Chapter 3

Orbits

3.1 Motion in a plane

You might wonder why there is a section on motion in two dimensions when vector methods can be readily used to study motion in three dimensions — or an arbitrary number of dimensions. The answer is that in the case of a particle moving under the influence of a central force (such as the gravitational field of a massive uniform spherical body or the electric field of a point charge), the motion takes place in a plane. We will prove that in section 3.2.

For a particle moving in a central force, the obvious coordinates to use are plane polar coordinates, with the origin at the centre of the force: for example, at the Sun (regarded as a point) in the case of planetary orbits. Before studying such orbits, we need some preliminary results.

3.1.1 Angular variables

It is useful, before we move to more general motion in the plane, to review the motion of a particle on a circle. In this simple situation, we meet some concepts that will defined more formally later in this section and in later chapters.

For a particle moving in a circle of (constant) radius $a$, the usual kinematic variables — distance, speed, acceleration and momentum — are not very convenient. Instead, the position of the particle on the circle can easily be determined in terms of the obvious angular variable, $\theta$. Similarly, the speed is $a|\dot{\theta}|$ which is determined in terms of $\dot{\theta}$, the angular velocity$^1$. The angular component of acceleration is $a\ddot{\theta}$, which is determined by $\dot{\theta}$, the angular acceleration.

We can write the kinetic energy in the form

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}ma^2\dot{\theta}^2 \equiv \frac{1}{2}I\dot{\theta}^2$$

where we have defined a quantity $I$, the moment of inertia of the particle, by

$$I = ma^2.$$ 

When angular velocity is used instead of (linear) velocity, the usual formula for kinetic energy holds provided mass is replaced by moment of inertia.$^2$ We will make extensive use of the idea of moment of inertia when we study the motion of rigid bodies in Chapter 5.

Instead of the (linear) momentum of the particle, we introduce a new quantity, the angular momentum, denoted by $H$, using the angular expressions corresponding to mass $\times$ velocity:

$$H = I\dot{\theta} \equiv ma^2\dot{\theta}. \quad (3.1)$$

Finally, suppose that there is a tangential force of magnitude $F$ acting on the particle. Newton’s second law in this situation is

$$F = m(a\ddot{\theta})$$

which we can write in the form

$$G \equiv aF = ma^2\ddot{\theta} \equiv I\ddot{\theta}.$$

$^1$Velocity rather than speed because it can be positive or negative. We are here considering essentially one-dimensional motion round a circle; in the next sections, we will investigate the velocity vector in two dimensions.

$^2$Moment of inertia could be thought of as ‘angular mass’, though this would sound a little odd.
so the angular version of Newton’s second law has $G$, or $aF$, on the left-hand side, which is called the moment of the force, or torque.

The table below summarises the correspondence.

<table>
<thead>
<tr>
<th>Motion on a straight line</th>
<th>Motion on a circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Displacement $x$</td>
<td>Angular displacement $\theta$</td>
</tr>
<tr>
<td>Velocity $\dot{x} = v$</td>
<td>Angular velocity $\dot{\theta} = \omega$</td>
</tr>
<tr>
<td>Acceleration $\ddot{x} = \ddot{v}$</td>
<td>Angular acceleration $\ddot{\theta} = \ddot{\omega}$</td>
</tr>
<tr>
<td>Mass $m$</td>
<td>Moment of inertia $I = ma^2$</td>
</tr>
<tr>
<td>Momentum $m\dot{x} = mv$</td>
<td>Angular momentum $I\dot{\theta} = I\omega$</td>
</tr>
<tr>
<td>Kinetic energy $\frac{1}{2}mv^2$</td>
<td>(Rotational) kinetic energy $\frac{1}{2}I\omega^2$</td>
</tr>
<tr>
<td>Newton’s second law $F = m\ddot{x}$</td>
<td>Newton’s second law $G = I\ddot{\theta}$</td>
</tr>
</tbody>
</table>

The idea of angular variables need not be confined to simple motion in a circle, as we shall see as this course progresses.

3.1.2 Acceleration: coordinate independent treatment

We will calculate the acceleration of a particle whose motion is confined to a plane in two ways: in this section, coordinate independently, using the tangent and normal vectors to the trajectory; and in the next section using plane polar coordinates.

Let $\mathbf{r}$ be the position vector of the particle at time $t$, with respect to a fixed origin, and let $\mathbf{t}$ be the unit tangent vector defined by

$$\mathbf{t} = \frac{d\mathbf{r}}{ds},$$

where $s$ is arc-length.

The velocity vector $\mathbf{v}$ is parallel to $\mathbf{t}$:

$$\mathbf{v} = \dot{s} \frac{d\mathbf{r}}{ds} \equiv \dot{s} \mathbf{t},$$

where $v = \dot{s}$ and we used the chain rule

$$\frac{d}{dt} = \dot{s} \frac{d}{ds}$$

for the first equality. We define the arc-length $s$ to increase in the direction that the curve is traversed so that $v \geq 0$.

Differentiating equation (3.3) gives

$$\ddot{\mathbf{r}} = \ddot{s} \mathbf{t} + v^2 \frac{d\mathbf{t}}{ds},$$

using the chain rule again. Now $\mathbf{t} \cdot \mathbf{t} = 1$ implies that

$$2\frac{d\mathbf{t}}{ds} \cdot \mathbf{t} = 0,$$

i.e., $\frac{d\mathbf{t}}{ds}$ is orthogonal to $\mathbf{t}$. Thus there is a scalar $\rho$ such that

$$\frac{d\mathbf{t}}{ds} = -\rho \mathbf{n},$$

or

$$\frac{d^2\mathbf{r}}{ds^2} = \rho \mathbf{n}.$$

3 The discussion above is a special case of a much grander scheme, which is presented in the Part II course Classical Dynamics. The starting point is a set of generalised coordinates $q_i$ (here, just $\theta$) and a function called the Lagrangian $L$ (here, just the kinetic energy), which is expressed in terms of $q_i$ and the generalised velocities $\dot{q}_i$ (here, just $\dot{\theta}$). The generalised momenta $p_i$, (here, the angular momentum) are defined by $\partial L/\partial \dot{q}_i$ and Newton’s laws translate to $\partial L/\partial q_i = -\rho_i$, the left-hand side corresponding to a generalised force expressed as the gradient of a generalised potential. It is all very neat.

4 There is an unfortunate duplication here in the use of the letter $t$; I will not mention time $t$ again to prevent confusion.
3.1. MOTION IN A PLANE

where \( \mathbf{n} \) is the normal to the curve and its direction is chosen to ensure that \( \rho \geq 0 \). (Recall that we are in two dimensions, so this last statement is unambiguous.)

Thus the acceleration of the particle \( (3.4) \) can be written in the form

\[
\mathbf{\ddot{r}} = \mathbf{v} \mathbf{t} + \frac{v^2}{\rho} \mathbf{n}.
\]  

(3.6)

The first term is the rate of change of speed, which is the component of acceleration along the curve; the second term is the acceleration in the normal direction. We know that the acceleration of a particle moving in a circle of radius \( R \) is \( v^2/R \), so we can identify \( \rho \) as the instantaneous radius of curvature of the trajectory.

You will have seen derivations similar to this in connection with the Serret-Frenet equations in Vector Calculus.

