+\mathrm{i}\beta y} + c.c. \ , \tag{2.2}$$

where Δ is an infinitesimal parameter and *c.c.* denotes complex conjugate (Heisenberg 1924, Potter 1966, Reynolds & Potter 1967, Hains 1967, Cowley & Smith 1985). At $R = R_{cr}$, it follows from Squire's (1933) transformation (see §6) that the neutral eigenmode is two-dimensional and has the form

$$[\tilde{u}, \tilde{v}, \tilde{w}] = \Delta \left[\psi'(z), 0, -\mathrm{i}\alpha_{cr}\psi(z)\right] \mathrm{e}^{\mathrm{i}\alpha_{cr}(x-c_{cr}t)} + c.c. \ , \tag{2.3}$$

where α_{cr} and c_{cr} are real, and $\psi(z)$ is the corresponding eigenfunction of the Orr-Sommerfeld problem (i.e. (6.10) with R_{cr} in place of $\hat{R}_{cr}$). In the neighbourhood of the neutral point in (R, α, β)-space, namely $R = R_{cr}$, $\alpha = \alpha_{cr}$ and $\beta = 0$, the complex growth rate of linearised disturbances can be expanded in the form

$$-\mathrm{i}\alpha c = -\mathrm{i}\alpha_{cr}c_{cr} + \mathrm{i}a_{1\mathrm{r}}(\alpha - \alpha_{cr}) - a_2(\alpha - \alpha_{cr})^2 - b_2\beta^2 + (R - R_{cr})d_1 + \dots \ , \tag{2.4}$$

where $-a_{1\mathrm{r}}$ is the (real) group velocity, while a_2, b_2 and d_1 are complex constants. Further, $\mathrm{Re}(d_1)$ must be positive for the flow to be linearly unstable for $R > R_{cr}$, while $\mathrm{Re}(a_2)$ and $\mathrm{Re}(b_2)$ must be positive for the growth rate, $\mathrm{Im}(\alpha c)$, to be maximised at $R = R_{cr}$ by $\alpha = \alpha_{cr}$, $\beta = 0$.† Numerical values of these constants for PPF are given by Davey *et al.* (1974); our calculated values for PPCF are given in table 1.

Stewartson & Stuart (1971) and Hocking *et al.* (1972) suppose that the flow is at a slightly supercritical Reynolds number, i.e. $0 < R - R_{cr} \ll 1$, and that at $t = 0$

† In principle, one or more of these quantities might be zero, but such exceptional cases will not concern us here; they do not arise in the case of PPCF.

Table 1. *Coefficients of (2.4) and (2.12a) for plane Poiseuille-Couette flow (2.1).*

u_w	R_{cr}	α_{cr}	c_{cr}	a_{1r}	a_{2r}	a_{2i}	b_{2r}
0.0	5 772.22	1.020 55	0.264 0	-0.383 1	0.186 7	0.027 5	0.004 66
0.01	6 024.83	1.007 90	0.260 1	-0.378 9	0.189 4	0.033 9	0.004 54
0.025	7 473.43	0.946 50	0.240 5	-0.355 6	0.203 3	0.057 5	0.004 00
0.05	12 792.65	0.804 41	0.191 7	-0.285 0	0.213 7	0.070 7	0.003 17
0.075	16 805.10	0.711 07	0.158 1	-0.230 0	0.189 1	0.069 0	0.003 41
0.1	17 384.56	0.655 52	0.138 3	-0.195 2	0.165 5	0.071 0	0.004 05
0.15	16 351.46	0.568 14	0.108 1	-0.139 8	0.131 5	0.073 4	0.005 32
0.2	16 197.29	0.480 20	0.078 3	-0.085 7	0.106 7	0.065 3	0.006 48
0.25	17 854.85	0.384 52	0.046 7	-0.030 4	0.088 5	0.041 0	0.007 41
0.3	23 824.72	0.272 25	0.012 0	0.026 6	0.077 4	-0.011 9	0.007 54
0.32	30 364.01	0.215 16	-0.003 7	0.050 7	0.076 8	-0.053 2	0.006 94
0.33	36 830.90	0.179 91	-0.012 3	0.063 3	0.079 1	-0.087 1	0.006 28
0.34	50 436.45	0.134 68	-0.021 5	0.076 5	0.087 9	-0.148 7	0.005 11
0.343 8	61 416.29	0.112 07	-0.025 3	0.081 8	0.097 1	-0.193 8	0.004 40
0.345	66 640.97	0.103 78	-0.026 6	0.083 5	0.101 9	-0.214 6	0.004 12
0.346 3	74 083.29	0.093 87	-0.027 9	0.085 4	0.109 1	-0.243 6	0.003 77
0.347 5	83 669.97	0.083 56	-0.029 2	0.087 2	0.118 7	-0.280 3	0.003 39

u_w	b_{2i}	d_r	d_i	k_r	k_i	q_r	q_i
0.0	0.080 83	1.683(-6)	8.113(-6)	30.96	-172.83	-1.27	29.07
0.01	0.081 46	1.532(-6)	7.589(-6)	37.86	-171.30	-1.31	28.48
0.025	0.083 63	9.584(-7)	5.475(-6)	62.71	-172.82	-1.40	25.27
0.05	0.083 01	3.205(-7)	2.526(-6)	53.93	-190.39	-0.92	16.43
0.075	0.078 70	2.054(-7)	1.691(-6)	26.37	-181.69	-0.42	10.23
0.1	0.074 88	2.001(-7)	1.559(-6)	22.29	-168.03	-0.24	7.36
0.15	0.067 69	2.102(-7)	1.572(-6)	36.42	-145.04	-0.18	5.00
0.2	0.059 13	1.845(-7)	1.464(-6)	58.86	-120.98	-0.20	3.84
0.25	0.048 33	1.227(-7)	1.152(-6)	85.36	-77.02	-0.24	3.01
0.3	0.034 25	4.694(-8)	6.540(-7)	62.13	-2.02	-0.27	2.22
0.32	0.026 86	2.116(-8)	4.145(-7)	31.54	8.80	-0.26	1.83
0.33	0.022 31	1.105(-8)	2.883(-7)	18.02	7.69	-0.25	1.57
0.34	0.016 55	3.679(-9)	1.587(-7)	7.48	4.11	-0.21	1.22
0.343 8	0.013 72	1.800(-9)	1.087(-7)	4.42	2.52	-0.18	1.03
0.345	0.012 68	1.347(-9)	9.280(-8)	3.57	2.02	-0.17	0.96
0.346 3	0.011 45	8.969(-10)	7.555(-8)	2.73	1.51	-0.16	0.87
0.347 5	0.010 18	5.665(-10)	5.957(-8)	2.02	1.06	-0.15	0.78

a small disturbance is introduced which is smooth and localized about the origin. In the case of a spanwise-independent ('line') disturbance which remains sufficiently small for linear theory to apply throughout, they show, following Benjamin (1961), Tam (1967), Gaster (1968) and others, that when $t \gg 1$ and $|x + a_{1r}t| \ll t$

$$\tilde{u} \sim \frac{\mathcal{A}_1 \psi'(z)}{t^{1/2}} \mathrm{e}^{\mathrm{i}\alpha_{cr}(x - c_{cr}t)} \exp\left[(R - R_{cr})d_1 t - \frac{(x + a_{1r}t)^2}{4a_2 t}\right] + c.c. \ , \qquad (2.5)$$

where $\mathcal{A}_1$ is a small (complex) constant. For a 'skewed line disturbance', that is, of the form

$$\mathrm{e}^{\mathrm{i}\alpha_{cr}(x - c_{cr}t)} \times \text{function of } (\hat{x}, z, t) \ , \quad \text{where} \quad \hat{x} = x\cos\hat{\theta} + y\sin\hat{\theta} \ , \qquad (2.6)$$

this generalizes to

$$\tilde{u} \sim \frac{\mathcal{A}_1\psi'(z)}{t^{1/2}} \mathrm{e}^{\mathrm{i}\alpha_{cr}(x-c_{cr}t)} \exp\left[(R-R_{cr})d_1 t - \frac{(\hat{x}+\hat{a}_{1\mathrm{r}}t)^2}{4\hat{a}_2 t}\right] + c.c.\ , \tag{2.7}$$

with $\hat{a}_{1\mathrm{r}} = a_{1\mathrm{r}}\cos\hat{\theta}$, $\hat{a}_2 = a_2\cos^2\hat{\theta} + b_2\sin^2\hat{\theta}$. Finally, if the initial disturbance is 'point-like', that is, localized in both the streamwise and spanwise directions, then (2.5) is modified to

$$\tilde{u} \sim \frac{\mathcal{A}_2\psi'(z)}{t} \mathrm{e}^{\mathrm{i}\alpha_{cr}(x-c_{cr}t)} \exp\left[(R-R_{cr})d_1 t - \frac{(x+a_{1\mathrm{r}}t)^2}{4a_2 t} - \frac{y^2}{4b_2 t}\right] + c.c.\,, \tag{2.8}$$

where $\mathcal{A}_2$ is a small (complex) constant, and $|y| \ll t$ in addition to $|x + a_{1\mathrm{r}}t| \ll t$.

(b) *Modulational amplitude equations*

In order to examine the weakly nonlinear stage subsequent to (2.5), (2.7) or (2.8), Stewartson & Stuart (1971), Hocking *et al.* (1972) and Davey *et al.* (1974) derived evolution equations for the modulation of the carrier wave

$$E = \mathrm{e}^{\mathrm{i}\alpha_{cr}(x-c_{cr}t)}\ . \tag{2.9}$$

They defined

$$\varepsilon = (R-R_{cr})d_{1\mathrm{r}}\ , \quad \tau = \varepsilon t\ , \quad \xi = \varepsilon^{1/2}(x+a_{1\mathrm{r}}t)\ , \quad \eta = \varepsilon^{1/2}y\ , \tag{2.10}$$

where the subscripts 'r' and 'i' are used to denote the real and imaginary parts respectively, and set

$$\tilde{u} = \varepsilon^{1/2}A(\xi,\eta,\tau)\psi'(z)E + c.c. + O(\varepsilon)\ , \tag{2.11}$$

where A is the modulation amplitude. (In order for this to be precisely defined, it is necessary to impose a suitable normalisation condition on ψ, conventionally $\psi(0) = 1$.) Similar expansions were posed for $\tilde{v}$, $\tilde{w}$ and the pressure perturbation $\tilde{p}$. A detailed analysis (see also Appendix B) led to the amplitude equations

$$A_\tau - a_2A_{\xi\xi} - b_2A_{\eta\eta} = \frac{d_1}{d_{1\mathrm{r}}}A + k|A^2|A + qAB\ , \tag{2.12a}$$

$$B_{\xi\xi} + B_{\eta\eta} = |A^2|_{\eta\eta}\ , \tag{2.12b}$$

where the complex constants k and q may be calculated as described in Appendix Appendix B. The values of these constants for PPCF are given in table 1†; we note in particular that for any wall velocity, $\mathrm{Re}(k) > 0$ and $\mathrm{Re}(q) < 0$. Higher-order corrections may be calculated if desired; these take the form of terms

$$\varepsilon^{1/2}\left[a_3A_{\xi\xi\xi} + b_3A_{\xi\eta\eta} + k_1A|A^2|_\xi + k_2|A^2|A_\xi + q_1AB_1 + q_2AB_\xi \right. \\ \left. + q_3BA_\xi + q_4A_\eta\int^\eta B_\xi\,(\xi,\eta',\tau)\,\mathrm{d}\eta'\right] + O\,(\varepsilon) \tag{2.13}$$

† The values of k can be obtained from Reynolds & Potter (1967), but allowance must be made for the different normalisation used there.

on the right-hand side of (2.12), where B_1 is the solution of

$$B_{1\xi\xi} + B_{1\eta\eta} = \sigma_1 \left(AA^*_\xi\right)_{\eta\eta} + \sigma_2 \left(AA^*_\eta\right)_{\xi\eta} + c.c. \tag{2.14}$$

(the asterisks denoting complex conjugate), and a_3, b_3 and the k_n, q_n, and σ_n are complex constants whose values need not concern us here. For a point disturbance, localized in both ξ and η, (2.12*b*) and (2.14) are to be solved subject to $B \to 0$, $B_1 \to 0$ as $\xi^2 + \eta^2 \to \infty$ (and the lower limit of the integral in (2.13) taken as $-\infty$). For a (skewed) line disturbance of the form (2.6), with A and B assumed to vanish as $|\hat{\xi}| \to \infty$, equations (2.12) reduce to the CGL equation

$$A_\tau - \hat{a}_2 A_{\hat{\xi}\hat{\xi}} = \frac{d_1}{d_{1\mathrm{r}}} A + \hat{k}|A^2|A \ , \tag{2.15}$$

where

$$\hat{\xi} = \varepsilon^{1/2}(\hat{x} + \hat{a}_{1\mathrm{r}} t) \equiv \xi \cos\hat{\theta} + \eta \sin\hat{\theta} \ , \quad \hat{k} = k + q \sin^2\hat{\theta} \ . \tag{2.16}$$

The correction terms (2.13) here take the form

$$\varepsilon^{1/2} \left(\hat{a}_3 A_{\hat{\xi}\hat{\xi}\hat{\xi}} + \hat{k}_1 A|A^2|_{\hat{\xi}} + \hat{k}_2 |A^2| A_{\hat{\xi}}\right) + O\left(\varepsilon\right) \tag{2.17}$$

where

$$\left.\begin{aligned} \hat{a}_3 &= a_3 \cos^3\hat{\theta} + b_3 \sin^2\hat{\theta}\cos\hat{\theta} \ , \\ \hat{k}_1 &= k_1 \cos\hat{\theta} + \left[q_1\left(\sigma_1 + \sigma_2\right) + q_2 + q_4\right] \sin^2\hat{\theta}\cos\hat{\theta} \ , \\ \hat{k}_2 &= k_2 \cos\hat{\theta} + \left[q_3 - 2\mathrm{i} q_1 \left(\sigma_{1\mathrm{i}} + \sigma_{2\mathrm{i}}\right)\right] \sin^2\theta \cos\hat{\theta} \ . \end{aligned}\right\} \tag{2.18}$$

According to Stewartson & Stuart (1971) and Hocking *et al.* (1972), the appropriate initial condition corresponding to (2.7) is

$$A \approx \frac{\Delta}{\tau^{1/2}} \exp\left(-\frac{\hat{\xi}^2}{4\hat{a}_2\tau}\right) \quad \text{as } \tau \to 0^+ \ , \tag{2.19}$$

where '$\Delta (\ll 1)$ is a constant of the order of the original infinitesimal disturbance' (Stewartson & Stuart 1971, p544). That Δ must be an *asymptotically small* quantity is emphasised by Hocking *et al.* (1972, p725): 'when $\Delta \approx 1$, [(2.19)] is inconsistent with [(2.15)], since the nonlinear term is not negligible in comparison with the linear term when $\tau \ll 1$'. For a point initial condition, Hocking *et al.* (1972) and Davey *et al.* (1974) propose that the system (2.12) should be solved subject to the initial condition

$$A \approx \frac{\Delta_2}{\tau} \exp\left(-\frac{\xi^2}{4a_2\tau} - \frac{\eta^2}{4b_2\tau}\right) \quad \text{as } \tau \to 0^+ \ , \tag{2.20}$$

with Δ_2 an asymptotically small constant. However, Hocking *et al.* (1972, p727) note that 'even if $\Delta_2 \ll 1$, [(2.20)] is not consistent with [(2.12)] for, at $\xi = \eta = 0$, the nonlinear term $\sim \Delta_2^3 \tau^{-3}$, while the linear terms $\sim \Delta_2 \tau^{-2}$ '. They claim that 'this non-uniformity can be overcome by supposing that the initial condition is imposed at a value of $\tau \sim \Delta_2^n$ where $\frac{1}{2} < n < 1$, or alternatively by making a shift in the origin of τ on the left-hand side of [(2.20)] by an equivalent amount'. In their numerical calculations, however, and in those of Hocking & Stewartson (1972, 1971) and Hocking (1974), an *ad hoc* approach is adopted, with Δ or Δ_2 taken to be small but finite, and the initial condition (2.19) or (2.20) applied at some finite positive value of τ. There remains the question of how to formulate the initial-value problem in a fully self-consistent way.

(*c*) *An alternative scaling*

With a view to resolving the matter, let us initially concentrate on (skewed) line disturbances. From consideration of successive quadratic interactions, we expect a weakly nonlinear interaction to develop when the disturbance amplitude reaches $O(\varepsilon^{1/2})$, that is, when $|\mathcal{A}_1|^2\exp(2\varepsilon t) = O(\varepsilon t)$. If t_1 is the solution to

$$|\mathcal{A}_1|^2\exp(2\varepsilon t_1) = \varepsilon t_1 \ , \tag{2.21}$$

then with $|\mathcal{A}_1| \ll 1$,

$$t_1 \sim \frac{1}{\varepsilon}\left(\ln|\mathcal{A}_1|^{-1} + \frac{1}{2}\ln\ln|\mathcal{A}_1|^{-1} + \dots\right) \gg \frac{1}{\varepsilon} \ . \tag{2.22}$$

This suggests that the scalings (2.10*a*), (2.16) should be modified to

$$t = t_1 + \varepsilon^{-1}\tilde{T} \ , \quad \tilde{\chi} = t_1^{-1/2}(\hat{x} + \hat{a}_{1\mathrm{r}}t) \ , \tag{2.23}$$

where $\varepsilon = (R - R_{cr})d_{1\mathrm{r}}$ as before. The key point to observe here is that the modulation length-scale has marginally increased. The amplitude scaling is unaltered, however, so that

$$\tilde{u} = \varepsilon^{1/2}\tilde{A}(\tilde{\chi},\tilde{T})\psi'(z)E + c.c. + O(\varepsilon) \ , \quad \text{etc.} \ . \tag{2.24}$$

The effect of the change in the scaling of the spatial coordinates is to reduce the importance of the linear dispersion term on the left-hand-side of (2.15). As a result the amplitude equation describing the first effects of nonlinearity becomes

$$\begin{aligned}\tilde{A}_{\tilde{T}} - \frac{d_1}{d_{1\mathrm{r}}}\tilde{A} - \hat{k}|\tilde{A}^2|\tilde{A} = \mu\hat{a}_2\tilde{A}_{\tilde{\chi}\tilde{\chi}} + \mu^{1/2}\varepsilon^{1/2}\hat{k}_1 A|A^2|_{\tilde{\chi}} + \mu^{1/2}\varepsilon^{1/2}\hat{k}_2|A^2|A_{\tilde{\chi}}&\\ +\mu^{3/2}\varepsilon^{1/2}\hat{a}_3 A_{\tilde{\chi}\tilde{\chi}\tilde{\chi}} + O(\varepsilon) \ ,&\end{aligned} \tag{2.25}$$

where, in view of (2.22),

$$\mu \equiv \frac{1}{\varepsilon t_1} \sim \frac{1}{\ln|\mathcal{A}_1|^{-1}} \ll 1 \ . \tag{2.26}$$

Thus, in the first instance, the asymptotically small right-hand side of (2.25) may be neglected, and the evolution of the disturbance is governed by a Landau equation, rather than a CGL equation. Moreover, as a result of the shift in origin of the time coordinate in (2.23), the initial condition for this equation is to be applied not as $\tau \to 0^+$ but as $\tilde{T} \to -\infty$. Specifically, by matching to (2.7) we obtain

$$\tilde{A} \sim \exp\left(\mathrm{i}\phi_1 + \frac{d_1}{d_{1\mathrm{r}}}\tilde{T} - \frac{\tilde{\chi}^2}{4\hat{a}_2}\right) \quad \text{as } \tilde{T} \to -\infty \ , \tag{2.27}$$

where

$$\phi_1 = \mu^{-1} d_{1\mathrm{i}}/d_{1\mathrm{r}} + \arg \mathcal{A}_1 \ . \tag{2.28}$$

