LAMINAR BOUNDARY-LAYER THEORY: A 20TH CENTURY PARADOX?

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Abstract Boundary-layer theory is crucial in understanding why certain phenomena occur. We start by reviewing steady and unsteady separation from the viewpoint of classical non-interactive boundary-layer theory. Next, interactive boundary-layer theory is introduced in the context of unsteady separation. This discussion leads onto a consideration of large-Reynolds-number asymptotic instability theory. We emphasise that a key aspect of boundary-layer theory is the development of singularities in solutions of the governing equations. This feature, when combined with the pervasiveness of instabilities, often forces smaller and smaller scales to be considered. Such a cascade of scales can limit the quantitative usefulness of solutions. We also note that classical boundary-layer theory may not always be the large-Reynolds-number limit of the Navier-Stokes equations. This is because of the possible amplification of short-scale modes, which are initially exponentially small, by a Rayleigh instability mechanism.

1. INTRODUCTION

Sectional lecturers were invited 'to weave in a bit more retrospective and/or prospective material [than normal] given the particular [Millennium] year of the Congress'. This invitation is reflected in the current, possibly idiosyncratic, article. For alternative viewpoints the reader is referred to Stewartson (1981), Smith (1982), Cowley and Wu (1994), Goldstein (1995) and Sychev *et al.* (1998). We begin with a deconstruction of the components of the title.

Boundary-Layer Theory. Prandtl (1904) proposed that viscous effects would be confined to thin shear layers adjacent to boundaries in the case of the 'motion of fluids with very little viscosity', i.e. in the case of flows for which the characteristic Reynolds number, Re, is large. In a more general sense we will use 'boundary-layer theory' (BLT) to refer to any large-Reynolds-number, $Re \gg 1$, asymptotic theory in which there are thin shear layers (whether or not there are boundaries).

20th Century. Prandtl (1904) published his seminal paper on the foundations of boundary-layer theory at the start of the 20th century, while the ICTAM 2000 was held at the end of the same century.

Laminar. Like Prandtl (1904) we will be concerned with laminar, rather than turbulent, flows. Flows that are in the process of laminar-turbulent transition will be viewed as unstable laminar flows.

A Paradox. Experimental flows at large Reynolds numbers are turbulent, yet useful comparisons with laminar-flow experiments at moderately large Reynolds numbers can sometimes be made with large-Reynolds-number asymptotic theories. We view as a paradox this seemingly contradictory result, i.e. that useful comparisons with laminar flow can be made with expansions made about Reynolds numbers when flows are almost invariably turbulent.

?. The question this paper will discuss is whether the final '?' is needed in the title. A subjective conclusion is given at the end.

2. CLASSICAL BOUNDARY LAYERS

2.1. FORMULATION

We consider incompressible flow of a fluid with constant density ρ and dynamic viscosity μ, past a body with typical length $\mathcal{L}$. We assume that a typical velocity scale is $\mathcal{U}$, and that the Reynolds number is given by

$$Re = \rho \mathcal{U} \mathcal{L} / \mu \gg 1 \,. \tag{1}$$

For simplicity we will, for the most part, consider two-dimensional incompressible flows, although many of our statements can be generalised to three-dimensional flows and/or compressible flows.

The Key Approximations. BLT applies to flows where there are extensive inviscid regions separated by *thin* shear layers, say, of typical width $\delta \ll \mathcal{L}$. For one such shear layer take local dimensional Cartesian coordinates $\hat{x}$ and $\hat{y}$ along and across the shear layer respectively. Denote the corresponding velocity components by $\hat{u}$ and $\hat{v}$ respectively, pressure by $\hat{p}$ and time by $\hat{t}$. On the basis of scaling arguments (e.g. Rosenhead 1936) it then follows that

$$\delta \sim Re^{-\frac{1}{2}} \mathcal{L} \ll \mathcal{L} \,. \tag{2}$$

Further, it can also be deduced that the key approximations in classical BLT are that the pressure is constant across the shear layer, i.e.

$$0 = -\hat{p}_{\hat{y}} \,, \tag{3}$$

and that streamwise diffusion is negligible, i.e. if $\bullet$ represents any variable

$$\bullet_{\hat{y}\hat{y}} \gg \bullet_{\hat{x}\hat{x}} \,. \tag{4}$$

The former approximation is more significant dynamically.

The Governing Equations. Using the transformations

$$(\hat{x}, \hat{y}, \hat{t}, \hat{u}, \hat{v}, \hat{p}) \to (\mathcal{L}\,x\,, Re^{-\frac{1}{2}}\mathcal{L}\,y\,, \mathcal{U}^{-1}\mathcal{L}\,t\,, \mathcal{U}u\,, Re^{-\frac{1}{2}}\mathcal{U}v\,, \rho\,\mathcal{U}^2\,p), \tag{5}$$

and taking the limit $Re \to \infty$, the BLT equations can be deduced from the Navier-Stokes equations:

$$u_t + uu_x + vu_y = -p_x + u_{yy} \,, \tag{6}$$

$$0 = -p_y \,, \quad u_x + v_y = 0 \,. \tag{7}$$

For flow past a rigid body the appropriate boundary conditions are

$$u = v = 0 \quad \text{on} \quad y = 0 \qquad \text{and} \qquad u \to U(x,t) \quad \text{as} \quad y \to \infty \,, \tag{8}$$

where $U(x,t)$ is the inviscid slip velocity past the body. Further, from (6) evaluated at the edge of the boundary layer

$$-p_x = U_t + UU_x \,. \tag{9}$$

We define the 'viscous blowing' velocity out of the boundary layer to be

$$v_b(x,t) = \lim_{y\to\infty} \left(v + U_x(x,t)y\right) \,. \tag{10}$$

v_b indicates the strength of blowing, or suction, at the edge of the boundary layer induced by viscous effects. It is a good diagnostic for dynamically significant effects within the boundary layer — much better than, say, the wall shear $u_y(x,0,t)$ which can remain regular while $v_b(x,t)$ becomes unbounded.

