Fluid Mechanics: Questions on Waves

1. Water fills a square container $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant a$ to an equilibrium depth h. Write down the equation and (exact) boundary conditions for the velocity potential and the motion of the free surface when it is disturbed from equilibrium. Explain how to linearise the free-surface conditions for small-amplitude disturbances. Seek separable solutions proportional to $\exp(-i\omega t)$ to the linearised equations, and thence obtain the frequencies of the 'normal modes'. Show the sign of the surface displacement in plan view for the five lowest frequency modes.

2. Fluid of density ρ_1 occupies the region $z > 0$ and overlies another fluid of density ρ_2 (with $\rho_2 > \rho_1$), which occupies the region $z < 0$. Show that small-amplitude oscillations with interfacial displacement $\zeta(x,t) \propto \exp[i(kx - \omega t)]$, $k > 0$, satisfy the dispersion relation

$$\omega^2 = gk\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right).$$

[*Hint: You will need different potentials ϕ_1 and ϕ_2 for the two regions and should apply the kinematic boundary condition to the flow in each region.*]

3. The dispersion relation for water waves of wavenumber k, including the effects of surface tension, is

$$\omega^2 = k(g + Tk^2/\rho)\tanh kh$$

Show that for sufficiently large k the group and phase velocities v_g and c become proportional to $k^{1/2}$ and independent of g and h, and that $v_g \sim \frac{3}{2}c$. What is 'sufficiently large'?

In ripple-tank experiments it is desired to keep v_g and c as constant as possible for smallish values of kh. By expanding ω^2 about $k = 0$, determine approximately what value of h, h_0 say, should be used. Show also that for $h > h_0$ there must exist a minimum value of the group velocity at some finite non-zero value of k.

4. Two semi-infinite layers of fluid with uniform densities $\rho_0 - \Delta\rho$ and $\rho_0 + \Delta\rho$ are separated by a layer of fluid in $-H \leqslant z \leqslant H$, where $\rho(z) = \rho_0 - (z/H)\Delta\rho$ and $\Delta\rho \ll \rho_0$. Write down the equation governing the vertical velocity of small-amplitude waves and use it to explain why w and $\partial w/\partial z$ should be expected to be continuous at $z = \pm H$.

Show that the dispersion relation for waves trapped by the stratification can be written

$$\left(\frac{N^2}{\omega^2}-1\right)^{1/2}\tan\left[\left(\frac{N^2}{\omega^2}-1\right)^{1/2}kH\right]=1$$

under the assumption that w is an even function of z (where N^2 is the middle-layer value).

5. An interface at $x=0$ separates fluid of density ρ_0 and sound speed c_0 in $x<0$ from fluid of density ρ_1 and sound speed c_1 in $x>0$. A plane harmonic sound wave is incident from $x<0$ with wavevector $\mathbf{k}=(k,0,0)$ and amplitude A (of its pressure perturbation). What is the frequency ω and the wavevector $\mathbf{k}'$ of the transmitted sound wave in $x>0$?

Write down the form of the pressure perturbation in $x<0$ and $x>0$, find the corresponding velocity potential and state the interfacial boundary conditions. Hence find the amplitudes of the reflected and transmitted waves.

Assume that $A=1$. Verify that the time-averaged acoustic energy flux is conserved. When is all the energy flux transmitted? How much is reflected if $\rho_0 \gg \rho_1$ and $c_0 \approx c_1$?

6. Find solutions to the wave equation of the form

$$\phi(x,y,t)=\exp(ikx-i\omega t)f(y) \tag{1}$$

for the case $k>\omega/c_0>0$. Hence find the solution in $y \geqslant 0$ in which there is no disturbance as $y\to\infty$ and waves are forced by the inhomogenous boundary condition

$$v=\mathrm{Re}\left[v_0\exp(ikx-i\omega t)\right] \quad \text{on} \quad y=0$$

Here $\nabla\phi=(u,v,0)$ and v_0 is a real constant. Over what lengthscale do the waves decay away from the boundary?

Calculate the time-averaged acoustic energy flux $\langle\,\mathbf{I}\,\rangle$ and verify that:

(i) the energy flux perpendicular to the boundary $y=0$ satisfies $\langle\, I_y\,\rangle=0$;

(ii) the energy flux parallel to the boundary satisfies $\langle\, I_x\,\rangle = c\,\langle\, E\,\rangle$ at any position y, where E is the acoustic energy density and $c=\omega/k$ is the phase velocity in the x-direction. [Since $c<c_0$, the disturbance and its energy travel subsonically along the boundary.]