Similarly, in the case of an initially linear point disturbance, we define t_2 to be the solution to

$$|\mathcal{A}_2|^2 \exp(2\varepsilon t_2) = \varepsilon t_2^2 \ ; \tag{2.29}$$

thus

$$t_2 \sim \frac{1}{\varepsilon}\left[\ln\left(|\mathcal{A}_2|^{-1}\varepsilon^{-1/2}\right) + \ln\ln\left(|\mathcal{A}_2|^{-1}\varepsilon^{-1/2}\right) + \ldots\right] \gg \frac{1}{\varepsilon} \ . \tag{2.30}$$

In this case the scaling (2.10) should be modified to

$$t = t_2 + \varepsilon^{-1}\tilde{T} \ , \quad \tilde{X} = t_2^{-1/2}(x + a_{1\mathrm{r}} t) \ , \quad \tilde{Y} = t_2^{-1/2} y \ , \tag{2.31}$$

and

$$\tilde{u} = \varepsilon^{1/2}\tilde{A}(\tilde{X},\tilde{Y},\tilde{T})\psi'(z)E + c.c. + O(\varepsilon) \ , \quad \text{etc.} \ . \tag{2.32}$$

Thus the modulational length-scale in both the streamwise and the spanwise direction has marginally increased. In consequence, the amplitude equations describing the first effects of nonlinearity for a three-dimensional localized disturbance become

$$\begin{aligned}
\tilde{A}_{\tilde{T}} - \frac{d_1}{d_{1\mathrm{r}}}\tilde{A} - k|\tilde{A}^2|\tilde{A} - q\tilde{A}\tilde{B} \ &= \hat{\mu}\left(a_2\tilde{A}_{\tilde{X}\tilde{X}} + b_2\tilde{A}_{\tilde{Y}\tilde{Y}}\right) \\
&+ (\mu\varepsilon)^{1/2}\Bigg[k_1\tilde{A}|A^2|_{\tilde{X}} + k_2|\tilde{A}^2|\tilde{A}_{\tilde{X}} + q_1\tilde{A}\tilde{B}_1 + q_2\tilde{A}\tilde{B}_{\tilde{X}} + q_3\tilde{B}\tilde{A}_{\tilde{X}} \\
&\qquad + q_4\tilde{A}_{\tilde{Y}}\int_{-\infty}^{\tilde{Y}}\tilde{B}_{\tilde{X}}\left(\tilde{X},\tilde{Y}',\tilde{T}\right)\mathrm{d}\tilde{Y}'\Bigg] \\
&+ \left(\mu^3\varepsilon\right)^{1/2}\left[a_3\tilde{A}_{\tilde{X}\tilde{X}\tilde{X}} + b_3\tilde{A}_{\tilde{X}\tilde{Y}\tilde{Y}}\right] + O\left(\varepsilon\right) \ ,
\end{aligned} \tag{2.33}$$

where $\tilde{B}$, $\tilde{B}_1$ are obtained from

$$\tilde{B}_{\tilde{X}\tilde{X}} + \tilde{B}_{\tilde{Y}\tilde{Y}} = |\tilde{A}^2|_{\tilde{Y}\tilde{Y}} \ , \tag{2.34a}$$

$$\tilde{B}_{1\tilde{X}\tilde{X}} + \tilde{B}_{1\tilde{Y}\tilde{Y}} = \sigma_1\left(\tilde{A}\tilde{A}^*_{\tilde{X}}\right)_{\tilde{Y}\tilde{Y}} + \sigma_2\left(\tilde{A}\tilde{A}^*_{\tilde{Y}}\right)_{\tilde{X}\tilde{Y}} + c.c. \ , \tag{2.34b}$$

$$\tilde{B} \to 0 \ , \ \tilde{B}_1 \to 0 \quad \text{as } \tilde{X}^2 + \tilde{Y}^2 \to \infty \ , \tag{2.34c}$$

and

$$\hat{\mu} \equiv \frac{1}{\varepsilon t_2} \sim \frac{1}{\ln\left(|\mathcal{A}_2|^{-1}\varepsilon^{-1/2}\right)} \ll 1 \ . \tag{2.35}$$

The appropriate initial condition for the leading-order problem is

$$\tilde{A} \sim \exp\left(\mathrm{i}\phi_2 + \frac{d_1}{d_{1\mathrm{r}}}\tilde{T} - \frac{\tilde{X}^2}{4a_2} - \frac{\tilde{Y}^2}{4b_2}\right) \quad \text{as } \tilde{T} \to -\infty \ , \tag{2.36}$$

where

$$\phi_2 = \hat{\mu}^{-1} d_{1\mathrm{i}}/d_{1\mathrm{r}} + \arg \mathcal{A}_2 \ . \tag{2.37}$$

3. Line Disturbances

For a (skewed) line disturbance, it is convenient to make the rescaling

$$\tilde{\chi} = 2|\hat{a}_2|\hat{a}_{2\mathrm{r}}^{-1/2}\chi \ , \quad \tilde{T} = -\tfrac{1}{2}\ln|\hat{k}_\mathrm{r}| + T \ , \tag{3.1}$$

$$\tilde{A}(\tilde{\chi},\tilde{T}) = |\hat{k}_\mathrm{r}|^{-1/2} e^{\mathrm{i}\phi_1 + \mathrm{i}d_{1\mathrm{i}}\tilde{T}/d_{1\mathrm{r}}} A(\chi, T) \ . \tag{3.2}$$

The amplitude equation (2.25) then reduces to

$$A_T - A - \mathrm{sgn}(\hat{k}_\mathrm{r})(1+\mathrm{i}\bar{\delta}_\mathrm{i})|A^2|A = \mu \, \frac{1+\mathrm{i}\bar{\alpha}_\mathrm{i}}{4(1+\bar{\alpha}_\mathrm{i}^2)} A_{\chi\chi}$$
$$+\mu^{1/2}\varepsilon^{1/2}\bar{k}_1 A|A^2|_\chi + \mu^{1/2}\varepsilon^{1/2}\bar{k}_2|A^2|A_\chi + \mu^{3/2}\varepsilon^{1/2}\bar{a}_3 A_{\chi\chi\chi} + O(\varepsilon) \ , \tag{3.3}$$

where

$$\bar{\alpha}_\mathrm{i} = \frac{\hat{a}_{2\mathrm{i}}}{\hat{a}_{2\mathrm{r}}} \ , \quad \bar{\delta}_\mathrm{i} = \frac{\hat{k}_\mathrm{i}}{\hat{k}_\mathrm{r}} \ , \quad \bar{k}_{1,2} = \frac{\hat{a}_{2\mathrm{r}}^{1/2}}{2|\hat{a}_2||\hat{k}_\mathrm{r}|}\hat{k}_{1,2} \ , \quad \bar{a}_3 = \frac{\hat{a}_{2\mathrm{r}}^{3/2}\hat{a}_3}{8|\hat{a}_2|^3} \ . \tag{3.4}$$

If the right-hand side of (3.3) is neglected, and the initial condition (2.27) is imposed in the rescaled form

$$A \sim \mathrm{e}^{T-(1-\mathrm{i}\bar{\alpha}_\mathrm{i})\chi^2} \quad \text{as} \quad T \to -\infty \ , \tag{3.5}$$

this problem has the well-known solution

$$A = \frac{\mathrm{e}^{T-(1-\mathrm{i}\bar{\alpha}_\mathrm{i})\chi^2}}{\left[1 - \mathrm{sgn}(\hat{k}_\mathrm{r})\mathrm{e}^{2T-2\chi^2}\right]^{(1+\mathrm{i}\bar{\delta}_\mathrm{i})/2}} \ . \tag{3.6}$$

In this paper, we shall be concerned for the most part with subcritical bifurcations, that is, with $\hat{k}_\mathrm{r} > 0$. However, we note in passing that for a supercritical case with $\hat{k}_\mathrm{r} < 0$, the solution (3.6) develops at large times into a pair of 'fronts' located around $\pm T^{1/2}$, with

$$A \sim \begin{cases} e^{-\mathrm{i}\bar{\delta}_\mathrm{i}T + \mathrm{i}(\bar{\alpha}_\mathrm{i}+\bar{\delta}_\mathrm{i})\chi^2} & |\chi| < T^{1/2} \\ e^{T-(1-\mathrm{i}\bar{\alpha}_\mathrm{i})\chi^2} & |\chi| > T^{1/2} \end{cases} \ , \tag{3.7}$$

and the transition taking place over a length-scale of $O(T^{-1/2})$. It follows that within the fronts, the first (dispersive) term on the right-hand side of (3.3) will be reinstated when $T = O(\mu^{-1})$, but we shall not pursue the details of this subsequent stage of development.

On the other hand, if $\hat{k}_\mathrm{r} > 0$, the solution (3.6) terminates in a localized singularity at $\chi = 0$ at time $T = 0$. Near $\chi = 0$ for $-1 \ll T < 0$, the singularity has the form

$$A \sim \frac{1}{\left[(-2T)\left(1+\bar{\xi}^2\right)\right]^{(1+\mathrm{i}\bar{\delta}_\mathrm{i})/2}} \ , \quad \text{where} \quad \bar{\xi} = \frac{\chi}{(-T)^{1/2}} \ . \tag{3.8}$$

Obviously, this solution cannot apply right up to infinite amplitude since weakly nonlinear theory becomes invalid when the (unscaled) disturbance amplitude rises to order one, or in other words when

$$|A| = O(\varepsilon^{-1}) \ , \quad \text{i.e.} \quad -T = O(\varepsilon) \ . \tag{3.9}$$

It is possible, however, that the solution (3.8) will cease to be a uniformly valid approximation before this happens, as may be seen by considering the $O(\mu)$ correction required to account for the first term on the right-hand side of (3.3). In fact, when

$$s = -\mu \ln(-T) \tag{3.10}$$

is of order one, it proves necessary to modify the form of the singularity structure (3.8) to

$$A \sim \frac{r(\xi,T)\mathrm{e}^{\mathrm{i}\phi(\xi,T)}}{(-2T)^{(1+\mathrm{i}\bar{\delta}_\mathrm{i})/2}} \ , \quad \xi = \frac{\chi - \chi_0(T)}{(-T)^{1/2}} \ . \tag{3.11}$$

where the origin shift

$$\chi_0(T) = -\mu^{1/2}\varepsilon^{1/2} d_0 \ln(-T) + \ldots \tag{3.12}$$

is produced by the terms on the second line of (3.3). On the other hand, if

$$-\mu \ln \varepsilon \ll 1 \ , \tag{3.13}$$

such a modification is unnecessary, since the disturbance attains order-one amplitude while s remains asymptotically small.

Substitution of (3.11)–(3.12) into (3.3) gives (cf. Hocking *et al.* 1972)

$$\begin{aligned} &\xi r_\xi - 2T r_T + r - r^3 \\ &\quad = \mu\left[\frac{1}{2(1+\bar{\alpha}_\mathrm{i}^2)}\left\{r_{\xi\xi} - r(\phi_\xi)^2\right\} - \frac{\bar{\alpha}_\mathrm{i}}{2(1+\bar{\alpha}_\mathrm{i}^2)}\left\{2r_\xi\phi_\xi + r\phi_{\xi\xi}\right\}\right] \\ &\qquad + \frac{\mu^{1/2}\varepsilon^{1/2}}{(-T)^{1/2}}\left[-2d_0 r_\xi + \left(2\bar{k}_{1\mathrm{r}} + \bar{k}_{2\mathrm{r}}\right) r^2 r_\xi - \bar{k}_{2\mathrm{i}} r^3 \phi_\xi\right] + \ldots \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\xi r\phi_\xi - 2T r\phi_T + \bar{\delta}_\mathrm{i} r - \bar{\delta}_\mathrm{i} r^3 \\ &\quad = \mu\left[\frac{1}{2(1+\bar{\alpha}_\mathrm{i}^2)}\left\{2r_\xi\phi_\xi + r\phi_{\xi\xi}\right\} + \frac{\bar{\alpha}_\mathrm{i}}{2(1+\bar{\alpha}_\mathrm{i}^2)}\left\{r_{\xi\xi} - r(\phi_\xi)^2\right\}\right] + \ldots \end{aligned} \tag{3.15}$$

where the linear growth term on the left-hand side of (3.3) has been ignored since it is smaller by a factor of order $(-T) = \mathrm{e}^{-s/\mu}$. We now expand r and ϕ as

$$r = r_{00}(\xi,s) + \mu r_{20}(\xi,s) + \mu^{1/2}\varepsilon^{1/2}(-T)^{-1/2} r_{11}(\xi,s) + \ldots \ , \tag{3.16}$$

$$\phi = \phi_{00}(\xi,s) + \mu\phi_{20}(\xi,s) + \mu^{1/2}\varepsilon^{1/2}(-T)^{-1/2}\phi_{11}(\xi,s) + \ldots \tag{3.17}$$

to obtain

$$r_{00} = \left[1 + C(s)\xi^2\right]^{-1/2} \ , \quad \phi_{00} = \Phi_0(s) - \tfrac{1}{2}\bar{\delta}_\mathrm{i} \ln\left[1 + C(s)\xi^2\right] \ , \tag{3.18}$$

$$\xi\frac{\partial r_{20}}{\partial\xi}+\left(1-\frac{3}{1+C\xi^2}\right)r_{20}= \tag{3.19}$$
$$\frac{\xi^2}{\left(1+C\xi^2\right)^{3/2}}\frac{\mathrm{d}C}{\mathrm{d}s}+\frac{1}{2\left(1+\bar{\alpha}_{\mathrm{i}}^2\right)}\left[\frac{\left(\bar{\alpha}_{\mathrm{i}}\bar{\delta}_{\mathrm{i}}-1\right)C}{\left(1+C\xi^2\right)^{3/2}}+\frac{FC^2\xi^2}{\left(1+C\xi^2\right)^{5/2}}\right]$$

$$\frac{\xi}{\left(1+C\xi^2\right)^{1/2}}\frac{\partial\phi_{20}}{\partial\xi}=-\bar{\delta}_{\mathrm{i}}\left(1-\frac{3}{1+C\xi^2}\right)r_{20}-\frac{2}{\left(1+C\xi^2\right)^{1/2}}\frac{\mathrm{d}\Phi_0}{\mathrm{d}s} \tag{3.20}$$
$$+\frac{\bar{\delta}_{\mathrm{i}}\xi^2}{\left(1+C\xi^2\right)^{3/2}}\frac{\mathrm{d}C}{\mathrm{d}s}-\frac{\left(\bar{\alpha}_{\mathrm{i}}+\bar{\delta}_{\mathrm{i}}\right)C}{2\left(1+\bar{\alpha}_{\mathrm{i}}^2\right)\left(1+C\xi^2\right)^{3/2}}+\frac{\left(3\bar{\alpha}_{\mathrm{i}}+4\bar{\delta}_{\mathrm{i}}-\bar{\alpha}_{\mathrm{i}}\bar{\delta}_{\mathrm{i}}^2\right)C^2\xi^2}{2\left(1+\bar{\alpha}_{\mathrm{i}}^2\right)\left(1+C\xi^2\right)^{5/2}}$$

$$\xi\frac{\partial r_{11}}{\partial\xi}+\left(2-\frac{3}{1+C\xi^2}\right)r_{11}=-\frac{2d_0C\xi}{\left(1+C\xi^2\right)^{3/2}}-\frac{\left(2\bar{k}_{1\mathrm{r}}+\bar{k}_{2\mathrm{r}}-\bar{\delta}_{\mathrm{i}}\bar{k}_{2\mathrm{i}}\right)C\xi}{\left(1+C\xi^2\right)^{5/2}} \tag{3.21}$$

where $C(s)$ and $\Phi_0(s)$ are arbitrary functions of integration, and we define

$$F=3-\bar{\delta}_{\mathrm{i}}^2-4\bar{\alpha}_{\mathrm{i}}\bar{\delta}_{\mathrm{i}}\ . \tag{3.22}$$

For r_{20}, ϕ_{20} and r_{11} to be analytic at $\xi=0$, we must have

$$\frac{\mathrm{d}C}{\mathrm{d}s}=-\frac{F}{2(1+\bar{\alpha}_{\mathrm{i}}^2)}C^2\ ,\quad \frac{\mathrm{d}\Phi_0}{\mathrm{d}s}=-\frac{\bar{\alpha}_{\mathrm{i}}(1+\bar{\delta}_{\mathrm{i}}^2)}{4(1+\bar{\alpha}_{\mathrm{i}}^2)}C\ ,\quad d_0=-\bar{k}_{1\mathrm{r}}-\tfrac{1}{2}\left(\bar{k}_{2\mathrm{r}}-\bar{\delta}_{\mathrm{i}}\bar{k}_{2\mathrm{i}}\right)\ . \tag{3.23}$$

Morover, matching to the previous stage described by (3.8) imposes the requirements that

$$C(0)=1\ ,\quad \Phi_0(0)=0\ . \tag{3.24}$$

It follows from (3.23) and (3.24) that

$$C=\frac{1}{1-s/s_0}\ ,\quad \Phi_0=\nu\ln\left(1-s/s_0\right)\ , \tag{3.25}$$

with

$$s_0=-\frac{2(1+\bar{\alpha}_{\mathrm{i}}^2)}{F}\ ,\quad \nu=-\frac{\bar{\alpha}_{\mathrm{i}}(1+\bar{\delta}_{\mathrm{i}}^2)}{2F}\ . \tag{3.26}$$

The behaviour of this solution depends on the sign of F (or equivalently s_0). If $F>0$ this slightly modified asymptotic description appears to be uniformly valid as the singularity is approached. Moreover, as s becomes large, the structure of the singularity goes over to that of the 'first species of burst' identified by Hocking *et al.* (1972) and Hocking & Stewartson (1972). For the particular case of PPF, this applies to skewed line disturbances of the form (2.6) with $\hat{\theta}>57.3°$. If $F<0$, however, C becomes infinite as $s\to s_0-$, and the foregoing description will not be uniformly valid if this happens while the amplitude is still small, that is, if

$$-\mu\ln\varepsilon>s_0\ . \tag{3.27}$$

Rather, as the blow-up time is approached, the length-scale of the singularity contracts rapidly, and the dispersive term on the right-hand side of (3.3) becomes comparable with the nonlinear term. By means of the rescaling

$$T = \Delta\hat{T} \ , \quad \chi = \frac{\varepsilon^{1/2} d_0 s_0}{\mu^{1/2}} + \frac{\mu^{1/2}\Delta^{1/2}}{2(1+\bar{\alpha}_{\mathrm{i}}^2)^{1/2}}\hat{\chi} \ , \quad A = \Delta^{-(1+\mathrm{i}\bar{\delta}_{\mathrm{i}})/2}\left(\frac{\mu}{s_0}\right)^{\mathrm{i}\nu}\hat{A}(\hat{\chi},\hat{T}) \ , \tag{3.28}$$

with $\Delta = \mathrm{e}^{-s_0/\mu}$, we recover the governing equation

$$\hat{A}_{\hat{T}} = (1+\mathrm{i}\bar{\alpha}_{\mathrm{i}})\hat{A}_{\hat{\chi}\hat{\chi}} + (1+\mathrm{i}\bar{\delta}_{\mathrm{i}})|\hat{A}^2|\hat{A} \ , \tag{3.29}$$

with boundary condition

$$\hat{A} \to 0 \quad \text{as } |\hat{\chi}| \to \infty \ , \tag{3.30}$$

and initial condition

$$\hat{A} \sim \left[\ln(-\hat{T})\right]^{\mathrm{i}\nu}\left[-2\hat{T} + \frac{\hat{\chi}^2}{|F|\ln(-\hat{T})}\right]^{-(1+\mathrm{i}\bar{\delta}_{\mathrm{i}})/2} \qquad \text{as } \hat{T} \to -\infty \ . \tag{3.31}$$

The subsequent development is thus described by the CGL equation (without a linear growth term). The initial condition is algebraic in $\hat{\chi}$ and applied at asymptotically large negative times (as opposed to the exponential-in-$\hat{\chi}$ initial condition applied at a finite time that has been used previously).