2.2. STEADY FLOWS

Steady Flow Past An Aligned Flat Plate: A Success. Probably the most famous solution to (6) and (7) is that of Blasius (1908) for flow past an aligned flat plate. A comparison between this similarity solution and Wortmann's visualisation of that flow (Van Dyke 1982) demonstrates that BLT seems to work in this case — at least for $R \approx 500$, where $R = \rho\,\mathcal{U}\delta/\mu = Re^{\frac{1}{2}}$.

Steady Flow Past A Circular Cylinder: A Failure. On the *assumption* that far from the boundary the velocity field is irrotational and inviscid to leading order, the inviscid slip velocity for steady flow past a circular cylinder is given by $U = 2\sin x$. Terrill (1960) showed numerically that the solution to (6)–(8) with this slip velocity terminates in a Goldstein (1948) singularity at $x = x_c \approx 104.5°$ with

$$v_b \sim k(x_c - x)^{-\frac{1}{2}} + \dots \quad \text{as} \quad x \to x_c\,, \tag{11}$$

for some constant k. This singularity occurs at the point where reversed flow is about to develop, i.e. $\min_y u \to 0$ as $x \to x_c$.

A Serious Problem. The occurrence of a singularity often indicates that there is a significant development in the flow physics, e.g. the formation of small scale structure. In such circumstances new physics can usually be included in the model by introducing an asymptotic scaling close to $x = x_c$, so enabling a solution to be found for $x > x_c$.

However, Stewartson (1970) showed that, in general, there is no inner rescaling which 'smoothes out' the Goldstein singularity (Smith and Daniels (1981) discuss an exception). As a result, no BLT solution exists for $x > x_c$. Moreover this result implies that both the BLT solution for $x < x_c$, and the inviscid solution far from the cylinder, are *incorrect.* BLT does not always work.

What Has Gone Wrong? The short answer is that the assumption that the flow far from the wall is irrotational is incorrect. Experimentally it is observed that, other than at very small Reynolds numbers, there is a rotational eddy behind the cylinder that is at least as large as the cylinder; moreover the flow is steady and symmetric only for $Re \leqslant Re_c \approx 47$ (for larger Reynolds numbers the flow is unsteady and asymmetric).

Further, Re_c is not an asymptotically large Reynolds number! Steady symmetric solutions for $Re \geqslant Re_c$ obtained numerically by specifically excluding the possibility of unsteadiness and asymmetry, suggest that the asymptotic regime for steady symmetric solutions is only reached for $Re \gtrsim 600$ (Fornberg 1985), i.e. at Reynolds numbers far larger than those at which the steady flow is stable.

One way forward is to study unsteady flows on the basis that these are what are observed experimentally (see § 2.3). Another is to seek the asymptotic form of the steady symmetric solution (even if it is experimentally unobservable) in the hope that the solution will shed light on the failure of classical BLT (although the true justification for studying the problem may be closer to that of mountaineers for climbing Everest, i.e. it's hard and it's there).

Steady Symmetric Flow Past A Circular Cylinder: An Answer. An asymptotic solution for steady symmetric separated flow past a bluff body at large Reynolds number must contain at least two important ingredients. First there must be a *local* solution at the point of separation of the boundary layer from the body surface. Second, an asymptotic model of the *global* wake is needed.

Local separation is described by Sychev's (1972) 'triple-deck' analysis (see also Smith 1977). This is based on the premise that, at the point of separation on a smooth surface, the pressure gradient is $\mathcal{O}(Re^{-\frac{1}{16}})$. In the context of Kirchhoff free-streamline theory this means that the appropriate free-streamline solution satisfies the Brillouin-Villat condition to leading order (e.g. see Sychev *et al.* (1998) for a discussion).

There have been a number of attempts to fit the above local description of separation into a consistent large-Reynolds-number asymptotic global solution for the flow past a circular cylinder. Building on the work of others, Chernyshenko (1988) proposed an asymptotic structure based on the special Sadovskii (1971) vortex where there is no velocity jump at the edge of the vortex. While this structure, in which both the length and the width of the wake are $\mathcal{O}(Re)$ in magnitude, may not be unique, it overcomes the technical shortcomings, especially as regards wake reattachment, of other proposals (e.g. see Chernyshenko 1998).

2.3. UNSTEADY SEPARATION

While the large-Reynolds-number asymptotic solution for *steady*, symmetric laminar flow past a bluff body is not experimentally realizable, this is not the case for fast *starting* flow past a smooth bluff body. In particular, when $t \ll 1$ the unsteady u_t term in (6) is much larger than the nonlinear uu_x term, and is balanced by the diffusive u_{yy} term (so that the boundary layers are very thin with $\delta \propto t^{\frac{1}{2}}$). Hence when $t \ll 1$ each point of the boundary layer looks locally like Rayleigh's solution for starting motion over a flat plate, and hence separation does not take place at sufficiently early times (e.g. Goldstein and Rosenhead 1936).

Impulsively Started Flow Past A Circular Cylinder. There have been a number of visualisations of this flow, e.g. Prandtl (1932), Coutanceau and Bouard (1977). Prandtl's (1932) film is particularly instructive as regards where separation of the boundary layer starts.

For many years research into unsteady separation focused on the rear stagnation point (e.g. Robins and Howarth 1972), since it is there that reversed flow first sets in. However, *reversed flow* is not the same as *separation/breakaway* of the boundary layer from the body surface —

there is plenty of reversed flow in Stokes' solution for flow over an oscillating plate, and yet this unsteady boundary layer remains attached to the plate for all Reynolds numbers. In contrast, Prandtl (1932) focused attention on a region approximately $\frac{3\pi}{4}$ from the front stagnation point, because it is approximately there that the boundary layer clearly first separates from the body surface. It is arguable that conventional wisdom, i.e. that the rear stagnation point was the place to look, delayed an understanding of unsteady separation by fifty years.

