LAMINAR BOUNDARY-LAYER THEORY: A 20TH CENTURY PARADOX?

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Keywords: Boundary layer, shear layer, separation, singularity, instability.

Abstract Laminar boundary-layer theory has made important contributions to aeronautics and other fields throughout this century. This paper begins by reviewing steady and unsteady separation from the viewpoint of classical non-interactive boundary-layer theory. Next, interactive boundary-layer theory is introduced in the context of unsteady separation. This discussion leads onto a consideration of large-Reynolds-number asymptotic instability theory.

A key aspect of boundary-layer theory is the development of singularities in solutions of the boundary-layer equations. This feature, when combined with the pervasiveness of instabilities, often forces smaller and smaller scales to be considered. Such a cascade of scales often limits the quantitative usefulness of solutions, although boundary-layer theory is crucial in understanding why certain phenomena occur.

We also highlight a recent result which suggests that classical boundary-layer theory may not always be the large-Reynolds-number limit of the Navier-Stokes equations. This is because of the possible amplification of short-scale modes, which are initially exponentially small, by a Rayleigh instability mechanism.

1. INTRODUCTION

Sectional lecturers were invited 'to weave in a bit more retrospective and/or prospective material [than normal] given the particular [Millennium] year of the Congress'. This invitation is reflected in the current article; indeed, some might argue that what follows has a rather idiosyncratic slant. The reader is referred to Stewartson (1981), Smith (1982), Cowley & Wu (1994), Goldstein (1995) and Sychev *et al.* (1998) for alternative viewpoints.

The Title. We begin with a deconstruction of the components of the title.

Boundary-Layer Theory. Prandtl (1904) proposed that viscous effects would be confined to thin shear layers adjacent to boundaries in the case of the 'motion of fluids with very little viscosity', i.e. in the case of the motion of fluids for which the characteristic Reynolds number, Re, is large. In a more general sense we will use 'boundary-layer theory' (BLT) to refer to any large-Reynolds-number, $Re \gg 1$, asymptotic theory in which there are thin shear layers (whether or not there are boundaries).

20^{th} Century. Prandtl (1904) published his seminal paper on the foundations of boundary-layer theory at the start of the $20^{\rm th}$ century, while the ICTAM 2000 was held at the end of the same century.[1]

Laminar. Like Prandtl (1904) we will be concerned with laminar, rather than turbulent, flows. Flows that are in the process of laminar-turbulent transition will be viewed as unstable laminar flows.

A Paradox. Experimental flows at large Reynolds numbers are turbulent, yet useful comparisons with laminar-flow experiments at moderately large Reynolds numbers can sometimes be made with large-Reynolds-number asymptotic theories. We view as a paradox this seemingly contradictory result, i.e. that useful comparisons with laminar flow can be made with expansions made about Reynolds numbers when flows are almost invariably turbulent.

?. The question this paper will discuss is whether the final '?' is needed in the title. A subjective conclusion is given at the end.

Notation And Assumptions. As a model consider the incompressible flow of a fluid with constant density ρ and dynamic viscosity μ past a body with typical length $\mathcal{L}$. Assume that typical velocity, pressure and time scales are $\mathcal{U}$, $\mathcal{P}$ and $\mathcal{T}$ respectively, and that the Reynolds number is given by

$$Re = \frac{\rho \mathcal{U} \mathcal{L}}{\mu} \gg 1 \,.$$

For simplicity we will, for the most part, consider two-dimensional incompressible flows, although many of our statements can be generalised to three-dimensional flows and/or compressible flows.

[1] We use the definition of the start of the $21^{\rm st}$ century as given by the Royal Observatory Greenwich (see `http://www.rog.nmm.ac.uk/mill/`).

2. CLASSICAL BOUNDARY-LAYER THEORY

2.1. FORMULATION

We consider flows where there are extensive inviscid regions separated by *thin* shear layers, say, of typical width $\delta \ll \mathcal{L}$. We concentrate on one such shear layer, and take local dimensional Cartesian coordinates with $\hat{x}$ and $\hat{y}$ along and across the shear layer respectively; we denote the corresponding velocity components by $\hat{u}(\hat{x},\hat{y},\hat{t})$ and $\hat{v}(\hat{x},\hat{y},\hat{t})$. Then, to leading order within the shear layer, conservation of mass can be expressed as (e.g. Rosenhead 1963)

$$\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} = 0\,. \tag{1}$$

If we denote a typical velocity across the shear layer by $\mathcal{V}$, then in an order-of-magnitude sense it follows from (1) that

$$\frac{\mathcal{U}}{\mathcal{L}} \sim \frac{\mathcal{V}}{\delta} \qquad \Rightarrow \qquad \mathcal{V} \sim \frac{\mathcal{U}\delta}{\mathcal{L}} \ll \mathcal{U}\,. \tag{2}$$

A similar scaling argument can be applied to the 'streamwise' momentum equation within the shear layer to deduce the magnitude of both the pressure and the thickness of the shear layer (using the fact that, by definition, the shear layer is a region where viscous effects are important):

$$\begin{gathered}\rho\,(\hat{u}_{\hat{t}} + \hat{u}\hat{u}_{\hat{x}} + \hat{v}\hat{u}_{\hat{y}}) = -\hat{p}_{\hat{x}} + \mu\,(\hat{u}_{\hat{y}\hat{y}} + \hat{u}_{\hat{x}\hat{x}})\,,\\ \tfrac{\rho\mathcal{U}}{\mathcal{T}} \sim \tfrac{\rho\mathcal{U}^2}{\mathcal{L}} \sim \tfrac{\rho\mathcal{V}\mathcal{U}}{\delta} \sim \tfrac{\mathcal{P}}{\mathcal{L}} \sim \tfrac{\mu\mathcal{U}}{\delta^2} \gg \tfrac{\mu\mathcal{U}}{\mathcal{L}^2}\,,\end{gathered} \tag{3}$$

$$\Rightarrow \quad \mathcal{P} \sim \rho\mathcal{U}^2\,, \quad \delta \sim Re^{-\frac{1}{2}}\mathcal{L} \ll \mathcal{L}\,. \tag{4}$$

Similarly, from the 'cross-stream' momentum equation, we deduce that, at leading order, the pressure does not vary across the shear layer:

$$\begin{gathered}\rho\,(\hat{v}_{\hat{t}} + \hat{u}\hat{v}_{\hat{x}} + \hat{v}\hat{v}_{\hat{y}}) = -\hat{p}_{\hat{y}} + \mu\,(\hat{v}_{\hat{y}\hat{y}} + \hat{v}_{\hat{x}\hat{x}})\,,\\ \tfrac{\rho\mathcal{V}}{\mathcal{T}} \sim \tfrac{\rho\mathcal{U}\mathcal{V}}{\mathcal{L}} \sim \tfrac{\rho\mathcal{V}^2}{\delta} \ll \tfrac{\mathcal{P}}{\delta} \gg \tfrac{\mu\mathcal{V}}{\delta^2} \gg \tfrac{\mu\mathcal{V}}{\mathcal{L}^2}\,.\end{gathered} \tag{5}$$

Key Assumptions. From the above scaling arguments it follows that the key assumptions in classical BLT can be summarised as:

(a) the pressure is constant across the shear layer, i.e.

$$0 = -\hat{p}_{\hat{y}}\,; \tag{6}$$

(b) streamwise diffusion is negligible, i.e.

$$\bullet_{\hat{y}\hat{y}} \gg \bullet_{\hat{x}\hat{x}} \,. \tag{7}$$

where $\bullet$ represents 'any' variable.

The former assumption, as we shall see, is the more dynamically significant.