According to Hocking & Stewartson (1972), equation (3.29) may or may not admit a finite-time singularity (a 'burst of the second species'), depending on the values of $\bar{\alpha}_{\mathrm{i}}$ and $\bar{\delta}_{\mathrm{i}}$. It remains to be confirmed that when this singularity is available, it is actually attained starting from the specific initial condition (3.31). On the other hand, if

$$-\mu\ln\varepsilon < s_0 \ , \quad \text{i.e.} \quad |\mathcal{A}_1| \ll \varepsilon^{1/s_0} \tag{3.32}$$

in view of (2.26), then the singularity structure (3.11)–(3.26) describes the development of the disturbance uniformly up to order-one amplitude, since this is attained before the stage described by the CGL equation (3.29) is reached. We conclude that if the initial amplitude $|\mathcal{A}_1|$ is sufficiently small, an initially localized line disturbance 'bursts' irrespective of the values of the coefficients $\hat{a}_2$ and $\hat{k}$ (provided only that $\hat{k}_{\mathrm{r}} > 0$).

4. Point Disturbances: $\mathrm{Re}(q) = 0$

We turn now to the weakly nonlinear evolution of localized 'point' disturbances, which is described by the reduced DHSS equations (2.33) and (2.34), with the right-hand-side of (2.33) taken to be zero because it is asymptotically small, and initial condition (2.36). In general, this problem requires numerical solution. An exception arises when $\mathrm{Re}(q) = 0$ (as is the case for the asymptotic suction boundary layer, as shown by Hocking 1975). In this case, with the rescaling

$$\tilde{X} = 2|a_2|a_{2\mathrm{r}}^{-1/2}X \ , \quad \tilde{Y} = 2|a_2|a_{2\mathrm{r}}^{-1/2}Y \ , \quad \tilde{T} = -\tfrac{1}{2}\ln|k_{\mathrm{r}}| + T \ , \tag{4.1}$$

$$\tilde{A} = |k_{\rm r}|^{-1/2} {\rm e}^{{\rm i}\phi_2 + {\rm i}d_{1{\rm i}}\tilde{T}/d_{1{\rm r}}} \bar{A}_0(X,Y,T) \ , \quad \tilde{B} = |k_{\rm r}|^{-1} \bar{B}_0(X,Y,T) \ , \tag{4.2}$$

the amplitude equations (2.33)–(2.34) reduce to

$$\frac{\partial \bar{A}_0}{\partial T} - \bar{A}_0 - \sigma(1+{\rm i}\bar{\delta}_{\rm i})|\bar{A}_0^2|\bar{A}_0 - {\rm i}\bar{q}_{\rm i}\bar{A}_0\bar{B}_0 = \hat{\mu}\left[\frac{1+{\rm i}\bar{\alpha}_{\rm i}}{4(1+\bar{\alpha}_{\rm i}^2)}\frac{\partial^2 \bar{A}_0}{\partial X^2} + \frac{\lambda(1+{\rm i}\bar{\beta}_{\rm i})}{4(1+\bar{\beta}_{\rm i}^2)}\frac{\partial^2 \bar{A}_0}{\partial Y^2}\right] + \dots \ , \tag{4.3}$$

$$\frac{\partial^2 \bar{B}_0}{\partial X^2} + \frac{\partial^2 \bar{B}_0}{\partial Y^2} = \frac{\partial^2 |\bar{A}_0^2|}{\partial Y^2} \ , \quad \bar{B}_0 \to 0 \quad \text{as } X^2+Y^2 \to \infty \ , \tag{4.4}$$

where $\sigma = {\rm sgn}(k_{\rm r})$, and

$$\bar{\delta}_{\rm i} = \frac{k_{\rm i}}{k_{\rm r}} \ , \quad \bar{q}_{\rm i} = \frac{q_{\rm i}}{|k_{\rm r}|} \ , \quad \bar{\alpha}_{\rm i} = \frac{a_{2{\rm i}}}{a_{2{\rm r}}} \ , \quad \bar{\beta}_{\rm i} = \frac{b_{2{\rm i}}}{b_{2{\rm r}}} \ , \quad \lambda = \frac{a_{2{\rm r}}|b_2|^2}{b_{2{\rm r}}|a_2|^2} \ . \tag{4.5}$$

With the right-hand side of (4.3) dropped, and initial condition

$$\bar{A}_0 \sim \exp\left[T - (1-{\rm i}\bar{\alpha}_{\rm i})X^2 - (1-{\rm i}\bar{\beta}_{\rm i})Y^2/\lambda\right] \quad \text{as } T \to -\infty \ , \tag{4.6}$$

corresponding to (2.36), the solution takes the form

$$|\bar{A}_0| = \left[\frac{R}{1-\sigma R}\right]^{1/2} \ , \quad R(X,Y,T) = {\rm e}^{2(T-X^2-Y^2/\lambda)} \ , \tag{4.7}$$

$$\bar{B}_0 = -\frac{2}{\pi\lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(Y-Y')Y'R(X',Y',T)\ {\rm d}X'{\rm d}Y'}{[(X-X')^2+(Y-Y')^2]\,[1-\sigma R(X',Y',T)]^2} \ , \tag{4.8}$$

$$\arg \bar{A}_0 = \bar{\alpha}_{\rm i}X^2 + \frac{\bar{\beta}_{\rm i}Y^2}{\lambda} - \tfrac{1}{2}\bar{\delta}_{\rm i}\ln(1-\sigma R) - \frac{\bar{q}_{\rm i}}{\pi\lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(Y-Y')Y'}{[(X-X')^2+(Y-Y')^2]}\frac{R(X',Y',T)}{[1-\sigma R(X',Y',T)]}\ {\rm d}X'{\rm d}Y' \ . \tag{4.9}$$

For a supercritical bifurcation with $k_{\rm r} < 0$ ($\sigma = -1$), this solution develops at large times into an advancing elliptical front located at $X^2 + Y^2/\lambda = T$, with the amplitude $|\bar{A}_0|$ close to unity inside and exponentially small outside; the width of the front decreases like $T^{-1/2}$, implying that the dispersive terms must be reinstated when $T = O(\hat{\mu}^{-1})$. Our main concern, however, is with the subcritical case, $k_{\rm r} > 0$ ($\sigma = 1$), for which the solution terminates in a localized singularity at the origin $(X,Y) = (0,0)$ at time $T = 0$. Locally,

$$\bar{A}_0 \sim \frac{\bar{r}(\bar{\xi},\bar{\eta}){\rm e}^{{\rm i}\bar{\phi}(\bar{\xi},\bar{\eta})}}{(-2T)^{(1+{\rm i}\bar{\kappa})/2}} \ , \quad \bar{B}_0 \sim \frac{\bar{b}(\bar{\xi},\bar{\eta})}{(-2T)} \ , \quad \bar{\xi} = \frac{X}{(-T)^{1/2}} \ , \quad \bar{\eta} = \frac{Y}{(-T)^{1/2}} \ , \tag{4.10}$$

where

$$\bar{r} = \left[1 + \bar{\xi}^2 + \bar{\eta}^2/\lambda\right]^{-1/2} \ , \tag{4.11}$$

$$\bar{b} = -\frac{1}{\pi\lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(\bar{\eta}-\bar{\eta}')\bar{\eta}'\ \mathrm{d}\bar{\xi}'\mathrm{d}\bar{\eta}'}{\left[(\bar{\xi}-\bar{\xi}')^2+(\bar{\eta}-\bar{\eta}')^2\right]\left[1+\bar{\xi}'^2+\bar{\eta}'^2/\lambda\right]^2}\ , \tag{4.12}$$

$$\bar{\phi} = -\tfrac{1}{2}\bar{\delta}_{\mathrm{i}}\ln\left[1+\bar{\xi}^2+\bar{\eta}^2/\lambda\right] - \frac{\bar{q}_{\mathrm{i}}}{2\pi\lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{H(\bar{\xi},\bar{\eta},\bar{\xi}',\bar{\eta}')}{1+\bar{\xi}'^2+\bar{\eta}'^2/\lambda}\ \mathrm{d}\bar{\xi}'\mathrm{d}\bar{\eta}'\ , \tag{4.13}$$

$$\bar{\kappa} = \bar{\delta}_{\mathrm{i}} + \bar{q}_{\mathrm{i}}\bar{b}(0,0) \equiv \bar{\delta}_{\mathrm{i}} + \bar{q}_{\mathrm{i}}/\left(\lambda^{1/2}+1\right)\ , \tag{4.14}$$

and

$$H(\bar{\xi},\bar{\eta},\bar{\xi}',\bar{\eta}') = \frac{(\bar{\eta}-\bar{\eta}')\bar{\eta}'}{(\bar{\xi}-\bar{\xi}')^2+(\bar{\eta}-\bar{\eta}')^2} + \frac{\bar{\eta}'^2}{\bar{\xi}'^2+\bar{\eta}'^2}\ . \tag{4.15}$$

As for line disturbances, consideration of the dispersive terms requires the modification

$$\bar{A}_0 \sim \frac{r(\xi,\eta,T)}{(-2T)^{1/2}}\exp\left[\frac{\mathrm{i}}{2\hat{\mu}}\int_0^S \kappa(S')\mathrm{d}S' + \mathrm{i}\phi(\xi,\eta,T)\right]\ ,\quad \bar{B}_0 \sim \frac{b(\xi,\eta,T)}{(-2T)}\ , \tag{4.16}$$

with

$$S = -\hat{\mu}\ln(-T)\ ,\quad \xi = \frac{X-X_0(T)}{(-T)^{1/2}}\ ,\quad \eta = \frac{Y}{(-T)^{1/2}}\ , \tag{4.17}$$

and

$$r = r_{00}(\xi,\eta,S) + \hat{\mu}r_{20}(\xi,\eta,S) + \ldots \quad \text{etc.}\ . \tag{4.18}$$

On substituting into (3.3), we find that at leading order

$$r_{00} = \left[1+C(S)\xi^2+D(S)\eta^2\right]^{-1/2}\ , \tag{4.19}$$

$$b_{00} = -\frac{D}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(\eta-\eta')\eta'\ \mathrm{d}\xi'\mathrm{d}\eta'}{\left[(\xi-\xi')^2+(\eta-\eta')^2\right]\left[1+C\xi'^2+D\eta'^2\right]^2}\ , \tag{4.20}$$

$$\phi_{00} = \Phi_0(S) - \tfrac{1}{2}\bar{\delta}_{\mathrm{i}}\ln\left[1+C\xi^2+D\eta^2\right] - \frac{\bar{q}_{\mathrm{i}}D}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{H(\xi,\eta,\xi',\eta')}{1+C\xi'^2+D\eta'^2}\ \mathrm{d}\xi'\mathrm{d}\eta'\ , \tag{4.21}$$

$$\kappa = \bar{\delta}_{\mathrm{i}} + \bar{q}_{\mathrm{i}}b_{00}(0,0,S) \equiv \bar{\delta}_{\mathrm{i}} + \bar{q}_{\mathrm{i}}D^{1/2}/\left(C^{1/2}+D^{1/2}\right)\ , \tag{4.22}$$

Matching to the previous stage (4.10)–(4.14) imposes the initial condition

$$C(0) = 1\ ,\quad D(0) = \lambda^{-1}\ ,\quad \Phi_0(0) = -\tfrac{1}{2}\bar{\kappa}\ln 2\ . \tag{4.23}$$

Further, at order $\hat{\mu}$, detailed calculation shows that for r_{20} to be analytic at $(\xi, \eta) = (0, 0)$, we must have

$$\frac{\mathrm{d}C}{\mathrm{d}S} = -\frac{C^2}{2(1+\bar{\alpha}_\mathrm{i}^2)}\left[F - \frac{2\bar{q}_\mathrm{i} D}{\left(C^{1/2}+D^{1/2}\right)^2}\, f(C,D)\right] , \tag{4.24}$$

$$\frac{\mathrm{d}D}{\mathrm{d}S} = -\frac{D^2}{2(1+\bar{\beta}_\mathrm{i}^2)}\left[G + \frac{2\bar{q}_\mathrm{i} C}{\left(C^{1/2}+D^{1/2}\right)^2}\, g(C,D)\right] , \tag{4.25}$$

where we define

$$f = \bar{\alpha}_\mathrm{i} + \bar{\delta}_\mathrm{i} + \frac{\left[\bar{\alpha}_\mathrm{i}(1+\bar{\beta}_\mathrm{i}^2) - \lambda\bar{\beta}_\mathrm{i}(1+\bar{\alpha}_\mathrm{i}^2)\right] D^{1/2}}{(1+\bar{\beta}_\mathrm{i}^2)\left(C^{1/2}+D^{1/2}\right)} + \frac{\bar{q}_\mathrm{i} D}{2\left(C^{1/2}+D^{1/2}\right)^2} ,$$

$$g = \lambda\left(\bar{\beta}_\mathrm{i} + \bar{\delta}_\mathrm{i} + \bar{q}_\mathrm{i}\right) + \frac{\left[\lambda\bar{\beta}_\mathrm{i}(1+\bar{\alpha}_\mathrm{i}^2) - \bar{\alpha}_\mathrm{i}(1+\bar{\beta}_\mathrm{i}^2)\right] C^{1/2}}{(1+\bar{\alpha}_\mathrm{i}^2)\left(C^{1/2}+D^{1/2}\right)} - \frac{\lambda\bar{q}_\mathrm{i} C}{2\left(C^{1/2}+D^{1/2}\right)^2} ,$$

$$F = 3 - \bar{\delta}_\mathrm{i}^2 - 4\bar{\alpha}_\mathrm{i}\bar{\delta}_\mathrm{i} , \quad G = \lambda\left[3 - \left(\bar{\delta}_\mathrm{i} + \bar{q}_\mathrm{i}\right)^2 - 4\bar{\beta}_\mathrm{i}\left(\bar{\delta}_\mathrm{i} + \bar{q}_\mathrm{i}\right)\right] . \tag{4.26}$$

We have not investigated the solutions of (4.23)–(4.26) in detail. However, in the special case with $\bar{q}_\mathrm{i} = 0$, which is appropriate to the asymptotic suction boundary layer (Hocking 1975), there is the analytic solution

$$C = \frac{1}{1 - S/S_0} , \quad D = \frac{1}{\lambda\left(1 - S/S_1\right)} , \tag{4.27}$$

with

$$S_0 = -\frac{2(1+\bar{\alpha}_\mathrm{i}^2)}{F} , \quad S_1 = -\frac{2(1+\bar{\beta}_\mathrm{i}^2)\lambda}{G} . \tag{4.28}$$

In this case, the solvability condition for θ_1 is found to be

$$\frac{\mathrm{d}\Phi_0}{\mathrm{d}s} = -\frac{\bar{\alpha}_\mathrm{i}(1+\bar{\delta}_\mathrm{i}^2)}{4(1+\bar{\alpha}_\mathrm{i}^2)}C - \frac{\lambda\bar{\beta}_\mathrm{i}(1+\bar{\delta}_\mathrm{i}^2)}{4(1+\bar{\beta}_\mathrm{i}^2)}D , \tag{4.29}$$

and hence

$$\Phi_0 = \nu_0 \ln\left(1 - S/S_0\right) + \nu_1 \ln\left(1 - S/S_1\right) - \tfrac{1}{2}\bar{\delta}_\mathrm{i} \ln 2 , \tag{4.30}$$

where

$$\nu_0 = -\frac{\bar{\alpha}_\mathrm{i}(1+\bar{\delta}_\mathrm{i}^2)}{2F} , \quad \nu_1 = -\frac{\lambda\bar{\beta}_\mathrm{i}(1+\bar{\delta}_\mathrm{i}^2)}{2G} . \tag{4.31}$$

If $F > 0$ and $G > 0$, the solution (4.27)–(4.31) is well behaved for all $S > 0$, and

$$C \sim \frac{|S_0|}{S} , \quad D \sim \frac{|S_1|}{\lambda S} , \quad \Phi_0 \sim (\nu_0 + \nu_1)\ln S \quad \text{as } S \to \infty , \tag{4.32}$$

which is the two-dimensional analogue of the 'first species of burst' discussed in the previous section, and whose existence was noted by Hocking *et al.* (1972) and

Hocking & Stewartson (1971). If, however, one or both of F and G is negative, the solution blows up at a finite value of S. For example, if F is negative and G is such that $S_0^{-1} > S_1^{-1}$, which is the case for the asymptotic suction boundary layer (Hocking 1975), C will become infinite at $S = S_0$, with D remaining finite. The upshot is that at times asymptotically close to S_0, the $\partial^2 \bar{A}_0/\partial X^2$ term will be reinstated at leading-order, but not the $\partial^2 \bar{A}_0/\partial Y^2$ term. Moreover, since the X length-scale is now asymptotically small compared to the Y length-scale, it follows from (4.4) that $\bar{B}_0$ is asymptotically small compared to $|\bar{A}_0^2|$. The subsequent evolution will thus be governed by a one-dimensional CGL equation, with the spanwise dependence entering only parametrically, cf. (5.86)–(5.87) below. On the other hand, if the initial amplitude, $\mathcal{A}_2$ in (2.8), is sufficiently small, with

$$|\mathcal{A}_2| \ll \varepsilon^{1/S_0}\varepsilon^{-1/2} \ , \tag{4.33}$$

then the disturbance will attain order-one amplitude before this happens, and the CGL stage is bypassed.

5. Point Disturbances: $\mathrm{Re}(q) \neq 0$

When considering the DHSS equations (2.12) with $q_\mathrm{r} \neq 0$, there is no loss of generality in taking $q_\mathrm{r} < 0$, since the substitution $\tilde{B} = |\tilde{A}^2| - B$ has the effect of replacing q by $-q$ and k by $k + q$ in (2.12*a*), and interchanging ξ and η in (2.12*b*) (Hocking 1974); in any case, as is evident from table 1, q_r is invariably negative in PPCF. With the rescaling

$$\tilde{X} = 2|a_2|a_{2\mathrm{r}}^{-1/2}X \ , \quad \tilde{Y} = 2|a_2|a_{2\mathrm{r}}^{-1/2}Y \ , \quad \tilde{T} = \tfrac{1}{2}\ln|q_\mathrm{r}| + \tau \ , \tag{5.1}$$

$$\tilde{A} = |q_\mathrm{r}|^{-1/2}e^{\mathrm{i}\theta_2 + \mathrm{i}d_{1\mathrm{i}}\tilde{T}/d_{1\mathrm{r}}}A(X,Y,\tau) \ , \quad \tilde{B} = |q_\mathrm{r}|^{-1/2}B(X,Y,\tau) \ , \tag{5.2}$$

equations (2.33)–(2.34) reduce to

$$\begin{aligned}\frac{\partial A}{\partial \tau} - A - \bar{k}_\mathrm{r}(1+\mathrm{i}\bar{\delta}_\mathrm{i})|A^2|A + (1+\mathrm{i}\bar{\gamma}_\mathrm{i})AB& \\ = \hat{\mu}\left[\frac{1+\mathrm{i}\bar{\alpha}_\mathrm{i}}{4(1+\bar{\alpha}_\mathrm{i}^2)}\frac{\partial^2 A}{\partial X^2} + \frac{\lambda(1+\mathrm{i}\bar{\beta}_\mathrm{i})}{4(1+\bar{\beta}_\mathrm{i}^2)}\frac{\partial^2 A}{\partial Y^2}\right]& + \ldots \ ,\end{aligned} \tag{5.3}$$

$$B_{XX} + B_{YY} = |A^2|_{YY} \ , \quad B \to 0 \quad \text{as } X^2 + Y^2 \to \infty \ , \tag{5.4}$$

where $\bar{k}_r = k_r/|q_r|$, $\bar{\gamma}_\mathrm{i} = q_\mathrm{i}/q_\mathrm{r}$, and the other parameters are defined in (4.5). In the first instance, the asymptotically small right-hand side of (5.3) is to be dropped, and equations (5.3)–(5.4) solved subject to the initial condition

$$A \sim \exp\left(\tau - (1-\mathrm{i}\bar{\alpha}_\mathrm{i})X^2 - (1-\mathrm{i}\bar{\beta}_\mathrm{i})Y^2/\lambda\right) \quad \text{as } \tau \to -\infty \ . \tag{5.5}$$

In fact, if only the amplitude $|A|$ of the disturbance is required, it is merely necessary to solve

$$|A|_\tau = |A| + \bar{k}_\mathrm{r}|A|^3 - |A|B \ , \tag{5.6a}$$

$$B_{XX} + B_{YY} = |A^2|_{YY} \ , \quad B \to 0 \quad \text{as } X^2 + Y^2 \to \infty \ , \tag{5.6b}$$

$$|A| \sim \exp(\tau - X^2 - Y^2/\lambda) \quad \text{as } \tau \to -\infty \ . \tag{5.7}$$

In §5 *a–c*, we shall confine attention to this reduced problem; the effect (if any) of the small dispersive terms on the right-hand side of (5.3) will be considered in §5 *d*.