Unsteady Separation: Physics. On the rear-side of an impulsively moved cylinder fluid particles are decelerating. These particles will tend to be squashed in the streamwise direction, with a compensating expansion in the direction normal to the boundary. In Navier-Stokes (NS) or Euler flows it is not possible to squash a particle to zero thickness in one direction and an infinite length in another because the rapid stretching of the fluid particles leads to the generation of a pressure gradient that inhibits the stretching. However, in classical BLT $p_y = 0$ (see (7)), and hence no pressure gradient can be induced in the direction normal to the wall. Unsteady separation occurs when a fluid particle is squashed to zero thickness in the direction parallel to the wall, so ejecting the fluid above it out of the boundary layer (van Dommelen 1981).

Unsteady Separation: Mathematics. Since unsteady separation is connected with the deformation of a particle, it is natural to seek a mathematical description in terms of Lagrangian co-ordinates, say $\boldsymbol{\xi} = (\xi, \eta)$ (Shen 1978, van Dommelen and Shen 1980). Then with $\mathbf{x} \equiv \mathbf{x}(\boldsymbol{\xi}, t)$, $\mathbf{u} \equiv \mathbf{u}(\boldsymbol{\xi}, t)$ and $U \equiv U(\boldsymbol{\xi}, t)$, the momentum equation (6) becomes

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial t} + \left(\frac{\partial x}{\partial \xi} \frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial}{\partial \xi} \right) \left(\frac{\partial x}{\partial \xi} \frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial}{\partial \xi} \right) u \, , \tag{12}$$

while kinematics and mass conservation yield

$$u = \frac{\partial x}{\partial t} \, , \quad v = \frac{\partial y}{\partial t} \, , \quad \text{and} \quad \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi} = 1 \, . \tag{13}$$

Van Dommelen and Shen (1980) made the key observation that (12) depends only on x and u, and hence that (12) and the first equation in (13) can be solved independently of the equations governing y and v. The solution for y can then be obtained from the third equation of (13), while the solution for v can be deduced subsequently from the second equation of (13).

For a given $x(\boldsymbol{\xi}, t)$, the Jacobian mass conservation relation in (13) is a hyperbolic equation for $y(\boldsymbol{\xi}, t)$ with a unit source term. If at some

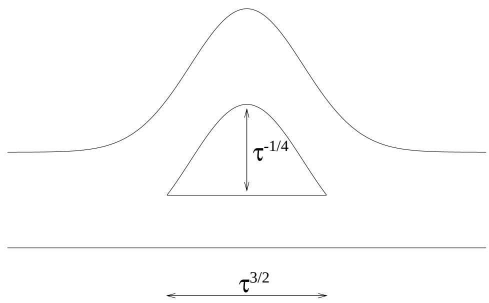

Figure 1 Schematic of a separating boundary layer ($\tau = t_s - t$).

time, say t_s, the solution evolves so that $\nabla x = 0$ (indicating that a particle has been squashed to zero thickness in the x-direction), then 'shock' singularities can form in y, and hence v (van Dommelen and Shen 1980).

Impulsively Started Circular Cylinder: Results. Numerical calculations show that a 'shock' singularity develops at a time $t = t_s \approx 1.5$ and a position $x = x_s(t_s) \approx 111°$. The position where the unsteady singularity forms, $x_s(t_s)$, is not the position where the Goldstein singularity forms, i.e. $x_c \approx 104.5°$.

As $\tau = (t_s - t) \to 0$, the unsteady singularity results in a rapid thickening of the boundary layer over a streamwise distance $(x - x_s(t)) = \mathcal{O}(\tau^{\frac{3}{2}})$, where $x_s(t)$ is the centre of the singularity structure, while the displacement thickness and blowing velocity vary like $\delta_b\,(x_s, t) \propto \tau^{-\frac{1}{4}}$ and $v_b\,(x_s, t) \propto \tau^{-\frac{7}{4}}$ respectively (see figure 1 for a schematic).

Three-Dimensional Unsteady Separation. The above analysis can be extended to describe three-dimensional separation (e.g. van Dommelen and Cowley 1990). When there are no symmetries the boundary-layer still thickens like $\tau^{-\frac{1}{4}}$ and the singularity is quasi-two-dimensional with a slower τ variation in a direction orthogonal to the more rapid $\tau^{\frac{3}{2}}$ variation. However, if there is a plane or axis of symmetry then the thickening of the boundary layer is more rapid than $\tau^{-\frac{1}{4}}$.

2.4. A SINGULARITY ... SURELY NOT?

Conventional wisdom is that finite-time singularities do not spontaneously develop in solutions to the NS equations (of course conventional wisdom may prove to be wrong — as illustrated by the aforementioned fifty years of misguided study of unsteady separation). Assuming that conventional wisdom is correct, the BLT singularity must be an artifact of the BLT approximation. Hence, at times very close to t_s, at least one of the terms that are usually asymptotically smaller than those included in the BLT equations, must grow to be so large that it cannot be neglected at leading order; e.g. in order to stop the extension of the squashed particle in the y-direction, we might anticipate that the $p_y = 0$ approximation will need to be refined.

