THE TWISTOR GEOMETRY OF A FAMILY OF ODES
INI TWISTOR THEORY 2024

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1. Introduction and Background

This talk concerns new constructions of complex hyper-Kähler metrics with a view to Joyce structures. The motivation for Joyce structures ultimately has its roots in Donaldson-Thomas (DT) theory. This talk will be mostly concrete but it is nice to situate this work in a bigger story so we start with some deliberately vague remarks: DT theory concerns the enumerative geometry of Calabi-Yau 3-folds. One may consider categories of:

- Holomorphic vector bundles / coherent sheaves
- Lagrangian submanifolds (objects in $\mathcal{F}(Y)$)

DT invariants are associated to subclasses in a category and "count" distinguished objects: stable bundles of a given Chern character and (maybe conjecturally) special Lagrangians of a fixed class in $H_3(Y)$ respectively. However the definitions of these notions depend on some input data from outside the category. For $\mathcal{F}(Y)$ the notion of special requires a choice of complex structure and an associated holomorphic volume form. The moduli of these extra pieces of data motivate the idea of spaces of *stability conditions*. Given certain assumptions, Bridgeland argues [4] the way the DT invariants vary as one moves around the space $M = \mathrm{Stab}(C)$ of stability conditions on a CY_3 category C should, after solving a holomorphically varying family of Riemann-Hilbert problems, define a geometric structure (named there a Joyce structure). In [6] this structure is interpreted as a complex hyper-Kähler metric with extra symmetry on the space $X = TM$.

Articles by Smith [14] and Smith together with Bridgeland [7] give a nice geometric realisation of spaces of stability conditions:

$$\mathrm{Stab}(\mathcal{F}(Y)) \leftrightarrow \mathrm{Quad}(g, \{m_1, ..., m_n\}) \tag{1.1}$$

when Y is a certain non-compact Calabi-Yau threefold fibreing over surface of genus g and $M = \mathrm{Quad}(g, \{m_1, ..., m_n\})$ is the moduli space of quadratic differentials on a Riemann surface of genus g with simple zeroes and poles of order $\{m_1, ..., m_n\}$.

Can we see the Joyce structure on X while starting on the right hand side of (1.1)?

The case of a pole of order seven on the Riemann sphere is detailed in [5]. In this talk I hope to get some way to explaining the construction for a single pole of order $2n + 5$ and in fact I hope the methods here can be adapted in a straightforward way to other moduli of meromorphic quadratic differentials on the Riemann sphere.

2. Complex hyper-Kähler metrics

We will be concerned with *complex* hyper-Kähler metrics on a complex manifold X of dimension $4n$. These are holomorphic non-degenerate sections g of $\odot^2 T^*X$ where TX

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denotes the holomorphic tangent bundle of X, with holomorphic endomorphisms I, J, K of TX satisfying

$$\begin{aligned} I^2 = J^2 = K^2 = IJK &= -\operatorname{Id}_{TX}, \\ I^*g = J^*g = K^*g &= g \\ \nabla I = \nabla J = \nabla K &= 0. \end{aligned} \tag{2.1}$$

This differs from (Riemannian) hyper-Kähler geometry and there is no notion of signature.

Let $F = X \times \mathbb{CP}^1$. An intermediate step on the way to constructing our metrics will be families of rank-$2n$ subbundles of TF depending on a *spectral parameter* which I will provocatively call $\hbar \in \mathbb{CP}^1$. Here $\hbar = u^1/u^0$ where $[u^0 : u^1] \in \mathbb{CP}^1$. These are *twistor distributions* that are of the form

$$L = \operatorname{span}\left\{ L_j := U_j + \frac{V_j}{\hbar} \right\}_{j=1}^{2n}. \tag{2.2}$$

where U_j, V_j are vector fields on X such that $TX = \operatorname{span}\{U_j, V_j\}_{j=1}^{2n}$. This setup defines a parabolic geometry with various names:

- $(2n, 2)$-paraconformal structure (Bailey-Eastwood [1])
- $(2n, 2)$-almost-Grassmannian structure [8], dropping the *almost* if L is integrable for all $\hbar \in \mathbb{CP}^1$ (which we will assume).

We have a canonical quaternionic structure I, J, K determined by

$$J(U_j) = V_j, \quad K(U_j) = iV_j \tag{2.3}$$

and the quaternion relations. The span of the U_i and V_i are the $\pm i$ eigenspaces of I.

L Frobenius integrable is equivalent to integrability of the $\pm i$-eigenspaces of the complex structures I, J, K and the existence of the twistor space $Z = (X \times \mathbb{CP}^1)/L$, potentially replacing X by a suitable open region to get something Hausdorff.

L also determines a family of metrics Hermitian for the quanternionic structure:

$$g = \sum_{i,j=1}^{2n} e_{ij} U^i \odot V^i. \tag{2.4}$$

Here (after choice of basis), each metric corresponds to a non-degenerate skew matrix e_{ij} of holomorphic functions on X. When does the family contain a hyper-Kähler metric? We will give a characterisation in terms of data on the twistor space.