To understand this better, we consider points \( \mathbf{r}(s) \) on the curve close to any given point \( \mathbf{r}(0) \), so that the distance \( s \) along the curve is small. The Taylor series for \( \mathbf{r}(s) \) is

\[
\mathbf{r}(s) = \mathbf{r}(0) + s \frac{d\mathbf{r}}{ds} + \frac{1}{2} s^2 \frac{d^2\mathbf{r}}{ds^2} + O(s^3)
\]

= \( \mathbf{r}(0) + st + \frac{1}{2} s^2 \mathbf{n}/\rho + + O(s^3) \) (using the definition \( (3.5) \))

where the derivatives, and \( \mathbf{t} \), \( \mathbf{n} \) and \( \rho \), are evaluated at \( s = 0 \). This shows that to first order in \( s \), the curve can be approximated by a straight line \( \mathbf{r}(s) = \mathbf{r}(0) + st \), which is what we were expecting.

It is not so obvious from this equation that, to second order in \( s \), the curve can be approximated by a circle. However, considering \( |\mathbf{r}(s) - \mathbf{r}(0) - \rho \mathbf{n}|^2 \) shows that to this approximation (i.e. ignoring \( s^3 \) and higher order terms) \( \mathbf{r}(s) \) describes a circle of radius \( \rho \) with centre at \( \mathbf{r}(0) - \rho \mathbf{n} \):

\[
|\mathbf{r}(s) - \mathbf{r}(0) - \rho \mathbf{n}|^2 = |st - \frac{1}{2} s^2 \mathbf{n}/\rho - \rho \mathbf{n}|^2
\]

= \( (st - \frac{1}{2} s^2 \mathbf{n}/\rho - \rho \mathbf{n}) \cdot (st - \frac{1}{2} s^2 \mathbf{n}/\rho - \rho \mathbf{n}) \)

= \( \rho^2 + O(s^3) \), (recall that \( \mathbf{n} \cdot \mathbf{t} = 0 \))

as required.

3.1.3 Example: car on bridge

We investigate the possibility of a small car taking off (leaving the ground) as it goes over a bridge. The equation of motion, using the expression \( (3.4) \) for the acceleration, is

\[
\mathbf{F} + \mathbf{R} + mg = mv \mathbf{t} + m \frac{v^2}{\rho} \mathbf{n}.
\]

(force = mass times acceleration)

The forces on the left-hand side are as follows. \( \mathbf{F} \) is the force of friction between the road and the car tyres; it is what pushes the car along and it is in the direction \( \mathbf{t} \). It is equal to the driving force provided by the car engine minus various losses due to friction of bearings and other moving parts. \( \mathbf{R} \) is the normal reaction of the road on the car wheels; it is in the direction \( \mathbf{n} \). Finally, \( g \) is, as always, the acceleration due to gravity, which is vertical. We have ignored air resistance. On the right-hand side, \( v \) is the speed of the car and \( \rho \) is the radius of curvature of the bridge.

The car will take off if the gravitational force is not sufficient to provide the acceleration required for the car to follow the curve of the bridge. We therefore look only at the normal component of the equation of motion:

\[
mg \cos \theta - R = m \frac{v^2}{\rho}.
\]

The normal reaction, of magnitude \( R \), is the difference between the component of the gravitational force in the downwards normal direction and the normal acceleration required (times the mass), and when this vanishes the car is on the point of taking off. The maximum speed \( v_{\text{max}} \) is therefore given by

\[
v_{\text{max}} = \sqrt{g\rho \cos \theta}.
\]

The maximum speed depends on the shape of the bridge. At points where the radius of curvature is infinite, which means that the bridge is not curved at all, the car can go as fast as it likes. The most dangerous points are where the bridge is steep (\( \cos \theta \) is small) and highly curved (\( \rho \) is small).

Putting in some typical figures: \( \rho = 40 \) metres, \( \cos \theta = 1 \) and \( g = 10 \) metres/sec/sec gives \( v_{\text{max}} = 20 \) metres/sec which is about 45 miles per hour.

End of example
3.1.4 Acceleration in polar coordinates

The previous calculation of acceleration used axes tied to the trajectory of the particle: namely the
tangent and normal. Instead we use plane polar coordinates and axes. The axes still depend on
the position of the particle (unlike Cartesian axes), as shown in the figure, but not on the direction
of the trajectory.

Let \( \hat{\mathbf{r}} \) and \( \hat{\theta} \) be unit vectors in the directions of \( r \) and \( \theta \) increasing, respectively. In Cartesian
axes, \( \hat{\mathbf{r}} = (\cos \theta, \sin \theta) \), \( \hat{\theta} = (-\sin \theta, \cos \theta) \)
so \( \frac{\partial \hat{\mathbf{r}}}{\partial \theta} = \hat{\theta} \), \( \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r} \). and \( \frac{\partial \hat{\mathbf{r}}}{\partial r} = \frac{\partial \hat{\theta}}{\partial r} = 0 \).

Starting from \( \mathbf{r} = r \hat{\mathbf{r}} \), we find that the velocity can be written in the form
\[
\dot{\mathbf{r}} = \frac{d(\mathbf{r})}{dt} = \dot{r} \hat{\mathbf{r}} + r \frac{d\hat{\mathbf{r}}}{dt} = \dot{r} \hat{\mathbf{r}} + r \frac{d\hat{\theta}}{dt} \frac{d\hat{\mathbf{r}}}{d\theta} = \dot{r} \hat{\mathbf{r}} + r \dot{\theta} \hat{\theta}. \quad (3.7)
\]
Differentiating again gives the acceleration:
\[
\ddot{\mathbf{r}} = (\ddot{r} \hat{\mathbf{r}} + \dot{r} \dot{\theta} \hat{\theta}) + (\dot{r} \dot{\theta} \hat{\theta} + r \ddot{\theta} \hat{\theta} - r \dot{\theta}^2 \hat{\mathbf{r}})
= (\ddot{r} - r \dot{\theta}^2) \hat{\mathbf{r}} + (2 \dot{r} \dot{\theta} + r \ddot{\theta}) \hat{\theta} \quad (3.8)
= (\ddot{r} - r \dot{\theta}^2) \hat{\mathbf{r}} + \frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) \hat{\theta}; \quad (3.9)
\]
which is the required (and very important) result.

Note that both the radial and the angular component of acceleration in the expression (3.8)
consist of two terms: the ‘obvious’ one and another that depends on first derivatives of the coor-
dinates. Later, we will identify these extra terms as corresponding to the centrifugal and coriolis
accelerations in a rotating frame. In the second expression (3.9), the quantity in the derivative will
later be identified as the angular momentum (per unit mass).

3.2 Angular momentum

3.2.1 Definition

We can extend the idea of angular momentum discussed in section 3.1.1 to the case of an arbitrarily
moving particle. We still need a fixed point corresponding to the centre of the circle in circular
motion. This point can be chosen arbitrarily, but the angular momentum will depend upon the
choice.

For a particle of mass \( m \) at position \( \mathbf{r} \) (relative to a given though irrelevant origin) moving
with velocity \( \dot{\mathbf{r}} \), we define the angular momentum \( \mathbf{H} \) about the fixed point \( a \) by
\[
\mathbf{H} = (\mathbf{r} - a) \times (m\dot{\mathbf{r}}). \quad (3.10)
\]
Taking lengths of both sides gives
\[
|\mathbf{H}| = mvd
\]
where \( d \) is the shortest distance between the line of the momentum and the fixed point \( a \) (see
diagram).\(^5\) In general, given a vector quantity, the cross product with the position vector relative
to \( a \) is called the moment of the quantity about \( a \). For example, the moment about the point \( a \) of
a force \( \mathbf{F} \) acting at a point \( \mathbf{r} \) is
\[
(\mathbf{r} - a) \times \mathbf{F},
\]
so angular momentum might equally be called moment of momentum.