An Upper Deck. For $\tau \ll 1$ it follows from (5) and the above scaling for $v_b(x_s, t)$ that the dimensional blowing velocity has magnitude

$$\hat{v}_b = \mathcal{O}(\mathcal{U}\, Re^{-\frac{1}{2}}\, \tau^{-\frac{7}{4}})\,. \tag{14}$$

This blowing velocity causes a perturbation to the inviscid flow in a region just above the boundary layer. This perturbation is both inviscid and irrotational; hence it is governed by Laplace's equation. The perturbation extends over a region sufficient for a pressure gradient normal to the wall to be felt (and so reduce the normal velocity to zero). Since the Laplacian is a 'smooth operator' and the extent of the variation in the $\hat{x}$-direction is $\mathcal{O}(\mathcal{L}\tau^{\frac{3}{2}})$, the extent of the variation in the $\hat{y}$-direction is also $\mathcal{O}(\mathcal{L}\tau^{\frac{3}{2}})$. From the continuity equation it follows that the perturbation velocity in the streamwise direction is of the same order of magnitude as the blowing velocity (14). Similarly it follows from the linearised version of the time-dependent Bernoulli equation that the dimensional pressure perturbation is $\mathcal{O}(\rho\mathcal{U}^2\, Re^{-\frac{1}{2}}\, \tau^{-\frac{7}{4}})$. Hence the dimensionless pressure-gradient perturbation has a magnitude

$$\tilde{p}_x = \mathcal{O}(Re^{-\frac{1}{2}}\, \tau^{-\frac{13}{4}})\,. \tag{15}$$

This induced perturbation pressure gradient can have a feedback effect on the boundary-layer flow when it is as large as the acceleration, $u_t = x_{tt} = \mathcal{O}(\tau^{-\frac{1}{2}})$, within the boundary layer. This occurs when $\tau = \mathcal{O}(Re^{-\frac{2}{11}})$. At such times a new asymptotic problem needs to be formulated involving four distinct asymptotic regions in the y-direction (Elliott *et al.* 1983). For this 'quadruple-deck' analysis to be valid, i.e. for the four asymptotic regions to be distinct, strictly we need $Re^{\frac{1}{22}} \gg 1$. This requirement of course raises the question of how large the Reynolds number has to be for the analysis to be valid. We do not address that issue here other than to note that 'large' can vary from approximately 10^0 (e.g. see Jobe and Burggraf 1974) to 10^8 (e.g. see Healey 1995).

The (Rescaled)^n Problem. The interaction problem allows for variations of the pressure gradient in the y-direction in the inviscid outer deck. Hence it might be hoped that the reformulation would be free of finite-time singularities because the induced pressure gradient would be sufficient to prevent a fluid particle being extended indefinitely in the y-direction. However, this is not the case. This rescaled interactive problem itself terminates in a finite-time singularity (Cassel *et al.* 1996).

It is then possible to formulate, at times close to this second singularity, another rescaled problem on an even shorter time-scale. This problem has not apparently been solved, although a model version has been studied by Li *et al.* (1998). They show that depending on the value of certain coefficients, the model problem may, or may not, terminate in yet another finite-time singularity.

The formation of a succession of singularities prompts the question as to whether something has gone wrong with the formulation and/or with the analysis. The answer, we believe, is 'not really'. As indicated earlier, in practice large-Reynolds-number flows are turbulent and hence, if they are to be modelled accurately, we can expect that there will be a natural tendency for structures with small length-scales and short time-scales to develop. The development of a succession of singularities with smaller and smaller length-scales and shorter and shorter time-scales just reflects this natural tendency.

The original singularity appears to be exciting instabilities that lead to small-scale turbulent structure. Thus it is difficult to envisage how it would be possible to find a detailed large-Reynolds-number asymptotic solution for order-one times beyond t_s — although that is not to say that some clever averaging or multiple-scales technique will not be found.

To summarise. The good news is that BLT predicts unsteady separation and the physical interactive effects that then come into play; the bad news is that it does not appear to provide a long-time predictor.

3. BLT: A NAVIER-STOKES LIMIT?

3.1. BRINCKMAN AND WALKER'S RESULTS

Conventional wisdom is that for, say, unsteady starting flows classical BLT is the $Re \gg 1$ asymptotic limit of solutions of the NS equations, or it is at least until the time at which BLT predicts a separation singularity. While this may be the case for many flows, recent numerical results by Brinckman and Walker (2001) suggest that it may not always be so.

In particular Brinckman and Walker (2001) study numerically a NS problem that standard arguments suggest should tend, in the $Re \gg 1$

limit, to the same BLT problem as unsteady flow past an impulsively started circular cylinder. However, as the Reynolds number is increased, Brinckman and Walker's (2001) calculations develop rapid oscillations in the solutions at times *before* the time at which a van Dommelen separation singularity develops in the BLT solution. This suggests that there are flows for which, at times before separation, the $Re \gg 1$ limit of solutions of the NS equations is *not* the BLT solution.[1]

Rayleigh Instabilities. The wavelength of the short-scale oscillations in Brinckman and Walker's (2001) calculations seems to vary like $Re^{-\frac{1}{2}}$. This suggests that the oscillations are a Rayleigh instability.

Consider unsteady classical BLT flow over a rigid surface in a region where there is an adverse pressure gradient, i.e. $p_x > 0$. On the rigid wall, i.e. where $u = v = 0$, it follows from (6) that $u_{yy} = p_x > 0$. However, if $u \to U > 0$ as $y \to \infty$ there is region away from the wall where $u_{yy} < 0$, in which case the velocity profile $u(x, y, t)$ has an inflection point in y.

The existence of an inflection point in a boundary-layer velocity profile implies that disturbances with short $Re^{-\frac{1}{2}}$ streamwise length-scales, i.e. streamwise length-scales comparable with the width of the boundary layer, can grow by means of a Rayleigh instability (Tollmien 1936). The amplitude of such a disturbance will locally behave like

$$\text{amplitude} \propto \exp\left(\lambda Re^{\frac{1}{2}} \textstyle\int \beta \, \mathrm{d}t\right) , \tag{16}$$

where $\lambda Re^{\frac{1}{2}}$ is the local [positive] wavenumber and $\beta(x, t) = \mathcal{O}(1)$ is a function of x and t (e.g. Tutty and Cowley 1986).