Non-Dimensional Form. Using the transformations

$$\hat{x} \to \mathcal{L}\, x\,, \quad \hat{y} \to \delta y = Re^{-\frac{1}{2}} \mathcal{L}\, y\,, \quad \hat{t} \to \mathcal{T} t = \mathcal{U}^{-1} \mathcal{L}\, t\,, \tag{8}$$

$$\hat{u} \to \mathcal{U} u\,, \quad \hat{v} \to \mathcal{V} v = Re^{-\frac{1}{2}} \mathcal{U} v\,, \quad \hat{p} \to \rho \mathcal{U}^2\, p\,, \tag{9}$$

the BLT equations can be non-dimensionalised to obtain

$$u_t + u u_x + v u_y = -p_x + u_{yy}\,, \tag{10}$$

$$0 = -p_y\,, \quad u_x + v_y = 0\,. \tag{11}$$

For flow past a rigid body the appropriate boundary conditions are

$$u = v = 0 \qquad \text{on} \qquad y = 0\,, \tag{12}$$

$$u \to U(x,t) \qquad \text{as} \qquad y \to \infty\,, \tag{13}$$

where $U(x,t)$ is the inviscid slip velocity past the body. Further, from (10) evaluated at the edge of the boundary layer

$$-p_x = U_t + U U_x\,. \tag{14}$$

We define the 'viscous blowing' velocity at the edge of the boundary layer to be

$$v_b(x,t) = \lim_{y \to \infty} \left(v + U_x(x,t) y \right)\,. \tag{15}$$

v_b indicates the strength of blowing, or suction, out of the boundary layer induced by viscous effects. As such it is a good diagnostic for dynamically significant effects within the boundary layer — much better than, say, the wall shear $u_y(x,0,t)$ which can remain regular while $v_b(x,t)$ becomes unbounded.

2.2. STEADY FLOWS

Steady Flow Past An Aligned Flat Plate: A Success. Probably the most famous solution to (10) and (11) is that of Blasius (1908) for flow past an aligned flat plate. A comparison between this similarity solution and, say, Wortmann's visualisation of that flow (Van Dyke 1982) demonstrates that BLT seems to work in this case ... at least for $R \approx 500$, where $R = \rho \mathcal{U} \delta / \mu = Re^{\frac{1}{2}}$.

Steady Flow Past A Circular Cylinder: A Failure. On the *assumption* that far from the boundary the velocity field is irrotational and inviscid to leading order, the inviscid slip velocity for steady flow past a circular cylinder is given by

$$U(x) = 2 \sin x \,. \tag{16}$$

Terrill (1960) showed numerically that the solution to (10)–(13) and (16) terminates in a Goldstein (1948) singularity at $x = x_c \approx 104.5^\circ$ with

$$v_b \sim k(x_c - x)^{-\frac{1}{2}} + \ldots \quad \text{as} \quad x \to x_c \,, \tag{17}$$

for some constant k.

An indication as to why this singularity occurs follows from the fact that $\min_y u \to 0$ as $x \to x_c$. Hence if the Goldstein singularity did not develop, then for $x > x_c$ there would be a region of reversed flow ($u < 0$) close to the wall. This would mean that for a range of y there would effectively be an 'ill-posed' region of negative diffusion[2] in the momentum equation (10), as can be seen by considering the key terms

$$u u_x \cdots = \ldots u_{yy} \,. \tag{18}$$

A Serious Problem. The occurrence of a singularity often indicates that there is a significant development in the flow physics, e.g. the formation of small scale structure. In such circumstances new physics can usually be included in the model by introducing an asymptotic scaling close to $x = x_c$, so enabling a solution to be found for $x > x_c$.

However, Stewartson (1970) showed that, in general, there is no inner rescaling which 'smoothes out' the Goldstein singularity[3] and, as a result, no BLT solution exists for $x > x_c$. Moreover this result implies that the BLT solution for $x < x_c$ and the inviscid solution far from the cylinder are *incorrect*. BLT does not always work.

What Has Gone Wrong? It is possibly helpful to consider this question from the viewpoints of first experimentalists, then computational scientists and finally theoreticians.

Experimentalists might respond on the lines that the steady, symmetric, attached, irrotational-flow solution proposed as the outer inviscid

[2] The full explanation is more subtle than this since it depends, *inter alia*, on whether the pressure gradient is specified or 'interactive'. In particular, if the pressure gradient is unknown before solution of the boundary-layer equations, e.g. as in triple-deck theory, then the onset of regions of reversed flow need not provoke the development of a singularity.

[3] Smith & Daniels (1981) discuss an exception.

flow is clearly incorrect (whether or not a BLT solution can be found),[4] since

1. other than at very small Reynolds numbers there is a region of detached flow which is at least as large as the cylinder, and

2. the observed flow is steady and symmetric only for $Re \leqslant Re_c \approx 47$, and that for larger Reynolds numbers the flow is unsteady and asymmetric.

Computational scientists would agree with the experimentalists on the shortcomings of the outer solution. However, computational scientists are able to calculate steady symmetric solutions for $Re \geqslant Re_c$ by specifically excluding the possibility of unsteadiness and asymmetry. Such calculations suggest that the asymptotic regime for steady symmetric solutions is only reached for $Re \gtrsim 600$ (Fornberg 1985), i.e. at Reynolds numbers far larger than those at which the steady flow is stable.

Similarly, *theoreticians* would also agree about the inadequacy of the outer solution. However, in addition to wishing to explain the experimentally observable flow, they often want to understand both the reason for the 'singularity' and the asymptotic form of the steady symmetric solution at large Reynolds numbers (even if it is experimentally unobservable). Theoreticians might justify these desires on the basis that an understanding will shed light on the real world but, like the fascination of mountaineers for Everest, they are also susceptible to wanting to answer such questions just because they are there.

Steady Symmetric Flow Past A Circular Cylinder: An Answer. In order to explain steady symmetric separated flow past a bluff body at large Reynolds number, at least two important ingredients are necessary. First there is a need to explain the *local* solution at the point of separation of the boundary layer from the body surface. Second, an asymptotic model of the *global* wake is needed.

Local separation is described by Sychev's (1972) 'triple-deck' analysis (see also Smith 1977). This is based on the premise that, at the point of separation on a smooth surface, the pressure gradient is $O(Re^{-\frac{1}{16}})$. In the context of Kirchoff free-streamline theory this means that the appropriate free-streamline solution satisfies the Brillouin-Villat condition to leading order (e.g. see Sychev *et al.* (1998) for a discussion).

There have been a number of attempts to fit the above local description of separation into a consistent large-Reynolds-number asymptotic

[4] 'Attached' as in non-separated, and 'outer' in the matched asymptotic sense.

global solution for the flow past a circular cylinder. Building on the work of others, Chernyshenko (1988) has proposed a convincing asymptotic structure based on the special Sadovskii (1971) vortex where there is no velocity jump at the edge of the vortex — in this case both the length and the width of the wake are $O(Re)$ in magnitude. This structure may, of course, not be unique, but many other proposals have technical shortcomings especially as regards reattachment at the end of the wake. For further references and details see Chernyshenko's (1998) review.

2.3. UNSTEADY SEPARATION

Although the asymptotic solution for steady, symmetric laminar large-Reynolds-number flow past a bluff body is not experimentally realizable, this is not the case for impulsively started flow past a smooth bluff body at 'small' times. The reason for this is that when $t \ll 1$, the unsteady u_t term in (10) is much larger than the nonlinear uu_x term, and is balanced by the diffusive u_{yy} term (the boundary layers are thus initially very thin with $\delta \propto t^{\frac{1}{2}}$). It follows that when $t \ll 1$ the solution at each point on the boundary looks locally like Rayleigh's solution for impulsively started motion over a flat plate, and hence separation will not take place at sufficiently early times (e.g. Goldstein & Rosenhead 1936).

Impulsively Started Flow Past A Circular Cylinder. There have been a number of visualisations of this flow, e.g. Prandtl (1932), Coutanceau & Bouard (1977). Prandtl's (1932) film is particularly instructive as regards where separation of the boundary layer starts.

For many years research into unsteady separation focused on the rear stagnation point (e.g. Robins & Howarth 1972), since it is there that reverse flow first sets in. However, *reverse flow* is not the same as *separation/breakaway* of the boundary layer from the body surface — there is plenty of reversed flow in Stokes' solution for flow over an oscillating plate, and yet this unsteady boundary layer remains attached to the plate for all Reynolds numbers. In contrast, Prandtl (1932) focused attention on a region approximately $\frac{3\pi}{4}$ from the front stagnation point. Indeed it is clear from the close-ups in Prandtl's (1932) film that the boundary layer separates from the body surface at approximately $\frac{3\pi}{4}$ from the front stagnation point. It is arguable that conventional wisdom, i.e. that the rear stagnation point was the place to look, delayed an understanding of unsteady separation by 50 years.