\(^5\)Note to self: need diagram.
### 3.2.2 Conservation of angular momentum

We now derive an important conservation law for angular momentum, corresponding to the conservation of linear momentum in directions orthogonal to the applied force. We have

\[
\frac{dH}{dt} = m \dot{\mathbf{r}} \times \ddot{\mathbf{r}} + m (\mathbf{r} - \mathbf{a}) \times \ddot{\mathbf{r}}
\]

(differentiating (3.10) using the product rule)

\[
= m (\mathbf{r} - \mathbf{a}) \times \ddot{\mathbf{r}}
\]

\[
= (\mathbf{r} - \mathbf{a}) \times \mathbf{F}
\]

(Neutral’s second law)

\[
\equiv G
\]

where the last equation defines the quantity \( G \), which is the moment of the force, or torque, acting on the particle.

For a central force, \( \mathbf{F} \) is parallel to \( \mathbf{r} \) (see section 2.1.6) so the moment of the force about the origin is zero. This the angular momentum about the centre of force is constant: taking \( \mathbf{a} = \mathbf{0} \) in the definition of angular momentum, we have

\[
G = \frac{dH}{dt} = 0.
\]

This will turn out to be an important result when we come to consider orbits in a central force.

In general (whether \( H \) is constant or not), it follows from the definition (3.10) that

\[
(\mathbf{r} - \mathbf{a}) \cdot \mathbf{H} = 0
\]

i.e.

\[
\mathbf{r} \cdot \mathbf{H} = \mathbf{a} \cdot \mathbf{H}.
\]

In the case when \( \mathbf{H} \) is a constant vector\(^6\) this equation describes a plane with normal \( \mathbf{H} \) containing the point \( \mathbf{a} \), so in this case the motion of the particle lies entirely in this plane. Using the results of section 3.1.4, for motion in a plane, and taking \( \mathbf{a} = \mathbf{0} \) for convenience, we have

\[
\mathbf{H} = m \mathbf{r} \times \dot{\mathbf{r}}
\]

\[
= m (\mathbf{r} \ddot{\mathbf{r}}) \times (\ddot{\mathbf{r}} + \dot{r} \dot{\mathbf{\hat{r}}})
\]

(\( \ddot{\mathbf{r}} = \dot{\mathbf{\hat{r}}} \))

\[
= m r^2 \dot{\mathbf{\hat{r}}} \times \dot{\mathbf{\hat{r}}}
\]

\[
= m r^2 \dot{\mathbf{\hat{r}}} \mathbf{\hat{z}}
\]

where \( \mathbf{\hat{z}} \) is the unit normal to the plane of motion. Note that the magnitude of this quantity is exactly the angular momentum of circular motion discussed in section 3.1.1.

### 3.3 Orbits in a central force

#### 3.3.1 Equations of motion

We recall that, by definition, a central force \( \mathbf{F}(\mathbf{r}) \) can be written in the form

\[
\mathbf{F}(\mathbf{r}) = f(r) \mathbf{\hat{r}}.
\]

(3.11)

The angular momentum, \( \mathbf{H} \), of a particle about the centre of force, which is here the origin, is given by

\[
\mathbf{H} = m \mathbf{r} \times \dot{\mathbf{r}}
\]

and is constant for the force (3.11), as was shown in the previous section. The motion takes place in the plane given by \( \mathbf{H} \cdot \mathbf{r} = 0 \), i.e. in the plane spanned by \( \mathbf{r} \) and \( \dot{\mathbf{r}} \), which is intuitively obvious: this is the plane in which the force acts, so there is no component of acceleration taking the particle out of the plane.

The equation of motion in plane polar coordinates and axes (3.9) of a particle moving in the force field (3.11) is

\[
(\ddot{r} - r \dot{\theta}^2) \mathbf{\hat{r}} + \frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) \mathbf{\hat{\theta}} = \frac{1}{m} f(r) \mathbf{\hat{r}}.
\]

\(^6\)Note that if \( \mathbf{H}_1 \) is the angular momentum of a particle about \( \mathbf{a}_1 \) and \( \mathbf{H}_2 \) is the angular momentum of the particle about \( \mathbf{a}_2 \), then \( \mathbf{H}_1 - \mathbf{H}_2 = (\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{p} \), where \( \mathbf{p} \) is the linear momentum of the particle. Therefore, the angular momentum about one point being constant does not imply that the angular momentum about another point is constant.
Taking components gives two equations for the two unknowns $\theta(t)$ and $r(t)$:

\[
\ddot{r} - r\dot{\theta}^2 = \frac{1}{m} f(r),
\]

\[
d(\dot{r^2}\dot{\theta}) = 0.
\]

These are the equations of motion of a particle subject to a central force. There are two main approaches to integrating these equations: the straightforward approach is to eliminate $\dot{\theta}$ to give a second-order non-linear differential equation for $r$ with $t$ as the independent variable; the alternative approach is to change variable to obtain a linear differential equation with $\theta$ as the independent variable. These two approaches are described in the next two sections.

In both cases, we start by integrating (3.13):

\[
r^2\dot{\theta} = h
\]

where $h$ is a constant, the angular momentum per unit mass. Now we use this equation to eliminate $\dot{\theta}$ from equation (3.12)

\[
\ddot{r} - \frac{h^2}{r^3} = \frac{1}{m} f(r).
\]

### 3.3.2 The $r$-$t$ orbital equation

Equation (3.15) is seemingly simple, but the non-linear $r^{-3}$ term makes it pretty intractable, as a second order differential equation, even without the force term on the right-hand side. However, it is a useful equation, and we can make some progress by obtaining a first integral:

\[
\frac{1}{2}\dot{r}^2 + \frac{h^2}{2r^2} = \frac{1}{m} \int^r f(r')dr' \equiv -\Phi(r) + A,
\]

where $\Phi(r)$ is the potential for $F(r)/m$ (see section 2.1.6) and $A$ is constant of integration. (Just differentiate equation (3.16) and compare with equation (3.15) to verify that it is correct.)

Rearranging the first integral (3.16) slightly gives

\[
\frac{1}{2}\dot{r}^2 + \frac{h^2}{2r^2} + \Phi(r) = A,
\]

which looks exactly like the usual equation of conservation of energy for a particle (of unit mass) moving in one-dimension, with the quantity

\[
\frac{h^2}{2r^2} + \Phi(r)
\]

acting as the potential. The first term in equation (3.17) is called the ‘centrifugal barrier’ potential and relates to the kinetic energy required to maintain, when the particle is at radius $r$, the constant angular momentum $h$. The quantity (3.17) is called the effective potential.

In the case of the gravitational field of a point particle, the effective potential is

\[
\frac{h^2}{2r^2} - \frac{GM}{r}
\]

and we can determine the motion of the particle qualitatively as in the example in section 2.1.4.

We can in principle integrate equation (3.17) again, starting with

\[
\dot{r} = \pm \sqrt{2A - 2\Phi(r) - h^2/r^2},
\]

the choice of sign ($\pm$) being determined by the direction of motion ($r$ increasing or decreasing). However, except in the case of a few special potentials, the function in square root will lead to a difficult integral.

When this integral can be evaluated, we obtain $r(t)$, i.e. $r$ as function of $t$, and we can then find $\theta(t)$ from $r^2\dot{\theta} = h$ (equation 3.13). This give the dynamical solution $(r(t), \theta(t))$ of the equations of motion.
3.3. ORBITS IN A CENTRAL FORCE

3.3.3 The \( u-\theta \) orbital equation

We can instead find the geometrical solution \( r(\theta) \) by means of a beautiful transformation of the differential equation (3.15) that makes it not just tractable but familiar. The transformation has two steps.