Associated to each metric of the form (2.4) we define a 2-form on F:

$$\begin{aligned} \Omega &= \sum_{i,j=1}^{2n} e_{ij} (V^i - U^i/\hbar) \wedge (V^j - U^j/\hbar) \\ &= \sum_{i,j=1}^{2n} \hbar^{-2} \underbrace{e_{ij} U^i \wedge U^j}_{\Omega_- :=} - 2\hbar^{-1} e_{ij} U^i \wedge V^j + \underbrace{e_{ij} V^i \wedge V^j}_{\Omega_+ :=}. \end{aligned} \tag{2.5}$$

One can think of this as an $\mathcal{O}(2)$-valued 2-form relative to the fibration $F \to \mathbb{CP}^1$ with kernel L. It is straightforward to see that if Ω is closed with respect to exterior differentiation on X for all $\hbar$ then Ω_- and Ω_+ are both closed two forms on X. Now $g(J\cdot, \cdot) = \Omega_+ + \Omega_-$ and $g(K, \cdot) = \Omega_+ - \Omega_-$. So $d\Omega_+ = d\Omega_- = 0$ implies $\nabla J = \nabla K = 0$ which implies $\nabla I = 0$ and the metric (2.4) is complex hyper-Kähler.

An $\mathcal{O}(2)$-valued relative 2-form of maximal rank on F while annihilating L is necessarily of the form (2.5) and determines a metric (2.4) which is hyper-Kähler if and only if it is relatively closed. Furthermore, in this case

$$\mathcal{L}_{L_i}\Omega = 0,$$

so Ω descends to an $\mathcal{O}(2)$-valued 2-form on Z.

This is basically the argument in [1] where a correspondence between relative symplectic $\mathcal{O}(2)$-valued 2-forms on Z and complex hyper-Kähler metrics on X is established. A similar correspondence for hyper-Kähler metrics (in the usual sense) appears in [11]. When one of Ω_- or Ω_+ is closed the geometry is called null-Kähler [9].

3. Schrödinger equations

Points in the space $F = X \times \mathbb{CP}^1$ will determine ODE in a complex parameter x of the following form

$$\hbar^2 \frac{d^2\psi}{dx^2} = Q(x)\psi \tag{3.1}$$

where

$$Q(x) = Q_0(x) + \hbar Q_1(x) + \hbar^2 Q_2(x) \tag{3.2}$$

here (the following may at first look mysterious)

$$Q_0(x) = x^{2n+1} + a_{2n}x^{2n-1} + ... + a_1 \tag{3.3}$$

$$Q_1(x) = \sum_{i=1}^{n} \frac{p_i}{x - q_i} + r_n x^{n-1} + ... + r_1 \tag{3.4}$$

where $p_i^2 = Q_0(q_i)$. Next $Q_2(x)$ is determined by the condition that

$$Q(x) = \frac{3}{4(x-q_i)^2} + \frac{u_i}{(x-q_i)} + u_i^2 + O((x-q_i)) \tag{3.5}$$

at $x = q_i$ for some functions u_i on F. This is set up so analytic continuation of solutions around the singularity corresponds to multiplication by -1.

Note that $a := (a_1, ..., a_{2n})$ gives local coordinates on the space $M = \mathrm{Quad}(\{2n+5\})$. To see this note there are $2n+3$ Möbius transforms that put any quadratic differential ϕ on $\mathbb{CP}^1$ with a single pole of order $2n+5$ into the form $Q_0(x)dx^2$ (these differ by multiplication by a $(2n+3)$rd root of unity).

We let X be the manifold with local coordinates (a, q, r), $q := (q_1, ..., q_n)$, $r := (r_1, ..., r_n)$, parametrising the choice of $Q_0(x)$ and $Q_1(x)$. There is a map $X \to \mathrm{Quad}(\{2n+5\})$ given by forgetting $Q_1(x)$. Time permitting, I will explain how each choice of $Q_1(x)$ up to swapping the q_i determines a unique point in $T_\phi M/\Gamma_\phi$ where Γ_ϕ is a lattice determined by the quadratic differential, and (basically by the Jacobi inversion theorem) that this map has open dense image. All the data we are about to define doesn't see the ordering of the q_i and so we can push it down to $T_\phi M/\Gamma_\phi$.

4. Isomonodromy

We will define a subbundle L by looking for deformations $(a(t), q(t), r(t))$ (fixing $\hbar$) that are *isomonodromic*. The theory of isomonodromic deformations for equations with irregular singular points was addressed by Jimbo, Miwa and Ueno in [12]. The end result in our case is the following:

Proposition 4.1. A vector field $W \in \Gamma(TX)$ generates a deformation for the ODE (3.1) that preserves the (generalised) monodromy data if and only if

$$W(Q(x)) = 2Q(x)\frac{\partial A(x)}{\partial x} + \frac{\partial Q(x)}{\partial x}A(x) - \frac{1}{2}\frac{\partial^3 A(x)}{\partial x^3} \tag{4.1}$$

for some meromorphic $A(x)$.