Assuming that surface tension and gravity are negligible, determine whether a non-zero solution can exist in which evanescent sound waves propagate along both sides of an (unforced) interface between two fluids with different physical properties in $y < 0$ and $y > 0$,

7. Find solutions to the wave equation of the form (1) for a region $0 < y < h$ with a rigid boundary at $y = 0$ and a free boundary at $y = h$. (Take $\omega > 0$, but make no *a priori* assumption about k.) Show that a wave can propagate in the x-direction only if ω exceeds a critical value ω_c. What happens if a disturbance is generated at $x = 0$ with frequency $\omega < \omega_c$?

8. Explain why the general spherically symmetric solution $\phi(r, t)$ to the wave equation can be written as

$$\phi = \frac{-1}{4\pi\rho_0}\left(\frac{Q_-(t - r/c_0)}{r} + \frac{Q_+(t + r/c_0)}{r}\right)$$

where $Q_\pm$ are arbitrary functions. Assume from now on that there are only outgoing waves. Calculate the radial velocity u_r and the pressure perturbation $\tilde{P}$.

(i) By considering the volume flux through a sphere of radius ϵ as $\epsilon \to 0$, show that $Q_-(t)$ is the mass flux out of $r = 0$. Show also that ϕ actually satisfies

$$\nabla^2\phi - \frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2} = Q_-(t)\delta(\mathbf{x})/\rho_0$$

where δ is the Dirac delta function. (*Hint*: integrate this differential equation over $r \leqslant \epsilon$ and let $\epsilon \to 0$.)

(ii) Show that in the far-field, i.e. for 'large' r, the kinetic energy density K, the potential energy density W, and the acoustic-energy flux $\mathbf{I} = \tilde{P}\mathbf{u}$, approximately satisfy the same equations, $K = W$ and $I = (K + W)c_0$, as in a plane wave. Similarly, show that the total power radiated across a 'large' sphere of radius R is approximately

$$\left(\dot{Q}_-(t - R/c_0)\right)^2/4\pi\rho_0 c_0$$

What does 'large r' mean for a time-harmonic source with $Q_-(t) = \mathrm{Re}(q_0 e^{i\omega t})$?

9. A bubble makes small spherically symmetric oscillations in a compressible inviscid fluid. When the radius $a(t)$ is perturbed slightly from its mean value a_0, the internal dynamics of the bubble are such that the bubble exerts a perturbation pressure $-\beta(a - a_0)$ on the fluid, where β is a constant. Derive the linearised equation of motion for the oscillations

$$\rho_0 a_0 \ddot{a} + \frac{\beta a_0}{c_0}\dot{a} + \beta(a - a_0) = 0$$

where ρ_0 is the undisturbed density of the fluid and c_0 is the sound speed (you may use results from Question 8). What is the mechanism of energy loss from the oscillations represented by the 'damping' term in this ODE for a?

10. At time $t = 0$ the velocity $u(x,t)$ in a one-dimensional simple wave, propagating in the positive x direction through a perfect gas, has the form $u = u_m \sin kx$, where u_m and k are positive constants. Find the time t^* at which shocks form. Sketch $u(x)$ at times $t = 0, t = \frac{1}{2}t^*$ and $t = t^*$. Show that in the time interval $(0, t^*)$ a single *wave-crest* (i.e. a local maximum of $u(x,t)$) travels a distance

$$\frac{1}{k}\left(\frac{2c_0}{(\gamma+1)u_m} + 1\right).$$

11. A perfect gas, initially at rest, occupies the region to the right of a piston whose position is $X(t) = \frac{1}{2}at^2$ for $t > 0$. Find the time and position where a shock first forms.

12. An artery is modelled as a long straight tube with elastic walls and cross-sectional area $A(x,t)$, which contains incompressible, inviscid blood of density ρ. On the assumption that the fluid velocity u and pressure P do not vary across the artery, conservation of mass and momentum imply that

$$A_t + (uA)_x = 0 \qquad \text{and} \qquad \rho u_t + \rho u u_x = -p_x.$$

The area A is related to the fluid pressure by an elastic 'tube law' of the form $P = P(A)$, where $P(A)$ is some given, strictly increasing function. Find the Riemann invariants and their corresponding propagation speeds.