- Change the independent variable, using the chain rule, from \( t \) to \( \theta \). This seems to be a small mathematical step but the effect is to change fundamentally the way we look at the problem: it was a dynamical equation for \( r \) as a function of \( t \); it becomes a geometrical equation for \( r \) as a function of \( \theta \), the solution of which is simply a plane curve.\(^7\)

- Change the dependent variable from \( r \) to \( u \) defined by \( u = r^{-1} \). The effect of this is to linearise the left-hand side.

The overall effect is to pass from a differential equation for \( r(t) \) to a simpler differential equation for \( u(\theta) \).

We work on these two steps together. First note that setting \( r = u^{-1} \) in (3.14) gives

\[
\dot{\theta} = hu^2. \tag{3.19}
\]

Then

\[
\frac{d}{dt} = \frac{d\theta}{d\theta} \frac{d}{d\theta} \tag{using the chain rule}
\]

so that

\[
\frac{dr}{dt} = hu^2 \frac{dr}{d\theta} = hu^2 \frac{d(1/u)}{d\theta} = -h \frac{du}{d\theta} \tag{3.20}
\]

and

\[
\frac{d^2r}{dt^2} = \frac{d}{dt} \left( -h \frac{du}{d\theta} \right) = hu^2 \frac{d}{d\theta} \left( -l \frac{du}{d\theta} \right) = -h^2 u^2 \frac{d^2u}{d\theta^2}. \tag{3.21}
\]

We have assumed throughout that \( \dot{\theta} \neq 0 \), i.e. that the motion is not purely radial: clearly if \( \theta \) is constant in the motion, it cannot be used to replace \( t \) as the parameter along the trajectory.

Substituting this into the equation of motion (3.15) gives

\[
-h^2 u^2 \frac{d^2u}{d\theta^2} - h^2 u^3 = \frac{1}{m} f(1/u) \tag{3.22}
\]

i.e.

\[
\frac{d^2u}{d\theta^2} + u = -\frac{f(1/u)}{mh^2 u^2} \tag{3.23}
\]

which is the geometric orbital equation.

The plan is to solve this equation for a given force \( f(1/u) \), then reintroduce time via

\[
\dot{\theta} = hu^2
\]

which can now be integrated to give \( \theta \) as a function of time, since \( u \) is known as a function of \( \theta \). Finally, now that we have \( \theta \) as a function of time, we can find \( r \) as a function of time from \( r = 1/u(\theta) \).

3.3.4 Kinetic energy

The expression for the kinetic energy of the particle in terms of \( u \) is a rather pleasing. We have:

\[
\frac{1}{2} \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \frac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) = \frac{1}{2} (h^2 u^2 + h^2 u^2) = \frac{1}{2} h^2 (u^2 + u^2), \tag{3.24}
\]

where the prime denotes differentiation with respect to \( \theta \). This is the kinetic energy per unit mass.

\(^7\)This change of variable is possible because the equations are autonomous, which means that there is no explicit \( t \) dependence in the equations.
3.4 Circular orbits

3.4.1 Existence

Closed orbits are those for which \( r(\theta + 2\pi n) = r(\theta) \) for some integer \( n \). Note that \( n \) need not be equal to 1: it is possible to imagine an orbit given by, say, \( r = 2 + \cos \frac{1}{2} \theta \), for which \( n = 2 \) (though this may correspond to a rather peculiar force law).8

A very special case of a closed orbit is a circular orbit. For any attractive \((f(r) < 0)\) central force, the orbital equation (3.23) admits circular orbits of any radius: the radius \( r \), and hence \( u \), is constant for a circular orbit so for any given \( u_0 \), it is only necessary to choose the value of \( h \) for which the orbital equation is satisfied:

\[
u_0 = -\frac{f(1/u_0)}{mh^2u_0^2} \quad \text{or} \quad \frac{1}{r_0} = -\frac{f(r_0)}{mh^2}, \tag{3.25}\]

where \( r_0 = 1/u_0 \). This equation determines a unique value of \( h^2 \) for any value of the radius of the circular orbit. In the rotating frame, this just means that the angular velocity is such that the centrifugal force exactly balances the central force.9

3.4.2 Stability

In order to determine whether an orbit is stable, we consider the evolution of a small perturbation. Since the orbit is determined by two variables, \( \theta(r) \) and \( r(t) \), there are two modes of perturbation: radial and tangential. However, the tangential direction is associated with the angular momentum, which is constant, so the effect of perturbation in the tangential direction can be related to a small change in angular momentum. Since angular momentum is conserved, this small change neither grows nor decays: it just remains small. We shall therefore confine attention to the radial direction.

We can investigate the evolution of the perturbation using orbital equation either in the \( r(t) \) form (3.15) or the \( u(\theta) \) form (3.23). Perhaps the \( r(t) \) form is easier to use.

Let \( r = r_0 + \eta \), where \( r_0 \) is the (constant) radius of the circular orbit, which satisfies (3.25), and \( \eta \) is a small time-dependent perturbation.

Substituting into the orbital differential equation (3.15) gives

\[rac{d^2\eta}{dt^2} = \frac{h^2}{(r_0 + \eta)^3} + \frac{1}{m} f(r_0 + \eta) \]

\[
= \frac{h^2}{r_0^3} - \frac{3h^2}{r_0^4} \eta + \frac{1}{m} f(r_0) + \frac{1}{m} f'(r_0) \eta + O(\eta^2) \quad \text{(using Taylor series for both expansions)}
\]

\[
= -\frac{3h^2}{r_0^4} \eta + \frac{1}{m} f'(r_0) \eta + O(\eta^2) \quad \text{(using (3.25))}
\]

\[
= \left( \frac{3f(r_0)}{mr_0} + \frac{1}{m} f'(r_0) \right) \eta \quad \text{(using (3.25) again and ignoring higher order terms in }\eta)\]

We can achieve this result a bit more slickly by writing the orbital equation in the form

\[
\ddot{r} = G(r),
\]

where \( G(r) = \frac{h^2}{r^3} + \frac{1}{m} f(r) \) and \( G(r_0) = 0 \). Then setting \( r = r_0 + \eta \) and taking the first non-zero term of the Taylor series for \( G(r_0 + \eta) \) gives

\[
\dot{\eta} = G'(r_0) \eta = \left( -\frac{3h^2}{r_0^4} + \frac{1}{m} f'(r_0) \right) \eta = \frac{1}{m} \left( 3r_0^{-1} f(r_0) + f'(r_0) \right) \eta
\]

as before.

8 Which could be found by substituting \( u = (2 + \cos \frac{1}{2} \theta)^{-1} \) into the left hand side of the geometric orbit equation (3.23), and then expressing it all in terms of \( r \) again; whatever appears there is \(-f(r)/mh^2\).

9 We are assuming that the circular orbit is centred on the centre of force, i.e. on the origin of polar coordinate. It is possible to imagine circular orbits centred on some point other than \( r = 0 \). If we choose the coordinate \( \theta \) such that the centre of the circle lies on the line \( \theta = 0 \), the equation of the circle is of the form

\[
r^2 - 2rd\cos \theta + d^2 = a^2 \quad \text{i.e.} \quad (d^2 - a^2)u^2 - 2ad\cos \theta + 1 = 0.
\]

Solving this quadratic equation to find \( u \) as function of \( \cos \theta \) and substituting into the geometric orbit equation (3.23) will reveal the force law that would allow such a circular orbit.
3.5. ORBITS IN AN INVERSE SQUARE FORCE

Now if \( 3r_0^{-1}f(r_0) + f'(r_0) < 0 \), this equation is simple harmonic; \( \eta \) will stay small and the orbit is stable to radial perturbations. Conversely, if \( 3r_0^{-1}f(r_0) + f'(r_0) > 0 \), one of the solutions of the equation is a growing exponential and the orbit is unstable to radial perturbations.

