Dynamics and Relativity: Example Sheet 2

1. In a system of particles, the ith particle has mass m_i and position vector $\mathbf{x}_i$ with respect to a fixed origin. The centre of mass of the system is at $\mathbf{R}$. Show that $\mathbf{L}$, the total angular momentum of the system about the origin, and $\mathbf{L}_{\text{CoM}}$, the total angular momentum of the system about the centre of mass, are related by

$$\mathbf{L}_{CoM} = \mathbf{L} - \mathbf{R} \times \mathbf{P}$$

where $\mathbf{P}$ is the total linear momentum of the system.

Given that $d\mathbf{P}/dt = \mathbf{F}$ where $\mathbf{F}$ is the total external force and $d\mathbf{L}/dt = \boldsymbol{\tau}$ where $\boldsymbol{\tau}$ is the total external torque about the origin, show that

$$\frac{d\mathbf{L}_{CoM}}{dt} = \boldsymbol{\tau}_{CoM},$$

where $\boldsymbol{\tau}_{CoM}$ is the total external torque about the centre of mass.

2. A system of particles with masses m_i and position vectors $\mathbf{x}_i$, $i = 1, \ldots, n$, moves under its own mutual gravitational attraction alone. Write down the equation of motion for $\mathbf{x}_i$. Show that a possible solution of the equations of motion is given by $\mathbf{x}_i = t^{2/3}\mathbf{a}_i$, where the vectors $\mathbf{a}_i$ are constant vectors satisfying

$$\mathbf{a}_i = \frac{9G}{2} \sum_{j \neq i} \frac{m_j(\mathbf{a}_i - \mathbf{a}_j)}{|\mathbf{a}_i - \mathbf{a}_j|^3}.$$

Show that, for this system, the total angular momentum about the origin and the total momentum both vanish. What is the angular momentum about any other fixed point?

3. A rocket, moving vertically upwards, ejects gas vertically downwards at speed u relative to the rocket. Derive the equation of motion

$$m\frac{dv}{dt} = -u\frac{dm}{dt} - gm$$

where v and m are the speed and total mass of the rocket (including fuel) at time t. If u is constant and the rocket starts from rest with total mass m_0, show that

$$m = m_0 e^{-(gt+v)/u}.$$

4. A firework of initial mass m_0 is fired vertically upwards from the ground. The rate of burning of fuel $dm/dt = -\alpha$ and the fuel is ejected at constant speed u relative to the firework. Show that the speed of the firework at time t, where $0 < t < m_0/\alpha$, is

$$v(t) = -gt - u\log\left(1 - \frac{\alpha t}{m_0}\right)$$

and that this is positive provided $u > m_0 g/\alpha$.

Suppose now that nearly all of the firework consists of fuel, the mass of the containing shell being negligible. Show that the height attained by the shell when all of the fuel is burnt is

$$\frac{m_0}{\alpha}\left(u - \frac{m_0 g}{2\alpha}\right)$$

5. A particle moves in a fixed plane and its position vector at time t is $\mathbf{r}$. Let (r, θ) be plane polar coordinates and let $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ be unit vectors in the direction of increasing r and increasing θ respectively. Show that

$$\dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

The particle moves outwards with speed v on the equiangular spiral $r = a\exp(\theta\cot\alpha)$, where a and α are constants, with $0 < \alpha < \frac{1}{2}\pi$. Show that

$$v\sin\alpha = r\dot{\theta}$$

and hence that

$$\dot{\mathbf{r}} = v\cos\alpha\,\hat{\mathbf{r}} + v\sin\alpha\,\hat{\boldsymbol{\theta}}$$

Give an expression for $\ddot{\mathbf{r}}$ and show that $|\ddot{\mathbf{r}}|^2 = \dot{v}^2 + v^2\dot{\theta}^2$.

If $\dot{\theta}$ takes a constant value ω, show that the acceleration has magnitude v^2/r and is directed at an angle 2α to the position vector.

6. For particle subject to an inverse square force given by $\mathbf{F} = -mk\hat{\mathbf{r}}/r^2$, the vectors $\mathbf{h}$ and $\mathbf{e}$ are defined by

$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}} \quad \text{and} \quad \mathbf{e} = \frac{\dot{\mathbf{r}} \times \mathbf{h}}{k} - \frac{\mathbf{r}}{r}$$

Show that $\mathbf{h}$ is constant and deduce that the particle moves in a plane through the origin. (Note that in the lectures, the vector $\mathbf{h}$ was called $\mathbf{l}$).

The vector $\mathbf{e}$ is known as the *Laplace-Runge-Lenz* vector. Show that it too is is constant and that

$$er\cos\theta = h^2/k - r$$

where $e = |\mathbf{e}|$, $h = |\mathbf{h}|$ and θ is the angle between $\mathbf{r}$ and $\mathbf{e}$. Deduce that the orbit is a conic section.

7. In these orbital questions, the particles move in a gravitational potential

$$V = -\frac{km}{r} \quad \text{with} \quad k > 0$$

You should answer the following questions *only* using the conservation of energy and angular momentum together with (for circular orbits) the radial component of the equation of motion.

(a) Show that the radius, R, of the orbit of a satellite in geostationary orbit (in the equatorial plane) is approximately $(28)^{-2/3}R_m$, where R_m is the radius of the moon's orbit round the Earth.

(b) A particle moves in a parabolic orbit and another particle moves in a circular orbit. Show that if they pass through the same point then the ratio of their speeds at this point is $\sqrt{2}$. For a satellite orbiting the Earth in a circular orbit, what is the relationship between its orbiting speed and its escape velocity?

If, instead of passing through the same point, the particles have the same angular momentum per unit mass, show that the perihelion distance of the parabola is half the radius of the circle.

(c) A particle moves with angular momentum l per unit mass in an ellipse, for which the distances from the focus to the periapsis (closest point to focus) and apoapsis (furthest point) are p and q, respectively. Show that

$$l^2\left(\frac{1}{p} + \frac{1}{q}\right) = 2k$$

Show also that the speed V of the particle at the periapsis is related to the speed v of a particle moving in a circular orbit of radius p by $V^2 = 2v^2(1 + p/q)^{-1}$.

(d) A particle P is initially at a very large distance from the origin moving with speed v on a trajectory that, in the absence of any force, would be a straight line for which the shortest distance from the origin is b. The shortest distance between P's actual trajectory and the origin is d. Show that $2kd = v^2(b^2 - d^2)$

8. Two particles of masses m_1 and m_2 move under their mutual gravitational attraction. Show from first principles that the quantity

$$\frac{1}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - \frac{GM}{r}$$

is constant, where $\mathbf{r}$ is the position vector of one particle relative to the other and $M = m_1 + m_2$.

The particles are released from rest a long way apart, and fall towards each other. Show that the position of their centre of gravity is fixed, and that when they are a distance r apart their relative speed is $\sqrt{2GM/r}$.

When the particles are a distance a apart, they are given equal and opposite impulses (change of momentum) each of magnitude I, and each perpendicular to the direction of motion. Show that subsequently $r^2\omega = aI/\mu$, where ω is the angular speed of either particle relative to the centre of mass and μ is the reduced mass of the system.

Show further that the minimum separation, d, of the two particles in the subsequent motion satisfies

$$(a^2 - d^2)I^2 = 2GM\mu^2 d$$