(*a*) *Limiting cases of the amplitude equations*

While we have been unable to find any exact analytic solution to the reduced form of (5.3)–(5.5), there are two limiting cases where an asymptotic description of the evolution may be obtained, namely when $|\bar{k}_\mathrm{r}| \gg 1$ and/or $\lambda \gg 1$.

(i) $|\bar{k}_\mathrm{r}| \gg 1$.

In this case, we make the rescaling

$$A = |\bar{k}_\mathrm{r}|^{-1/2}\bar{A}(X,Y,T) \ , \quad B = |\bar{k}_\mathrm{r}|^{-1}\bar{B}(X,Y,T) \ , \quad \tau = -\tfrac{1}{2}\ln|\bar{k}_\mathrm{r}| + T \ . \tag{5.8}$$

We also allow the imaginary part of q to be comparable with $\bar{k}_\mathrm{r}$, and so write

$$\bar{\gamma}_\mathrm{i} = -|\bar{k}_\mathrm{r}|\bar{q}_\mathrm{i} \ , \tag{5.9}$$

with $\bar{q}_\mathrm{i}$ of order one; this situation may be of relevance to PPF, for which $\bar{k}_\mathrm{r} \approx 24.25$, $\bar{\gamma}_\mathrm{i} \approx -22.91$. The problem then becomes

$$\bar{A}_T = \bar{A} + \mathrm{sgn}(\bar{k}_\mathrm{r})(1+\mathrm{i}\bar{\delta}_\mathrm{i})|\bar{A}^2|\bar{A} + \left(\mathrm{i}\bar{q}_\mathrm{i} - |\bar{k}_\mathrm{r}|^{-1}\right)\bar{A}\bar{B} \,, \tag{5.10}$$

$$\bar{B}_{XX} + \bar{B}_{YY} = |\bar{A}^2|_{YY} \ , \quad \bar{B} \to 0 \quad \text{as } X^2+Y^2 \to \infty \ , \tag{5.11}$$

$$\bar{A} \sim \exp\left(T - (1-\mathrm{i}\bar{\alpha}_\mathrm{i})X^2 - (1-\mathrm{i}\bar{\beta}_\mathrm{i})Y^2/\lambda\right) \quad \text{as } T \to -\infty \ . \tag{5.12}$$

With the expansions

$$\bar{A} = \bar{A}_0 + |\bar{k}_\mathrm{r}|^{-1}\bar{A}_1 + \dots \ , \quad \bar{B} = \bar{B}_0 + |\bar{k}_\mathrm{r}|^{-1}\bar{B}_1 + \dots \ , \tag{5.13}$$

the leading-order problem for $\bar{A}_0$ and $\bar{B}_0$ is precisely that considered in 4 above; when $\bar{k}_\mathrm{r} > 0$, the solution terminates in a singularity at $T = 0$ of the form (4.10)–(4.15). As we have noted, this structure cannot describe the evolution right up to infinite amplitude since it is modified by the dispersive terms when $\ln(-T) = O(\hat{\mu}^{-1})$. On the other hand, if $\bar{k}_\mathrm{r}^{-1} \gg \hat{\mu}$, the asymptotic description will cease to be valid sooner, or to be specific when

$$\bar{s} \equiv -\bar{k}_\mathrm{r}^{-1}\ln(-T) \tag{5.14}$$

is of order one. For this new phase of evolution, the leading-order solutions are as in (4.16)–(4.22) with S replaced by $\bar{s}$, but the next correction is forced by the asymptotically small final term, $-\bar{k}_\mathrm{r}^{-1}\bar{A}\bar{B}$, in equation (5.10) rather than by the dispersive terms as in §4. The solvability conditions at $O\left(\bar{k}_\mathrm{r}^{-1}\right)$ are found to give

$$\frac{\mathrm{d}C}{\mathrm{d}\bar{s}} = \frac{C^{3/2}D^{1/2}}{(C^{1/2}+D^{1/2})^2} \ , \quad \frac{\mathrm{d}D}{\mathrm{d}\bar{s}} = -\frac{C^{1/2}D^{3/2}}{(C^{1/2}+D^{1/2})^2} \ . \tag{5.15}$$

Matching to the previous stage imposes the initial conditions

$$C(0) = 1 \ , \quad D(0) = \lambda^{-1} \ . \tag{5.16}$$

The solution has

$$CD = \lambda^{-1} \ , \tag{5.17}$$

and C given implicitly by

$$\lambda^{1/2} C + 2 \ln C - \lambda^{-1/2} C^{-1} = \bar{s} + \lambda^{1/2} - \lambda^{-1/2} \ . \tag{5.18}$$

It follows that for large $\bar{s}$

$$C \sim \lambda^{-1/2} \bar{s} \ , \quad D \sim \lambda^{-1/2} \bar{s}^{-1} \ , \tag{5.19}$$

and hence the 'terminal' form of the singularity (in original variables) has

$$|A| \sim \left[2\bar{k}_{\mathrm{r}} \left(\tau_s - \tau\right)\right]^{-1/2} \left[1 + C_0 \left(\tilde{\xi}^2 + \bar{k}_{\mathrm{r}}^2 \tilde{\eta}^2\right)\right]^{-1/2} \ , \tag{5.20}$$

with

$$\tau_s = -\tfrac{1}{2} \ln \bar{k}_{\mathrm{r}} \ , \quad C_0 = \bar{k}_{\mathrm{r}}^{-1} \lambda^{-1/2} \ , \tag{5.21}$$

and

$$\tilde{\xi} = \frac{X \sigma^{1/2}}{\left(\tau_s - \tau\right)^{1/2}} \ , \quad \tilde{\eta} = \frac{Y}{\left(\tau_s - \tau\right)^{1/2} \sigma^{1/2}} \ , \quad \sigma = -\ln\left(\tau_s - \tau\right) \ . \tag{5.22}$$

This does not take account of the dispersive terms, which may come into play at asymptotically large values of $\bar{s}$. Consideration of this matter is deferred to §5 *d*.

(ii) $\lambda \gg 1$.

In this case the solution is only weakly dependent on Y, and B is small compared to $|A^2|$, or more precisely $B/|A^2| = O(\lambda^{-1/2})$ (cf. Hocking 1974). It is convenient once again to make the rescaling (5.8), but this time the solution expands in the form

$$\left.\begin{aligned} \bar{A} &\sim A_0(X,y,T) + \lambda^{-1/2} A_1(X,y,T) + \dots \ , \\ \bar{B} &\sim \lambda^{-1/2} B_1(y,T) + \lambda^{-1} B_2(X,y,T) + \dots \ , \end{aligned}\right\} \quad \text{with } y = \lambda^{-1} Y \ . \tag{5.23}$$

The leading-order amplitude A_0 satisfies the Landau equation

$$\frac{\partial A_0}{\partial T} = A_0 + \mathrm{sgn}(\bar{k}_{\mathrm{r}})(1 + \mathrm{i}\bar{\delta}_{\mathrm{i}})|A_0^2| A_0 \ , \tag{5.24}$$

and initial condition

$$A_0 \sim \exp\left[T - (1 - \mathrm{i}\bar{\alpha}_{\mathrm{i}})X^2 - (1 - \mathrm{i}\bar{\beta}_{\mathrm{i}})y^2\right] \quad \text{as } T \to -\infty \ , \tag{5.25}$$

so that

$$A_0 = \frac{\exp\left[T - (1 - \mathrm{i}\bar{\alpha}_\mathrm{i})X^2 - (1 - \mathrm{i}\bar{\beta}_\mathrm{i})y^2\right]}{\left[1 - \mathrm{sgn}(\bar{k}_\mathrm{r})\mathrm{e}^{2(T - X^2 - y^2)}\right]^{(1+\mathrm{i}\bar{\delta}_\mathrm{i})/2}} \ . \tag{5.26}$$

Substitution of (5.23) into (5.4) leaves B_1 undetermined, but gives

$$\frac{\partial^2 B_2}{\partial X^2} = \frac{\partial^2 |A_0^2|}{\partial y^2} \ , \tag{5.27}$$

from which we have

$$B_2 \sim X c_\pm(y, T) \quad \text{as } X \to \pm\infty \ , \tag{5.28}$$

with

$$c_+ - c_- = \frac{\partial J}{\partial y} \ , \quad J(y, T) = \frac{\partial}{\partial y} \int_{-\infty}^{\infty} |A_0^2|(X', y, T)\mathrm{d}X' \ . \tag{5.29}$$

This indicates the need for an outer region where the X scale is comparable with the slow Y scale. In this region, $\bar{A}$ is exponentially small, and

$$\bar{B} \sim \lambda^{-1/2} b_1(x, y, T) + \ldots \ , \quad x = \lambda^{-1} X \ , \tag{5.30}$$

with

$$\frac{\partial^2 b_1}{\partial x^2} + \frac{\partial^2 b_1}{\partial y^2} = 0 \ , \quad b_1 \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty \ , \tag{5.31}$$

and b_1 continuous across $x = 0$, but

$$\frac{\partial b_1}{\partial x}(0^+, y, T) - \frac{\partial b_1}{\partial x}(0^-, y, T) = \frac{\partial J}{\partial y} \ . \tag{5.32}$$

The solution is given by

$$b_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(y - y')\, J(y', T)\mathrm{d}y'}{x^2 + (y - y')^2} \ , \tag{5.33}$$

and matching to the inner region determines

$$B_1 = b_1(0, y, T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{J(y', T)\ \mathrm{d}y'}{y - y'} \ . \tag{5.34}$$

For $\bar{k}_\mathrm{r} > 0$, this solution terminates in a finite-time singularity at $T = 0$, in the neighbourhood of which

$$A_0 \sim \frac{1}{[(-2T)\,(1 + \xi^2 + \eta^2)]^{(1+\mathrm{i}\bar{\delta}_\mathrm{i})/2}} \ , \tag{5.35a}$$

$$B_1 \sim \frac{1}{(-2T)} \left[\frac{1}{1 + \eta^2} - \frac{\eta}{(1 + \eta^2)^{3/2}} \tanh^{-1}\left(\frac{\eta}{\sqrt{1 + \eta^2}} \right) \right] \ , \tag{5.35b}$$

with

$$\xi = \frac{X}{(-T)^{1/2}} \ , \quad \eta = \frac{y}{(-T)^{1/2}} \ , \tag{5.35c}$$

However, the expansion (5.23) ceases to be valid close to the singularity when

$$\tilde{s} \equiv -\lambda^{-1/2} \ln(-T) \tag{5.36}$$

is of order one (assuming that $\lambda^{-1/2} \gg \hat{\mu}$). It then becomes necessary to modify the asymptotic form (5.23) to

$$\bar{A} \sim \frac{r(\xi,\eta,\tilde{s})\mathrm{e}^{\mathrm{i}\phi(\xi,\eta,\tilde{s})}}{(-2T)^{(1+\mathrm{i}\bar{\delta}_{\mathrm{i}})/2}} \ , \quad \bar{B} \sim \frac{b(\xi,\eta,\tilde{s})}{(-2T)} \ , \tag{5.37}$$

and expand

$$r = r_0(\xi,\eta,\tilde{s}) + \lambda^{-1/2} r_1(\xi,\eta,\tilde{s}) + \dots \ , \tag{5.38}$$

$$\phi = \phi_0(\xi,\eta,\tilde{s}) + \lambda^{-1/2} \phi_1(\xi,\eta,\tilde{s}) + \dots \ , \tag{5.39}$$

$$b = \lambda^{-1/2} b_1(\eta,\tilde{s}) + \lambda^{-1} b_2(\xi,\eta,\tilde{s}) + \dots \ . \tag{5.40}$$

At leading order,

$$r_0 = \left[1 + C(\tilde{s})\xi^2 + D(\tilde{s})\eta^2\right]^{-1/2} \ , \tag{5.41}$$

$$\phi_0 = \Phi_0(\tilde{s}) - \tfrac{1}{2}\bar{\delta}_{\mathrm{i}} \ln\left[1 + C(\tilde{s})\xi^2 + D(\tilde{s})\eta^2\right] \ , \tag{5.42}$$

$$b_1 = \left(\frac{D}{C}\right)^{1/2} \left[\frac{1}{1+D\eta^2} - \frac{D^{1/2}\eta}{(1+D\eta^2)^{3/2}} \tanh^{-1}\left(\frac{D^{1/2}\eta}{\sqrt{1+D\eta^2}}\right)\right] \ . \tag{5.43}$$

Further, the solvability conditions at order $\lambda^{-1/2}$ give

$$\frac{\mathrm{d}C}{\mathrm{d}\tilde{s}} = (CD)^{1/2} \ , \quad \frac{\mathrm{d}D}{\mathrm{d}\tilde{s}} = -\frac{D^{3/2}}{C^{1/2}} \ , \tag{5.44}$$

and in addition

$$C(0) = 1 \ , \quad D(0) = 1 \ , \tag{5.45}$$

to match to (5.35). It follows that

$$C = 1 + \tilde{s} \ , \quad D = (1+\tilde{s})^{-1} \ , \tag{5.46}$$

and hence the large-$\tilde{s}$ form of the singularity is once again given by (5.20)–(5.22), at least as long as the dispersive terms remain negligible.

(b) A candidate singularity structure for $\bar{k}_\mathrm{r} > 0$

Both the limiting cases discussed above terminate in a finite-time singularity (at $\tau = \tau_s$, say) whose ultimate form (5.20)–(5.22) has the X length-scale contracting like $[(\tau - \tau_s)/\ln(\tau_s - \tau)]^{1/2}$, and the Y length-scale contracting more slowly by a factor of $-\ln(\tau_s - \tau)$. We now show that this singularity structure represents a possible terminal form for *any* set of parameter values with $\bar{k}_\mathrm{r} > 0$. A similar type of singularity, though with different powers of the logarithm, was proposed by Hocking (1974) for the full DHSS equations; his analysis can be taken over with minor modification to the present case.

On the basis that the singularity develops at the origin $(X, Y) = (0, 0)$ at a time $\tau = \tau_s$, we propose

$$A \sim \frac{\sigma^{\mathrm{i}\kappa} r(\tilde{\xi}, \tilde{\eta}, \sigma) e^{\mathrm{i}\phi(\tilde{\xi}, \tilde{\eta}, \sigma)}}{\left[2\bar{k}_\mathrm{r}(\tau_s - \tau)\right]^{(1+\mathrm{i}\bar{\delta}_\mathrm{i})/2}} \ , \quad B \sim \frac{b(\tilde{\xi}, \tilde{\eta}, \sigma)}{2\bar{k}_\mathrm{r}(\tau_s - \tau)} \ , \tag{5.47}$$

with

$$\tilde{\xi} = \frac{X\sigma^{1/2}}{(\tau_s - \tau)^{1/2}} \ , \quad \tilde{\eta} = \frac{Y}{(\tau_s - \tau)^{1/2}\sigma^{1/2}} \ , \quad \sigma = -\ln(\tau_s - \tau) \ . \tag{5.48}$$

Substituting into the reduced form of (5.3) and (5.4), and neglecting the linear growth term which is smaller by a factor of order $(\tau_s - \tau) = \mathrm{e}^{-\sigma}$, we obtain

$$r + \left[1 + \sigma^{-1}\right] \tilde{\xi} r_{\tilde{\xi}} + \left[1 - \sigma^{-1}\right] \tilde{\eta} r_{\tilde{\eta}} = r^3 - \bar{k}_\mathrm{r}^{-1} r b \ , \tag{5.49}$$

$$2\kappa\sigma^{-1} + \bar{\delta}_\mathrm{i} + \left[1 + \sigma^{-1}\right] \tilde{\xi} \phi_{\tilde{\xi}} + \left[1 - \sigma^{-1}\right] \tilde{\eta} \phi_{\tilde{\eta}} = \bar{\delta}_\mathrm{i} r^2 - \bar{\gamma}_\mathrm{i} \bar{k}_\mathrm{r}^{-1} b \ , \tag{5.50}$$

and

$$b_{\tilde{\xi}\tilde{\xi}} + \sigma^{-2} b_{\tilde{\eta}\tilde{\eta}} = \sigma^{-2} \left(r^2\right)_{\tilde{\eta}\tilde{\eta}} \ . \tag{5.51}$$

As in the analysis for $\lambda \gg 1$ in the previous subsection, and in Hocking's (1974) singularity analysis, we expect that when $\sigma \gg 1$ the solution expands in the form

$$\begin{aligned} r(\tilde{\xi}, \tilde{\eta}, \sigma) &= r_0(\tilde{\xi}, \tilde{\eta}) + \sigma^{-1} r_1(\tilde{\xi}, \tilde{\eta}) + \dots \ , && (5.52)\\ \phi(\tilde{\xi}, \tilde{\eta}, \sigma) &= \phi_0(\tilde{\xi}, \tilde{\eta}) + \sigma^{-1} \phi_1(\tilde{\xi}, \tilde{\eta}) + \dots \ , && (5.53)\\ b(\tilde{\xi}, \tilde{\eta}, \sigma) &= \sigma^{-1} b_1(\tilde{\eta}) + \sigma^{-2} b_2(\tilde{\xi}, \tilde{\eta}) + \dots \ , && (5.54) \end{aligned}$$

with

$$r_0(\tilde{\xi}, \tilde{\eta}) = (1 + C_0\tilde{\xi}^2 + D_0\tilde{\eta}^2)^{-1/2} \ , \tag{5.55}$$

$$\phi_0(\tilde{\xi}, \tilde{\eta}) = \Phi_0 - \tfrac{1}{2}\bar{\delta}_\mathrm{i} \ln(1 + C_0\tilde{\xi}^2 + D_0\tilde{\eta}^2) \ , \tag{5.56}$$

$$b_1(\tilde{\eta}) = \left(\frac{D_0}{C_0}\right)^{1/2} \left[\frac{1}{1 + D_0\tilde{\eta}^2} - \frac{D_0^{1/2}\tilde{\eta}}{(1 + D_0\tilde{\eta}^2)^{3/2}} \tanh^{-1}\left(\frac{D_0^{1/2}\tilde{\eta}}{\sqrt{1 + D_0\tilde{\eta}^2}}\right)\right] \ . \tag{5.57}$$

At $O(\sigma^{-1})$, a solution for r_1 and ϕ_1 analytic at $(\tilde{\xi}, \tilde{\eta}) = (0,0)$ can be found provided that

$$D_0 = \bar{k}_{\mathrm{r}}^2 C_0 \quad \text{and} \quad \kappa = \frac{1}{2}\left(\bar{\delta}_{\mathrm{i}} - \bar{\gamma}_{\mathrm{i}}\right) \ . \tag{5.58}$$

The value of C_0 is undetermined in this local analysis, and in general has to be found from a numerical solution of (5.6)–(5.7) as

$$C_0 = \lim_{\tau \to \tau_s} \frac{\left(|A|_{XX}|A|_{YY}\right)^{1/2}}{2\bar{k}_{\mathrm{r}}|A|^3} \ . \tag{5.59}$$

However, in the limiting cases of $\bar{k}_{\mathrm{r}} \gg 1$ and/or $\lambda \gg 1$ discussed above, we have the asymptotic result

$$C_0 \sim \bar{k}_{\mathrm{r}}^{-1} \lambda^{-1/2} \ . \tag{5.60}$$

For reference we mention that other singularity structures have been sought. In particular a structure similar to that of the 'second species of burst' described by Hocking & Stewartson (1971) was considered, for which

$$|A| \sim \frac{\bar{a}(\xi,\eta)}{(\tau_s - \tau)^{(1+\mathrm{i}\gamma)/2}} + \dots \ , \quad B \sim \frac{\bar{b}(\xi,\eta)}{\tau_s - \tau} + \dots \ , \tag{5.61}$$

with

$$\xi = \frac{X}{(\tau_s - \tau)^{1/2}} \ , \quad \eta = \frac{Y}{(\tau_s - \tau)^{1/2}} \ , \tag{5.62}$$

and γ a constant. However, neither analytical nor numerical solutions could be found for the resulting similarity equations for $\bar{a}$ and $\bar{b}$. Further, a singularity of the form (5.47) but with more general similarity variables

$$\tilde{\xi} = \frac{X}{(\tau_s - \tau)^{1/2}\sigma^{\alpha}} \ , \quad \tilde{\eta} = \frac{Y}{(\tau_s - \tau)^{1/2}\sigma^{\beta}} \ , \quad \sigma = -\ln(\tau_s - \tau) \tag{5.63}$$

did not appear viable except for the case $\alpha = -1/2$, $\beta = 1/2$ discussed above.