In the case of a power law force, \( f(r) = -kr^n \), where we require \( k > 0 \) since the force must be attractive,

\[
3f(r_0) + r_0 f'(r_0) = -k(3 + n)r_0^n
\]

so the forces that allow stable circular orbits must have \( n > -3 \). This includes (fortunately) the inverse square law.

At this point, we can ask an interesting question: which forces \( f(r) \) will provide not only stable but also closed orbits? The above equation for the perturbation \( \eta \) is linearised: it will give a necessary and sufficient condition for the orbit to be stable and a necessary condition for the orbit to be closed. Over many orbits, quadratic and higher order terms may become important and it is quite difficult to determine whether these terms will mean that the orbit is not in fact closed. There is a theorem due to Bertrand, which states (using second order perturbations) that the only force laws that permits closed and stable orbits are \( f(r) \propto r^{-2} \) (inverse square) and \( f(r) \propto r \) (Hooke’s law)!

3.5 Orbits in an inverse square force

3.5.1 The orbits as conic sections

Let \( f(r) = -\frac{mk}{r^2} \equiv -mku^2 \) (3.26)

where \( k = \left\{ \begin{array}{ll} GM & \text{gravitational force for a spherical body of mass } M \\ -\frac{qQ}{4\pi\varepsilon_0 m} & \text{electrostatic force between two point electric charges } q \text{ and } Q. \end{array} \right. \) (3.27)

Substituting this into the orbital equation (3.23) gives

\[
\frac{d^2u}{d\theta^2} + u = \frac{k}{\ell^2} \tag{3.28}
\]

and (magic!) we have an equation for which we can write down the solution using the standard complementary function plus particular integral method:

\[
u = A \cos(\theta - \theta_0) + \frac{k}{\ell^2}, \tag{3.29}
\]

where \( A \) can be chosen, without loss of generality, to be non-negative by setting \( \theta_0 \to \theta_0 + \pi \) if necessary.

The largest value of \( u \), which corresponds to the smallest value of \( r \) (‘closest approach’), is given by

\[
u_{\text{max}} = A + \frac{k}{\ell^2} \tag{3.30}
\]

and, without loss of generality, we choose the \( \theta \) coordinate such that this corresponds to \( \theta = 0 \), which is equivalent to setting \( \theta_0 = 0 \) in the general solution (3.29):

\[
u = A \cos \theta + \frac{k}{\ell^2} \tag{3.31}
\]

Now we set

\[
\frac{|k|}{\ell^2} = \frac{1}{e} \tag{3.32}
\]

and

\[
A = \frac{e}{\ell} \tag{3.33}
\]

so that the orbital equation (3.30) becomes

\[
u = \frac{1}{e}(e \cos \theta \pm 1) \quad \text{or} \quad r = \frac{\ell}{e \cos \theta \pm 1} \tag{3.34}
\]
where the + sign corresponds to attractive forces (gravitation, force between unlike electric charges) and the − sign corresponds to repulsive\footnote{Repulsive, like massive, is a word that physicists use in a very literal sense: a repulsive force is one that repels rather than one that is horrid.} forces (force between like electric charges) as in the definition (3.27) of \( k \).

In the form (3.34) is recognisable immediately as a conic section,\footnote{Conic sections are discussed in detail in section 3.10.} so the orbits are be hyperbolae, ellipses or parabolae as follows.

- **Ellipse**: \( 0 \leq e < 1 \), and the + sign; the circle is a special case corresponding to \( e = 0 \).
- **Parabola**: the borderline case \( e = 1 \).
- **Hyperbola**: \( e > 1 \), and either the + sign if the centre of the force is inside the hyperbola (corresponding to an attractive force) or the − sign if the centre of force is outside the hyperbola (corresponding to a repulsive force). The two cases are illustrated in the figures below.

\[ \theta = 0 \]

\[ \theta = 0 \]

The figures show the hyperbolic trajectories of a particle in an attractive inverse square potential (left-hand figure) and in a repulsive inverse square potential (right-hand figure). In both cases, the solid blob is the centre of the force and also the origin of polar coordinates with \( \theta = 0 \) corresponding to closest approach as shown. The dashed lines are parallel to the asymptotes.

You don’t have to remember whether \( e > 1 \) corresponds to an ellipse or a hyperbola: it is obvious. The gross distinguishing feature is that an ellipse is bounded (\( r \) does not go off to infinity) and a hyperbola is unbounded. If \( 0 \leq e < 1 \), the factor \((e \cos \theta \pm 1)\) that appears in the solution (3.34) is non-zero for all \( \theta \) so the orbit is bounded and hence an ellipse if we take the positive sign, corresponding to an attractive force, and we get nothing if we take the negative sign, because then \( r < 0 \) (which is expected: we could not imagine an elliptical orbit in a repulsive force).

The asymptotes of the hyperbolic trajectories are determined by \( u \to 0 \), so \( \cos \theta = -1/e \) in the attractive case and \( \cos \theta = 1/e \) in the repulsive case.

The total energy per unit mass is all cases is given by

\[
E = KE + PE = \frac{1}{2}h^2(u'^2 + u^2) - ku
\]

(\( u'^2 + u^2 \equiv (u + k/h)^2 - 2ku \))

\[
= \frac{1}{2}h^2((A \sin \theta)^2 + (A \cos \theta + k/h^2)^2) - k(A \cos \theta + k/h^2)
\]

(from (3.31))

\[
= \frac{1}{2}h^2A^2 - \frac{1}{2}k^2/h^2
\]

(3.35)

\[
= \frac{1}{2}\left(\frac{h^2e^2}{l^2} - \frac{h^2}{l^2}\right) = \frac{h^2}{2l^2}(e^2 - 1).
\]

(\( e^2 - 1 \) can be written in terms of \( A \) and \( k \) using the definitions (3.33) and (3.32))

We see that:

- the energy is constant (of course);
- the energy is positive if \( e > 1 \), which means that the particle has more than the escape velocity at each point on its trajectory — this corresponds to the hyperbolic orbits;
- the energy is negative if \( 0 \leq e < 1 \) meaning that the orbit is bound — this corresponds to the elliptic orbits;
- the energy is zero if \( e = 1 \), so the particle has exactly the escape velocity at each point on its trajectory — this corresponds to the parabolic orbits.

Thus everything fits together nicely.
3.5.2 Rutherford scattering

A particle with a positive charge $q$ and mass $m$ moves in the electric field produced by a positive charge $Q$ which is fixed at the origin. The mutual gravitational attraction of the charges is negligible compared with the repulsive electrostatic Coulomb force.

Initially, the particle is approaching from a very large distance (effectively $r = \infty$) at speed $V$ along a path which, in the absence of $Q$, would pass a distance $b$ from the origin: $b$ is called the impact parameter. Our task is to find the angle through which the particle is deflected in terms of the parameters $h$ and $b$.

The angular momentum per unit mass is given by $h = -Vb$, because $h = \pm|\mathbf{r} \times \dot{\mathbf{r}}|$ (angular momentum is ‘moment of momentum’ so this is just like taking the moment of a force: you multiply the magnitude of the force by the shortest distance between the line of action of the force and the point). The $\pm$ is to take into account the sense of the angular momentum, i.e. whether it is clockwise or anti-clockwise. In this case, the minus sign is correct because, as can be seen from the diagram, the moment is clockwise; to put it another way, $h = r^2\dot{\theta}$ and $\dot{\theta} < 0$.