Unsteady Separation: Physics. At the front/rear of an impulsively moved cylinder, fluid particles are accelerating/decelerating. Hence at

the rear of the cylinder fluid particles will tend to be squashed in the streamwise direction, with a compensating expansion in the direction normal to the boundary. In Navier-Stokes (NS) or Euler flows it is not possible to squash a particle to zero thickness in one direction and an infinite length in another because the rapid stretching of the fluid particles leads to the generation of a pressure gradient that inhibits the stretching. However, in classical BLT

$$p_y = 0\,,$$

and hence no pressure gradient can be induced in the direction normal to the wall. Unsteady separation occurs when a fluid particle is squashed to zero thickness in the direction parallel to the wall, so ejecting the fluid above it out of the boundary layer (van Dommelen 1981).

Unsteady Separation: Mathematics. Since unsteady separation is connected with the deformation of a particle, it is more natural to seek a mathematical description in terms of Lagrangian, rather than Eulerian, co-ordinates (Shen 1978, van Dommelen & Shen 1980). Let $\boldsymbol{\xi} = (\xi, \eta)$ denote Lagrangian co-ordinates, then with $\mathbf{x} \equiv \mathbf{x}\,(\boldsymbol{\xi}, t)$, $\mathbf{u} \equiv \mathbf{u}\,(\boldsymbol{\xi}, t)$ and $U \equiv U\,(\boldsymbol{\xi}, t)$, the momentum equation (10) becomes

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial t} + \left(\frac{\partial x}{\partial \xi}\,\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\,\frac{\partial}{\partial \xi}\right)\left(\frac{\partial x}{\partial \xi}\,\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\,\frac{\partial}{\partial \xi}\right) u\,, \qquad (19)$$

while the kinematic equation yields

$$u = \left(\frac{\partial x}{\partial t}\right)_{\boldsymbol{\xi}}\,, \quad v = \left(\frac{\partial y}{\partial t}\right)_{\boldsymbol{\xi}}\,. \qquad (20)$$

Van Dommelen & Shen (1980) made the key observation that (19) depends only on x and u, and hence that (19) and the first of (20) can be solved independently of the equations governing y and v. The solution for y can then be obtained from the mass conservation Jacobian

$$\frac{\partial x}{\partial \xi}\,\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\,\frac{\partial y}{\partial \xi} = 1\,, \qquad (21)$$

while that for v can be deduced subsequently from the second of (20).

For a given $x(\boldsymbol{\xi}, t)$, (21) is a hyperbolic equation for $y(\boldsymbol{\xi}, t)$ with a unit source term. If at some time, say t_s, the solution for x evolves so that $\boldsymbol{\nabla} x = 0$ (which is a mathematical statement that a particle has been squashed to zero thickness in the x direction), then 'shock' singularities can form in y, and hence v (van Dommelen & Shen 1980).

Impulsively Started Circular Cylinder: Results. Numerical calculations show that a 'shock' singularity develops after about 3/4 of a diameter movement at

$$t = t_s \approx 1.5\,, \quad x = x_s(t_s) \approx 111^\circ\,. \tag{22}$$

Note that the position where the unsteady singularity forms, $x_s(t_s)$, is not the position where the Goldstein singularity forms, i.e. $x_c \approx 104.5^\circ$.

The unsteady singularity manifests itself by a rapid thickening of the boundary layer over a streamwise distance $(x - x_s(t)) = O\,(t_s - t)^{\frac{3}{2}}$, where $x_s(t)$ is the centre of the singularity structure. Further, it can be shown that as $t \to t_s$ the displacement thickness δ_b and blowing velocity v_b vary like

$$\delta_b\,(x_s, t) \propto (t_s - t)^{-\frac{1}{4}} \quad \text{and} \quad v_b\,(x_s, t) \propto (t_s - t)^{-\frac{7}{4}}\,. \tag{23}$$

A schematic of this structure is illustrated in figure 1.

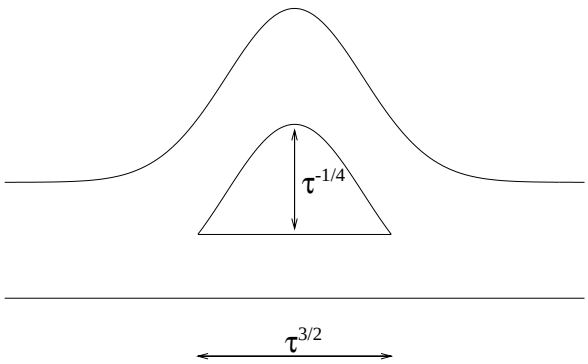

Figure 1 Schematic of a separating boundary layer ($\tau = t_s - t$).

Three-Dimensional Unsteady Separation. It is straightforward to extend the two-dimensional analysis to describe three-dimensional separation (e.g. van Dommelen & Cowley 1990). When there are no symmetries present the singularity structure is quasi-two-dimensional with a slower $(t_s - t)$ variation in a direction orthogonal to the more rapid $(t_s - t)^{\frac{3}{2}}$ variation — the boundary-layer still thickens at a rate proportional to $(t_s - t)^{-\frac{1}{4}}$. However, if there is a plane or axis of symmetry then the thickening of the boundary layer is more rapid than $(t_s - t)^{-\frac{1}{4}}$.

2.4. A SINGULARITY ... SURELY NOT?

Conventional wisdom[5] is that finite-time singularities do not spontaneously develop in solutions to the NS equations. This suggests that at times very close to t_s at least one of the terms that are usually asymptotically smaller than those included in the BLT equations, grows to be of a size such that it cannot be neglected at leading order. In particular, in order to stop the extension of the squashed particle in the y-direction, we might anticipate that the $p_y = 0$ approximation will need to be refined.

An Upper Deck. For $\tau = (t_s - t) \ll 1$ it follows from (3) and (23) that the dimensional blowing velocity has magnitude

$$\hat{v}_b = O(\mathcal{U}\, Re^{-\frac{1}{2}}\, \tau^{-\frac{7}{4}})\,. \tag{24}$$

This blowing velocity causes a perturbation to the inviscid flow in a region just above the boundary layer. This perturbation is both inviscid and irrotational; hence it is governed by Laplace's equation. The perturbation extends over a region sufficient for a pressure gradient normal to the wall to be felt (and so reduce the normal velocity to zero). Since the Laplacian is a 'smooth operator' and the extent of the variation in the $\hat{x}$-direction is $O(\mathcal{L}\tau^{\frac{3}{2}})$, the extent of the variation in the $\hat{y}$-direction is also $O(\mathcal{L}\tau^{\frac{3}{2}})$. From the continuity equation it follows that the perturbation velocity in the streamwise direction is of the same order of magnitude as the blowing velocity (24). Similarly it follows from the linearised version of the time-dependent Bernoulli equation that the dimensional pressure perturbation is $O(\rho\mathcal{U}^2\, Re^{-\frac{1}{2}}\, \tau^{-\frac{7}{4}})$. Hence the dimensionless pressure-gradient perturbation has a magnitude

$$\tilde{p}_x = O(Re^{-\frac{1}{2}}\, \tau^{-\frac{13}{4}})\,. \tag{25}$$

This induced perturbation pressure gradient can have a feedback effect on the boundary-layer flow when it is as large as the acceleration, $u_t = x_{tt} = O(\tau^{-\frac{1}{2}})$, within the boundary layer. This occurs when $\tau = O(Re^{-\frac{2}{11}})$. At such times a new asymptotic problem needs to be formulated involving four distinct asymptotic regions in the y-direction (Elliott *et al.* 1983). For this 'quadruple-deck' analysis to be valid, i.e. for the four asymptotic regions to be distinct, strictly we need $Re^{\frac{1}{22}} \gg 1$. This of course raises the question of how large the Reynolds number has

[5] Conventional wisdom of course may eventually prove to be wrong, as in the aforementioned example where studies of unsteady separation for flow past a circular cylinder concentrated on the rear stagnation point.

to be for the analysis to be valid. We do not address that issue here other than to note that 'large' can vary from approximately 10^0 (e.g. see Jobe & Burggraf 1974) to 10^8 (e.g. see Healey 1995).

The (Rescaled)^n Problem. It might be surmised that since the interaction problem allows for variations of the pressure gradient in the y-direction (even if only in the inviscid outer deck), then this reformulation would be free of finite-time singularities because the induced pressure gradient would be sufficient to prevent a fluid particle being extended indefinitely in the y-direction. However, this is not the case. This rescaled interactive problem itself terminates in a finite-time singularity (Cassel *et al.* 1996).