(*c*) *Numerical solutions*

Our first attempt at a numerical solution of (5.6)–(5.7) used a simple finite-difference method with a fixed non-uniform grid in X and Y and a standard fourth-order Runge-Kutta scheme for time-marching. It was clear from these computations that when $\bar{k}_{\mathrm{r}} > 0$, a singularity develops at the origin $X = Y = 0$ at a finite time τ_s. However, the results did not provide convincing evidence for the structure of this singularity. We therefore adopted the two-stage numerical scheme described in Appendix A. The first stage is the simple fixed-grid method outlined above, which is used to integrate from a starting time (typically $\tau = -5.0$) up to a time τ_1, shortly before the singularity develops (typically $\tau_s - \tau_1 \approx 10^{-2}$). For times beyond τ_1, we use an adaptive-grid method based on that described by Landman, Papanicolaou, Sulem, Sulem & Wang (1991) and Papanicolaou, Sulem, Sulem & Wang (1994). This second scheme allows us to integrate accurately up to times with $\tau_s - \tau \approx 10^{-10}$, and so confirm some details of the singularity.

We shall concentrate here on the results of three particular integrations:

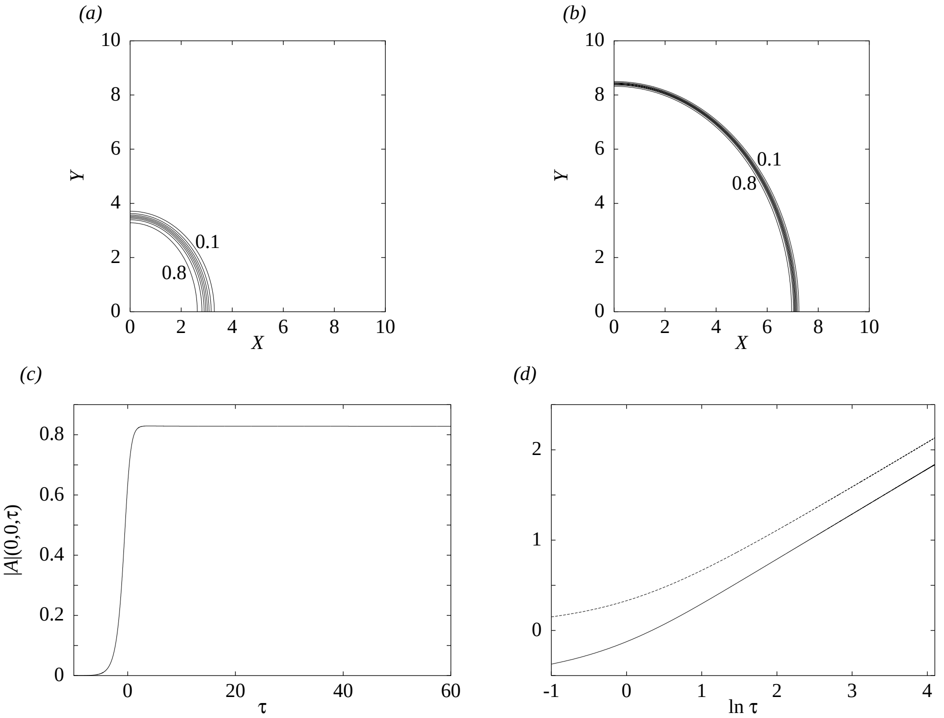

Figure 1. Numerical solution of (5.6)–(5.7) with $\bar{k}_\mathrm{r} = -1.0$, $\lambda = 1.0$.
(a) Contour plot of $|A|$ at time $\tau = 10.0$. The contour spacing is 0.1.
(b) The same, at $\tau = 60.0$.
(c) The value of $|A|$ at the origin, versus time.
(d) Log-log plots of the maximum of $-|A|_Y$ on the Y-axis, and the position of this maximum, versus time. Solid line: $\ln[\max_Y \{-|A|_Y(0,Y,\tau)\}] \equiv \ln[-|A|_Y(0,Y_m,\tau)]$. Dashed line: $\ln Y_m$.

(i) $\bar{k}_\mathrm{r} = -1.0$, $\lambda = 1.0$;

(ii) $\bar{k}_\mathrm{r} = 0.2$, $\lambda = 1.0$;

(iii) $\bar{k}_\mathrm{r} = 24.25$, $\lambda = 7.36$.

The third values are those appropriate to PPF (Davey *et al.* 1974).

Our numerical results suggest that when $\bar{k}_\mathrm{r} \leq 0$ a finite-time singularity does not form (see figure 1). The behaviour is similar to the supercritical cases of §3 and §4. At any fixed time, $|A|$ is approximately constant in a region surrounding the origin. This region, which is roughly circular but slightly elongated in the Y direction, is bounded by an expanding 'front' outside which A is close to zero (figure 1a,b). The value of $|A|$ inside the front quickly rises to a maximum, and then declines slowly as the system evolves in time (figure 1c). To demonstrate that overturning does not occur in finite time, figure 1d shows log-log plots of the maximum of $-|A|_Y$ on the Y-axis (solid line), and the position of this maximum (dashed line), against τ. In each case, the graph appears to be tending to a straight line of slope approximately $\frac{1}{2}$, suggesting that both quantities may be increasing like $\tau^{1/2}$.

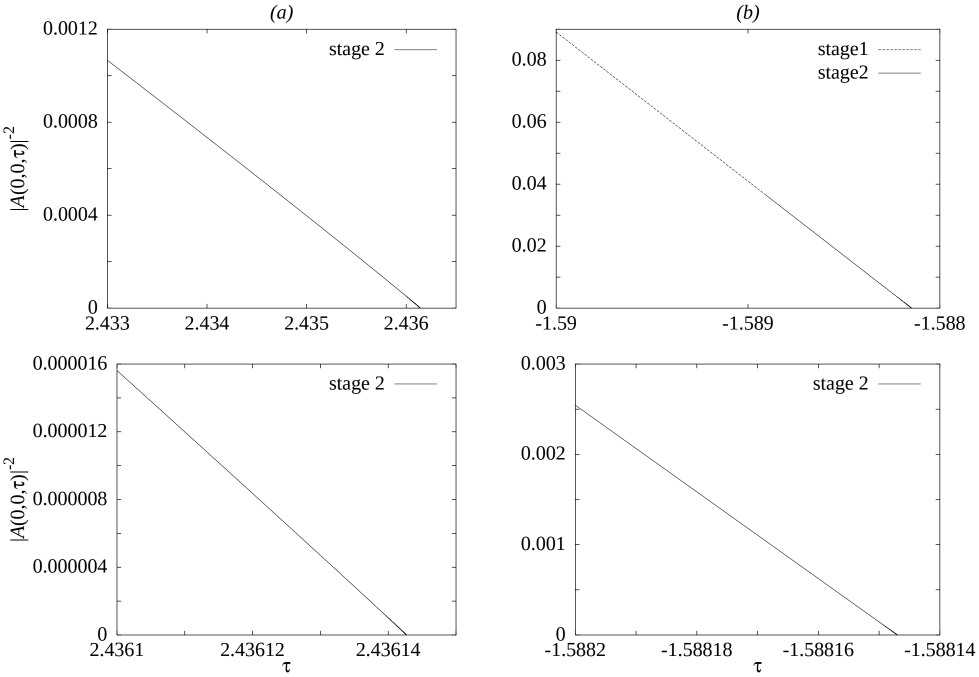

Figure 2. (a) Numerical solution of (5.6)–(5.7) with $\bar{k}_{\mathrm{r}} = 0.2$, $\lambda = 1.0$: the value of $|A|^{-2}$ at the origin versus time, for two different ranges of time. 'Stage 1' and 'stage 2' refer to the numerical method used (see Appendix A). (b) The same, for the PPF case $\bar{k}_{\mathrm{r}} = 24.25$, $\lambda = 7.36$.

On the other hand, for the two cases with $\bar{k}_{\mathrm{r}} > 0$, our computations suggest that a singularity develops at the origin, and has the structure proposed in the previous subsection. To verify this, we monitored the evolution of $|A|^{-2}$ and $|A^2|_{YY}/|A^2|_{XX}$ at the origin, for which we have the asymptotic predictions

$$|A|^{-2} \sim 2\bar{k}_{\mathrm{r}}\,(\tau_s - \tau) \tag{5.64a}$$

and

$$\frac{|A^2|_{YY}}{|A^2|_{XX}} \sim \left[\frac{\bar{k}_{\mathrm{r}}}{\ln(\tau_s - \tau)}\right]^2 . \tag{5.64b}$$

In figures 2a,b we plot the value of $|A|^{-2}$ at the origin for the parameter pairs (ii) and (iii) above. In each case the result is an excellent approximation to a straight line. Extrapolation to $|A| = 0$ gives estimates of the singularity time τ_s. As a further confirmation of the asymptotic behaviour of $|A|$, in figures 3 a,b we plot $\ln|A|/\ln(\tau_s - \tau)$ at the origin using the extrapolated values of τ_s. Given that the plots are sensitive to small changes in τ_s, both graphs seem consistent with a limiting value of $-\frac{1}{2}$, as required by (5.64a).

In order to indicate the spatial structure of the singularity, in figures 4a,b we plot the value of $|A^2|_{YY}/|A^2|_{XX}$ at the origin against time. For the $\bar{k}_{\mathrm{r}} = 0.2$, $\lambda = 1.0$ case, it seems clear that this quantity is tending to zero as $\tau \to \tau_s$. However, for the larger value of $\bar{k}_{\mathrm{r}}$ appropriate to PPF, this trend is not observed until τ is very close

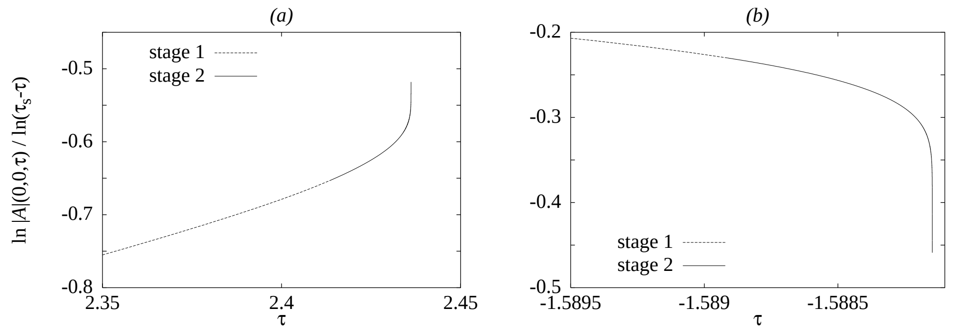

Figure 3. (a) The value of $\ln|A|/\ln(\tau_s-\tau)$ at the origin versus time, for $\bar{k}_{\rm r}=0.2$, $\lambda=1.0$. (b) The same, for $\bar{k}_{\rm r}=24.25$, $\lambda=7.36$ (PPF)

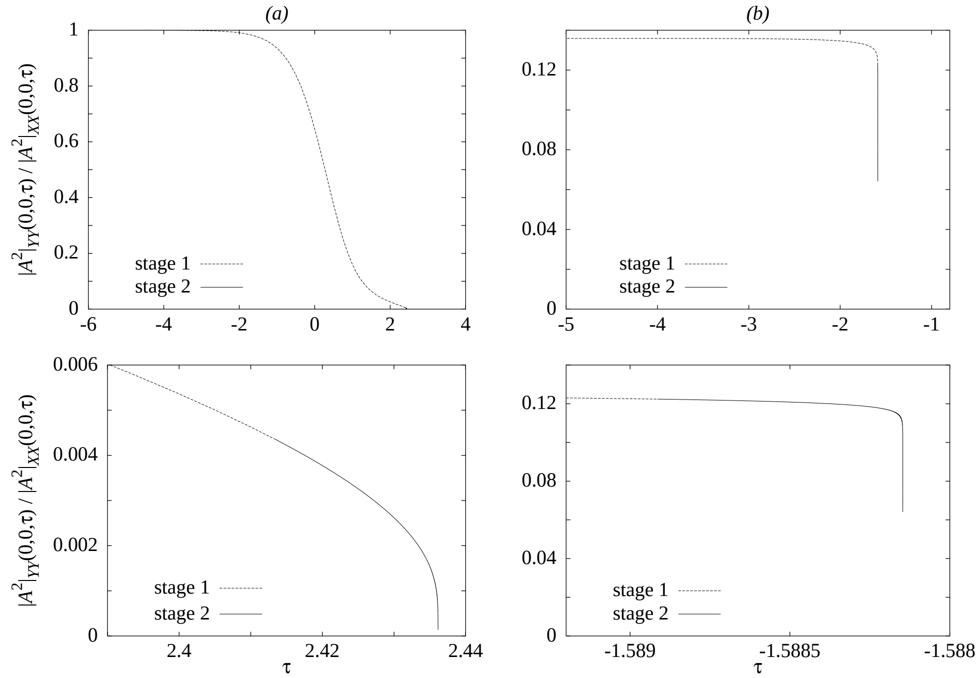

Figure 4. (a) The value of $|A^2|_{YY}/|A^2|_{XX}$ at the origin versus time, for $\bar{k}_{\rm r}=0.2$, $\lambda=1.0$. The first graph shows the full range of integration, the second a short range of times close to singularity formation. (b) The same, for $\bar{k}_{\rm r}=24.25$, $\lambda=7.36$ (PPF).

to τ_s. This was to be expected since the asymptotic structure is only recovered when $|\ln(\tau_s-\tau)|\gg\bar{k}_{\rm r}$; for the PPF case, we were unable to obtain numerical results at such times. It would thus seem desirable, in order to test the asymptotic prediction, that $\bar{k}_{\rm r}$ should be chosen as small as possible. On the other hand, for small values of $\bar{k}_{\rm r}$, front-like structures similar to those in figure 1a,b develop away from the origin well before the singularity time τ_s, and spurious oscillations develop if these fronts are not resolved. In practice, we failed to obtain reliable results close to the singularity with $\bar{k}_\mathrm{r}$ smaller than about 0.2.

We conclude that for at least two cases with $\bar{k}_\mathrm{r} > 0$, the numerical solution to (5.6)–(5.7) terminates in a finite-time singularity whose structure is consistent with that proposed in § 5 *b*.

(*d*) *Do point disturbances 'burst'?*

The question again arises as to whether solutions to the reduced form of (5.3)–(5.5) remain uniformly valid in time. We do not consider the case of $\bar{k}_\mathrm{r} < 0$ in detail here, and merely note that, as in §3 and §4, a non-uniformity may develop in the neighbourhood of a front, leading to the reinstatement of the asymptotically small dispersion terms on the right-hand side of (5.3).

In the subcritical case, $\bar{k}_\mathrm{r} > 0$, the solution appears to terminate in a singularity whose X length-scale is contracting faster than $(\tau_s - \tau)^{1/2}$ by a factor of $\sigma^{-1/2}$, where $\sigma = -\ln(\tau_s - \tau)$. It follows that the term proportional to $\hat{\mu} A_{XX}$ on the right-hand side of (5.3) will affect the solvability condition for r_1 in (5.52) when

$$\hat{\mu}\sigma = O\left(\sigma^{-1}\right) \ , \tag{5.65}$$

or in other words when

$$\hat{s} = -\hat{\mu}^{1/2} \ln(\tau_s - \tau) \tag{5.66}$$

is of order one. To describe this new phase of evolution, we write

$$A = \frac{\hat{\mu}^{-\mathrm{i}\kappa/2} \hat{r}(\hat{\xi}, \hat{\eta}, \tau) \mathrm{e}^{\mathrm{i}\hat{\phi}(\hat{\xi},\hat{\eta},\tau)}}{\left[2\bar{k}_\mathrm{r}(\tau_s - \tau)\right]^{(1+\mathrm{i}\bar{\delta}_\mathrm{i})/2}} \ , \quad B = \frac{\hat{b}(\hat{\xi}, \hat{\eta}, \tau)}{2\bar{k}_\mathrm{r}(\tau_s - \tau)} \ , \tag{5.67}$$

$$\hat{\xi} = \frac{X - X_0(\tau)}{\hat{\mu}^{1/4} (\tau_s - \tau)^{1/2}} \ , \quad \hat{\eta} = \frac{\hat{\mu}^{1/4} Y}{(\tau_s - \tau)^{1/2}} \ , \quad X_0 = -\hat{\mu}^{1/2} \varepsilon^{1/2} \hat{d}_0 \ln(\tau_s - \tau) + \dots \ , \tag{5.68}$$

and expand

$$\begin{aligned}
\hat{r} &= \hat{r}_{00}(\hat{\xi}, \hat{\eta}, \hat{s}) + \hat{\mu}^{1/2} \hat{r}_{20}(\hat{\xi}, \hat{\eta}, \hat{s}) + \hat{\mu}^{1/4} \varepsilon^{1/2} (\tau_s - \tau)^{-1/2} \hat{r}_{11}(\hat{\xi}, \hat{\eta}, \hat{s}) + \dots \ , && (5.69)\\
\hat{\phi} &= \hat{\phi}_{00}(\hat{\xi}, \hat{\eta}, \hat{s}) + \hat{\mu}^{1/2} \hat{\phi}_{20}(\hat{\xi}, \hat{\eta}, \hat{s}) + \dots \ , && (5.70)\\
\hat{b} &= \hat{\mu}^{1/2} \hat{b}_{20}(\hat{\eta}, \hat{s}) + \hat{\mu} b_{40}(\hat{\xi}, \hat{\eta}, \hat{s}) + \dots \ , && (5.71)
\end{aligned}$$

The leading-order solutions take the form

$$\hat{r}_{00} = \left[1 + C(\hat{s})\hat{\xi}^2 + D(\hat{s})\hat{\eta}^2\right]^{-1/2} \ , \tag{5.72}$$

$$\hat{\phi}_{00} = \hat{\Phi}(\hat{s}) - \tfrac{1}{2}\bar{\delta}_\mathrm{i} \ln\left[1 + C(\hat{s})\hat{\xi}^2 + D(\hat{s})\hat{\eta}^2\right] \ , \tag{5.73}$$

$$\hat{b}_{20} = \left(\frac{D}{C}\right)^{1/2} \left[\frac{1}{1 + D\hat{\eta}^2} - \frac{D^{1/2}\hat{\eta}}{(1 + D\hat{\eta}^2)^{3/2}} \tanh^{-1}\left(\frac{D^{1/2}\hat{\eta}}{\sqrt{1 + D\hat{\eta}^2}}\right)\right] \ . \tag{5.74}$$

The solvability conditions for $\hat{r}_{20}$ are found to be

$$C(\hat{s}) = C_0 l^{-1} f(\hat{S}) \ , \quad D(\hat{s}) = \bar{k}_{\mathrm{r}}^2 C_0 l\, g(\hat{S}) \ , \quad \hat{S} = l\hat{s} \ , \quad l = \frac{|F|^{1/2} C_0^{1/2}}{2(1+\bar{\alpha}_{\mathrm{i}}^2)^{1/2}} \ , \tag{5.75}$$

where F is again specified by (3.23) and

$$f' = -2\,\mathrm{sgn}(F) f^2 + f^{1/2} g^{1/2} \ , \quad g' = -f^{-1/2} g^{3/2} \ . \tag{5.76}$$

Matching to (5.47) requires that

$$f(\hat{S}) \sim \hat{S} \ , \quad g(\hat{S}) \sim \hat{S}^{-1} \quad \text{as } \hat{S} \to 0^+ \ . \tag{5.77}$$

The solvability condition for $\hat{\phi}_{20}$ is

$$\frac{\mathrm{d}\hat{\Phi}}{\mathrm{d}\hat{s}} = -\frac{\bar{\alpha}_{\mathrm{i}}(1+\bar{\delta}_{\mathrm{i}}^2)}{4(1+\bar{\alpha}_{\mathrm{i}}^2)} C + \frac{\kappa}{\bar{k}_{\mathrm{r}}} \left(\frac{D}{C}\right)^{1/2} \ , \tag{5.78}$$

and we must have

$$\hat{\Phi}(\hat{s}) \sim \kappa \ln \hat{s} + \Phi_0 \quad \text{as } \hat{s} \to 0 \ , \tag{5.79}$$

in order to match to (5.47) and (5.55)–(5.56). The origin-shift parameter $\hat{d}_0$ may be determined as described in §3 for line disturbances.