Is There Anything To Grow? The next question is whether there are any inherent high-wavenumber modes in the solution with wavelengths of $\mathcal{O}(Re^{-\frac{1}{2}})$ that might be amplified by a Rayleigh instability. We emphasise that by 'modes' we do *not* mean disturbances introduced through noise or, in the case of numerical calculations, rounding error.

Let $u(x, y, t)$ be a solution to the unsteady classical BLT equations that develops a singularity at $t = t_s$. Analytically continue u into the complex x-plane. For times $t < t_s$ there will almost certainly be singularities of u in the complex x-plane. These singularities will move around the complex x-plane and intersect the real x-axis at $t = t_s$ (cf. a similar situation for vortex sheets as explained by, say, Krasny 1986).

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[1] It is arguable that the 'upper' boundary condition used in the calculations is not consistent. This casts a slight doubt on the results, as possibly does the relatively low iteration tolerance for the Poisson solver. Nevertheless the calculations seem to pose an important question.

Suppose that at time $t < t_s$ the singularity in the complex x-plane nearest to the real x-axis is a distance $\alpha(t)$ from that axis. If $\widetilde{u}(k,y,t)$ is the k^{th} term of the Fourier series of u, then as explained in Carrier *et al.* (1983)

$$\widetilde{u}(k,y,t) \propto \exp(-\alpha|k|) \quad \text{as} \quad |k| \to \infty \,. \tag{17}$$

We now *hypothesise* that this exponential decay for wavenumbers on the 'body length-scale' holds for all large wavenumbers up to the $k = \mathcal{O}(Re^{\frac{1}{2}})$ Rayleigh scale. If this *hypothesis*, which needs verifying, is correct then it follows from (17) that there are modes, albeit with exponentially small amplitudes, that might be amplified by a Rayleigh instability.

A Race. Naïvely combining (16) and (17) we argue that the high-wavenumber modes generated by nonlinear interactions can in principle be amplified by the Rayleigh instability according to

$$\widetilde{u}(\lambda Re^{\frac{1}{2}}, y, t) \propto \exp\left(-\lambda Re^{\frac{1}{2}}\left(\alpha - \textstyle\int \beta \, \mathrm{d}t\right)\right) \,. \tag{18}$$

The key point is that (18) suggests that within an order-one time the high-wavenumber modes can grow to be comparable with the basic BLT solution; this is not inconsistent with Brinckman and Walker's (2001) numerical results. Thus there is a 'race' between the growth of the modes amplified by the Rayleigh instability, and the development of a van Dommelen singularity. Whether the instability or the singularity develops first probably depends on particular circumstances.

Comments. Clearly this [very] heuristic argument needs to be placed on a firmer footing by means of an analysis based on, say, asymptotics beyond all orders. Nevertheless it is worth noting that

- no high-wavenumber modes are needed in the boundary or initial conditions — self-induced nonlinear effects seem to be sufficient to fill out the spectrum;
- in order to track the amplification of the short Rayleigh-scale modes there is apparently a need to consider terms that are initially exponentially small;
- there is apparently no hint from the BLT asymptotics that a short-scale disturbance can grow to 'infect' the BLT solution.

Of course we have not proved that the short-scale instabilities observed by Brinckman and Walker (2001) are generated by the above mechanism. Indeed we have not considered convective effects and it is possible that disturbances that begin to grow can be convected into regions where they subsequently decay (although in Brinckman and Walker's (2001) problem this effect may be less important because the existence of the 'rear' stagnation point tends to confine any disturbance). However, we believe that there is a case to answer. In the next section we consider a model problem where we predict a similar effect on the basis of an analogous scaling argument.

3.2. A MODEL PROBLEM

Consider the Kuramoto-Sivashinsky (KS) equation

$$u_t + uu_x = -\varepsilon u_{xx} - \Delta u_{xxxx}\,, \tag{19}$$

with, say, the initial condition $u = \cos x$ at $t = 0$. For the case $\Delta = \varepsilon = 0$, i.e. the kinematic wave equation, there is a known analytic solution, $u_0(x,t)$, that develops a singularity at $t = 1$. From this analytic solution it is straightforward to show (e.g. Sulem *et al.* 1983) that for $t > 0$ and $k \gg 1$,

$$\widetilde{u}_0(k,t) \propto \exp(-\alpha(t)k)\,, \tag{20}$$

where $\widetilde{u}_0(k,t)$ denotes the Fourier series of $u_0(x,t)$, and $\alpha(t) > 0$.

For $\Delta > 0$ and $\varepsilon > 0$ the KS equation is known to be well-posed and have regular solutions for all time. We will study the particular scaling $0 < \Delta \ll \varepsilon \ll 1$ and consider for what times u_0 is the leading-order solution for u in an expansion in powers of Δ and ε. We argue that u_0 is analogous to the BLT solution in that it develops a singularity within a finite time, while if $0 < \Delta \ll \varepsilon \ll 1$ there are rapidly growing high-wavenumber instabilities analogous to Rayleigh instabilities, e.g. when $k^2 = \varepsilon/2\Delta \gg 1$ small amplitude instabilities grow like

$$\text{amplitude} \propto \exp(\varepsilon^2\, t/2\Delta)\,. \tag{21}$$

An order-of-magnitude argument from a comparison of (20) and (21) suggests that short-scale instabilities can grow to be comparable with $u_0(x,t)$ when $t \sim \Delta^{\frac{1}{2}}\varepsilon^{-\frac{3}{2}}$. Hence if $\Delta \ll \varepsilon^3$ short-scale instabilities should develop before the singularity forms at $t = 1$.

KS Equation: Numerical Solutions. As a preliminary confirmation of the above prediction we have solved the KS equation numerically for $\varepsilon = 10^{-1}$, $\Delta = 10^{-5}$, and with $u = \cos x$ at $t = 0$. The initial condition, and numerical solution at $t = 0.46$, are shown in figure 2; the development of a short-scale instability well before the time at which a singularity develops in u_0 is evident.