It is then possible to formulate, at times close to this second singularity, another rescaled problem on an even shorter time-scale. To the best of our knowledge this problem has not been solved, although a model version has been studied by Li *et al.* (1998). They show that depending on the value of certain coefficients, this model problem may, or may not, terminate in yet another finite-time singularity.

The formation of a succession of singularities prompts the question as to whether something has gone wrong with the formulation and/or with the analysis. The answer, we believe, is 'not really'. As indicated earlier, in practice large-Reynolds-number flows are turbulent and hence, if they are to be modelled accurately, we can expect that there will be a natural tendency for structures with small length-scales and time-scales to develop. The development of a succession of singularities with smaller and smaller length-scales and shorter and shorter time-scales just reflects this natural tendency.

In some sense the original singularity is exciting instabilities that lead to small-scale turbulent structure. As a result it is difficult to envisage how it would be possible to obtain a detailed large-Reynolds-number asymptotic solution for order-one times beyond t_s — although that is not to say that some clever averaging or multiple-scales technique will not be found.

Thus, to summarise, the good news is that BLT predicts both unsteady separation and also the physical interactive effects that then come into play; the bad news is that it does not appear to provide a long-time predictor.

3. IS BLT THE ASYMPTOTIC LIMIT OF THE NAVIER-STOKES EQUATIONS?

3.1. BRINCKMAN & WALKER'S CALCULATIONS

Conventional wisdom is that for, say, unsteady flows starting from rest, classical unsteady BLT is the $Re \gg 1$ asymptotic limit of solutions of the NS equations (or, to be more precise, it is the limit until the time at which a separation singularity is predicted to develop). While this may be the case for many flows, recent numerical results by Brinckman & Walker (2001) suggest that it may not always be so.

In particular Brinckman & Walker (2001) study numerically a NS problem that standard arguments suggest should tend, in the large-Reynolds-number limit, to the same BLT problem as unsteady flow past an impulsively started circular cylinder. However, as the Reynolds number is increased, Brinckman & Walker's (2001) calculations develop rapid oscillations in the solutions at times *before* the time at which a van Dommelen separation singularity develops in the BLT solution.[6] This suggests that there are flows for which, at times before separation, the $Re \gg 1$ limit of solutions of the NS equations is *not* the BLT solution.

Rayleigh Instabilities. Brinckman & Walker's (2001) calculations suggest that the wavelength of the short-scale oscillations varies like $Re^{-\frac{1}{2}}$, which in turn suggests that the oscillations may be related to a Rayleigh instability.

Consider unsteady classical BLT flow over a rigid surface in a region where there is an adverse pressure gradient, i.e. $p_x > 0$, but where the slip velocity $U(x,t)$ is positive. It follows from (10) that on the rigid wall (where $u = v = 0$)

$$u_{yy} = p_x > 0\,.$$

Moreover, if $u \to U$ as $y \to \infty$ there must be region away from the wall where

$$u_{yy} < 0\,.$$

It follows that the velocity profile $u(x,y,t)$ has an inflection point in y.

[6] It is arguable that Brinckman & Walker's (2001) treatment of their 'upper' boundary condition is not consistent. This casts a slight doubt on their calculations, as possibly does a relatively low tolerance for the convergence of their iterative Poisson solver. Nevertheless Brinckman & Walker's (2001) calculations seem to pose an important question.

The existence of an inflection point in a boundary-layer velocity profile implies that disturbances with short $Re^{-\frac{1}{2}}$ streamwise length-scales, i.e. streamwise length-scales comparable with the width of the boundary layer, can grow by means of a Rayleigh instability (Tollmien 1936). The amplitude of such a disturbance will locally behave like

$$\text{amplitude} \propto \exp\left(\lambda Re^{\frac{1}{2}} \int \beta \, \mathrm{d}t\right), \tag{26}$$

where $\lambda Re^{\frac{1}{2}}$ is the local [positive] wavenumber and $\beta(x,t) = O(1)$ is a function of x and t. Further discussion of this point can be found in the appendix to Tutty & Cowley (1986).

Is There Anything To Grow? The next question is whether there are any inherent high-wavenumber modes in the solution with wavelengths of $O(Re^{-\frac{1}{2}})$ that might be amplified by a Rayleigh instability. Here we emphasise that by 'modes' we do *not* mean disturbances introduced through noise or, in the case of numerical calculations, rounding error.

Suppose that $u(x,y,t)$ is a solution to the unsteady classical BLT equations that develops a singularity at $t = t_s$. Analytically continue $u(x,y,t)$ into the complex x-plane. For times $t < t_s$ there will almost certainly be singularities of $u(x,y,t)$ in the complex x-plane. These singularities will move around the complex x-plane and intersect the real x-axis at $t = t_s$ (cf. a similar situation for vortex sheets as explained by, say, Krasny 1986).

Suppose that at time $t < t_s$ the singularity in the complex x-plane nearest to the real x-axis is a distance $\alpha(t)$ from that axis. If $\widetilde{u}(k,y,t)$ is the k^{th} term of the Fourier series of $u(x,y,t)$, then as explained in Carrier, Krook & Pearson (1983)

$$\widetilde{u}(k,y,t) \propto \exp(-\alpha|k|) \quad \text{as} \quad |k| \to \infty\,. \tag{27}$$

We now *hypothesise* that this exponential decay for wavenumbers on the 'body length-scale' holds for all large wavenumbers up to the $k = O(Re^{\frac{1}{2}})$ Rayleigh scale. We emphasise that this is a *hypothesis* that needs verifying. However, if the hypothesis is correct then it follows from (27) that there are modes, albeit with exponentially small amplitudes, that might be amplified by a Rayleigh instability.

A Race. Naïvely combining (26) and (27) we argue that the high-wavenumber modes generated by nonlinear interactions can in principle

be amplified by the Rayleigh instability according to

$$\widetilde{u}(\lambda Re^{\frac{1}{2}}, y, t) \propto \exp\left(-\lambda Re^{\frac{1}{2}}\left(\alpha - \int \beta \,\mathrm{d}t\right)\right) . \tag{28}$$

The key point is that (28) suggests that within an order-one time the high-wavenumber modes can grow to be comparable with the basic BLT solution — this is not inconsistent with numerical results of Brinckman & Walker (2001). Thus there is a 'race' between the growth of the modes amplified by the Rayleigh instability, and the development of a van Dommelen singularity. It appears that whether the Rayleigh instability or the van Dommelen singularity develops first will depend on particular circumstances.

Comments. Clearly this [very] heuristic argument needs to be placed on a firmer footing by means of an analysis based on, say, asymptotics beyond all orders. Nevertheless it is worth noting that

- no high-wavenumber modes need to be present in the boundary or initial conditions — self-induced nonlinear effects seem to be sufficient to fill out the spectrum;
- in order to track the amplification of the short Rayleigh-scale modes there is apparently a need to consider terms that are initially exponentially small;
- there is apparently no hint from the BLT asymptotics that a short-scale disturbance can grow to 'infect' the BLT solution.

Of course we have not proved that the short-scale instabilities observed by Brinckman & Walker (2001) are generated by the above mechanism. Indeed we have not considered convective effects and it is possible that disturbances that begin to grow can be convected into regions where they subsequently decay (although in Brinckman & Walker's (2001) problem this effect may be less important because the existence of the 'rear' stagnation point tends to confine any disturbance). However, we believe that there is a case to answer. In the next section we consider a model problem where we predict a similar effect on the basis of an analogous scaling argument.

3.2. A MODEL PROBLEM

Consider the Kuramoto-Sivashinsky (KS) equation

$$u_t + uu_x = -\varepsilon u_{xx} - \Delta u_{xxxx} , \tag{29}$$

with initial condition

$$u = \sin x \quad \text{at} \quad t = 0\,. \tag{30}$$

For the case $\Delta = \varepsilon = 0$, i.e. the kinematic wave equation, there is a known analytic solution, $u_0(x,t)$, that develops a singularity at $t = 1$. From this analytic solution it is straightforward to show (e.g. Sulem, Sulem & Frisch 1983) that for $t > 0$ and $k \gg 1$,

$$\widetilde{u}_0(k,t) \propto \exp(-\alpha(t)k)\,, \tag{31}$$

where $\widetilde{u}_0(k,t)$ denotes the Fourier series of $u_0(x,t)$ and $\alpha(t) > 0$.