The problem (5.76)–(5.77) was solved numerically for the two cases $F > 0$ and $F < 0$. If $F > 0$, a solution exists for all $\hat{s} > 0$, and

$$C(\hat{s}) \sim 8(1+\bar{\alpha}_{\mathrm{i}}^2) F^{-1} \hat{s}^{-1} \ , \quad D(\hat{s}) \sim 72 \bar{k}_{\mathrm{r}}^2 (1+\bar{\alpha}_{\mathrm{i}}^2) F^{-1} \hat{s}^{-3} \quad \text{as} \quad \hat{s} \to \infty \ . \tag{5.80}$$

This large-$\hat{s}$ limit is in fact the Hocking (1974) singularity for the full DHSS equations. On the other hand, if $F < 0$, the solution terminates in a finite-time singularity at $\hat{S} = \hat{S}_0 \approx 1.056$, with

$$C(\hat{s}) \sim \frac{2(1+\bar{\alpha}_{\mathrm{i}}^2)}{|F|} \frac{1}{\hat{s}_0 - \hat{s}} \quad \text{as } \hat{s} \to \hat{s}_0 \equiv \frac{2(1+\bar{\alpha}_{\mathrm{i}}^2)^{1/2}}{|F|^{1/2} C_0^{1/2}} \hat{S}_0 \ , \tag{5.81}$$

but D remaining finite; in fact

$$D \to D(\hat{s}_0) \equiv \frac{\bar{k}_{\mathrm{r}}^2 |F|^{1/2} C_0^{3/2} g_0}{2(1+\bar{\alpha}_{\mathrm{i}}^2)^{1/2}} \ , \quad g_0 = g(\hat{S}_0) \approx 1.225 \ . \tag{5.82}$$

Equation (5.78) then gives

$$\hat{\Phi}(\hat{s}) \sim \nu \ln(1 - \hat{s}/\hat{s}_0) + \phi_0 \quad \text{as } \hat{s} \to \hat{s}_0 \ , \tag{5.83}$$

where ν is given by (3.26), and ϕ_0 is a constant whose value is of no concern. It follows that at times asymptotically close to $\hat{s}_0$, the A_{XX} term will be reinstated in the amplitude equations at leading order. However, the A_{YY} term will still be negligible, and moreover B will be small compared to $|A|^2$. As a result, the next stage of evolution will be governed by a one-dimensional CGL equation (without the linear growth term), with the Y dependence appearing parametrically. To be specific, the rescaling

$$\tau = \tau_s + \hat{\Delta}\hat{\tau} \ , \quad X = \varepsilon^{1/2}\hat{d}_0\hat{s}_0 + \frac{\mu^{1/2}\hat{\Delta}^{1/2}}{2(1+\bar{\alpha}_{\mathrm{i}}^2)^{1/2}}\hat{X} \ , \quad Y = \frac{\mu^{-1/4}\hat{\Delta}^{1/2}}{[2D(\hat{s}_0)]^{1/2}}\hat{Y} \ , \tag{5.84}$$

$$A = \hat{\Delta}^{-(1+\mathrm{i}\bar{\delta}_{\mathrm{i}})/2}\left(\frac{\hat{\mu}}{\hat{s}_0}\right)^{\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}\phi_0}\hat{A}(\hat{X},\hat{Y},\hat{\tau}) \ , \quad \text{with} \quad \hat{\Delta} = \exp\left(-\frac{\hat{s}_0}{\hat{\mu}^{1/2}}\right) \ , \tag{5.85}$$

leads to the amplitude equation

$$\hat{A}_{\hat{\tau}} = (1+\mathrm{i}\bar{\alpha}_{\mathrm{i}})\hat{A}_{\hat{X}\hat{X}} + (1+\mathrm{i}\bar{\delta}_{\mathrm{i}})|\hat{A}^2|\hat{A} \ , \tag{5.86}$$

and initial condition

$$\hat{A} \sim [\ln(-\hat{\tau})]^{\mathrm{i}\nu}\left[-2\hat{\tau} + \frac{\hat{X}^2}{|F|\ln(-\hat{\tau})} + \hat{Y}^2\right]^{-(1+\mathrm{i}\bar{\delta}_{\mathrm{i}})/2} \qquad \text{as } \hat{\tau} \to -\infty \ . \tag{5.87}$$

Except for the additional $\hat{Y}$-dependence in the initial condition, this is the same as the corresponding problem for line disturbances, (3.29)–(3.31).

As before, there is a caveat to the above conclusions. The analysis leading to (5.86) ceases to be valid if the amplitude of the disturbance rises to order one before the rescaling is necessary. This occurs if

$$-\hat{\mu}^{1/2}\ln\varepsilon \sim \frac{\ln\varepsilon^{-1}}{\left[\ln\left(|\mathcal{A}_2|^{-1}\varepsilon^{-1/2}\right)\right]^{1/2}} < \hat{s}_0 \ . \tag{5.88}$$

This is a more stringent requirement than the corresponding conditions (3.32) for line disturbances and (4.33) for point disturbances with $\mathrm{Re}(q) = 0$: if $\hat{s}_0$ is of order one, (5.88) demands that the initial amplitude $|\mathcal{A}_2|$ be smaller than any power of ε.

6. Three-dimensional Poiseuille-Couette flow

In the above analysis we have followed previous authors and concentrated on two-dimensional basic flows. However, the analyses of Stewartson & Stuart (1971), Davey *et al.* (1974), etc., and the modifications described in this paper, are readily extended to a three-dimensional parallel flow of the form $[\bar{U}(z), \bar{V}(z), 0]$; details are given in Appendix B.† For definiteness, we shall concentrate on the particular case of three-dimensional Poiseuille-Couette flow

$$\left[1 - z^2 + U_{\mathrm{w}}z, -V_{\mathrm{w}}z, 0\right] \ . \tag{6.1}$$

If U_{w} and V_{w} are both non-zero, the marginally stable mode at the critical Reynolds number $R = R_{cr}$ is in general an *oblique* mode

$$\mathrm{e}^{\mathrm{i}\alpha_{cr}\cos\theta(x-c_{cr}t)+\mathrm{i}\alpha_{cr}\sin\theta y} \ . \tag{6.2}$$

† Some related results for Ekman layers can be found in Zaff (1987), Dwarka (1993) and Dwarka & Herron (1996).

It is convenient to make Squire's (1933) transformation, that is, to introduce rotated axes which are aligned with and orthogonal to this neutral mode:

$$\hat{x} = x\cos\theta + y\sin\theta\ , \quad \hat{y} = -x\sin\theta + y\cos\theta\ , \tag{6.3}$$

and scaled variables

$$\hat{u} = u + v\tan\theta\ , \quad \hat{v} = -u\tan\theta + v\ , \quad \hat{w} = w\sec\theta\ , \tag{6.4}$$

$$\hat{p} = p\sec^2\theta\ , \quad \hat{t} = t\cos\theta\ . \tag{6.5}$$

In terms of the new (hatted) variables, the Navier-Stokes equations are unaltered in form except that the Reynolds number R is replaced by

$$\hat{R} = R\cos\theta\ . \tag{6.6}$$

The unperturbed flow is given by

$$\hat{u} = \bar{U}(z) \equiv 1 - z^2 + u_{\mathrm{w}}z\ , \quad \hat{v} = \bar{V}(z) \equiv -\Gamma\left(1 - z^2\right) - v_{\mathrm{w}}z\ , \quad \hat{w} = 0\ , \tag{6.7}$$

with

$$u_{\mathrm{w}} = U_{\mathrm{w}} - V_{\mathrm{w}}\tan\theta\ , \quad v_{\mathrm{w}} = V_{\mathrm{w}} + U_{\mathrm{w}}\tan\theta\ , \quad \Gamma = \tan\theta\ . \tag{6.8}$$

At the critical Reynolds number, $\hat{R} = \hat{R}_{cr}$, the linear neutral mode takes the form

$$\begin{aligned}[\tilde{u}, \tilde{v}, \tilde{w}] &\equiv \left[\hat{u} - (1 - z^2 + u_{\mathrm{w}}z), \hat{v} + \Gamma\left(1 - z^2\right) + v_{\mathrm{w}}z, \hat{w}\right] \\ &= \left[\psi'(z), \chi(z), -i\alpha_{cr}\psi(z)\right]\mathrm{e}^{\mathrm{i}\alpha_{cr}(\hat{x} - c_{cr}\hat{t})} + c.c.\end{aligned} \tag{6.9}$$

where

$$\left.\begin{aligned}\left(1 - z^2 + u_{\mathrm{w}}z - c_{cr}\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \alpha_{cr}^2\right)\psi + 2\psi - \frac{1}{\mathrm{i}\alpha_{cr}\hat{R}_{cr}}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \alpha_{cr}^2\right)^2\psi = 0\ , \\ \psi(\pm 1) = 0\ , \quad \psi'(\pm 1) = 0\ ,\end{aligned}\right\} \tag{6.10}$$

$$\left.\begin{aligned}\left(1 - z^2 + u_{\mathrm{w}}z - c_{cr}\right)\chi - \frac{1}{\mathrm{i}\alpha_{cr}\hat{R}_{cr}}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \alpha_{cr}^2\right)\chi = \left(-2\Gamma z + v_{\mathrm{w}}\right)\psi\ , \\ \chi(\pm 1) = 0\ .\end{aligned}\right\} \tag{6.11}$$

(6.10) is precisely the Orr-Sommerfeld problem for the *plane* Poiseuille-Couette flow (PPCF)

$$\left[1 - z^2 + u_{\mathrm{w}}z, 0, 0\right]\ . \tag{6.12}$$

If the critical Reynolds number for this flow is denoted by $\mathcal{R}_c(u_{\mathrm{w}})$, it follows from (6.6) that for a particular U_{w} and V_{w}, θ is the value of γ which minimises

$$\sec\gamma\,\mathcal{R}_c(U_{\mathrm{w}} - V_{\mathrm{w}}\tan\gamma)\ . \tag{6.13}$$

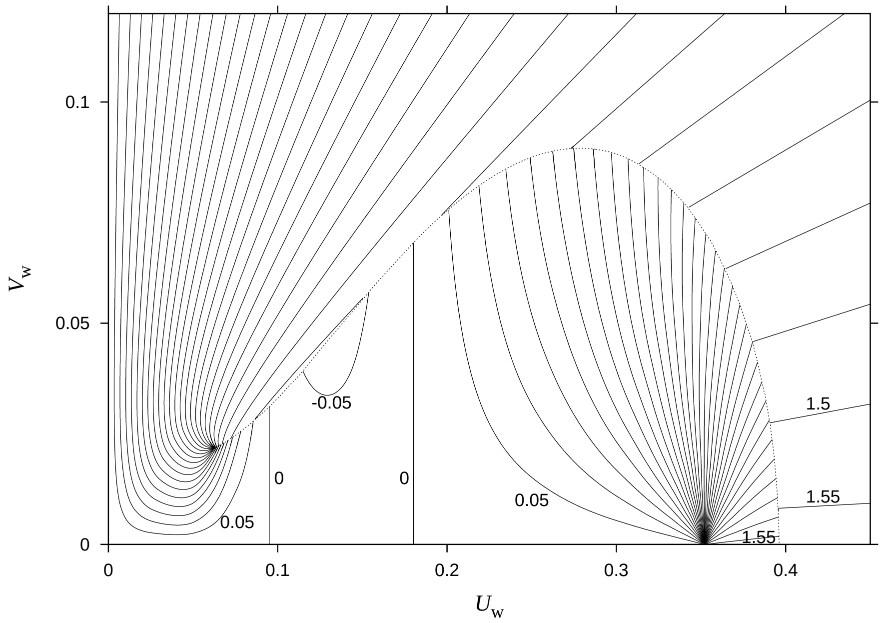

Figure 5. Contour plot of the angle θ between the Poiseuille-flow direction and the wavevector of the critical neutral mode for the three-dimensional Poiseuille-Couette flow (6.1), as a function of U_{w} and V_{w}. The spacing between adjacent contours is 0.05 rad. The dotted curve represents parameter values for which there are two neutral modes at the critical Reynolds number.

Figure 5 shows θ as a function of U_{w} and V_{w}.

At slightly supercritical Reynolds number, with $0 < \hat{R} - \hat{R}_{cr} \ll 1$, the dispersion relation for a near-marginal mode $\mathrm{e}^{\mathrm{i}\alpha(\hat{x}-c\hat{t})+\mathrm{i}\beta\hat{y}}$ with $|\alpha - \alpha_{cr}| \ll 1$ and $|\beta| \ll 1$ expands as

$$-\mathrm{i}\alpha c = -\mathrm{i}\alpha_{cr}c_{cr} + \mathrm{i}a_{1\mathrm{r}}\left(\alpha - \alpha_{cr}\right) + \mathrm{i}\hat{b}_{1\mathrm{r}}\beta - a_2\left(\alpha - \alpha_{cr}\right)^2 - \hat{c}_2\left(\alpha - \alpha_{cr}\right)\beta - \hat{b}_2\beta^2 + (\hat{R} - \hat{R}_{cr})d_1 + \dots \; . \tag{6.14}$$

The coefficients may be calculated as described in Appendix B (and in more detail by Jennings 1997). In particular, it is found that $a_{1\mathrm{r}}$, a_2 and d_1 take the same values as for the two-dimensional basic flow (6.12), and thus are functions of u_{w} only, as given in table 1. The remaining coefficients depend also on the transverse mean-flow component, but this dependence takes the simple form

$$\hat{b}_{1\mathrm{r}} = -\Gamma\hat{b}_{1\mathrm{r}}^{(P)}(u_{\mathrm{w}}) - v_{\mathrm{w}}\hat{b}_{1\mathrm{r}}^{(C)}(u_{\mathrm{w}}) \; , \tag{6.15a}$$

$$\hat{b}_2 = b_2(u_{\mathrm{w}}) + \Gamma^2\hat{b}_2^{(P)}(u_{\mathrm{w}}) + v_{\mathrm{w}}^2\hat{b}_2^{(C)}(u_{\mathrm{w}}) + \Gamma v_{\mathrm{w}}\hat{b}_2^{(PC)}(u_{\mathrm{w}}) \; , \tag{6.15b}$$

$$\hat{c}_2 = -\Gamma\hat{c}_2^{(P)}(u_{\mathrm{w}}) - v_{\mathrm{w}}\hat{c}_2^{(C)}(u_{\mathrm{w}}) \; , \tag{6.15c}$$

where b_2 is given in table 1, and the remaining coefficients are presented in table 2.

Table 2. *Coefficients of (6.15) and (6.25) for three-dimensional Poiseuille-Couette flow (6.7).*

u_{w}	$\hat{b}_{1\mathrm{r}}^{(P)}$	$\hat{b}_{1\mathrm{r}}^{(C)}$	$\hat{b}_{2\mathrm{r}}^{(P)}$	$\hat{b}_{2\mathrm{i}}^{(P)}$	$\hat{b}_{2\mathrm{r}}^{(C)}$	$\hat{b}_{2\mathrm{i}}^{(C)}$	$\hat{b}_{2\mathrm{r}}^{(PC)}$
0.0	0.218 1	0.0	0.002 2	-0.019 7	4.009 4	-4.750 5	0.0
0.01	0.215 7	-0.098 8	0.002 0	-0.020 0	3.708 8	-4.969 2	0.013 5
0.025	0.203 8	-0.260 0	0.000 4	-0.020 8	2.067 0	-5.422 6	0.087 0
0.05	0.174 7	-0.464 8	-0.004 2	-0.020 6	-0.248 7	-3.311 2	0.216 6
0.075	0.157 1	-0.520 0	-0.004 5	-0.019 8	0.478 9	-1.529 5	0.107 4
0.1	0.149 8	-0.528 2	-0.001 3	-0.021 9	1.167 3	-1.141 3	-0.038 1
0.15	0.143 1	-0.534 9	0.008 7	-0.032 1	1.549 5	-1.156 1	-0.237 4
0.2	0.138 2	-0.546 3	0.022 9	-0.049 4	1.604 8	-1.263 6	-0.383 5
0.25	0.133 8	-0.562 1	0.044 0	-0.077 5	1.690 8	-1.432 8	-0.539 8
0.3	0.129 7	-0.583 4	0.082 0	-0.133 2	2.000 3	-1.848 6	-0.791 3
0.32	0.128 0	-0.594 4	0.112 6	-0.181 8	2.352 9	-2.274 8	-0.998 7
0.33	0.127 0	-0.600 9	0.139 4	-0.225 9	2.710 6	-2.692 1	-1.188 0
0.34	0.125 8	-0.608 3	0.192 1	-0.314 1	3.485 4	-3.572 0	-1.573 9
0.343 8	0.125 3	-0.611 5	0.233 3	-0.383 5	4.126 8	-4.287 0	-1.883 5
0.345	0.125 2	-0.612 5	0.252 8	-0.416 2	4.435 7	-4.628 9	-2.031 2
0.346 3	0.125 0	-0.613 7	0.280 5	-0.462 8	4.879 1	-5.117 3	-2.242 1
0.347 5	0.124 8	-0.614 8	0.316 0	-0.522 6	5.454 8	-5.748 6	-2.514 7

u_{w}	$\hat{b}_{2\mathrm{i}}^{(PC)}$	$\hat{c}_{2\mathrm{r}}^{(P)}$	$\hat{c}_{2\mathrm{i}}^{(P)}$	$\hat{c}_{2\mathrm{r}}^{(C)}$	$\hat{c}_{2\mathrm{i}}^{(C)}$	$\hat{s}^{(P)}$	$\hat{s}^{(C)}$
0.0	0.0	0.054 2	0.089 1	0.0	0.0	-19.57	0.0
0.01	0.084 6	0.051 0	0.091 6	0.466 5	0.029 8	-19.54	-2.33
0.025	0.191 0	0.036 9	0.104 8	1.071 2	-0.164 0	-19.67	-5.51
0.05	0.146 3	0.012 6	0.128 4	1.258 6	-0.688 9	-22.27	-10.23
0.075	0.055 6	-0.008 4	0.132 4	1.107 5	-0.734 7	-27.34	-14.62
0.1	0.062 8	-0.027 7	0.132 2	1.006 0	-0.679 4	-31.32	-17.38
0.15	0.162 3	-0.057 5	0.128 9	0.864 6	-0.601 9	-33.43	-18.98
0.2	0.282 5	-0.077 9	0.117 2	0.755 3	-0.559 9	-32.22	-18.53
0.25	0.433 3	-0.092 1	0.093 5	0.675 0	-0.545 9	-30.51	-17.71
0.3	0.702 6	-0.107 5	0.051 6	0.651 3	-0.592 7	-29.03	-16.96
0.32	0.936 2	-0.121 2	0.023 3	0.692 8	-0.668 8	-28.57	-16.71
0.33	1.151 2	-0.135 3	0.002 2	0.753 8	-0.751 9	-28.37	-16.60
0.34	1.586 0	-0.167 0	-0.032 5	0.908 7	-0.939 2	-28.21	-16.50
0.343 8	1.930 9	-0.194 1	-0.055 8	1.047 1	-1.097 7	-28.16	-16.47
0.345	2.094 2	-0.207 4	-0.066 1	1.115 4	-1.174 8	-28.14	-16.46
0.346 3	2.326 5	-0.226 5	-0.080 2	1.214 8	-1.285 8	-28.12	-16.45
0.347 5	2.625 4	-0.251 5	-0.097 5	1.345 3	-1.430 8	-28.11	-16.44