Proposition 4.2 ([10])**.** There is are $2n$-linearly independent vector fields that satisfy the equation (4.1) for each $\hbar$ that define a twistor distribution L of the form (2.2). They correspond to flows with

$$A(x) = \sum_{i=1}^{n} \frac{\zeta_i}{(x - q_i)} \tag{4.2}$$

for some functions ζ_i on X.

5. An intersection form as a twistor 2-form

How do we see the hyper-Kähler metric? We need to realise L as the kernel of a closed relative $\mathcal{O}(2)$-valued 2-form. I will sketch the construction:

Note that for generic $(p, \hbar) \in X \times \mathbb{CP}^1$, $\Phi := y dx$ defines a 1-form on a genus-$2n$ Riemann surface $\Sigma_{p,\hbar}$ defined by $y^2 = Q(x)$ with residues that are *constant* on X (it has a pole at infinity and simple poles at the points corresponding to $x = q_i$). Accordingly differentiating Φ with respect to the coordinates on X will produce a 1-form with no residues and thus represent a cohomology class. We therefore get a map

$$\mu_{p,\hbar} : T_p X \to H^1(\Sigma_{p,\hbar}, \mathbb{C}) \tag{5.1}$$

given by

$$W \mapsto W(\Phi). \tag{5.2}$$

Inspect (4.1) and note that the first two terms on the right hand side resemble the product rule but with the wrong factors. This suggests we should write (4.1) in terms of Ψ by substituting $y^2 = Q(x)$. We get the nice simplification:

$$W(\Phi) = d\big(y \cdot A\big) - \frac{1}{4y}\frac{\partial^3 A}{\partial x^3}dx. \tag{5.3}$$

This says that the derivative of Φ in the direction of an isomonodromic flow $W \in L_\hbar$ is equal in cohomology $H^1(\Sigma_{p,\hbar}, \mathbb{C})$ to

$$\kappa_A := \frac{1}{4y}\frac{\partial^3 A}{\partial x^3}dx. \tag{5.4}$$

We have the usual intersection form

$$H^1(\Sigma_{p,\hbar}, \mathbb{C}) \times H^1(\Sigma_{p,\hbar}, \mathbb{C}) \to \mathbb{C} \tag{5.5}$$

given by, for *smooth* cohomology representatives α, β

$$\langle \alpha | \beta \rangle = \int_{\Sigma_{p,\hbar}} \alpha \wedge \beta$$

or, via Stokes' theorem, given *meromorphic* representatives μ, ν with vanishing residues (also called differentials of the second kind) this can be evaluated as a finite sum

$$\langle \mu | \nu \rangle = 2\pi i \sum_{x \in \Sigma_{p,\hbar}} \mathrm{Res}_x(\mu\ d^{-1}\nu). \tag{5.6}$$

Here by $d^{-1}\nu$ I mean we take *local* meromorphic antiderivatives of ν around $x \in \Sigma_{p,\hbar}$.

Proposition 5.1. *κ_A is orthogonal to the image of $\mu_{p,\hbar} := T_pX \to H^1(\Sigma_{p,\hbar}, \mathbb{C})$ and hence $L_\hbar = \ker \Omega_\hbar$ where $\Omega_\hbar := \hbar^{-2}\mu^*\langle\cdot|\cdot\rangle$.*

Explicitly we may write

$$\Omega_\hbar(W_1, W_2) = \sum_{x\in\Sigma_{p,\hbar}} 2\pi i \operatorname{Res}_x \Big(W_1(\Psi)\ d^{-1}W_2(\Psi)\Big). \tag{5.7}$$

This formula implies Ω is closed for fixed $\hbar$ by the Schwarz rule and integration by parts. We can also check it is of maximal rank while annihilating L.

The proof of Proposition 5.1 uses the fact κ_A vanishes to high enough order at infinity, as well as the fact the Laurent expansion of y at the two points on $\Sigma_{p,\hbar}$ corresponding to $x = q_i$ is determined by (3.5).

Proposition 5.2. *Ω defines a $\mathcal{O}(2)$-valued 2-form on $F = X \times \mathbb{CP}^1$.*

The proof of Proposition 5.2 can be sketched as follows: Sufficiently close to the pole $x = \infty$ we have the expansion

$$\Psi = \sqrt{Q_0(x)}dx + \hbar\frac{Q_1(x)}{2\sqrt{Q_0(x)}}dx + \hbar^2\frac{4Q_0(x)Q_2(x) - Q_1(x)}{8Q_0(x)^{3/2}}dx + O(\hbar^3). \tag{5.8}$$

The omitted terms have zeroes of high enough order at infinity to not contribute to any residue at ∞ in (5.7). Meanwhile a product of terms coming from each of the $\hbar$ and $\hbar^2$ terms above does not contribute to (5.7) for the same reason so the residue at infinity only picks up terms of order $\hbar^0, \hbar^1, \hbar^2$. We also need to check we only get contribution to these orders at the pair of poles corresponding to each $x = q_i$ separately but using (3.5) this is a direct calculation which is not too hard.

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