Now suppose that

$$P(A) = p_0 + \frac{\rho c_0^2}{2\kappa}\left(\frac{A}{A_0}\right)^{2\kappa}$$

where p_0, A_0, c_0 and κ are positive constants. For $t < 0$ the artery has uniform area A_0 and there is no flow. Blood is then pumped into the artery ($x > 0$) with velocity $U(t)$ at $x = 0$, where

$$U(t) = \frac{U_0 t}{t_1}\left(2 - \frac{t}{t_1}\right) \quad \text{when} \quad 0 \leqslant t \leqslant 2t_1$$

and $U(t) = 0$ for $t > 2t_1$, where $U_0(1-\kappa) < c_0$. Calculate the time and place at which a 'shock' first forms.

13. A piston moves with constant positive velocity u_1 into a perfect gas of specific heat ratio $\gamma > 1$, generating a shock wave which moves ahead of the piston. Show that a possible solution of all the relevant equations is one in which the gas is at rest beyond the shock, at pressure p_0, and is moving with constant velocity u_1 in the region between the piston and the shock, throughout which region the density and pressure also take constant values ρ_1, p_1 which are determined by

$$\frac{\rho_1}{\rho_0} = \frac{2\gamma + (\gamma+1)\beta}{2\gamma + (\gamma-1)\beta} \quad \text{and} \quad \frac{1}{\beta^2} + \frac{\gamma+1}{2\gamma\beta} = \frac{c_0^2}{\gamma^2 u_1^2}$$

where β is the shock strength defined as $(p_1 - p_0)/p_0 > 0$, and ρ_0 and c_0 are the density and sound speed of the undisturbed gas. Show also that the shock speed

$$V = c_0 \left(1 + \frac{1+\gamma}{2\gamma}\beta\right)^{1/2}$$

14. Assume that the speed of cars down a long straight (one-way) road is a known, monotonically decreasing function $u(\rho)$ of the local density ρ of traffic. The flux of cars is thus given by $q(\rho) = \rho u$. From conservation of cars deduce that ρ is constant on characteristics $dx/dt = c(\rho)$, where $c = dq/d\rho$. Deduce also that if a shock develops between regions of density ρ_1 and ρ_2 then it propagates with speed $[q(\rho_1) - q(\rho_2)]/(\rho_1 - \rho_2)$.

Consider the case $u(\rho) = U(1 - \rho/\rho_0)$ where U is (10% faster than) the speed limit and ρ_0 is the density of a nose-to-tail traffic jam. Sketch the functions $q(\rho)$ and $c(\rho)$. Explain why shocks only form when light traffic is behind heavy traffic, and why the shocks can travel either forwards or backwards depending on the density of traffic.

A queue of cars with density ρ_0 is waiting in $-L < x < 0$ behind a red traffic light at $x = 0$. There are no other cars on the road. The light turns green at $t = 0$. Find the time T when the last car starts to move, and determine the velocity of the last car for $t > T$.

15. A bistable system with diffusion is given by

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} - p(p-r)(p-1),$$

where $0 < r < 1$. Seek a travelling wave solution by setting $\xi = x - ct$ and $p(x,t) = f(\xi)$, and find the differential equation satisfied by f.

a) Rewrite your differential equation as two first order equations. Suppose that c takes the exact value that allows a travelling wave solution (you will need to consider $r < 1/2$ and $r > 1/2$ separately). Sketch the phase plane for the system, marking the trajectory that corresponds to the travelling wave.

b) Impose the (slightly odd requirement) that the solution to the original second-order differential equation also satisfies $f' = af(f-1)$. What values of a and c yield a valid solution? By solving this first-order equation for f, give the corresponding solution for $p(x,t)$.

16. The spread of an disease in one spatial dimension can be modelled by considering the susceptible population $S(t,x)$ and infected population $I(t,x)$, which obey

$$\frac{\partial S}{\partial t} = -\beta IS + D\frac{\partial^2 S}{\partial x^2}$$
$$\frac{\partial I}{\partial t} = +\beta IS - \nu I + D\frac{\partial^2 I}{\partial x^2}.$$

Suppose that an disease wave arrives in a previously uninfected region (so $S \approx N$, the total population, and $I \approx 0$). Consider the dynamics near this wave front by taking

$$S = N - u(\xi) \quad \text{and} \quad I = v(\xi)$$

with $\xi = x - ct$, and linearise in u and v. You may assume that the system will settle to the slowest possible wave speed. Find the wave speed of the epidemic.