For an initially small point disturbance, the large-time limit of the linear stage has

$$\tilde{u} \sim \frac{\mathcal{A}_2 \psi'(z)}{\hat{t}} \mathrm{e}^{\mathrm{i}\alpha_{cr}(\hat{x}-c_{cr}\hat{t})} \exp\left[(\hat{R}-\hat{R}_{cr})d_1\hat{t} - \frac{Q(\hat{x},\hat{y},\hat{t})}{(4a_2\hat{b}_2-\hat{c}_2^2)\hat{t}}\right] + c.c., \tag{6.16}$$

where

$$Q = \hat{b}_2(\hat{x}+a_{1\mathrm{r}}\hat{t})^2 - \hat{c}_2(\hat{x}+a_{1\mathrm{r}}\hat{t})(\hat{y}+\hat{b}_{1\mathrm{r}}\hat{t}) + a_2(\hat{y}+\hat{b}_{1\mathrm{r}}\hat{t})^2 \ , \tag{6.17}$$

and $|\mathcal{A}_2| \ll 1$. This leads to a weakly nonlinear régime for which

$$[\tilde{u}, \tilde{v}, \tilde{w}] \sim \hat{\varepsilon}^{1/2} \hat{A}(\hat{X}, \hat{Y}, T)\, [\psi'(z), \chi(z), -\mathrm{i}\alpha_{cr}\psi(z)]\, \mathrm{e}^{\mathrm{i}\alpha_{cr}(\hat{x}-c_{cr}\hat{t})} + c.c. + O(\hat{\varepsilon}) \;, \tag{6.18}$$

with

$$\hat{\varepsilon} = \left(\hat{R} - \hat{R}_{cr}\right) d_{1\mathrm{r}} \;, \quad T = \hat{\varepsilon}\left(\hat{t} - \hat{t}_2\right) \;, \quad (\hat{X}, \hat{Y}) = \hat{t}_2^{-1/2}\left(\hat{x} + a_{1\mathrm{r}}\hat{t}, \hat{y} + \hat{b}_{1\mathrm{r}}\hat{t}\right) \;, \tag{6.19}$$

and $\hat{t}_2$ defined as before by

$$|\mathcal{A}_2|^2 \exp(2\hat{\varepsilon}\hat{t}_2) = \hat{\varepsilon}\hat{t}_2^2 \;. \tag{6.20}$$

From a standard weakly nonlinear analysis (see Appendix B), the evolution of the modulation amplitude $\hat{A}$ is found to be governed by

$$\hat{A}_T = \hat{A} + k|\hat{A}^2|\hat{A} + q\hat{A}\hat{B} + \hat{\mu}\left(a_2\hat{A}_{\hat{X}\hat{X}} + \hat{c}_2\hat{A}_{\hat{X}\hat{Y}} + \hat{b}_2\hat{A}_{\hat{Y}\hat{Y}}\right) + \dots \;, \tag{6.21}$$

$$\hat{B}_{\hat{X}\hat{X}} + \hat{B}_{\hat{Y}\hat{Y}} = \hat{s}|\hat{A}^2|_{\hat{X}\hat{Y}} + |\hat{A}^2|_{\hat{Y}\hat{Y}} \;, \tag{6.22}$$

where k and q are the coefficients given in table 2 for plane Poiseuille-Couette flow (6.12) and

$$\hat{\mu} = \frac{1}{\hat{\varepsilon}\hat{t}_2} \sim \frac{1}{\ln\left(|\mathcal{A}_2|^{-1}\hat{\varepsilon}^{-1/2}\right)} \ll 1 \;, \tag{6.23}$$

$$\hat{s} = -\frac{\mathrm{Im}\left(\int_{-1}^{1} z\psi\chi^* \,\mathrm{d}z\right)}{\mathrm{Im}\left(\int_{-1}^{1} z\psi\psi'^* \,\mathrm{d}z\right)} \;. \tag{6.24}$$

It is evident from (6.11) and (6.24) that

$$\hat{s} = -\Gamma\hat{s}^{(P)}(u_{\mathrm{w}}) - v_{\mathrm{w}}\hat{s}^{(C)}(u_{\mathrm{w}}) \;; \tag{6.25}$$

the values of $\hat{s}^{(P)}$ and $\hat{s}^{(C)}$ are given in table 2.

The system (6.21)–(6.22) can be simplified by the further rotation of axes

$$X = \hat{X}\cos\tilde{\theta} + \hat{Y}\sin\tilde{\theta} \;, \quad Y = -\hat{X}\sin\tilde{\theta} + \hat{Y}\cos\tilde{\theta} \;, \quad \tilde{\theta} = -\tfrac{1}{2}\arctan\hat{s} \,, \tag{6.26}$$

together with the substitutions

$$\hat{A} = (\cos 2\tilde{\theta})^{1/2} A(X, Y, T) \;, \quad \hat{B} = -\sin^2\tilde{\theta}|A^2| + B(X, Y, T) \;, \tag{6.27}$$

which give

$$A_T = A + \tilde{k}|A^2|A + qAB + \hat{\mu}\left(\tilde{a}_2 A_{XX} + \tilde{c}_2 A_{XY} + \tilde{b}_2 A_{YY}\right) + \dots \;, \tag{6.28}$$

$$B_{XX} + B_{YY} = |A^2|_{YY} \;, \tag{6.29}$$

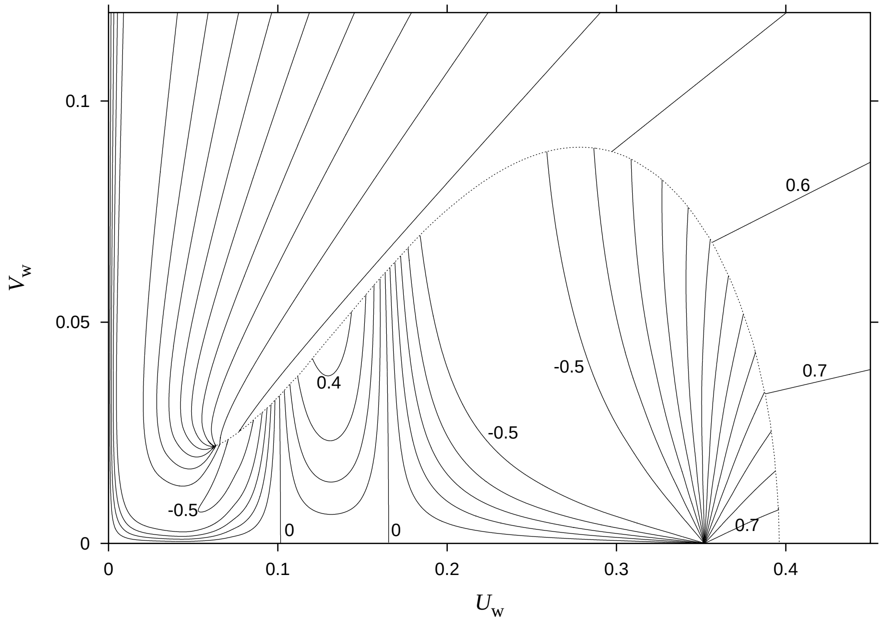

Figure 6. Contour plot of the angle $\theta + \tilde{\theta}$ between the Poiseuille-flow direction and the rotated X-axis specified by (6.3) and (6.26) for the three-dimensional Poiseuille-Couette flow (6.1), as a function of $U_{\rm w}$ and $V_{\rm w}$. The spacing between adjacent contours is 0.1 rad.

where

$$\tilde{a}_2 = a_2 \cos^2\tilde{\theta} + \hat{b}_2 \sin^2\tilde{\theta} + \tfrac{1}{2}\hat{c}_2 \sin 2\tilde{\theta} \ , \quad \tilde{b}_2 = a_2 \sin^2\tilde{\theta} + \hat{b}_2 \cos^2\tilde{\theta} - \tfrac{1}{2}\hat{c}_2 \sin 2\tilde{\theta} \ ,$$

$$\tilde{c}_2 = \hat{c}_2 \cos 2\tilde{\theta} + \left(\hat{b}_2 - a_2\right)\sin 2\tilde{\theta} \ , \quad \tilde{k} = k\cos 2\tilde{\theta} - q\sin^2\tilde{\theta} \ . \tag{6.30}$$

The inclination of the new X-axis to the Poiseuille-flow direction, i.e. $\theta+\tilde{\theta}$, is shown in figure 6 as a function of $U_{\rm w}$ and $V_{\rm w}$.

When the terms of $O(\hat{\mu})$ are neglected, the amplitude equations (6.28)–(6.29) have the same form as for a two-dimensional basic state. Moreover, since $\mathrm{Re}(k)$ and $\mathrm{Re}(-q)$ are always positive for PPCF, the same is true of $\mathrm{Re}(\tilde{k})$. The problem is not fully equivalent to that for a two-dimensional basic flow since matching to (6.16) requires

$$A \sim \exp\left(\mathrm{i}\phi_2 + \frac{d_1}{d_{1\mathrm{r}}}T - \frac{X^2}{4p_2} + \frac{XY}{4r_2} - \frac{Y^2}{4q_2}\right) \quad \text{as } T \to -\infty \ , \tag{6.31}$$

with

$$\phi_2 = \hat{\mu}^{-1} d_{1\mathrm{i}}/d_{1\mathrm{r}} + \arg \mathcal{A}_2 \ , \quad \left[p_2, q_2, r_2\right] = \left(\tilde{a}_2\tilde{b}_2 - \tfrac{1}{4}\tilde{c}_2^2\right)\left[\frac{1}{\tilde{b}_2}, \frac{1}{\tilde{a}_2}, \frac{1}{\tilde{c}_2}\right] . \tag{6.32}$$

In general $\mathrm{Re}(r_2) \neq 0$, and the initial condition for $|A|$ does not reduce to the form (5.7). Nevertheless, it seems plausible that the solution of (6.28)–(6.29), with the

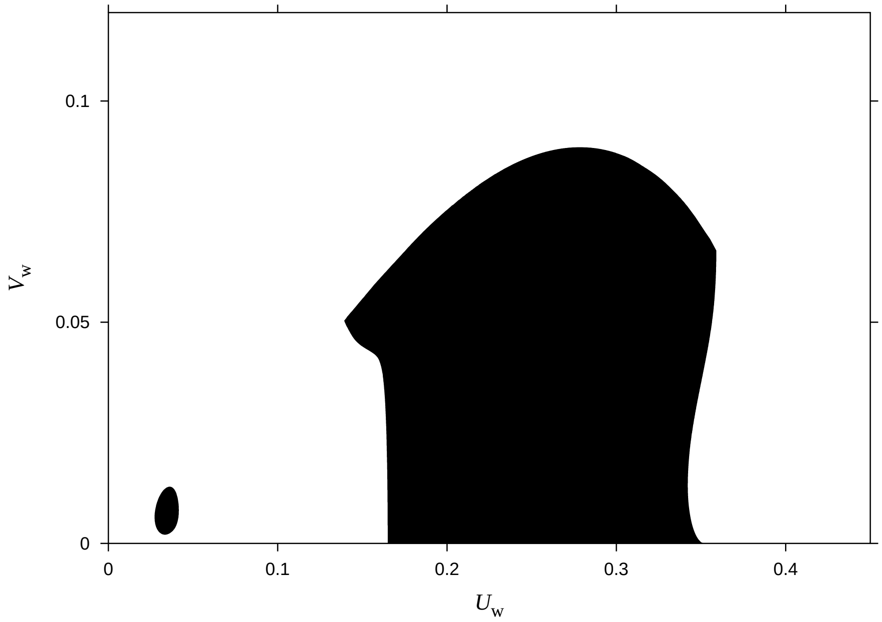

Figure 7. The region of the $(U_\mathrm{w}, V_\mathrm{w})$-plane (shaded) where the quantity $\tilde{F}$ defined by (6.33) is positive, and hence the Hocking (1974) singularity is available.

$O(\hat{\mu})$ terms dropped, will still terminate in the type of singularity described in §5*b* above, with flattening along the Y-direction (which in this case is generally not perpendicular to the direction of the wave). If so, the conclusions of §5*d* carry over essentially unchanged, with the ultimate behaviour dependent on the sign of

$$\tilde{F} = 3 - \left(\frac{\tilde{k}_\mathrm{i}}{\tilde{k}_\mathrm{r}}\right)^2 - 4\,\frac{\tilde{a}_{2\mathrm{i}}}{\tilde{a}_{2\mathrm{r}}}\,\frac{\tilde{k}_\mathrm{i}}{\tilde{k}_\mathrm{r}}\ . \tag{6.33}$$

Figure 7 shows the region of the $(U_\mathrm{w}, V_\mathrm{w})$-plane where $\tilde{F} > 0$, i.e. the region where the Hocking (1974) singularity is available. It is to be expected that in this region of parameter space, any small localized disturbance can burst to order-one amplitude. If $\tilde{F} < 0$, however, then unless the initial amplitude satisfies a condition of the form (5.88), the $\partial^2 A/\partial X^2$ term will presumably be reinstated at leading order in the amplitude equation while the disturbance is still small, leading to a second weakly nonlinear stage governed by a one-dimensional CGL equation.

7. Conclusions

The aim of this paper has to examine the growth of initially small localized disturbances in a marginally unstable system. Conventionally it has been supposed that when weakly nonlinear effects become important, the evolution of the modulated wavetrain can be described by the complex Ginzburg-Landau (CGL) equation or one of its relatives, such as the complex Davey-Stewartson (DHSS) equations in the case of three-dimensional disturbances in Poiseuille-Couette flow. However, as

Hocking *et al.* (1972) noted, a difficulty with this approach is that direct matching with the earlier linear solution is not possible.

We have shown that matching can be achieved by means of a logarithmic shift in the origin of time, but that when weakly nonlinear effects first become important, the amplitude is governed by the Landau equation (line disturbances) or a reduced form of the DHSS equations (point disturbances). An important consequence is that whenever the bifurcation is subcritical, a finite-time singularity is encountered on this scaling. Further, if the disturbance is sufficiently small initially, it can 'burst' to order-one amplitude. Indeed, if the coefficients are such that the quantity F defined by (3.22) is positive, this will happen regardless of the initial size of disturbance. If $F < 0$, however, and the initial amplitude is small but does not satisfy the restriction (3.32), (4.33) or (5.88) as appropriate, a dispersion term will be reinstated at leading order in the evolution equation before the burst can occur. The subsequent evolution of both line and point initial disturbances is then governed by the one-dimensional CGL equation (but without the linear growth term). We have posed, but not solved, the generic initial-value problem. However, it is known from the work of Hocking & Stewartson (1972) that for certain values of the coefficients a finite-time singularity ('of the second species') is available, in which case the disturbance may still burst to order-one amplitude.

These results slightly revise our understanding of the instability of PPF. Previous studies have suggested that point disturbances do not burst, while line disturbances only burst if they are skewed at angles greater than 57.3° to the direction of flow. We have found that if the initial disturbance amplitude is sufficiently small then both line and point disturbances can burst through to order-one amplitude. However, for slightly larger (though still small) initial amplitudes, the disturbances experience a period of explosive growth, followed by a second weakly nonlinear stage governed by a one-dimensional CGL equation. The values of the coefficients of this equation suggest that a point disturbance will remain bounded (Hocking & Stewartson 1972).

As an example of a three-dimensional shear flow, we have studied the stability of three-dimensional Poiseuille-Couette flow. We have shown that when a point disturbance first becomes weakly nonlinear, the appropriate amplitude equations can be reduced to the same form as those for a two-dimensional shear flow (namely, the reduced DHSS equations).

Finally, we wish to emphasise that the full CGL or DHSS equations may indeed be valid models for the evolution of certain types of initial disturbance. For example, if *periodic* boundary conditions are imposed, there is no difficulty in finding solutions of these equations which go to zero at large negative times, and so match directly to an earlier linear stage. On the other hand, for an initially small *localized* disturbance, at least the *first* weakly nonlinear stage is governed by a reduced form of these equations, without the dispersive terms.

We are indebted to Prof. L.M. Hocking and Dr X. Wu for stimulating discussions. Our thanks also go to the referees for their helpful comments, to Dr J.M. Rallison for alerting us to an error in an earlier draft of this paper, and to Dr M.O. de Souza for the use of his Orr-Sommerfeld solver. This work was presented at the British Applied Mathematics Colloquium, 10–13 April 1995, Birmingham, UK, and at the IUTAM Symposium on Nonlinear Instability and Transition in Three-Dimensional Boundary Layers, 17–20 July 1995, Manchester, UK.

Appendix A.
Numerical Procedures

In §5*c*, for the subcritical cases ($\bar{k}_{\mathrm{r}} > 0$) which terminate in a finite-time singularity, we use a two-stage method to solve equations (5.6) numerically. For the first of these two stages ($\tau < \tau_1$), we rescale (5.6) using the stretching

$$X = C \sinh(\alpha\hat{\xi}) , \qquad Y = C \sinh(\alpha\hat{\eta}) , \tag{A 1}$$

and take a uniform square grid in $\hat{\xi}$ and $\hat{\eta}$. Since the solution is symmetric about the axes, it is only necessary to solve in the first quadrant. For the solutions shown in figures 2–4 we take $C = 1$ and $\alpha = 0.36$, the outer boundaries are set at $X = 1000$ and $Y = 1000$, and about 550^2 grid points are used. A standard fourth-order Runge-Kutta scheme is used to integrate (5.6*a*) forward in time. The function B is found from (5.6*b*) using a fourth-order finite-difference elliptic solver (*sepeli* from the *slatec-fishpack* library), with B set equal to zero on the outer boundaries and a symmetry condition on the axes; the right-hand side is evaluated using fourth-order differences (centred wherever possible).

