

# Classical Dynamics: Example Sheet 3

Copyright 2025: Faculty of Mathematics, University of Cambridge.

1. Show that the effect of three rotations by Euler angles results in the relationship  $\mathbf{e}_a = R_{ab}\tilde{\mathbf{e}}_b$  between the body frame axes  $\{\mathbf{e}_a\}$  and the space frame axes  $\{\tilde{\mathbf{e}}\}$  where the orthogonal matrix  $R$  is

$$R = \begin{pmatrix} \cos \psi \cos \phi - \cos \theta \sin \phi \sin \psi & \sin \phi \cos \psi + \cos \theta \sin \psi \cos \phi & \sin \theta \sin \psi \\ -\cos \phi \sin \psi - \cos \theta \cos \psi \sin \phi & -\sin \psi \sin \phi + \cos \theta \cos \psi \cos \phi & \sin \theta \cos \psi \\ \sin \theta \sin \phi & -\sin \theta \cos \phi & \cos \theta \end{pmatrix}$$

Use this to confirm that the angular velocity  $\boldsymbol{\omega}$  can be expressed in terms of Euler angles as

$$\boldsymbol{\omega} = [\dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi] \mathbf{e}_1 + [\dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi] \mathbf{e}_2 + [\dot{\psi} + \dot{\phi} \cos \theta] \mathbf{e}_3 \quad (1)$$

in the body frame  $\{\mathbf{e}_a\}$ . Or, alternatively, as

$$\boldsymbol{\omega} = [\dot{\psi} \sin \theta \sin \phi + \dot{\theta} \cos \phi] \tilde{\mathbf{e}}_1 + [-\dot{\psi} \sin \theta \cos \phi + \dot{\theta} \sin \phi] \tilde{\mathbf{e}}_2 + [\dot{\phi} + \dot{\psi} \cos \theta] \tilde{\mathbf{e}}_3 \quad (2)$$

in the space frame  $\{\tilde{\mathbf{e}}_a\}$ .

2. The physicist Richard Feynman tells the following story:

*“I was in the cafeteria and some guy, fooling around, throws a plate in the air. As the plate went up in the air I saw it wobble, and I noticed the red medallion of Cornell on the plate going around. It was pretty obvious to me that the medallion went around faster than the wobbling.*

*I had nothing to do, so I start figuring out the motion of the rotating plate. I discover that when the angle is very slight, the medallion rotates twice as fast as the wobble rate – two to one. It came out of a complicated equation!*

*I went on to work out equations for wobbles. Then I thought about how the electron orbits start to move in relativity. Then there’s the Dirac equation in electrodynamics. And then quantum electrodynamics. And before I knew it....the whole business that I got the Nobel prize for came from that piddling around with the wobbling plate.”*

Feynman was right about quantum electrodynamics. But what about the plate?

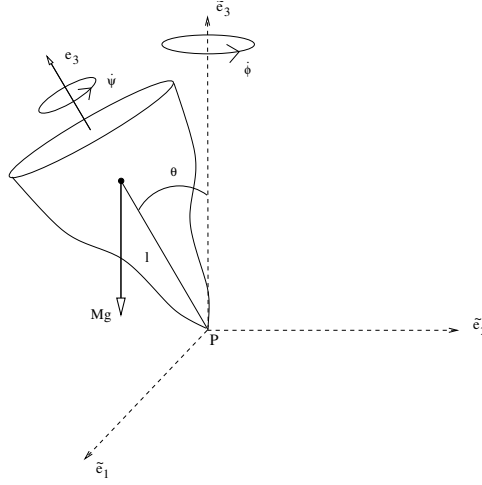


Figure 1: The Euler angles for the heavy symmetric top

**3.** Consider a heavy symmetric top of mass  $M$ , pinned at point  $P$  which is a distance  $l$  from the centre of mass. The principal moments of inertia about  $P$  are  $I_1$ ,  $I_1$  and  $I_3$  and the Euler angles are shown in the figure. The top is spun with initial conditions  $\dot{\phi} = 0$  and  $\theta = \theta_0$ . Show that  $\theta$  obeys the equation of motion,

$$I_1 \ddot{\theta} = -\frac{\partial V_{\text{eff}}(\theta)}{\partial \theta} \quad (3)$$

where

$$V_{\text{eff}}(\theta) = \frac{I_3^2 \omega_3^2 (\cos \theta - \cos \theta_0)^2}{2I_1 \sin^2 \theta} + Mgl \cos \theta \quad (4)$$