For $\Delta > 0$ and $\varepsilon > 0$ the KS equation is known to be well-posed and have regular solutions for all time. We will study the particular scaling $0 < \Delta \ll \varepsilon \ll 1$ and consider for what times u_0 is the leading-order solution for u in an expansion in powers of Δ and ε. We argue that u_0 is analogous to the BLT solution in that it develops a singularity within a finite time, while if $0 < \Delta \ll \varepsilon \ll 1$ there are rapidly growing high-wavenumber instabilities analogous to Rayleigh instabilities, e.g. when $k^2 = \varepsilon/2\Delta \gg 1$ small amplitude instabilities grow like

$$\text{amplitude} \propto \exp(\varepsilon^2\, t/2\Delta)\,. \tag{32}$$

An order-of-magnitude argument from a comparison of (31) and (32) suggests that short-scale instabilities can grow to be comparable with $u_0(x,t)$ when

$$t \sim \Delta^{\frac{1}{2}} \varepsilon^{-\frac{3}{2}}\,. \tag{33}$$

Hence if $\Delta \ll \varepsilon^3$ short-scale instabilities should develop before the singularity forms at $t = 1$.

KS Equation: Numerical Solutions As a preliminary confirmation of the above prediction we have solved the KS equation numerically for initial condition (30), $\varepsilon = 10^{-1}$ and $\Delta = 10^{-5}$. Both the initial condition, and numerical solution at $t = 0.46$, are shown in figure 2 — the development of a short-scale instability well before the time at which a singularity develops in u_0 is evident.

Comments On The KS Problem. Our crude heuristic arguments are apparently supported by the numerical experiments. Further, the use of a single-mode initial condition emphasises the fact that the 'exponentially small' higher modes are generated by nonlinear interactions. In addition, preliminary analysis suggests that the crucial need to consider

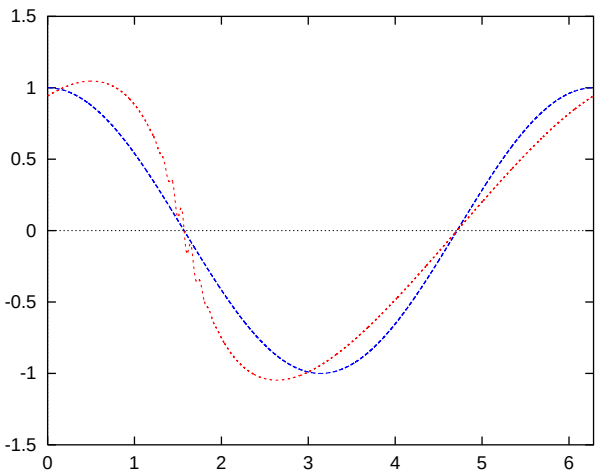

Figure 2 Numerical solution with $\varepsilon = 10^{-1}$, $\Delta = 10^{-5}$ and $u = \sin x$ at $t = 0$. – · –: solution at $t = 0$, – – –: solution at $t = 0.46$.

'exponentially small' terms is not hinted at by solving for higher-order terms of a regular perturbation expansion in powers of Δ and ε (i.e. regular perturbation theory fails, and fails spectacularly).

While there is clearly a need to tighten up the analysis (and work is underway with that in mind), it seems that there is an *a priori* case for believing that the correct mechanism for the growth of the short-scale disturbances has been identified in the KS model, and that these short-scale disturbances can alter the leading-order solution by an order-one amount. A similar, although not identical, change to the leading-order solution caused by the growth of exponentially small terms has previously been reported in a Saffman-Taylor Hele-Shaw problem by Siegel, Tanveer & Dai (1996). As in the Hele-Shaw problem it may be possible to place our analysis on a firmer footing by analytically continuing into the complex x-plane.

Of course the KS model does not contain all the dynamics of BLT theory, e.g. it does not include spatial regions of both growth and decay of the instability. However we believe that KS model does include key aspects of the mathematics that are similar to those in the BLT problem.

4. ASYMPTOTIC INSTABILITY THEORY

4.1. INTRODUCTION

We have seen that laminar $Re \gg 1$ asymptotic analysis is problematic in that it is sometimes successful (e.g. Blasius solution at $R \approx 500$), but

other times not (e.g. unsteady flow past a circular cylinder at least at times past singularity formation, if not before). We have also noted two 'features' of classical BLT.

1. The leading-order problem can lead to a succession of singularities forcing consideration of extremely short time scales with the result that it is impossible to obtain solutions an order-one time after the first appearance of a singularity.

2. It appears possible for terms that start exponentially small to be able to grow to alter the leading-order solution; moreover, as yet there is no means of identifying whether or not this will occur by means of predictive asymptotic analysis.

There is thus a tendency for short-scale phenomena to occur naturally in BLT. As a result, a main strand of research that has developed in BLT over the last thirty years has been the study of instabilities and transition to turbulence in shear layers.

4.2. 'PARALLEL' FLOWS: ORR-SOMMERFELD THEORY

The stability of thin shear layers has been the subject of research for well over a century. In particular, thin almost-parallel shear layers have often been idealised as exactly parallel so that the underlying flow is given by

$$\mathbf{u} = (U(y), 0, 0) \equiv \mathbf{U}\,. \tag{34}$$

A linear stability analysis of such a flow is then performed based on normal mode perturbations of the form

$$\mathbf{u} = \mathbf{U} + \epsilon\widetilde{\mathbf{u}}(y)\exp(\mathrm{i}\alpha x + \mathrm{i}\beta z - \mathrm{i}\alpha ct) + \dots\,, \tag{35}$$

where α, β are here the wavenumbers in the x and z directions respectively, and c is the phase-speed. Substitution into the linearised NS equations and solution of the resulting Orr-Sommerfeld (OS) equation yields a Re-dependent dispersion relation relating α, β and c:

$$F(\alpha, \beta, c; Re) = 0\,. \tag{36}$$

In the case of linear two-dimensional Tollmien-Schlichting (TS) waves on a flat plate, the predictions of OS theory are in very good agreement with experiment (Ross *et al.* 1970, Klingmann *et al.* 1993). However, good agreement is not invariably obtained, and in the case of Görtler rolls and cross-flow instability, OS theory can yield misleading results.

A drawback of OS theory, and possibly the reason that it does not always work, is that the theory is mathematically inconsistent. On the one hand, for the basic shear layer flow is to be 'almost' parallel then, formally, it is necessary to assume that the Reynolds number is asymptotically large, i.e. $Re \gg 1$. On the other hand, for the derivation of the linearised OS equation and the resulting dispersion relation (36), it is necessary to assume that the Reynolds number is, formally, an order-one quantity, i.e. $Re = O(1)$.

We emphasise that the distinction between 'asymptotically large' and 'order one' does not depend on whether or not the Reynolds number is numerically large. The distinction concerns the approximations made in the analysis.[7] Moreover, it was not until there was a proper appreciation of the incompatibility of the two different treatments of the Reynolds number in OS theory that it became clear how to deal with non-parallel and/or nonlinear effects in a consistent manner.

4.3. ASYMPTOTIC LINEAR THEORY

An alternative to OS theory is to assume consistently that $Re \gg 1$. The drawback of this approach is that almost all flows first become unstable at moderate Reynolds numbers where it is not clear *a priori* that results derived on the basis of an asymptotically large Reynolds number will hold. Moreover it is difficult, if not impossible, to study the fastest-growing disturbances with this approach. Nevertheless, *sometimes* these difficulties are not show stoppers.

Triple-Deck Theory (TDT) The most significant advance in BLT after Prandtl's original formulation was the simultaneous discovery of TDT by Messiter (1970), Neiland (1969) and Stewartson (1969). This theory applies to disturbances that change 'rapidly' in the downstream direction, that is on a length scale short compared with that over which the underlying boundary layer varies, though still long compared with the boundary-layer thickness. This relatively rapid change means that

[7] As an illustration consider flow over an aligned flat plate.