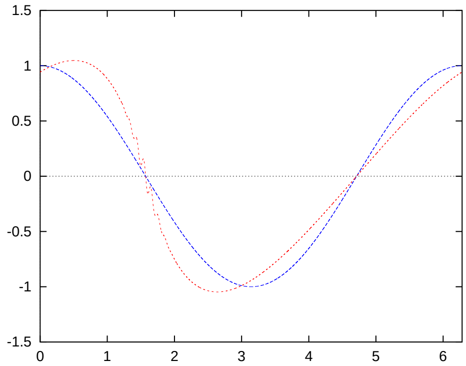

Figure 2 Numerical solution with $\varepsilon = 10^{-1}$, $\Delta = 10^{-5}$ and $u = \cos x$ at $t = 0$. $- \cdot -$: solution at $t = 0$, $- - -$: solution at $t = 0.46$.

Comments On The KS Problem. Our heuristic arguments are apparently supported by the numerical experiments. Further, the use of a single-mode initial condition emphasises the fact that the 'exponentially small' higher modes are generated by nonlinear interactions. In addition, preliminary analysis suggests that the crucial need to consider 'exponentially small' terms is not hinted at by solving for higher-order terms of a regular perturbation expansion in powers of Δ and ε (i.e. regular perturbation theory fails, and fails spectacularly).

While there is clearly a need to improve the analysis (and work is underway with that in mind), it seems that there is an *a priori* case for believing that a mechanism for the growth of short-scale disturbances has been identified in the KS model, and that these short-scale disturbances can alter the leading-order solution by an order-one amount. A similar, although not identical, change to the leading-order solution caused by the growth of exponentially small terms has been reported in a Saffman-Taylor Hele-Shaw problem by Siegel, Tanveer and Dai (1996). As in the Hele-Shaw problem it may be possible to place our analysis on a firmer footing by analytically continuing into the complex x-plane.

Of course the KS model does not contain all the dynamics of BLT theory, e.g. it does not include spatial regions of both growth and decay of the instability. However we believe that KS model does include key aspects of the mathematics that are similar to those in the BLT problem.

4. ASYMPTOTIC INSTABILITY THEORY

4.1. INTRODUCTION

We have seen that laminar $Re \gg 1$ asymptotic analysis is problematic in that it is sometimes successful (e.g. Blasius solution at $R \approx 500$), and other times not (e.g. unsteady flow past a circular cylinder at times past singularity formation). We have also noted two 'features' of BLT.

First, the leading-order problem can lead to a succession of singularities forcing consideration of extremely short time scales with the result that it is impossible to obtain solutions an order-one time after the first appearance of a singularity. Second, it appears possible that terms that are initially exponentially small can grow to alter the leading-order solution; moreover, as yet there seems to be no way of identifying whether or not this will occur by means of predictive asymptotic analysis.

There is thus a tendency for short-scale phenomena to occur naturally in BLT. As a result, a main strand of research that has developed in BLT over the last thirty years has been the study of instabilities and transition to turbulence in shear layers.

4.2. ALMOST PARALLEL FLOWS

The stability of thin shear layers has been studied for well over a century. Thin almost-parallel shear layers have often been idealised as exactly parallel so that the underlying flow is given by

$$\mathbf{u} = (U(y), 0, 0) \equiv \mathbf{U}\,. \tag{22}$$

A linear stability analysis of such a flow is then performed based on normal mode perturbations of the form

$$\mathbf{u} = \mathbf{U} + \epsilon\widetilde{\mathbf{u}}(y)\exp(\mathrm{i}\alpha x + \mathrm{i}\beta z - \mathrm{i}\alpha c t) + \dots\,, \tag{23}$$

where α, β are here the wavenumbers in the x and z directions respectively, and c is the phase-speed. Substitution into the linearised NS equations and solution of the resulting Orr-Sommerfeld (OS) equation yields a Re-dependent dispersion relation relating α, β and c:

$$F(\alpha, \beta, c; Re) = 0\,. \tag{24}$$

In the case of linear two-dimensional Tollmien-Schlichting (TS) waves on a flat plate, the predictions of OS theory are in very good agreement with experiment (e.g. Klingmann *et al.* 1993). However, good agreement is not invariably obtained, and in the case of Görtler rolls and cross-flow instability, OS theory can yield misleading results.

A drawback of OS theory, and possibly the reason that it does not always work, is that the theory is mathematically inconsistent. On the one hand, for the basic shear layer flow to be 'almost' parallel then, formally, it is necessary to assume that the Reynolds number is asymptotically large, i.e. $Re \gg 1$. On the other hand, for the derivation of the linearised OS equation and the resulting dispersion relation (24), it is necessary to assume that the Reynolds number is, formally, an order-one quantity, i.e. $Re = \mathcal{O}(1)$.

We emphasise that the distinction between 'asymptotically large' and 'order one' does not depend on whether or not the Reynolds number is numerically large. The distinction concerns the approximations made in the analysis. OS theory tries to have its cake (i.e. by approximating the basic flow with the *asymptotic* Blasius solution), and eat it (i.e by taking the Reynolds number to be *order one* in the OS equation). Moreover, it was not until there was a proper appreciation of the incompatibility of the two different treatments of the Reynolds number in OS theory that it became clear how to deal with non-parallel and/or nonlinear effects in a consistent manner.

4.3. ASYMPTOTIC LINEAR THEORY

An alternative to OS theory is to assume consistently that $Re \gg 1$. The drawback of this approach is that almost all flows first become unstable at moderate Reynolds numbers where it is not clear *a priori* that results derived on the basis of an asymptotically large Reynolds number will hold. Moreover it is difficult, if not impossible, to study the fastest-growing disturbances with this approach. Nevertheless, *sometimes* these difficulties are not show stoppers.