Suppose that the top is spinning very fast so that

$$I_3 \omega_3 \gg \sqrt{Mgl I_1} \quad (5)$$

Show that  $\theta_0$  is close to the minimum of  $V_{\text{eff}}(\theta)$ . Use this fact to deduce that the top nutates with frequency

$$\Omega \approx \frac{\omega_3 I_3}{I_1} \quad (6)$$

and draw the subsequent motion.

**4.** Throw a book in the air. If the principal moments of inertia are  $I_1 > I_2 > I_3$ , convince yourself that the book can rotate in a stable manner about the principal axes  $\mathbf{e}_1$  and  $\mathbf{e}_3$ , but not about  $\mathbf{e}_2$ .

Use Euler's equations to show that the energy  $E$  and the total angular momentum  $\mathbf{L}^2$  are conserved. Suppose that the initial conditions are such that

$$\mathbf{L}^2 = 2I_2E \quad (7)$$

with the initial angular velocity  $\boldsymbol{\omega}$  perpendicular to the intermediate principal axes  $\mathbf{e}_2$ . Show that  $\boldsymbol{\omega}$  will ultimately end up parallel to  $\mathbf{e}_2$  and derive the characteristic time taken to reach this steady state.

5. A rigid lamina (i.e. a two dimensional object) has principal moments of inertia about the centre of mass given by,

$$I_1 = (\mu^2 - 1) \quad I_2 = (\mu^2 + 1) \quad , \quad I_3 = 2\mu^2 \quad (8)$$

Write down Euler's equations for the lamina moving freely in space. Show that the component of the angular velocity in the plane of the lamina (i.e.  $\sqrt{\omega_1^2 + \omega_2^2}$ ) is constant in time.

Choose the initial angular velocity to be  $\boldsymbol{\omega} = \mu N \mathbf{e}_1 + N \mathbf{e}_3$ . Define  $\tan \alpha = \omega_2/\omega_1$ , which is the angle the component of  $\boldsymbol{\omega}$  in the plane of the lamina makes with  $\mathbf{e}_1$ . Show that it satisfies

$$\ddot{\alpha} + N^2 \cos \alpha \sin \alpha = 0 \quad (9)$$

and deduce that at time  $t$ ,

$$\boldsymbol{\omega} = [\mu N \operatorname{sech} Nt] \mathbf{e}_1 + [\mu N \tanh Nt] \mathbf{e}_2 + [N \operatorname{sech} Nt] \mathbf{e}_3 \quad (10)$$

6. The Lagrangian for the heavy symmetric top is

$$L = \frac{1}{2}I_1 \left( \dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) + \frac{1}{2}I_3 (\dot{\psi} + \dot{\phi} \cos \theta)^2 - Mgl \cos \theta \quad (11)$$

Obtain the momenta  $p_\theta$ ,  $p_\phi$  and  $p_\psi$  and the Hamiltonian  $H(\theta, \phi, \psi, p_\theta, p_\phi, p_\psi)$ .

7. A system with two degrees of freedom  $x$  and  $y$  has the Lagrangian,

$$L = x\dot{y} + y\dot{x}^2 + \dot{x}\dot{y} \quad (12)$$

Derive Lagrange's equations. Obtain the Hamiltonian  $H(x, y, p_x, p_y)$ . Derive Hamilton's equations and show that they are equivalent to Lagrange's equations.