Conservation of energy shows that when the particle has been bounced back by the repulsive field of the central charge and is heading out to $\infty$ its speed tends back to $V$; then conservation of angular momentum shows that the ‘backwards’ impact parameter is also $b$. The trajectory has a reflection symmetry as shown in the diagram.

\[
\begin{align*}
\dot{\mathbf{r}} &= \dot{r} \hat{r} + r \dot{\theta} \hat{\theta} \\
&= \dot{r} \hat{r} + \frac{h}{r} \hat{\theta} \\
&= -h \frac{du}{d\theta} \hat{r} + hu \hat{\theta} \\
&= hA \sin \theta \hat{r} + h(A \cos \theta + \frac{k}{h^2}) \hat{\theta}.
\end{align*}
\]

The figure shows the hyperbolic trajectory of a particle undergoing Rutherford scattering. The distance $b$ is the impact parameter. The angle $\alpha$ is equal to the asymptotic value of $\theta$. The angle $\phi$ is the angle through which the particle, coming in from the top right, is deflected.

The orbital equation for like electric charges is

\[
\frac{d^2u}{d\theta^2} + u = \frac{k}{h^2},
\]

where $k = -qQ/4\pi\varepsilon_0 < 0$. The solution to this is

\[
u = A \cos \theta + \frac{k}{h^2} \tag{3.36}
\]

There are two unknowns in this solution, $A$ and $h$, and these can be expressed in terms of $V$ and $b$. However, for present purposes we are only trying to find $\alpha$, which can be done easily by considering the velocity:
As $\theta \rightarrow \alpha$, $\dot{r} \rightarrow -V\hat{r}$ (there is a minus sign because the particle is coming inwards from $r = \infty$) so

$$hA \sin \alpha = -V \quad \text{and} \quad A \cos \alpha + \frac{k}{h^2} = 0.$$ 

Eliminating $A$ from these two equations gives

$$\tan \alpha = \frac{Vh}{k} = -\frac{V^2b}{k}.$$ 

The angle we are after is angle through which the particle is deflected, which is $\phi$ on the diagram. We have

$$\phi = \pi - 2\alpha \implies \frac{1}{2}\phi = \frac{1}{2}\pi - \alpha$$

$$\implies \tan \frac{1}{2}\phi = \cot \alpha = -\frac{k}{V^2b}$$ 

so we obtain the pleasingly simple result\(^{12}\) that

$$\phi = 2 \tan^{-1} \left( \frac{|k|}{V^2b} \right).$$

The same result is obtained for the deflection of, say, an interstellar comet (i.e., not a periodic comet such as Halley’s comet) by the sun. For a comet, though, $V$ and $b$ are probably not measurable but the distance $d$ of closest approach to the sun at which $\theta = 0$, and the speed $U$ at this point may be measurable.\(^{13}\) We can then find $h$ and $A$ from

$$\frac{1}{d} = A + \frac{k}{h^2}; \quad h = dU.$$ 

### 3.5.3 Kepler’s laws

Kepler\(^{14}\) formulated his three laws in about 1605, on the basis of remarkably accurate observations by Tycho Brahe.\(^{15}\)

They may be stated as follows:

K1: Each planet moves on an ellipse with the sun at one focus.

K2: The radius vector to the planet sweeps out equal areas in equal times.

K3: The period of the orbit is proportional to ($\text{mean radius}^3$).

We have already established K1: the planets move in ellipses as a consequence of the inverse square law of gravitational attraction.\(^{16}\)

---

\(^{12}\)Ernest (later Lord) Rutherford used this result to explain the way that $\alpha$-particles (positively charged) are scattered by atoms. In a series of experiments on gold leaf, he observed that scattering angles greater than $\pi/2$ were sometimes found, which he calculated was inconsistent with the current plum pudding model due to J.J. Thompson in which the electrons were the plums and the pudding was smeared out positive charge. He realised that his results were consistent with a model of the atom as a small, heavy positively charged nucleus surrounded by orbital electrons. This work was carried out in 1911, eight years before he came to Cambridge as head of the Cavendish laboratory (and three years after he was awarded the Nobel prize for Chemistry for his work on the physics of alpha particles.\(^{17}\)

\(^{13}\)If one simply replaces $V$ by $c$, the speed of light, in the deflection angle, one gets a formula that would give the deflection of star light by the sun if Newtonian dynamics applied to photons. Interestingly, Einstein’s first effort at General Relativity, in 1911, gave exactly this result. It took into account the principle of equivalence (gravity is indistinguishable from acceleration) but not the fact that space-time is curved by the presence of massive bodies. He was impatient for observers to prove him right, but since the effect is too small to measure unless the light just grazes the sun, observations could only be made at a solar eclipse. Attempts to observe were thwarted first by cloudy conditions and then by the war. By the time of Eddington’s expedition to Principe in 1919, Einstein had evolved his full theory of relativity, which gave a prediction of exactly twice the ‘Newtonian’ value, and this was triumphantly corroborated by Eddington (with suspicious accuracy). It is interesting to speculate how General Relativity would have fared without the hold-up to these observations.

\(^{14}\)Johannes Kepler (1571-1630)

\(^{15}\)1546 — 1601. Interesting factoid of the day: While a student, Tycho lost part of his nose in a duel. For the rest of his life, he was said to have worn a realistic replacement made of silver and gold, using a paste to keep it attached. When Tycho’s tomb was opened in 24 June 1901 green marks were found on his skull, suggesting the false nose also had copper. It is possible that he had a number of different noses for different occasions.

\(^{16}\)Curiously, there is only one other power law that gives rise to ellipses, namely $f(r) = r$, which is Hooke’s law; but in this case the sun would be at the centre of the ellipse, not at the focus.
K2 is a consequence of the law of conservation of the law of conservation of angular momentum and therefore holds for any central force. This can be seen as follows. Let $\delta A$ be the area swept out by the radius vector in time $\delta t$, during which the polar angle changes by $\delta \theta$. Then

$$\delta A = \frac{1}{2} r^2 \delta \theta$$

by the usual geometrical argument. Thus

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} \equiv \frac{1}{2} \hbar.$$

Since $\hbar$ is constant, $\frac{dA}{dt}$ is constant, which is the required result.

It is not clear, and it seems that Kepler was not very clear about it either, what precisely he meant by ‘mean radius’. He certainly would not have been able to average $r$ over an orbit. Since planetary orbits are more or less circular, it doesn’t much matter what it means. However, if we calculate the period of an orbit of a particle subject to an inverse square force, we see that K3 does in fact hold is a perfectly acceptable sense.

In order to calculate the period, $T$, we need an equation relating $\theta$ and $t$. Equation (3.19) is just the thing. We have

$$\dot{\theta} = hu^2$$

so

$$T \equiv \int_0^T dt = \int_0^{2\pi} \frac{d\theta}{\dot{\theta}} = \int_0^{2\pi} \frac{d\theta}{hu^2}$$

$$= \int_0^{2\pi} \frac{d\theta}{h(1 + e \cos \theta)^2/\ell^2} \quad \text{(from the geometrical solution (3.34))}$$

$$= \frac{\ell^2}{h} \frac{2\pi}{(1 - e^2)^{3/2}} \quad \text{(take my word for this.)}$$

The result of the last integral is of interest, though actually doing the integral is not.\footnote{The integral is doable by elementary means, starting for example with the substitution $t = \tan \frac{1}{2} \theta$; it takes about half an hour. It is quicker by means of a contour integral in the complex plane; about 10 minutes. \textsc{Mathematica} can do it in 0.47 seconds, but makes a terrible mess of it.}

Now the smallest value of $r$ (at the perihelion) $\theta = 0$, and the largest value of $r$ (at the aphelion) $\theta = \pi$ satisfy (see the geometric form of the orbit (3.34))

$$\frac{1}{2}(r_{\text{min}} + r_{\text{max}}) = \frac{1}{2} \left( \frac{\ell}{1 + e} + \frac{\ell}{1 - e} \right) = \frac{\ell}{1 - e^2}$$

so

$$T = \frac{2\pi \ell^2}{h} \left( \frac{1}{2}(r_{\text{min}} + r_{\text{max}}) \right)^{3/2} = \frac{2\pi}{\sqrt{k}} \left( \frac{1}{2}(r_{\text{min}} + r_{\text{max}}) \right)^{3/2}$$

using the definition (3.32) to eliminate $h$. This certainly counts as being proportional to (mean radius)$^{3/2}$.