Triple-Deck Theory (TDT). The most significant advance in BLT after Prandtl's original formulation was the simultaneous discovery of TDT by Messiter (1970), Neiland (1969) and Stewartson (1969). This theory applies to disturbances that change 'rapidly' in the downstream direction, that is on a length scale short compared with that over which the underlying boundary layer varies, though still long compared with the boundary-layer thickness. This relatively rapid change means that viscous effects associated with the disturbances are confined to a thin sublayer close to the wall (the 'lower deck'), while the bulk of the underlying boundary layer adjusts through an inviscid, rotational displacement (the 'middle deck'). The fluid ejected from the middle deck induces a flow in an 'upper deck' above the boundary layer that is inviscid and irrotational. In turn this irrotational flow induces a dynamically significant pressure gradient in the lower deck. There is thus a feedback loop whereby fluid motion in the lower deck can change the pressure gradient felt in the lower deck (albeit indirectly by means of the flow generated in the upper deck). In contrast, in classical BLT the pressure gradient is fixed by the slip velocity and is not influenced by induced motions in the boundary layer. Finally we note that while at leading order $p_y = 0$ in the lower and middle decks, at leading order $p_y \neq 0$ in the upper deck.

Linear TS Waves. Whilst TDT was first formulated in terms of formal asymptotic expansions in the late 60s, the key ideas can be found in the linearised analysis of shock/boundary-layer interactions (Lighthill 1953) and lower-branch TS waves (Tollmien 1929, Lin 1945). However it was Smith (1979) who realised that lower-branch TS could be placed in the TDT framework, and so make it possible to study laminar-turbulent transition using a large-Reynolds-number asymptotic approach.

For the TS lower branch Smith (1979) expanded the dynamical variables in powers of $Re^{-\frac{1}{8}}$, and used multiple-scales in x and matched asymptotic expansions in y. His asymptotic expansion for the neutral curve for flow over a flat plate is in reasonably good agreement with experiments at moderate Reynolds number. However, the equivalent asymptotic expansion for the TS neutral upper branch only provides a good approximation to the neutral curve at Reynolds numbers when the flow would, in practice, be fully turbulent (Healey 1995).

4.4. ASYMPTOTIC NONLINEAR THEORY

A major advantage of the $Re \gg 1$ asymptotic approach is that there is a consistent way to examine nonlinear effects. A disadvantage is that there is a plethora of possible scalings, and it is difficult to identify *a priori* which, if any, will give good agreement with experiment. For instance there are different types of modes (e.g. TS, Rayleigh, Klebanoff) and different sources of instability (e.g. 2D/3D, localised/global, linear/nonlinear, noisy/controlled disturbances that are internal/external to the shear layer). There are also different types of analysis. For instance there have been many studies of uniform or modulated wavetrains of almost neutral linear modes, but there are other possibilities such as studies of wavetrains of almost neutral nonlinear modes or studies of algebraically growing modes (e.g. Klebanoff modes). There have also been numerical studies of modes with order-one growth rates. See the review by Cowley and Wu (1994) for other examples.

Particularly in the case of studies of wavetrains of almost neutral modes, two key ideas reoccur, namely wave/mean-flow interactions and critical-layer effects.

Wave/Mean-Flow Interactions. Such effects arise many circumstances. An archetypal example occurs if there are two 'carrier' modes propagating at equal and opposite directions to the mean flow, e.g.

$$\mathrm{e}^{\mathrm{i}\alpha x}\mathrm{e}^{\mathrm{i}\beta z}\mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.} \quad \text{and} \quad \mathrm{e}^{\mathrm{i}\alpha x}\mathrm{e}^{-\mathrm{i}\beta z}\mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.}\,, \tag{25}$$

where c.c. denotes complex conjugate. Nonlinear interactions between such modes through the quadratic terms in the NS equations, generate a steady mean flow:

$$\underset{\text{mode 1}}{\left(\mathrm{e}^{\mathrm{i}\alpha x}\mathrm{e}^{\mathrm{i}\beta z}\mathrm{e}^{-\mathrm{i}\omega t}\right)}\;\underset{\text{c.c. of mode 2}}{\left(\mathrm{e}^{-\mathrm{i}\alpha x}\mathrm{e}^{\mathrm{i}\beta z}\mathrm{e}^{\mathrm{i}\omega t}\right)} = \mathrm{e}^{2\mathrm{i}\beta z}\;. \tag{26}$$

If the spanwise scale, β^{-1}, and 'slow' streamwise scale, $\mathcal{L}$, are disjoint, i.e. $\beta\mathcal{L} \ll 1$, then modest spanwise motions generate large streamwise mean flows, as a scaling argument based on the continuity equation shows:

$$\left.\begin{array}{llll} u_x & +\, v_y + & w_z & = 0 \\ \frac{\mathcal{U}_m}{\mathcal{L}} & \sim & \beta\mathcal{W}_m & \end{array}\right\} \quad \Rightarrow \quad \mathcal{U}_m \sim \beta\mathcal{L}\mathcal{W}_m \gg \mathcal{W}_m\,. \tag{27}$$

This mechanism is one reason why relatively strong longitudinal vortices are observed in transitional (and fully turbulent) boundary-layers (e.g. Jang *et al.* 1986, Hall and Smith 1989).

Critical Layers. Often in a weakly nonlinear perturbation analysis the $u_t + uu_x$ part of the NS equations reduces at leading-order to $\mathrm{i}\alpha(U-c)\widetilde{u}$, where U is the underlying mean flow and $\widetilde{u}$ is the perturbation velocity. If $U = c$ at $y = y_c$, then y_c is said to be a critical level. Critical levels are important since linear inviscid solutions almost always have singularities there, e.g. for 3D disturbances $\widetilde{u} \propto (y - y_c)^{-1}$.

This singularity is smoothed out by one or more effects (e.g. viscosity, unsteadiness, nonlinearity) in a thin 'critical layer' surrounding $y = y_c$. Critical layers tend to be dynamically important since nonlinear effects are largest within them. For this reason many analytic studies have focused on 'phase-locked' nonlinear interactions, that is, interactions among modes with the same phase speed c, since nonlinear interactions are strongest when the critical layers coincide (e.g. Goldstein 1995).