For $\tau > \tau_1$, we use the 'dynamic rescaling' method of Landman, Papanicolaou, Sulem, Sulem & Wang (1991) and Papanicolaou, Sulem, Sulem & Wang (1994), and make the change of variables

$$\tau - \tau_1 = \int_0^\sigma L^2(\sigma')\mathrm{d}\sigma' \ , \quad X = \lambda_1(\sigma)\xi_1 \ , \quad Y = \lambda_2(\sigma)\xi_2 \ , \tag{A 2}$$

$$|A(X, Y, \tau)| = L(\sigma)^{-1} u\,(\xi_1, \xi_2, \sigma) \ , \quad B(X, Y, \tau) = L(\sigma)^{-2} v\,(\xi_1, \xi_2, \sigma) \ , \tag{A 3}$$

where

$$\lambda_1 = \left(\frac{\int\int_{-\infty}^{\infty} X^2 |A|^{2p} \mathrm{d}X\mathrm{d}Y}{\int\int_{-\infty}^{\infty} |A|^{2p} \mathrm{d}X\mathrm{d}Y} \right)^{\frac{1}{2}} , \quad \lambda_2 = \left(\frac{\int\int_{-\infty}^{\infty} Y^2 |A|^{2p} \mathrm{d}X\mathrm{d}Y}{\int\int_{-\infty}^{\infty} |A|^{2p} \mathrm{d}X\mathrm{d}Y} \right)^{\frac{1}{2}} , \tag{A 4}$$

for some specified integer p, and

$$\frac{2}{L^2} = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} \ . \tag{A 5}$$

It turns out that the accuracy of the resultant solutions is insensitive to the choice of p, provided that $3 \le p \le 20$. For the results presented we have chosen $p = 10$. The evolution is governed by

$$\frac{\partial u}{\partial \sigma} = \sum_{n=1}^{2} a_n \xi_n \frac{\partial u}{\partial \xi_n} + bu + \bar{k}_{\mathrm{r}} u^3 - uv \ , \quad \frac{\mathrm{d}\lambda_n}{\mathrm{d}\sigma} = \lambda_n a_n \quad \text{for } n = 1, 2 \ , \tag{A 6}$$

$$\lambda_1^{-2} \frac{\partial v}{\partial \xi_1^2} + \lambda_2^{-2} \frac{\partial v}{\partial \xi_2^2} = \lambda_2^{-2} \frac{\partial (u^2)}{\partial \xi_2^2} \ , \quad v \to 0 \quad \text{as } \xi_1^2 + \xi_2^2 \to \infty \ , \tag{A 7}$$

where

$$a_n = \frac{p\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\xi_n^2-1)u^{2p}(\bar{k}_r u^2 - v)d\xi_1 d\xi_2}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u^{2p}\mathrm{d}\xi_1\mathrm{d}\xi_2} \quad \text{for } n=1,2\ , \tag{A 8}$$

$$b = \frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}\sigma} + L^2 \equiv \frac{a_1\lambda_2^2 + a_2\lambda_1^2 + 4\lambda_1^2\lambda_2^2}{2\,(\lambda_1^2+\lambda_2^2)}\ . \tag{A 9}$$

A uniform square grid in (ξ_1, ξ_2) is used with spacing chosen so that in (X, Y)-space the density of points near the origin is initially comparable with that for the first scheme. Typically about 150^2 grid points are used. The reduction in the number of grid points and the absence of a stretch mean that the computational region is now significantly smaller. At the changeover time, the values of $|A|$ at the new grid points are interpolated using a system of fourth-order B-splines, and initial values of $\lambda_{1,2}$ calculated from (A 4).

Equations (A 6)–(A 9) are integrated forward in (scaled) time using a fourth-order Adams-Bashforth-Moulton scheme (which proved superior to a Runge-Kutta method). The spatial derivatives are evaluated by fourth-order differences, centred where possible. The Poisson equation (A 7) is solved by the same method as before, except that the computational region is now too small to impose $v = 0$ on the outer boundaries. Instead, we assume that the solution does not vary rapidly at the edge of the grid, and set

$$B(X_n, Y_n, \tau_n) = B(X_n, Y_n, \tau_{n-1})\ , \tag{A 10}$$

where τ_n is the value of τ at the n-th σ-step, and (X_n, Y_n) is a typical point of the boundary at this time. Since it turns out that the computational region is always shrinking in time, the right-hand side can be found by interpolation, which is again performed using fourth-order B-splines. This is a somewhat crude approach, but the results obtained appear sufficiently consistent for our purposes.

Appendix B.

Amplitude equations for a point disturbance in a marginally unstable parallel flow

We consider the basic flow

$$\left[\bar{U}(z), \bar{V}(z), 0\right] \tag{B 1}$$

between rigid boundaries at $z = \pm 1$, where variables have been non-dimensionalised as in §2a. To describe the evolution of an unstable point disturbance at slightly supercritical Reynolds number, we follow Davey *et al.* (1974) and propose the expansions

$$\begin{aligned}
\tilde{u} &= \left[\hat{\varepsilon}^{1/2}u_{11}(\xi,\eta,z,\tau)+\hat{\varepsilon}u_{21}(\xi,\eta,z,\tau)+\hat{\varepsilon}^{3/2}u_{31}(\xi,\eta,z,\tau)+\ldots\right]E \\
&\quad +\left[\hat{\varepsilon}u_{22}(\xi,\eta,z,\tau)+\ldots\right]E^2+c.c.+\hat{\varepsilon}u_{2M}(\xi,\eta,z,\tau)+\ldots \qquad \text{(B 2)}\\
\tilde{v} &= \left[\hat{\varepsilon}^{1/2}v_{11}(\xi,\eta,z,\tau)+\hat{\varepsilon}v_{21}(\xi,\eta,z,\tau)+\hat{\varepsilon}^{3/2}v_{31}(\xi,\eta,z,\tau)+\ldots\right]E \\
&\quad +\left[\hat{\varepsilon}v_{22}(\xi,\eta,z,\tau)+\ldots\right]E^2+c.c.+\hat{\varepsilon}v_{2M}(\xi,\eta,z,\tau)+\ldots \qquad \text{(B 3)}\\
\tilde{w} &= \left[\hat{\varepsilon}^{1/2}w_{11}(\xi,\eta,z,\tau)+\hat{\varepsilon}w_{21}(\xi,\eta,z,\tau)+\hat{\varepsilon}^{3/2}w_{31}(\xi,\eta,z,\tau)+\ldots\right]E \\
&\quad +\left[\hat{\varepsilon}w_{22}(\xi,\eta,z,\tau)+\ldots\right]E^2+c.c.+\hat{\varepsilon}^{3/2}w_{3M}(\xi,\eta,z,\tau)+\ldots \qquad \text{(B 4)}\\
\tilde{p} &= \left[\hat{\varepsilon}^{1/2}p_{11}(\xi,\eta,z,\tau)+\hat{\varepsilon}p_{21}(\xi,\eta,z,\tau)+\hat{\varepsilon}^{3/2}p_{31}(\xi,\eta,z,\tau)+\ldots\right]E \\
&\quad +\left[\hat{\varepsilon}p_{22}(\xi,\eta,z,\tau)+\ldots\right]E^2+c.c.+\hat{\varepsilon}^{1/2}p_{1M}(\xi,\eta,\tau)+\ldots \qquad \text{(B 5)}
\end{aligned}$$

for perturbation velocity and pressure, where, in terms of the transformed variables (6.3)–(6.6),

$$E \equiv \mathrm{e}^{\mathrm{i}\alpha(\hat{x}-c\hat{t})} \tag{B 6}$$

represents the neutral mode at the (effective) critical Reynolds number $\hat{R}_{cr}$†, and

$$\hat{\varepsilon}=\left(\hat{R}-\hat{R}_{cr}\right)d_{1\mathrm{r}} \ll 1\ ,\quad \xi=\hat{\varepsilon}^{1/2}(\hat{x}+a_{1\mathrm{r}}\hat{t})\ ,\quad \eta=\hat{\varepsilon}^{1/2}(\hat{y}+\hat{b}_{1\mathrm{r}}\hat{t})\ ,\quad \tau=\hat{\varepsilon}\hat{t}\ . \tag{B 7}$$

(Although we believe that these scalings need modification in the case of a initially small localized disturbance, this can be handled by taking a suitable limit of the final amplitude equations.)

On substituting (B 2)–(B 5) into the Navier-Stokes equations, and equating coefficients of $\hat{\varepsilon}^{m/2}E^n$, we obtain from the coefficients of $\hat{\varepsilon}^{1/2}E$ that

$$u_{11}=A(\xi,\eta,\tau)\psi'(z)\ ,\quad v_{11}=A\chi(z)\ ,\quad w_{11}=-\mathrm{i}\alpha A\psi\ , \tag{B 8}$$

$$p_{11}=A\left[-(\bar{U}-c)\psi'+\bar{U}'\psi+(\mathrm{i}\alpha\hat{R}_{cr})^{-1}\left(\psi'''-\alpha^2\psi'\right)\right]\ , \tag{B 9}$$

where $\psi(z)$ is the (suitably normalised) eigenfunction of the Orr-Sommerfeld problem

$$\left.\begin{aligned}\hat{L}\psi \equiv (\bar{U}-c)\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\alpha^2\right)\psi-\bar{U}''\psi-\frac{1}{\mathrm{i}\alpha\hat{R}_{cr}}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\alpha^2\right)^2\psi=0\ ,\\ \psi(\pm1)=0\ ,\quad \psi'(\pm1)=0\ ,\end{aligned}\right\} \tag{B 10}$$

and $\chi(z)$ is found from the Squire problem

$$(\bar{U}-c)\,\chi-\frac{1}{\mathrm{i}\alpha\hat{R}_{cr}}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\alpha^2\right)\chi=-\bar{V}'\psi\ ,\quad \chi(\pm1)=0\ . \tag{B 11}$$

† For comparison with Section 6, note that in this Appendix we write α, c instead of α_{cr}, c_{cr}.

From the coefficients of $\hat{\varepsilon}E$ we have *inter alia* that

$$w_{21} = -\mathrm{i}\alpha A_1(\xi,\eta,\tau)\psi(z) - \frac{\partial A}{\partial \xi}\psi_a(z) - \frac{\partial A}{\partial \eta}\psi_b(z) \ , \tag{B 12}$$

with A_1 arbitrary and $\psi_{a,b}$ satisfying

$$\hat{L}\psi_a = -\hat{M}_a\psi \ , \quad \hat{L}\psi_b = -\hat{M}_b\psi \ , \quad \psi_{a,b}(\pm 1) = \psi_{a,b}(\pm 1) = 0 \ , \tag{B 13}$$

where we define

$$\hat{M}_a\psi = \left\{\bar{U} + a_{1\mathrm{r}} - 4\mathrm{i}\alpha\hat{R}_{cr}^{-1}\right\}\left(\psi'' - \alpha^2\psi\right) - \left\{2\alpha^2(\bar{U}-c) + \bar{U}''\right\}\psi \tag{B 14}$$

$$\hat{M}_b\psi = \left\{\bar{V} + \hat{b}_{1\mathrm{r}}\right\}\left(\psi'' - \alpha^2\psi\right) - \bar{V}''\psi \ . \tag{B 15}$$

The group velocity $(-a_{1\mathrm{r}}, -\hat{b}_{1\mathrm{r}})$ is determined by the solvability conditions for ψ_a and ψ_b, namely

$$\int_{-1}^{1} \psi^\dagger\left(\hat{M}_a\psi\right)\,\mathrm{d}z = \int_{-1}^{1} \psi^\dagger\left(\hat{M}_b\psi\right)\,\mathrm{d}z = 0 \ , \tag{B 16}$$

where $\psi^\dagger(z)$ is the eigenfunction adjoint to (B 10).

The second harmonics u_{22}, etc., take the form

$$u_{22} = A^2\psi_2'(z) \ , \quad v_{22} = A^2\chi_2(z) \ , \quad w_{22} = -2\mathrm{i}\alpha A^2\psi_2(z) \ , \tag{B 17}$$

where ψ_2 is specified by

$$\left(\bar{U}-c\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - 4\alpha^2\right)\psi_2 - \bar{U}''\psi_2 - \frac{1}{2\mathrm{i}\alpha\hat{R}_{cr}}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - 4\alpha^2\right)^2\psi_2 = \frac{1}{2}\left(\psi\psi''' - \psi'\psi''\right) \ ,$$

$$\psi_2(\pm 1) = 0 \ , \quad \psi_2'(\pm 1) = 0 \ , \tag{B 18}$$

while χ_2 is not required for the present purpose. The mean-flow corrections u_{2M}, etc., satisfy

$$\frac{\partial u_{2M}}{\partial \xi} + \frac{\partial v_{2M}}{\partial \eta} + \frac{\partial w_{3M}}{\partial z} = 0 \ , \tag{B 19}$$

$$\hat{R}_{cr}^{-1}\frac{\partial^2 u_{2M}}{\partial z^2} - \frac{\partial p_{1M}}{\partial \xi} = \mathrm{i}\alpha|A|^2\left[\psi^*\psi' - \psi\psi'^*\right]' \ , \tag{B 20}$$

$$\hat{R}_{cr}^{-1}\frac{\partial^2 v_{2M}}{\partial z^2} - \frac{\partial p_{1M}}{\partial \eta} = \mathrm{i}\alpha|A|^2\left[\psi^*\chi - \psi\chi^*\right]' \ , \tag{B 21}$$

where an asterisk denotes complex conjugate. The solutions of (B 20)-(B 21) with the boundary conditions $u_{2M} = 0$, $v_{2M} = 0$ on $z = \pm 1$ are

$$u_{2M} = -\tfrac{1}{2}\hat{R}_{cr}\frac{\partial p_{1M}}{\partial \xi}\left(1 - z^2\right) - |A|^2\left[S_1(z) - \tfrac{1}{2}S_1(1)(1+z)\right] \ , \tag{B 22}$$

$$v_{2M} = -\tfrac{1}{2}\hat{R}_{cr}\frac{\partial p_{1M}}{\partial \eta}\left(1 - z^2\right) - |A|^2\left[S_2(z) - \tfrac{1}{2}S_2(1)(1+z)\right] \ , \tag{B 23}$$

where we define

$$S_1(z) = -\operatorname{Re}\left(\mathrm{i}\alpha\hat{R}_{cr}\int_{-1}^{z}\psi\psi'^{*}\mathrm{d}z'\right)\ ,\quad S_2(z) = -\operatorname{Re}\left(\mathrm{i}\alpha\hat{R}_{cr}\int_{-1}^{z}\psi\chi^{*}\mathrm{d}z'\right)\ . \tag{B 24}$$

Then from (B 19) and the boundary conditions $w_{3M}=0$ on $z=\pm 1$, we obtain

$$0 = \int_{-1}^{1}\left(\frac{\partial u_{2M}}{\partial \xi}+\frac{\partial v_{2M}}{\partial \eta}\right)\mathrm{d}z = -\tfrac{2}{3}\hat{R}_{cr}\left(\frac{\partial^2 p_{1M}}{\partial \xi^2}+\frac{\partial^2 p_{1M}}{\partial \eta^2}\right) - I_1\left(\frac{\partial}{\partial \xi} - s\frac{\partial}{\partial \eta}\right)|A^2|\ , \tag{B 25}$$

where

$$I_1 = \int_{-1}^{1} S_1(z)\ \mathrm{d}z - S_1(1)\ ,\quad I_2 = \int_{-1}^{1} S_2(z)\ \mathrm{d}z - S_2(1)\ ,\quad s = -\frac{I_2}{I_1}\ . \tag{B 26}$$

If $B(\xi,\eta,\tau)$ is specified by

$$\frac{\partial^2 B}{\partial \xi^2}+\frac{\partial^2 B}{\partial \eta^2} = \left(s\frac{\partial^2}{\partial\xi\partial\eta}+\frac{\partial^2}{\partial\eta^2}\right)|A^2|\ ,\quad B\to 0\quad \text{as}\quad \xi^2+\eta^2\to\infty\ , \tag{B 27}$$

then from (B 22) and (B 25) we have

$$u_{2M} = |A|^2F'(z) - \tfrac{3}{4}I_1\left(1-z^2\right)B\ , \tag{B 28}$$

with

$$F'(z) = \tfrac{3}{4}I_1\left(1-z^2\right) - S_1(z) + \tfrac{1}{2}S_1(1)(1+z)\ . \tag{B 29}$$

At $O(\hat{\varepsilon}^{3/2})$, we find that

$$\begin{aligned}\hat{L}w_{31} = \frac{\partial A}{\partial T}\left(\psi''-\alpha^2\psi\right) - \frac{\mathrm{i}}{\alpha}\Bigg[&\frac{\partial^2 A}{\partial \xi^2}f_1(z)+\frac{\partial^2 A}{\partial\xi\partial\eta}f_2(z)+\frac{\partial^2 A}{\partial\eta^2}f_3(z) \\ &+\frac{A}{d_{1\mathrm{r}}}f_4(z)+\mathrm{i}\alpha\frac{\partial A_1}{\partial \xi}\hat{M}_a\psi+\mathrm{i}\alpha\frac{\partial A_1}{\partial\eta}\hat{M}_b\psi+|A|^2Ag(z)+ABh(z)\Bigg]\ ,\end{aligned} \tag{B 30}$$

where

$$f_1(z) = \hat{M}_a\psi_a - \alpha^2\left(\bar{U}-c\right)\psi - 2\alpha^2\left(\bar{U}+a_{1\mathrm{r}}\right)\psi - 2\mathrm{i}\alpha\hat{R}_{cr}^{-1}\left(\psi''-3\alpha^2\psi\right) \tag{B 31}$$

$$f_2(z) = \hat{M}_a\psi_b + \hat{M}_b\psi_a - 2\alpha^2\left(\bar{V}+\hat{b}_{1\mathrm{r}}\right)\psi\ , \tag{B 32}$$

$$f_3(z) = \hat{M}_b\psi_b - \alpha^2\left(\bar{U}-c\right)\psi - 2\mathrm{i}\alpha\hat{R}_{cr}^{-1}\left(\psi''-\alpha^2\psi\right)\ , \tag{B 33}$$

$$f_4(z) = \frac{\mathrm{i}\alpha}{\hat{R}_{cr}^2}\left[\psi^{(iv)}-2\alpha^2\psi''+\alpha^4\psi\right]\ , \tag{B 34}$$

$$\begin{aligned}g(z) = &-2\alpha^2\psi'^{*}\left(\psi_2''-2\alpha^2\psi_2\right)-\alpha^2\psi^{*}\left(\psi_2''-2\alpha^2\psi_2\right)'+\alpha^2\psi_2'\left(\psi^{*\prime\prime}+\alpha^2\psi^{*}\right) \\ &+2\alpha^2\psi_2\left(\psi^{*\prime\prime}+\alpha^2\psi^{*}\right)'-\alpha^2F'\left(\psi''-\alpha^2\psi\right)+\alpha^2\psi F'''\ ,\end{aligned} \tag{B 35}$$

$$h(z) = \tfrac{3}{4}\alpha^2I_1\left[\left(1-z^2\right)\left(\psi''-\alpha^2\psi\right)+2\psi\right]\ . \tag{B 36}$$

The solvability condition for w_{31} is thus

$$\frac{\partial A}{\partial \tau} = a_2 \frac{\partial^2 A}{\partial \xi^2} + \hat{c}_2 \frac{\partial^2 A}{\partial \xi \partial \eta} + \hat{b}_2 \frac{\partial^2 A}{\partial \eta^2} + \frac{d_1}{d_{1\mathrm{r}}} A + k|A|^2 A + qAB \ , \tag{B 37}$$

with

$$\left[a_2, \hat{c}_2, \hat{b}_2, d_1, k, q\right] = \frac{\mathrm{i} \int_{-1}^{1} \psi^\dagger(z) \left[f_1(z), f_2(z), f_3(z), f_4(z), g(z), h(z)\right] \mathrm{d}z}{\alpha \int_{-1}^{1} \psi^\dagger(z) \left(\psi'' - \alpha^2 \psi\right) \ \mathrm{d}z} \ . \tag{B 38}$$

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ADDENDUM

After the publication of this paper, it came to our attention that much of §2 and §3 repeats an earlier analysis of this problem by Haberman (1973, 1977). We apologize unreservedly to Prof. Haberman for our failure (through ignorance) to acknowledge his precedence.