### 3.6 Circular orbits and quantum mechanics

This little aside arises because Keppler’s laws reminded me of another law, also based on observations, which is also relevant — but not very relevant, first because the law is wrong\footnote{We shouldn’t be too discouraged by this: the results are correct even though the theory is wrong.} and second because the effect is so small\footnote{You may be a bit discouraged when you see the magnitude of the effect}. I have in mind Bohr’s model of the atom, which postulated that the orbital angular momentum of an electron orbiting a nucleus is quantised, which means that it is only allowed to take certain values:

$$\text{Angular momentum} \equiv M_e R v = nh.$$  \hspace{1cm} (3.37)

In this formula, $M_e$ is the mass of the Earth, $R$ is the radius of the Earth’s orbit (assumed circular), $v$ is the speed of the Earth, $n$ is an integer and $\hbar$ is Planck’s constant (divided by $2\pi$).\footnote{The integral is doable by elementary means, starting for example with the substitution $t = \tan \frac{1}{2} \theta$; it takes about half an hour. It is quicker by means of a contour integral in the complex plane; about 10 minutes. \textsc{Mathematica} can do it in 0.47 seconds, but makes a terrible mess of it.}
Together with the Newton’s second law for a circular orbit:

\[
\frac{GM_e M_s}{R^2} = \frac{M_e v^2}{R}
\]

(3.38)

(‘gravitational force equals centrifugal force’), equation (3.37) show that \( R \) is also quantised; it can only take certain values. Eliminating \( v \) from (3.37) and (3.38) gives

\[
R = \frac{n^2 h^2}{GM_e^2 M_s}
\]

Substituting

\[
h = 1 \times 10^{-34} \text{ kg m}^2 \text{ s}^{-1}
\]

(3.39)

\[
G = 6.7 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}
\]

(3.40)

\[
M_e = 2 \times 10^{30} \text{ kg}
\]

(3.41)

\[
M_s = 6 \times 10^{24} \text{ kg}
\]

(3.42)

\[
R = 1.5 \times 10^{10} \text{ m}
\]

(3.43)

shows that \( n \approx 8 \times 10^{70} \). There are therefore \( 10^{70} \) allowed values for the radius of orbits closer to the Sun than the Earth’s orbit. The allowed orbits are tightly spaced: the difference in radius between the \( n \)th and \((n + 1)\)th allowed orbits is

\[
\frac{[(n + 1)^2 - n^2]h^2}{GM_e^2 M_s} \approx 2n h^2 \frac{2nR}{n^2} \approx 10^{-62} \text{ km}.
\]

3.7 Hohmann Transfer

**********Section under construction**********

A very important problem for the Apollo mission\(^{20}\) was how to transfer the spacecraft from low orbit Earth orbit (the parking orbit) to the high Earth orbit of the Moon using as little fuel as possible. The solution to the problem turns out to be a Hohmann transfer, in which a roughly elliptical orbit interpolates between the two approximately circular orbits. This requires a burst of engine fire at both extremes of the roughly elliptical orbit, these two manoeuvres being called TransLunar Injection and TransEarth Injection.

To a good approximation, we can regard the roughly elliptical orbit as being a perfect ellipse with the Earth at one focus for most of the orbit and a perfect ellipse with the Moon at one focus for the remainder. The matching between the two ellipses takes place when the gravitational force of the moon is comparable with that of the Earth. The two accelerations are

\[
\frac{GM_e}{r_e^2}, \quad \frac{GM_m}{r_m^2}
\]

\(^{20}\)The aim of the Apollo programme was to put a man on the moon. It was conceived in the presidency of Dwight D. Eisenhower but really got going after President Kennedy’s 1961 address to a joint session of Congress promising to land a man on the Moon by the end of the decade. This was achieved by Apollo 11 in July 1969. The record for the greatest distance from the Earth is held by the three astronauts of the Apollo 13 mission, which malfunctioned on the outward journey. Instead of entering a bound (elliptical) moon orbit, a small adjustment was made that put the spacecraft into an unbound (hyperbolic) moon orbit which would automatically (without further use of the engines) return to Earth (free return trajectory). In the event, a small correction had to be made on the far side of the moon.
where \( r_e \) and \( r_m \) are the distances from the Earth and Moon respectively, shows that they will be roughly equal when
\[
\frac{r_e}{r_m} = \sqrt{\frac{M_m}{M_e}} = 9
\]
since \( M_e = 81 \times M_m \).

Need numbers for Apollo and parameters and the basic transfer equation. Should be able to do it geometrically as well as dynamically.

The major axis is obviously \( R_1 + R_2 \) and the min and max distance are \( R_1 \) and \( R_2 \), which gives
\[
\ell = \frac{2R_1R_2}{R_1 + R_2}, \quad e = \frac{R_2 - R_1}{R_2 - R_1} + 1
\]

### 3.8. General forces

The general orbital equation (3.23) may be tackled in two ways:

- direct integration of the second order differential equation;
- integration of the first order energy equation

\[
\left( \frac{du}{d\theta} \right)^2 + u^2 = \phi(u)
\]

where \( 2\frac{d\phi}{du} = -\frac{f(1/u)}{m\hbar^2 u^2} \).

We may also find that some problems can be tackled using only the conservation of energy and angular momentum.

The problem with direct integration is that in general (in fact, except in the case of inverse square and inverse cube forces), the differential equation is non-linear in \( u \) and therefore very unlikely to have a solution in terms of standard functions. However, there are well-established approximation techniques that can be applied. The problem with the integral
\[
\int \frac{du}{\sqrt{\phi(u) - u^2}}
\]

arising from the first order equation is that it is again unlikely to yield standard functions and might be hard to approximate. It is exceptionally fortunate that we live in a universe to which the inverse law of gravitational attraction applies (at least approximately)!

### 3.8.1 Example: General relativistic orbits

The purpose of this example is to show how that presence of a small perturbation to an inverse square law affects elliptical orbits. We will find that the orbits remain almost elliptical but the axes of the ellipse precess (rotate) slowly. We will set this problem in the context of a famous historical calculation that provided a triumph for the newly hypothesised theory of General Relativity.

The solution of Einstein’s field equations corresponding to a spherically symmetric gravitating body (the Sun in our case) turns out to be relatively simple.\(^{21}\) It is possible to write down the geodesic equations, the solutions of which describe the trajectories of free particles, and of light rays. As in the Newtonian case, setting \( u = 1/r \) makes the equation tractable, though it is not soluble in terms of elementary functions.\(^{22}\) The equation is
\[
\frac{d^2u}{d\theta^2} + u = \frac{GM}{\hbar^2} \left( 1 + \frac{3\hbar^2 u^2}{c^2} \right), \quad (3.44)
\]

\(^{21}\)The solution is named after Karl Schwarzschild (1873–1916) who discovered it while serving in the German army on the Russian front in 1915, which was the year in which Einstein first introduced general relativity. Schwarzschild died the year after (though not as a result of his military service).

