Fluid Dynamics I: Questions on Inviscid Flows

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1. Consider the two-dimensional flow $u = 1/(1+t)$, $v = 1$ in $t > -1$. Find and sketch

(i) the streamline at $t = 0$ which passes through the point $(1, 1)$,

(ii) the path of a fluid particle which is released from $(1, 1)$ at $t = 0$.

2, A steady two-dimensional flow (pure straining) is given by $u = \alpha x$, $v = -\alpha y$ with $\alpha > 0$ constant.

(i) Find the equation for a general streamline of the flow, and sketch some of them.

(ii) At $t = 0$ the fluid on the curve $x^2 + y^2 = a^2$ is marked (by an electro-chemical technique). Find the equation for this material fluid curve for $t > 0$.

(iii) Does the area within the curve change in time, and why?

3. Repeat question 2(ii) for the two-dimensional flow (simple shear) given by $u = \gamma y$, $v = 0$ with $\gamma > 0$ constant. Sketch the streamlines and the material curve at $\gamma t \approx 0, 1, 2$.

4. An incompressible two-dimensional flow is represented by a streamfunction $\psi(x, y)$ with $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Show that

(i) the streamlines are given by $\psi =$ constant.

(ii) $|\mathbf{u}| = |\nabla\psi|$, so that the flow is faster where the streamlines are closer,

(iii) the volume flux crossing any curve from $\mathbf{x}_0$ to $\mathbf{x}_1$ is given by $\psi(\mathbf{x}_1) - \psi(\mathbf{x}_0)$,

(iv) $\psi =$ constant on any *fixed* (i.e. stationary) boundary.

5. Verify that the two-dimensional flow given in Cartesian coordinates by

$$u = \frac{y - b}{(x-a)^2 + (y-b)^2}, \qquad v = \frac{a - x}{(x-a)^2 + (y-b)^2}$$

satisfies $\nabla \cdot \mathbf{u} = 0$, and then find the streamfunction $\psi(x, y)$ such that $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Sketch the streamlines.

6. Verify that the two-dimensional flow given in polar coordinates by

$$u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \qquad u_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

satisfies $\nabla \cdot \mathbf{u} = 0$ and find the streamfunction $\psi(r, \theta)$. Sketch the streamlines, starting with $\psi = 0$.

7. Verify that the axisymmetric flow (uniaxial straining) given in cylindrical polar coordinates by $u_r = -\frac{2}{\alpha}r$, $u_z = \alpha z$ satisfies $\nabla \cdot \mathbf{u} = 0$, and find the Stokes streamfunction $\Psi(r, z)$. Sketch the streamlines in the (r, z)-plane.

[*For axisymmetric flow in coordinates* (r, θ, z)

$$\nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z}, \quad \textit{and} \quad u_r = -\frac{1}{r}\frac{\partial \Psi}{\partial z}, \quad u_z = \frac{1}{r}\frac{\partial \Psi}{\partial r}\,.]$$

8. An axisymmetric jet of water of speed $U = 1\,\mathrm{m\,s^{-1}}$ and cross-section $A = 6\times 10^{-4}\,\mathrm{m}^2$ strikes a wall at right angles and spreads out over it. By using the momentum integral equation over a suitable control volume, and neglecting gravity, calculate the force on the wall due to the jet.

9. Starting from the Euler momentum equation for an incompressible fluid of density ρ with a potential force $-\nabla\chi$, show that for a fixed volume V enclosed by surface ∂V

$$\frac{d}{dt}\int_V dV\ \frac{1}{2}\rho u^2\, dV + \int_{\partial V} dA\ H\,\mathbf{u}\cdot\mathbf{n} = 0\,,$$

where $H = \rho u^2/2 + P + \chi$ is the Bernoulli quantity, so concluding that $H\mathbf{u}$ is the energy flux and H is the transportable energy. Comment on the interpretation of $\mathbf{u}\cdot\nabla H = 0$ in steady flow.

10. A cylindrical tank of radius a is filled to a depth h_0 with fluid of density ρ. The tank is rotated about its axis with angular velocity Ω for a long time, until the fluid rotates uniformly with it and $\mathbf{u} = (\Omega y, -\Omega x, 0)$. Use the Euler equation and the free-surface boundary condition to determine the pressure distribution $P(r, z)$ and the height of the free surface $h(r)$ for the case $h_0 \geqslant \Omega^2 a^2/4g$. Comment on the physical significance of the term $\mathbf{u}\cdot\nabla\mathbf{u}$.

11. How high can water rise up one's arm hanging in the river from a lazily moving ($1\,\mathrm{m\,s^{-1}}$) punt? [*Hint: Use Bernoulli on a surface streamline.*]

12. Waste water flows into a large open-topped tank with volume flux Q and out through a small exit pipe of cross-sectional area A into the air. In steady state, how high above the pipe is the water in the tank?

13. A water clock is an axisymmetric vessel with a small exit pipe in the bottom. Find the shape for which the water level falls equal heights in equal intervals of time.

14 A flat-bottomed barge closely fits a canal, so that while it travels slowly it still generates a fast current with speed U under it. Estimate how much lower the barge sits in the water as a result of this current when $U = 5\,\mathrm{m\,s^{-1}}$. [*Hint: Use Archimedes when stationary. Flow reduces pressure, so have to sit deeper for same pressure on long bottom.*]

15. Calculate the vorticity of the velocity field

$$u = -\alpha x - yrf(t), \quad v = -\alpha y + xrf(t), \quad w = 2\alpha z$$

where $r^2 = x^2 + y^2$. Use the (inviscid) vorticity equation to deduce that $f(t) \propto \mathrm{e}^{3\alpha t}$. Explain the nature of this flow and describe the physical principle illustrated by your result. (Why is the growth rate 3α?)