If the Reynolds number is assumed to be consistently asymptotically large then the underlying steady flow is described by the Blasius solution, while the linearised equation describing the evolution of small perturbations with wavelengths comparable with the thickness of the boundary-layer, is just the Rayleigh equation.

If alternatively the Reynolds number is assumed to be consistently order one, then the underlying steady flow will be described by the nonlinear steady NS equations, while the perturbation will be described by the linearised unsteady NS equations.

OS theory tries to have its cake (i.e. by approximating the basic flow with the asymptotic Blasius solution), and eat it (i.e by taking the Reynolds number to be order one in the OS equation).

viscous effects associated with the disturbances are confined to a thin sublayer close to the wall (the 'lower deck'), while the bulk of the underlying boundary layer adjusts through an inviscid, rotational displacement (the 'middle/main deck'). The fluid ejected from the middle deck induces a flow in an 'upper deck' above the boundary layer that is inviscid and irrotational. In turn this irrotational flow induces a dynamically significant pressure gradient in the lower deck. There is thus a feedback loop whereby fluid motion in the lower deck can change the pressure gradient felt in the lower deck (albeit indirectly by means of the flow generated in the upper deck). In contrast, in classical BLT the pressure gradient is fixed by the slip velocity and is not influenced by induced motions in the boundary layer. Finally we note that while at leading order $p_y = 0$ in the lower and middle decks, at leading order $p_y \neq 0$ in the upper deck.

Linear TS Waves Whilst TDT was first formulated in terms of formal asymptotic expansions in the late 60s, the key ideas can be found in the linearised analysis of shock/boundary-layer interactions (Lighthill 1953) and lower-branch TS waves (Tollmien 1929, Lin 1945). However it was Smith (1979) who realised that lower-branch TS could be placed in the TDT framework, and that this might enable laminar-turbulent transition to be studied by means of a large-Reynolds-number asymptotic approach.

Smith (1979) expanded the dynamical variables in powers of $Re^{-\frac{1}{8}}$, and used multiple-scales in x and matched asymptotic expansions in y. His asymptotic expansion for the TS neutral lower branch for flow over a flat plate is in reasonably good agreement with experiments at moderate Reynolds number. However, the equivalent asymptotic expansion for the TS neutral upper branch only provides a good approximation to the neutral curve at Reynolds numbers when the flow would, in practice, be fully turbulent (Healey 1995).

4.4. ASYMPTOTIC NONLINEAR THEORY

One of the major advantages of a large-Reynolds-number, asymptotic approach is that there is a consistent way to examine nonlinear effects. A disadvantage is that there is a plethora of possible scalings, and it is difficult to identify in advance which, if any, will give good agreement with experiment. For instance there are different types of modes (e.g. TS, Rayleigh, Klebanoff) and different sources of instability (e.g. 2D/3D, localised/global, linear/nonlinear, whether the disturbances is introduced in a controlled manner or arises from noise, whether the disturbance is

introduced within or outside the shear layer). There are also different types of analysis. For instance there have been many studies of uniform or modulated wavetrains of almost neutral linear modes, but there are other possibilities such as studies of wavetrains of almost neutral nonlinear modes or studies of algebraically growing modes (e.g. Klebanoff modes). There have also been numerical studies of modes with order-one growth rates. An extensive (though not exhaustive) review can be found in Cowley & Wu (1994).

Particularly in the case of studies of wavetrains of almost neutral modes, two key ideas reoccur, namely wave/mean-flow interactions and critical-layer effects.

Wave/Mean-Flow Interactions. Such effects can arise in a number of circumstances, but an archetypal example is when there are multiple 'carrier' modes two of which are propagating at equal and opposite directions to the mean flow, e.g.

$$\mathrm{e}^{\mathrm{i}\alpha x}\mathrm{e}^{\mathrm{i}\beta z}\mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.} \quad \text{and} \quad \mathrm{e}^{\mathrm{i}\alpha x}\mathrm{e}^{-\mathrm{i}\beta z}\mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.}\,, \tag{37}$$

where c.c. denotes complex conjugate. Nonlinear interactions between such modes through the quadratic terms in the NS equations, generate a steady mean: flow

$$\underset{\text{mode 1}}{\left(\mathrm{e}^{\mathrm{i}\alpha x}\mathrm{e}^{\mathrm{i}\beta z}\mathrm{e}^{-\mathrm{i}\omega t}\right)}\ \underset{\text{c.c. of mode 2}}{\left(\mathrm{e}^{-\mathrm{i}\alpha x}\mathrm{e}^{\mathrm{i}\beta z}\mathrm{e}^{\mathrm{i}\omega t}\right)} = \mathrm{e}^{2\mathrm{i}\beta z}\,. \tag{38}$$

Moreover, if the spanwise scale, β^{-1}, and 'slow' streamwise scale, $\mathcal{L}$, are disjoint, i.e. $\beta\mathcal{L} \ll 1$, then modest spanwise motions generate large streamwise mean flows as can be seen from a scaling argument based on the continuity equation:

$$\left.\begin{array}{rcl} u_x + v_y + & w_z & = 0 \\ \frac{\mathcal{U}_m}{\mathcal{L}} & \sim & \beta\mathcal{W}_m \end{array}\right\} \quad \Rightarrow \quad \mathcal{U}_m \sim \beta\mathcal{L}\mathcal{W}_m \gg \mathcal{W}_m\,. \tag{39}$$

This mechanism is one reason why relatively strong longitudinal vortices are observed in transitional (and fully turbulent) boundary-layers (e.g. Jang *et al.* 1986, Hall & Smith 1989).

Critical Layers/Levels. Often in a weakly nonlinear perturbation analysis the $u_t + uu_x$ part of the NS equations reduces at leading-order to

$$\mathrm{i}\alpha(U-c)\widetilde{u}\,, \tag{40}$$

where U is the underlying mean flow and $\widetilde{u}$ is the perturbation velocity. Suppose that $U = c$ at $y = y_c$, then y_c is said to be a critical level. Critical levels are important since linear inviscid solutions almost always have singularities there, e.g. for 3D disturbances

$$\widetilde{u} \propto \frac{1}{y - y_c}\,. \tag{41}$$

This singularity is smoothed out by one or more effects (e.g. viscosity, unsteadiness, nonlinearity) in a thin 'critical layer' surrounding $y = y_c$. Moreover, critical layers tend to be dynamically important since nonlinear effects are largest within them. It is for this reason that many analytic studies have focused on 'phase-locked' nonlinear interactions, that is, interactions among modes with the same phase speed c, since nonlinear interactions are strongest when the critical layers coincide (e.g. Goldstein 1995).

TS Resonant-Triad Instability. One of the more intriguing aspects of laminar-turbulent transition over a flat plate is the appearance of subharmonics (e.g. Knapp & Roache 1968, Kachanov *et al.* 1977). In order to explain these observations, Craik (1971) proposed a weakly nonlinear theory involving a *phase-locked resonant-triad* interaction (see also Raetz 1959), while Herbert (1988) used a Floquet approach to demonstrate secondary instability of fully nonlinear TS waves (or rather, OS solutions) to subharmonic (and other) perturbations. Although these analyses identified key aspects of the physics, the approach was somewhat heuristic. A firm asymptotic description of the resonant-triad mechanism, including a qualitative explanation of the observed super-exponential growth, was eventually given by Goldstein & Lee (1992) and Mankbadi, Wu & Lee (1993). A central feature of their analyses was nonlinear interactions within critical layers. An important revelation of the asymptotic approach was that wave/mean-flow interactions can be as important as the resonant-triad interaction.

As with other large-Reynolds-number analyses, the first nonlinear scaling in the resonant-triad problem predicts that time/length scales of the modulation amplitude rapidly shorten.[8] Consequently, in order to follow the evolution of the flow it is necessary to consider a succession of asymptotic problems with shorter and shorter time/length scales.

[8] We emphasise that it is the time/length scales of the *modulation amplitude* that shorten. While such an occurrence may well be a prelude to transition, the time and/or length scales are still long compared with the period and/or wavelength of the carrier wave[s], i.e. singularity development, or similar, in the modulational amplitude does not mean that fine-scale structure on the length of an instability wave has developed.

As with unsteady separation a potential drawback of the analysis is, therefore, that an asymptotic description may not be possible at times much beyond that at which the first nonlinear interaction takes place. Moreover, while the asymptotic theory is in qualitative agreement with experiment, quantitative agreement has yet to be achieved (at least for an asymptotic theory that is correct).

