Electrodynamics: Example Sheet 1

1 Show that the Lagrangian density $\mathcal{L}$, where $S = \int \mathcal{L}\, dt\, d^3x$, for the electromagnetic action

$$S = -\frac{1}{4\mu_0 c} \int d^4x \; F^{\mu\nu} F_{\mu\nu} + \frac{1}{c} \int d^4x \; A^\mu J_\mu$$

can be written as

$$\mathcal{L} = \frac{\varepsilon_0}{2} |\nabla\phi + \partial\mathbf{A}/\partial t|^2 - \frac{1}{2\mu_0} |\nabla \times \mathbf{A}|^2 - \rho\phi + \mathbf{J} \cdot \mathbf{A},$$

where $A^\mu = (\phi/c, \mathbf{A})$ and $J^\mu = (\rho c, \mathbf{J})$. Vary the action with respect to ϕ and $\mathbf{A}$ directly to obtain the sourced Maxwell equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

2. Consider the action

$$S = \frac{1}{c} \int d^4x \; \left(-\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu \right)$$

for a prescribed 4-current J^μ with $\partial_\mu J^\mu = 0$. Assuming

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,$$

show that requiring $\delta S = 0$ for arbitrary variations δA^μ that vanish at infinity implies one half of the Maxwell equations

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu.$$

Show also that S is gauge-invariant.

Now consider

$$S_P = \frac{1}{c} \int d^4x \; \left(\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\mu_0} F^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) + J^\mu A_\mu \right)$$

which reduces to S if $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Regarding A^μ and $F^{\mu\nu}$ as independent quantities, show that requiring $\delta S_P = 0$ for arbitrary variations δA^μ that vanish at infinity gives the same Maxwell equations as before. Show also that $\delta S_P = 0$ for arbitrary $\delta F^{\mu\nu}$ implies $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, and hence gives the other half of the Maxwell equations.

3. A particle of rest mass m and charge q moves in constant uniform fields $\mathbf{E} = (0, E, 0)$ and $\mathbf{B} = (0, 0, E/c)$, starting from rest at the origin. Show that

$$\frac{dt}{d\tau} - \frac{1}{c}\frac{dx}{d\tau} = 1,$$

and that

$$t = \tau + \frac{1}{6c^2}\alpha^2\tau^3, \qquad x = \frac{1}{6c}\alpha^2\tau^3, \qquad y = \frac{1}{2}\alpha\tau^2, \qquad z = 0,$$

where $\alpha = qE/m$. By projecting the orbit in the (t, x),(t, y), and (x, y)-planes, give a qualitative description of the motion.

4. The fields on either side of a physical boundary S with unit normal $\hat{n}$ (pointing from region 1 to 2 are $(\mathbf{E}_1, \mathbf{B}_1)$ and $(\mathbf{E}_2, \mathbf{B}_2)$. The discontinuities across S of the electromagnetic fields are

$$\mathbf{B}_2 - \mathbf{B}_1 = \mu_0 \mathbf{J}_S \times \hat{n} \quad \text{and} \quad \mathbf{E}_2 - \mathbf{E}_1 = \frac{\sigma_S \hat{n}}{\varepsilon_0},$$

where $\mathbf{J}_S$ is the surface current density and σ_S is the surface charge density. Verify that the net rate at which electromagnetic momentum flows into the discontinuity per unit area, $f_{S,i} = \sigma^1_{ij}\hat{n}_j - \sigma^2_{ij}\hat{n}_j$, is given by

$$\mathbf{f}_S = \frac{1}{2}\left[\mathbf{J}_S \times (\mathbf{B}_1 + \mathbf{B}_2) + \sigma_S(\mathbf{E}_1 + \mathbf{E}_2)\right],$$

so that $\mathbf{f}_S$ is the force per area acting on the surface.

[*Hint: You may find it easier to consider the electric and magnetic parts, and the parallel and perpendicular components, seperately.*]

5. Show that the equation $\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$ is equivalent to

$$\partial_\nu F_{\rho\sigma} + \partial_\rho F_{\sigma\nu} + \partial_\sigma F_{\nu\rho} = 0.$$

Using this and $\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu$, show that the electromagnetic stress-energy tensor

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(F^\mu{}_\rho F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)$$

satisfies $\eta_{\mu\nu}T^{\mu\nu} = 0$ and $\partial_\mu T^{\mu\nu} = -F^\nu{}_\rho J^\rho$. Verify that

$$T^{00} = \frac{1}{2\mu_0}\left(\frac{|\mathbf{E}|^2}{c^2} + |\mathbf{B}|^2\right) \quad \text{and} \quad T^{0i} = \frac{1}{\mu_0 c}(\mathbf{E} \times \mathbf{B})_i$$

and construct the components of the Maxwell stress tensor σ_{ij}.

[*Hint: You may wish to use* $\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\sigma} = -6\delta_\alpha{}^{[\mu}\delta_\beta{}^{\nu}\delta_\gamma{}^{\rho]}$ *where square brackets denote antisymmetrisation on the enclosed indices.*]

6. If J^μ is a conserved current (i.e. $\partial_\mu J^\mu = 0$), verify that the corresponding charge $Q = \int d^3x (J^0/c)$ is conserved.

If $T^{\mu\nu} = T^{\nu\mu}$ is a conserved stress-energy tensor (i.e. $\partial_\nu T^{\mu\nu} = 0$), verify, by considering $S^{\mu\nu\rho} = T^{\mu\rho}x^\nu - T^{\mu\nu}x^\rho$ or otherwise, that

$$M^{\mu\nu} = \int d^3x \, (x^\mu T^{0\nu} - x^\nu T^{0\mu})$$

is conserved.

Let $M^{ij} = c\varepsilon^{ijk}J_{\text{em},k}$. Show that for the electromagnetic field

$$\mathbf{J}_{\text{em}} = \varepsilon_0 \int d^3x \; \mathbf{x} \times (\mathbf{E} \times \mathbf{B}) \; .$$

By expressing the rate of change of $\mathbf{J}_{\text{em}}$ in terms of the charge and current densities, show that $\mathbf{J}_{\text{em}}$ may be interpreted as the angular momentum of the electromagnetic field.

7. A hypothetical magnetic monopole is fixed at the origin and has a magnetic field:

$$\mathbf{B}(\mathbf{x}) = \frac{g\mu_0\mathbf{x}}{4\pi|\mathbf{x}|^3}.$$

A particle of charge q is situated at position $\mathbf{r}$. Show that the angular momentum of the electromagnetic field can be written as

$$\mathbf{J}_{\text{em}} = \int d^3x \; \mathbf{x} \times \left(\frac{g\mu_0\mathbf{x}}{4\pi|\mathbf{x}|^3} \times \nabla \frac{q}{4\pi|\mathbf{x} - \mathbf{r}|} \right) = -\frac{gq\mu_0}{4\pi}\frac{\mathbf{r}}{|\mathbf{r}|},$$

after integrating by parts and neglecting a surface integral.

For non-relativistic motion of the electric charge, treat its electric field as due to a static charge at its location and ignore its magnetic field. Show directly that the total angular momentum $\mathbf{J} = \mathbf{r} \times \mathbf{p} + \mathbf{J}_{\text{em}}$ is constant, using $\dot{\mathbf{p}} = q\dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r})$.