TS Resonant-Triad Instability. One of the more intriguing aspects of laminar-turbulent transition is the appearance of subharmonics (e.g. Knapp and Roache 1968, Kachanov *et al.* 1977). Craik (1971) proposed a weakly nonlinear theory involving a *phase-locked resonant-triad* interaction to explain these observations (see also Raetz 1959), while Herbert (1988) used a Floquet approach to demonstrate secondary instability of approximate TS wave solutions to subharmonic (and other) perturbations. Although these analyses identified key aspects of the physics, the approaches were somewhat heuristic. An asymptotic description of the resonant-triad mechanism, including a qualitative explanation of the observed super-exponential growth, was eventually given by Goldstein and Lee (1992) and Mankbadi *et al.* (1993). A central feature of their analyses was nonlinear interactions within critical layers. An important revelation of the asymptotic approach was that wave/mean-flow interactions can be as important as the resonant-triad interaction.

As with many other asymptotic analyses the first nonlinear scaling in the resonant-triad problem predicts that the time/length scales of the modulation amplitude rapidly shorten. Consequently, in order to follow the evolution of the flow, it is necessary to consider a succession of asymptotic problems with shorter and shorter time/length scales. As a result it is not possible, at present, to obtain an asymptotic description at times much beyond that at which the first nonlinear interaction takes place (cf. unsteady separation). Moreover, while the asymptotic theory is in qualitative agreement with experiment, quantitative agreement has yet to be achieved (at least for a *correct* asymptotic theory).

Further, note that it is the time/length scales of the *modulation amplitude* that shorten. While such an occurrence may be a prelude to transition, the time and/or length scales are still longer than the period and/or wavelength of the carrier wave[s]. Singularity formation, or similar, in the modulational amplitude does not necessarily imply the development of fine-scale structure on the length of an instability wave.

Receptivity: How A Disturbance Penetrates A Shear Layer. Another success for asymptotic theory has been an explanation of how sound waves can interact with a 'rivet' protruding from an otherwise smooth surface, and so generate TS waves (Ruban 1984, Goldstein 1985). A key qualitative observation is that the rivet length should match the triple-deck lengthscale. A related analysis for TS wave generation by a curvature discontinuity on a surface shows quantitative agreement with experiment (Goldstein and Hultgren 1987).

Wu (1999) has also explained how sound waves and a vorticity or entropy gust can interact in the upper deck to generate TS waves. While at first sight the required asymptotic scaling between the length and time scales of the sound wave and gust appears to rule out general applicability, Wu (1999) shows how the analysis can be applied to broad-band spectra.

Other Successes. Other than receptivity, there are relatively few examples where asymptotic theory has obtained good *quantitative* agreement with experiment. Hultgren's (1992) theoretical explanation of the 2D nonlinear roll-up of a shear layer is one notable exception, while the asymptotic description of Görtler instability is another (e.g. Hall 1990).

5. CONCLUSIONS

An undoubted strength and success of BLT is its ability to explain, qualitatively, fundamental concepts such as separation, nonlinear instability and receptivity. However, there are surprisingly few *reliable* calculations where good quantitative agreement has been obtained between asymptotic theory and experiment. Some of the best examples with good quantitative agreement have been mentioned above. There are also reports that the Russian space shuttle Buran was designed using large-Reynolds-number hypersonic theory. Unfortunately that work is not for the most part available in the open literature (although, assuming that the work is available to some western agencies, some credence to the claim might be gleaned from NASA's interest in asymptotic theory in the mid-80s).

A drawback of nonlinear large-Reynolds-number asymptotic instability theory is that the analysis can become complicated, e.g. a resonant-triad interaction of TS waves requires a 'septuple-deck' structure (Mankbadi *et al.* 1993). As a result it is arguable that the payoff does not always justify the effort. Further, there are a number of examples where the technical difficulties of the analysis have lead to erroneous results — some of which have nevertheless agreed with experiment! For instance see the discussions in Wu *et al.* (1996) and Moston *et al.* (2000).

We also recall the pervasiveness of singularities in BLT. Often the development of a singularity indicates an important physical feature, e.g. unsteady separation or the onset of short time/length scales in laminar-turbulent transition. However, after the formation of an initial singularity a succession of problems with increasingly short time/length scales can result, making it difficult to obtain an asymptotic description for order-one times beyond the formation of the initial singularity. Moreover we have also seen that in order to obtain the correct asymptotic solution, it maybe necessary to include the effects of terms that are initially exponentially small using a 'beyond-all-orders' asymptotic analysis.

In the light of these comments we return to the question of the title.

Laminar BLT: A Paradox. Laminar BLT *is* a paradox, in that it is based on the assumption that $Re \gg 1$, whereas almost all flows are turbulent if $Re \gg 1$. As a result, in order to obtain laminar solutions it is necessary to suppress instabilities. Sometimes this is possible (e.g. the Blasius flat-plate solution and receptivity), but other times it is not (e.g. for a 'long-time' description of unsteady separation).

Laminar BLT: A 20th Century Paradox? But is BLT a *20th century* paradox? On the one hand one might argue that the answer to this question is *no*, since BLT is still good for explaining fundamental mechanisms and obtaining scalings. On the other hand one might argue that the answer is *yes*, since for quantitative agreement with experiment BLT will be outgunned by computational fluid dynamics (CFD) in the 21^{st} century.

I argue that the answer is *yes*. With the rise of modern computers and codes, good engineering answers for laminar flows can be obtained with CFD for the Reynolds numbers when asymptotic theory might be applicable. This is not to say that BLT does not have a rôle in explaining fundamental mechanisms, but many, if not all, of the fundamental questions in BLT have now been answered.

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