Receptivity: How A Disturbance Penetrates A Shear Layer. Another success for asymptotic theory has been an explanation of how sound waves can interact with a 'rivet' protruding from an otherwise smooth surface, and so generate TS waves (Ruban 1984, Goldstein 1985). A key qualitative observation is that the rivet length should match the triple-deck lengthscale. A related analysis for TS wave generation by a curvature discontinuity on a surface shows quantitative agreement with experiment (Goldstein & Hultgren 1987).

Similarly, Wu (1999) has explained how sound waves and a vorticity or entropy gust can interact quadratically in the upper deck to generate TS waves. While at first sight the required asymptotic scaling between the length and time scales of the sound wave and gust appears to rule out general applicability, Wu (1999) shows how the analysis can be applied to a broad-band spectrum.

Other Successes. Other than receptivity, there are relatively few examples where asymptotic theory has obtained good *quantitative* agreement with experiment. Hultgren's (1992) theoretical explanation of the 2D nonlinear roll-up of a shear layer is one notable exception, while the asymptotic description of Görtler instability is another (e.g. Hall 1990).

5. CONCLUSIONS

An undoubted strength and success of BLT is its ability to explain, qualitatively, fundamental concepts such as separation, nonlinear instability and receptivity. However, given the number of papers that have been published in the field there are relatively few *reliable* calculations where good quantitative agreement has been obtained between asymptotic theory and experiment. Many of the best examples where there is good quantitative agreement have been mentioned above, here we also note that there are reports that the Russian space shuttle Buran was designed using large-Reynolds-number hypersonic asymptotic the-

ory. Unfortunately that work is not for the most part available in the open literature.[9]

One of the drawbacks of nonlinear large-Reynolds-number asymptotic instability theory is that the analysis can become complicated, e.g. a resonant-triad interaction of TS waves requires a 'septuple-deck' structure (Mankbadi, Wu & Lee 1993). As a result it is arguable that the payoff does not always justify the effort. Further, there are a number of examples where the technical difficulties of the analysis have lead to erroneous results,[10] e.g. see the discussions in Wu *et al.* (1996) and Moston *et al.* (2000).

We also recall that a feature of BLT is the formation of singularities. Often the development of a singularity indicates an important feature of the physical flow, e.g. unsteady separation or the formation of short time/length scale features in laminar-turbulent transition. However, after the formation of an initial singularity a succession of problems with increasingly short time/length scales can result, again indicating that it is difficult to obtain an asymptotic description for order-one times beyond the formation of the initial singularity. Moreover we have also seen that in order to obtain the correct asymptotic solution, it is maybe necessary to include the effects of terms that are initially exponentially small using a 'beyond-all-orders' asymptotic analysis.

In the light of these comments we return to the question posed in the title.

Laminar BLT: A Paradox. First we note that it is certainly true that laminar BLT *is* a paradox, in that it is based on the assumption that $Re \gg 1$, whereas almost all flows are turbulent if $Re \gg 1$. As a result, in order to obtain laminar solutions it is necessary to suppress instabilities. Sometimes this is possible (e.g. the Blasius flat-plate solution, receptivity), but other times it is not (e.g. for a 'medium-term' description of unsteady separation).

Laminar BLT: A $20^{\mathbf{th}}$ Century Paradox? But is BLT a *20^{th} century* paradox? On the one hand one might argue that the answer to this question is *no*, since BLT is still good for explaining fundamental mechanisms and obtaining scalings. On the other hand one might argue that the answer is *yes*, since for quantitative agreement with experiment

[9] However, on the assumption that the work is available to some western agencies, some corroboration might possibly be gleaned from NASA's interest in asymptotic theory in the mid-80s.

[10] Remarkably, some of these results have agreed very well with experiment!

BLT will be outgunned by computational fluid dynamics (CFD) in the 21^{st} century.

I would argue that the answer is *yes*. With the rise of modern computers and codes, good engineering answers for laminar flows can be obtained with CFD for the Reynolds numbers when asymptotic theory might be applicable. This is not to say that BLT does not have a rôle in explaining fundamental mechanisms, but many, if not all, of the fundamental questions in BLT have now been answered. I would be surprised if there were another ICTAM lecture on BLT.

Acknowledgments

I am very grateful to Philip Stewart for his help in preparing this manuscript — both his advice on content, and his assiduousness when it came to details, were invaluable. The work in § 3 on the KS equation is joint with him, while the opinions, rhetoric and mistakes are wholly mine. I would also like to thank James Phillips and Hassan Aref for their patience.

References

Blasius, H. (1908). Grenzschichten in Flüssigkeiten mit kleiner Reibung. *Z. Math. Phys.* **56**(1), 1–37; Engl. transl. in NACA TM 1256.

Brinkman, K.W. & Walker, J.D.A. (2001). Instability in a viscous flow driven by streamwise vortices. To appear in *J. Fluid Mech.*.

Carrier, G.F., Krook, M. & Pearson, C.E. (1983). Functions of a Complex Variable: Theory and Technique. Hod Books, Ithaca, NY.

Cassel, K.W., Smith, F.T. & Walker, J.D.A. (1996). The onset of instability in unsteady boundary-layer separation. *J. Fluid Mech.* **315**, 223–256.

Chernyshenko, S. (1988). The asymptotic form of the stationary separated circumference of a body at high Reynolds number *Prikh. Mat. Mekh.* **52**(6), 958–966; Engl. transl. *J. Appl. Math. Mech.* **52**, 746.

Chernyshenko, S. (1998). Asymptotic theory of global separation. *Appl. Mech. Reviews* **9**, 523–536.

Coutanceau, M. & Bouard, R. (1977). Experimental determination of the main features of the viscous flow in the wake of a circular cylinder in uniform translation. Part 2. Unsteady flow. *J. Fluid Mech.* **79**, 257–272.

Cowley, S.J. & Wu, X. (1994). Asymptotic approaches to transition modelling. *Progress in Transition Modelling, AGARD Rep.* 793, Chap. 3, 1–38.

Craik, A.D.D. (1971). Non-linear resonant instability in boundary layers. *J. Fluid Mech.* **50**, 393–413.

van Dommelen, L.L. (1981). Unsteady boundary-layer separation. Ph.D. thesis, Cornell University, Ithaca, NY.

van Dommelen, L.L. & Shen, S.F. (1980). The spontaneous generation of the singularity in a separating laminar boundary layer. *J. Comput. Phys.* **38**, 125.

van Dommelen, L.L. & Cowley, S.J. (1990). On the Lagrangian description of unsteady boundary layer separation. Part 1. General theory. *J. Fluid Mech.* **210**, 593–626.

Elliott, J.W., Cowley, S.J. & Smith, F.T. (1983). Breakdown of boundary layers: (i) on moving surfaces; (ii) in semi-similar unsteady flow; (iii) in fully unsteady flow. *Geophys. Astrophys. Fluid Dyn.* **25**, 77–138.

Fornberg, B. (1985). Steady viscous flow past a circular cylinder up to Reynolds number 600. *J. Comp. Phys.* **61**(2), 297–320.

Goldstein, M.E. (1985). Scattering of acoustic-waves into Tollmien-Schlichting waves by small streamwise variations in surface geometry. *J. Fluid Mech.* **154**, 509–529

Goldstein, M.E. (1995). The role of nonlinear critical layers in boundary-layer transition. *Phil. Trans. Roy. Soc. Lond.* **352**, 425–442.

Goldstein, M.E. & Hultgren, L.S. (1987). A note on the generation of Tollmien-Schlichting waves by sudden surface-curvature change. *J. Fluid Mech.* **181**, 519–525.

Goldstein, M.E. & Lee, S.S. (1992). Fully coupled resonant-triad interaction in an adverse-pressure-gradient boundary layer. *J. Fluid Mech.* **245**, 523–551.

Goldstein, S. (1948). On laminar boundary-layer flow near a position of separation. *Q.J.M.A.M* **1**(1), 43–69.

Goldstein, S. & Rosenhead L.N. (1936). Boundary-layer growth. *Proc. Camb. Phil. Soc.* **32**, 392–401.

Hall, P. (1990). Görtler vortices in growing boundary layers: the leading edge receptivity problem, linear growth and the nonlinear breakdown stage. *Mathematika* **37**, 151–189.