16. If $\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{x}$ (uniform rotation with angular velocity $\boldsymbol{\Omega}$) show that $\boldsymbol{\omega} = 2\boldsymbol{\Omega}$.

For a two-dimensional flow $(u(x,y), v(x,y), 0)$ show that $\boldsymbol{\omega} = (0, 0, -\nabla^2\psi)$, where ψ is the streamfunction.

A long cylinder filled with water has elliptical cross-section with major and minor semi-axes a and b. While $t < 0$ both the cylinder and the water within it rotate about the axis of the cylinder with uniform angular velocity $(0, 0, \Omega)$. What is the vorticity of the flow? Sketch the streamlines noting that they intersect the elliptical boundary of the cylinder. (Why?).

At $t = 0$ the cylinder is suddenly brought to rest. What is the vorticity for $t > 0$? Verify that the flow can be described by

$$\psi = \frac{a^2 b^2 \Omega}{a^2 + b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

in suitable coordinates and sketch the streamlines.

17. A sphere of radius a moves with constant velocity U through inviscid fluid otherwise at rest. How far ahead of the sphere is there a disturbance of magnitude $\frac{1}{20}U$? Show that the acceleration of a fluid particle at distance x ahead of the centre of the sphere is

$$3U^2\left(\frac{a^3}{x^4} - \frac{a^6}{x^7}\right).$$

18. Write down the velocity potential $\phi(x,y)$ for the two-dimensional flow produced by a point source of strength q located at the origin in a uniform stream $(U,0)$. Show that there is a stagnation point at $(-a,0)$, where $a = q/2\pi U$. Sketch the streamlines. Show that the streamfunction is given by $\psi = Uy + Ua\theta$, where θ is the polar angle from the positive x axis. From the sketch and the streamfunction show that ϕ represents the flow past a semi-infinite body whose width tends to $2\pi a$ far downstream.

19. The velocity in the far-field of steady uniform flow past a stationary two-dimensional aerofoil with circulation κ takes the form

$$\mathbf{u} = (U,0) + \frac{\kappa}{2\pi r}(-\sin\theta, \cos\theta) + O(1/r^2)\,,$$

where the $O(1/r^2)$ dipole term depends on the detailed shape of the object. Determine the pressure $P(r,\theta)$ in the far-field to the same level of approximation. Use the momentum integral equation to show that the aerofoil experiences a force $(0, -\rho U\kappa)$.

20. An orifice in the side of an open vessel containing water leads smoothly into a horizontal tube of uniform cross-section and length L. The diameter of the tube is small compared with L, with the horizontal dimensions of the free surface, and with the depth h of the orifice below the free surface. A plug at the end of the tube is suddenly removed and the water begins to flow. Show, using the expression for the pressure in unsteady irrotational flow, that the outflow velocity at subsequent times t is approximately

$$\sqrt{2gh}\tanh\left(\frac{t\sqrt{2gh}}{2L}\right).$$

Estimate the time scale for the flow in a garden hose to accelerate to its maximum velocity (Assume that tap pressure is equivalent to $\rho g h$ with $h = 5\,\mathrm{m}$.)

21. A rigid circular disc of radius R is at a height $h(t)$ above a fixed horizontal plane $z = 0$, and inviscid incompressible fluid fills the gap $0 < z < h(t)$, $r < R$ between them. Assume that $h \ll R$ and that the axisymmetric flow in the thin gap has radial component $u_r(r,t)$ independent of z. Use conservation of mass and the boundary conditions to deduce that the velocity in the gap is given by

$$\mathbf{u} = \nabla\phi \quad \text{with} \quad \phi = \frac{\dot{h}}{4h}\left(2z^2 - r^2\right)\,.$$

Assuming that the pressure at the edge of the disc is a constant P_0 (as velocities and pressures are much larger in the thin gap than elsewhere), find the pressure distribution in the gap and hence determine the force on the plane due to the motion.

22. A rigid sphere of radius a executes small-amplitude oscillations with velocity $U(t)\mathbf{e}_z$ about the centre $r = 0$ of a larger fixed sphere of radius b. By linearising the boundary condition on the smaller sphere onto $r = a$, find the velocity potential for the induced irrotational motion of fluid that fills the gap between the two spheres and, again neglecting terms quadratic in the amplitude, show that the (dynamic) pressure on the surface of the inner sphere is

$$\frac{a^3 + \frac{2}{b}^{3}}{b^3 - a^3}\,\rho \dot{U} a \cos\theta\,,$$

where θ is the angle from $\mathbf{e}_z$. Hence find the force exerted by the fluid on the inner sphere. Why is the force on the outer fixed sphere different? Comment on the case of a tight fit.

23. A U-tube consists of two long uniform vertical tubes of different cross-sectional areas A_1, A_2 connected at the base by a short tube of large cross-section, and contains an inviscid, incompressible fluid whose surface, in equilibrium, is at height h above the base. Derive the equation governing the nonlinear oscillations of the displacement $\zeta(t)$ of the surface in the tube of cross-section A_2

$$(h + r\zeta)\frac{d^2\zeta}{dt^2} + \frac{r}{2}\left(\frac{d\zeta}{dt}\right)^2 + g\zeta = 0 \quad \text{where} \quad r = 1 - A_2/A_1.$$