Electrodynamics: Example Sheet 3

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1. A dielectric sphere of radius a and permittivity ε is placed at rest at the origin in a constant electric field which takes a uniform value $\mathbf{E}_0$ far from the origin. Show that Maxwell's equations with the appropriate boundary conditions are solved by an electric field which is uniform inside the sphere, with value:

$$\mathbf{E}_{\text{int}} = \frac{3}{2 + \varepsilon_r} \mathbf{E}_0$$

where $\varepsilon_r = \varepsilon/\varepsilon_0$, and a field outside the sphere which is the superposition of $\mathbf{E}_0$ and the field

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \left(\frac{3(\mathbf{p} \cdot \hat{\mathbf{x}})\hat{\mathbf{x}} - \mathbf{p}}{|\mathbf{x}|^3} \right)$$

of an electric dipole at the origin, with dipole moment

$$\mathbf{p} = 4\pi\varepsilon_0 \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right) a^3 \mathbf{E}_0.$$

By considering the macroscopic polarisation $\mathbf{P}$ inside the sphere, calculate the bound surface charge density σ. Verify explicitly that the electric dipole moment of σ is $\mathbf{p}$.

2. A permanent bar magnet can be modelled as a long cylinder of length l and radius a with uniform macroscopic magnetisation $\mathbf{M}$ along its axis. By considering the bound surface current due to $\mathbf{M}$, or otherwise, show that for $a \ll l$, the magnetic field on axis at the centre of the cylinder is $\mathbf{B} = \mu_0 \mathbf{M}$. Show further that the magnetic field on axis at the end of the cylinder is $\mathbf{B} = \mu_0 \mathbf{M}/2$. Sketch the field lines of the magnetic field $\mathbf{B}$ and the magnetising field $\mathbf{H}$ both inside and outside the cylinder.

3. Two plane, semi-infinite dielectric slabs with refractive index n are separated by a vacuum region of width d. A plane wave of wavenumber k is incident on the vacuum region from one of the slabs at an angle of incidence θ_I to the normal, with $n \sin \theta_I > 1$. The electric field of the incident radiation is normal to the plane of incidence (normal polarisation).

Show that some of the radiation is transmitted across the gap, and that the ratio of the amplitudes of the electric fields of the transmitted and incident waves satisfies:

$$\left| \frac{E_T}{E_I} \right|^2 = \frac{1}{1 + \alpha^2 \sinh^2 \kappa d},$$

where
$$\alpha = \frac{k(1 - 1/n^2)}{2\kappa \cos \theta_I} \quad \text{and} \quad \kappa = k\sqrt{\sin^2 \theta_I - 1/n^2}.$$
(This process is known as frustrated total internal reflection and is analogous to quantum tunnelling.)

[Hint: you will need to consider both growing and decaying evanescent waves in the gap since it is of finite width.]

4. Consider a Gaussian wavepacket propagating in a dispersive medium, taking the form
$$E(t, z) = \text{Re} \left[\int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} E_0(k) e^{i(kz - \omega(k)t)} \right]$$
where $E_0(k) \propto \exp[-(k - k_0)^2/(2\sigma^2)]$. Ignore any dissipative effects so that k and $\omega(k)$ are real-valued. Expanded to second order about k_0, the dispersion relation takes the form
$$\begin{aligned} \omega(k) &\approx \omega(k_0) + (k - k_0)\omega'(k_0) + \frac{1}{2}(k - k_0)^2\omega''(k_0) \\ &= k_0 v_p(k_0) + (k - k_0)v_g(k_0) + \frac{1}{2}(k - k_0)^2 v_g'(k_0) \end{aligned}$$
where primes denote derivatives with respect to k, and v_p and v_g are the phase and group velocities at k_0. Show that:
$$E(t, z) \propto \sqrt{\frac{\sigma}{\sigma_z(t)}} e^{-\frac{(z - v_g t)^2}{2\sigma_z^2(t)}} \cos\left[k_0(z - v_p t) + \frac{\sigma^2 v_g' t(z - v_g t)}{2\sigma_z^2(t)} - \frac{\phi(t)}{2} \right]$$
where $\sigma_z^2(t) = 1/\sigma^2 + (\sigma v_g' t)^2$ and $\tan \phi(t) = \sigma^2 v_g' t$. Note that the Gaussian modulation of the wavepacket has a width $\sigma_z(t)$ that spreads in time for non-zero v_g'.

Show that while the fractional increase in the width of the wavepacket is small, the terms involving v_g' have a negligible effect on the phase of the oscillation for $|z - v_g t| \leq \sigma_z(t)$.

5. Suppose that space is filled with a uniform, non-dispersive dielectric medium such that the refractive index is n. Write the macroscopic Maxwell equations in terms of the magnetic vector potential $\mathbf{A}$ (such that $\mathbf{B} = \nabla \times \mathbf{A}$) and electric potential ϕ (with $\mathbf{E} = -\partial \mathbf{A}/\partial t - \nabla \phi$), to show that, with a suitable choice of gauge:
$$\nabla^2 \mathbf{A} - \frac{n^2}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}_{\text{free}}, \quad \text{and} \quad \nabla^2 \phi - \frac{n^2}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho_{\text{free}}}{\varepsilon_0 n^2}.$$
Construct the Green's function for the wave operator in these equations to show that the (retarded) vector potential generated by the free current is:
$$\mathbf{A}(t, \mathbf{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\mathbf{J}_{\text{free}}(t - n|\mathbf{x} - \mathbf{x}'|/c, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

A point charge q moves through the dielectric with velocity $\mathbf{v}$, so that $\mathbf{J}_{\text{free}}(t, \mathbf{x}) = q\mathbf{v}\delta^{(3)}(\mathbf{x} - \mathbf{v}t)$. Show that

$$\mathbf{A}(t, \mathbf{x}) = \frac{\mu_0 q}{4\pi} \mathbf{v} f(\mathbf{x} - \mathbf{v}t)$$

where

$$f(\mathbf{x}) = \int d^3 r \, \frac{1}{|\mathbf{r}|} \delta^{(3)}(\mathbf{x} + \mathbf{r} + n\mathbf{v}|\mathbf{r}|/c) \ .$$

(Note: the integration variable here is $\mathbf{r} = \mathbf{x}' - \mathbf{x}$.) Taking the velocity to be along the positive z-direction, $\mathbf{v} = \beta c \hat{\mathbf{z}}$, show that

$$f(\rho, z) = \int_{-\infty}^{\infty} dr_{\parallel} \, \frac{1}{\sqrt{\rho^2 + r_{\parallel}^2}} \delta\left(z + r_{\parallel} + n\beta\sqrt{\rho^2 + r_{\parallel}^2}\right)$$

where $\rho = \sqrt{x^2 + y^2}$. For the case $n\beta > 1$ (i.e. the speed of the particle exceeds the speed of light in the dielectric), show that $f(\rho, z)$ vanishes unless $z \leq -\rho\sqrt{n^2\beta^2 - 1}$. Show further that

$$\mathbf{A}(t, \mathbf{x}) = \mathbf{v}\frac{\mu_0 q}{4\pi} \frac{2}{\sqrt{(z - vt)^2 + \rho^2(1 - n^2\beta^2)}} \quad \text{for} \quad z - vt \leq -\rho\sqrt{n^2\beta^2 - 1}$$

where $v = |\mathbf{v}|$. These shock-like fields describe *Cherenkov radiation*.

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