Hall, P. & Smith, F.T. (1989). Nonlinear Tollmien-Schlichting vortex interaction in boundary-layers. *Euro. J. Mech. B-Fluids* **8**, 179-205; for corrections see Smith, F.T. & Blennerhassett, P. in *Proc. R. Soc. Lond.* A**436**, 585–602 (1992).

Healey, J.J. (1995). On the neutral curve of the flat-plate boundary-layer — comparison between experiment, Orr-Sommerfeld theory and asymptotic theory. *J. Fluid Mech.* **288**, 59–73.

Herbert, T. (1988). Secondary instability of boundary layers. *Ann. Rev. Fluid Mech.* **20**, 487–526.

Hultgren, L.S. (1992). Nonlinear spatial equilibration of an externally excited instability wave in a free shear layer. *J. Fluid Mech.* **236**, 635–664.

Jang, P.S., Benney, D.J. & Gran, R.L. (1986). On the origin of streamwise vortices in a turbulent boundary layer. *J. Fluid Mech.* **169**, 109–123.

Jobe, C.E. & Burggraf, O.R. (1974) The numerical solution of the asymptotic equations of trailing edge flow. Proc. R. Soc. **A340**, 91–111.

Klingmann, B.G.B., Boiko, A.V., Westin, K.J.A., Kozlov, V.V. & Alfredsson, P.H. (1993). Experiments on the stability of Tollmien-Schlichting waves. *Euro. J. Mech. B* **12**, 493–514.

Kachanov, Yu.S., Kozlov, V.V. & Levchenko, V.Ya. (1977). Nonlinear development of a wave in a boundary layer. *Izv. Akad. Nauk SSSR, Mekh. Zhidk. i Gaza* (3), 49–58; Engl. transl. *Fluid Dyn.* **12**, 383–390 (1978).

Knapp, C.F. & Roache, P.J. (1968). A combined visual and hot wire anemometer investigation of boundary-layer transition. *AIAA J.* **6**, 29–36.

Krasny, R. (1986). A study of singularity formation in a vortex sheet by the point-vortex approximation, *J. Fluid Mech.* **167**, 65–93.

Li, L., Walker, J.D.A., Bowles, R.I. & Smith, F.T. (1998). Short-scale break-up in unsteady interactive layers: local development of normal pressure gradients and vortex wind-up. *J. Fluid Mech.* **374**, 335–378.

Lighthill, M.J. (1953). On boundary layers and upstream influence. II. Supersonic flows without separation. *Proc. Roy. Soc. Lond.* A**217**, 478–507.

Lin, C.C. (1945). On the stability of two-dimensional parallel flows. *Quart. Appl. Math.* **3**, 117–142, 213–234, 277–301 (1946).

Mankbadi, R.R., Wu, X. & Lee, S.S. (1993). A critical-layer analysis of the resonant triad in boundary-layer transition - nonlinear interactions. *J. Fluid Mech.* **256**, 85–106.

Messiter, A.F. (1970). Boundary-layer flow near the trailing edge of a flat plate. *SIAM J. Appl. Math.* **18**, 241–257.

Moston, J., Stewart, P.A. & Cowley, S.J. (2000). On the nonlinear growth of two-dimensional Tollmien-Schlichting waves in a flat-plate boundary layer. *J. Fluid Mech.* **425**, 259–300.

Neiland, V.Ya. (1969). Towards a theory of separation of the laminar boundary layer in a supersonic stream. *Izv. Akad. Nauk SSSR, Mekh. Zhidk. i Gaza* (4), 53–57; Engl. transl. *Fluid Dyn.* **4**(4), 33–35.

Prandtl, L. (1904). Über Flüssigkeitsbewegung bei sehr kleiner Reibung. *Verh. III. Intern. Math. Kongr., Heidelberg, 1904,* S. 484–491, Teubner, Leipzig, 1905; Engl. transl. in NACA TM 452.

Prandtl, L. (1932). Production of vortices (film, 10 minutes, silent, black and white).

Raetz, G.S. (1959). A new theory of the cause of transition in fluid flows. *Norair Rep.* NOR-59-383, Hawthorne, California.

Robins, A.J. & Howarth, J.A. (1972). Boundary-layer development at a two-dimensional rear stagnation point, *J. Fluid Mech.* **56**, 161–171.

Rosenhead, L. (1963). *Laminar boundary layers.* Dover.

Ross, J.A., Barnes, F.H., Burns, J.G. & Ross, M.A.S. (1970). The flat plate boundary layer. Part 3. Comparison of theory with experiment. *J. Fluid Mech.* **43**, 819–832.

Ruban, A.I. (1984). On the generation of Tollmien-Schlichting waves by sound. *Izv. Akad. Nauk SSSR, Mekh. Zhidk. i Gaza* (5), 44–52; Engl. transl. *Fluid Dyn.* **19**(5), 709–716.

Sadovskii, V.S. (1971). Vortex regions in a potential flow wiht a jump in the Bernouilli constant at the boundary. *Prikh. Mat. Mekh.* **35**(3), 773–779; Engl. transl. *J. Appl. Math. Mech.* **35**(5), 729–779.

Shen, S.F. (1978). Unsteady separation according to the boundary-layer equation. *Adv. Appl. Mech.* **18**, 177–220.

Siegel, M., Tanveer, S. & Dai, W.-S. (1996). Singular effects of surface tension in evolving Hele-Shaw flows. *J. Fluid Mech.* **323**, 201–236.

Smith F.T. (1977). The laminar separation of an incompressible fluid streaming past a smooth surface. *Proc. R. Soc. Lond.* A**356**, 433–463.

Smith F.T. (1979). On the non-parallel flow stability of the Blasius boundary layer. *Proc. R. Soc. Lond.* A**366**, 91–109.

Smith F.T. (1982). On the high-Reynolds-number thoery of laminar flows. *IMA J. Appl. Math.* **28**, 207–281.

Smith, F.T. & Daniels, P.G. (1981). Removal of Goldstein's singularity at separation, in flow past obstacles in wall layers. *J. Fluid Mech.* **110**, 1–37.

Stewartson, K. (1969). On the flow near the trailing edge of a flat plate. Part II. *Mathematika* **16**, 106–121.

Stewartson, K. (1970). Is the singularity at separation removable? *J. Fluid Mech.* **44**, 347–364.

Stewartson, K. (1981). D'Alembert's paradox. *SIAM Rev.* **23**, 308–343.

Sulem, C, Sulem, P.L & Frisch, H. (1983). Tracing complex singularities with spectral methods. *J. Comp. Phys.* **50**(1), 138–161.

Sychev, V.V. (1972). On laminar separation. *Izv. Akad. Nauk SSSR, Mekh. Zhidk. i Gaza* (3), 47–59; Engl. transl. *Fluid Dyn.* **7**, 407–419 (1974).

Sychev, V.V., Ruban, A.I., Sychev, V.V. & Korolev, G.L. (1998). Asymptotic theory of separated flows. Cambridge University Press.

Terrill, R.M. (1960). Laminar boundary-layer flow near separation with and without suction. *Phil. Trans. Roy. Soc. Lond.* A**253**, 55–100.

Tollmien, W. (1929). Über die Entstehung der Turbulenz. Nachr. Ges. Wiss. Göttingen, 21–44; Engl. transl. in NACA TM 609 (1931).

Tollmien, W. (1935). Ein allgemeines Kriterium der Instabilität laminarer Geschwindigkeitsverteilungen. *Nachr. Ges. Wiss. Fachgruppe, Göttingen* **1**, 79–114; see also in NACA TM 792 (1936).

Tutty, O.R. & Cowley, S.J. (1986). On the stability and the numerical solution of the unsteady interactive boundary-layer equation. *J. Fluid Mech.* **168**, 431–456.

Van Dyke, M. (1982). *An Album of Fluid Motion.* Parabolic Press, Stanford, CA.

Wu, X. (1999). Generation of Tollmien–Schlichting waves by convecting gusts interacting with sound. *J. Fluid Mech.* **397**, 285–316.

Wu, X., Stewart, P.A. & Cowley, S.J. (1996). On the weakly nonlinear development of Tollmien-Schlichting wavetrains in boundary layers. *J. Fluid Mech.* **323**, 133–171.