\(^{22}\)Multiplying by \( du/d\theta \) and integrating gives an energy-like first integral of the form:
\[
\frac{1}{2} \frac{du}{d\theta}^2 = E - \frac{1}{2} u^2 - \frac{GM}{\hbar^2} u + \frac{GM}{c^2} u^3,
\]

where \( E \) is a constant of integration. Taking square-roots and integrating gives an elliptic integral.
where \( h \) is as usual the angular momentum per unit mass. This differs from the Newtonian orbital equation only in the last term and is in the standard form for an orbit in a central force.

One interesting affect of the extra term is that for \( h^2 < 12GM/c^2 \), there are no circular orbits; this can easily be seen by setting \( d^2u/d\theta^2 = 0 \) and solving the resulting quadratic equation for \( u \). This contrasts with the Newtonian case where for any angular momentum there is a circular orbit.\(^{23}\)

For most astrophysical situations, the extra term is small. Nevertheless, planetary orbits have been observed for many centuries and even very small non-Newtonian affects are detectable.

We will calculate the advance of the perihelion of Mercury. We can estimate the magnitude of the extra term in equation (3.44) as follows. The mean radius of the orbit about the Sun is about \( 6 \times 10^{10} \) metres. The period is 88 days, which is about \( 7.6 \times 10^6 \) seconds, so \( \dot{\theta} = 2\pi(7.6 \times 10^6)^{-1} \approx 10^{-6} \) radians per second. Thus

\[
h = r^2\dot{\theta} \approx 36 \times 10^{20} \times 10^{-6} = 3.6 \times 10^{15} \text{ metres squared per second}
\]

and

\[
\frac{3h^2u^2}{c^2} = \frac{3 \times 3.6^2 \times 10^{30}}{36 \times 10^{20} \times 9 \times 10^{16}} \approx 10^{-7}
\]

which is a very small perturbation. We can therefore approximate the solution by iteration.

The plan is to set

\[
u(\theta) = u_0(\theta + \lambda u_1(\theta) + \lambda^2 u_2(\theta) + z \cdots ,
\]

substitute this expression into the differential equation (3.44). We can do this term by term.

The unperturbed solution is the Newtonian solution

\[
u_0(\theta) = \ell^{-1}(1 + e \cos \theta)
\]

where \( \ell = h^2/GM \).

To obtain the first iteration, we substitute the Newtonian solution \( u_0(\theta) \) into the extra term on the right of equation (3.44) (setting \( MG = h^2/\ell \)):

\[
d^2u_1/d\theta^2 + u_1 = \frac{1}{\ell} + \frac{3h^2}{\ell^3c^2}(1 + 2e \cos \theta + e^2 \cos^2 \theta).
\]

This we can solve by the usual particular integral/complementary function method. However, it is clear that only the \( 2e \cos \theta \) term is of interest: this is resonant and will give a non-periodic particular integral, whereas all the other terms are periodic (with periods either \( 2\pi \) or \( \pi \)) and cannot affect the perihelion advance; indeed, they cancel out when averaged over several orbits.

The corresponding term of the particular integral is

\[
\frac{\lambda e}{\ell} \theta \sin \theta, \quad \text{where } \lambda = \frac{3h^2}{\ell^3c^2}.
\]

Taking into account only this term of the particular integral gives the first iteration:

\[
u_0 + \lambda u_1 \approx \ell^{-1}(1 + e \cos \theta + \lambda e \theta \sin \theta) \approx \ell^{-1}(1 + e \cos((1 - \lambda)\theta)
\]

where for the last approximation we have used \( \cos(\lambda \theta) \approx 1 \) and \( \sin \lambda \theta \approx \lambda \theta \) (and ignored the periodic terms in \( u_1 \)).

At the perihelions, \( \cos((1 - \lambda)\theta) = 1 \). If the first is when \( \theta = 0 \), then the second is when \( (1 - \lambda)\theta = 2\pi \), i.e., when \( \theta \approx 2\pi(1 + \lambda) \). The perihelion advance is therefore \( 2\pi \lambda \) radians per orbit.

Putting in the data for Mercury gives an advance of 43 arc second per century. Remarkably, it was known several decades before general relativity was formulated that out of a total observed precession of 5000 arc seconds per century, only 43 arc seconds are unexplained by Newtonian effects (such as the influence of other planets).

### 3.9 Conic sections

Conic sections are plane curves formed by the intersections of a plane in \( \mathbb{R}^3 \) with a double cone. There are three types, ignoring the degenerate cases of a point, a line, and a pair of lines, that arise if the plane passes through the apex of the cones. Suppose that the common axis of the cones is vertical and the semi-angle (the angle between a straight line on the surface of the cone and the axis) is \( \alpha \). Let the acute angle between the normal to the plane and the vertical be \( \pi/2 - \theta \). Then the three cases are as follows.

\(^{23}\)As shown in section ??, for any attractive force circular orbit of any radius can be found, by choosing \( h \) appropriately.
3.9. CONIC SECTIONS

- Ellipses, which are closed curves with $\theta > \alpha$. Circles are special cases with $\theta = \pi/2$.
- Hyperbolae, which are open curves with $\theta < \alpha$. Each consists of two branches, corresponding to the plane intersecting the two cones.
- Parabolae, which are open curves with $\theta = \alpha$.

That is how to picture conics, but it is easier to obtain properties of the conic sections by using a different, two-dimensional, defining property. Let $O$ be the origin of Cartesian coordinates, and let $L$ be a fixed line (called the directrix of the conic). The point $P$ moves on a conic if $OP = ePB$, where $B$ is the point on $L$ closest to $P$ and $e$ is a fixed positive number called the eccentricity of the conic, as shown in the diagrams.

We can work out the Cartesian and polar equations of conics as follows. Let the equation of the directrix be $x = \ell/e$, where $\ell$ is a (fixed) positive constant.\(^{24}\) Then the point $(x, y)$ lies on the conic defined by $e$ and $\ell$ if

$$x^2 + y^2 = e^2(\ell/e - x)^2$$

It is easy to rearrange this to obtain, after a translation along the $x$-axis, the three standard forms corresponding to $e < 1$ (ellipse), $e > 1$ (hyperbola) and $e = 1$ parabola:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$4x - cy^2 = 0$$

where

$$a^2 = \frac{\ell^2}{(1 - e^2)^2}, \quad b^2 = \frac{\ell^2}{1 - e^2}, \quad c = -\ell/2.$$

Much more useful to us are the equations in polar coordinates. Setting $\sqrt{x^2 + y^2} = +r$ and $x = r\cos \theta$ gives, in cases (i) and (ii) in the diagram,

$$r = e(\ell/e - r\cos \theta) \quad \text{i.e.} \quad r = \frac{\ell}{1 + e\cos \theta} \quad (3.45)$$

\(^{24}\ell$ is the length of the line parallel to the directrix from the focus (here, the origin) to the curve; the line is called the semi latus rectum from the Latin half + side + straight.
and in case (iii)

\[ r = e(r \cos \theta - \ell/e) \quad \text{i.e.} \quad r = \frac{\ell}{-1 + e \cos \theta} \quad \text{(3.46)} \]

Note that it is easy to distinguish between the closed (and hence bounded) ellipses and the open (and hence unbounded) hyperbolae/parabolae: \( r \) can go to infinity only if the denominator of equation (3.45) goes to zero, which can only happen when \( e \geq 1 \). In the case \( e > 1 \), the hyperbolae tend to the pair of straight lines (the asymptotes) given by \( \cos \theta = -1/e \) in case (ii) (given by equation (3.45)) and \( \cos \theta = 1/e \) in case (iii) (given by equation (3.46)).

### 3.10 Some useful numbers

Mass of Earth
- Mass of Sun
- Newton’s Gravitational constant \((G)\)
- Radius of Earth
- Earth’s orbital aphelion
- Earth’s orbital perihelion