Principles of Quantum Mechanics: Example Sheet 2

1. Let $\hat{A}$ and $\hat{B}$ be any operators that commute with $[\hat{A},\hat{B}]$, and let $\lambda \in \mathbb{C}$.

i) Prove that $[\hat{A},\hat{B}^n] = n\hat{B}^{n-1}[\hat{A},\hat{B}]$ for all $n \in \mathbb{Z}^+$, and that $[\hat{A}, e^{\hat{B}}] = e^{\hat{B}}[\hat{A},\hat{B}]$.

ii) Define $F(\lambda) = e^{\lambda\hat{A}}e^{\lambda\hat{B}}e^{-\lambda(\hat{A}+\hat{B})}$. Show that $F'(\lambda) = \lambda[\hat{A},\hat{B}]F(\lambda)$. Hence deduce that

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]} = e^{\hat{B}}e^{\hat{A}}e^{[\hat{A},\hat{B}]} \ .$$

iii) Now let $\hat{A}$ and $\hat{B}$ be any operators (not necessarily commuting with $[\hat{A},\hat{B}]$. Prove that

$$\frac{d}{d\lambda}\left(e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}\right) = e^{\lambda\hat{A}}[\hat{A},\hat{B}]e^{-\lambda\hat{A}}.$$

Hence deduce that

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2}[\hat{A},[\hat{A},\hat{B}]] + \cdots .$$

2. Let $\hat{x}(t) = e^{i\hat{H}t/\hbar}\hat{x}e^{-i\hat{H}t/\hbar}$ and $\hat{p}(t) = e^{i\hat{H}t/\hbar}\hat{p}e^{-i\hat{H}t/\hbar}$, where $\hat{x}$ and $\hat{p}$ are the usual position and momentum operators, and $\hat{H}$ is the Hamiltonian of the 1D harmonic oscillator. Show that

$$\hat{x}(t) = \hat{x}\cos(\omega t) + \frac{1}{m\omega}\hat{p}\sin(\omega t), \quad \hat{p}(t) = \hat{p}\cos(\omega t) - m\omega\hat{x}\sin(\omega t).$$

Interpret this result. Evaluate $[\hat{x}(t),\hat{p}(t)]$.

3. A Fermi oscillator has Hilbert space $\mathcal{H} = \mathbb{C}^2$ and Hamiltonian $\hat{H} = \hat{B}^\dagger\hat{B}$, where $\hat{B}^2 = 0$ and $\hat{B}^\dagger\hat{B} + \hat{B}\hat{B}^\dagger = 1$.

Find the eigenvalues of $\hat{H}$. If $|0\rangle$ satisfies $\hat{H}|0\rangle = 0$ and $\langle 0|0\rangle = 1$, find $\hat{B}|0\rangle$ and $\hat{B}^\dagger|0\rangle$. Obtain a matrix representation of $\hat{B}$, $\hat{B}^\dagger$ and $\hat{H}$.

4. Let $\hat{\mathbf{p}}/\hbar$ and $\hat{\mathbf{L}}/\hbar$ be the generators of translations and rotations. By considering the effect of a rotation and translation on $\mathbf{v} \in \mathbb{R}^3$, show that $[\hat{J}_i,\hat{p}_j] = i\hbar\epsilon_{ijk}\hat{p}_k$.

5. A quantum particle in $\mathcal{H} = \mathcal{L}^{\in}(\mathbb{R}^{\ni})$ has Hamiltonian $\hat{H} = \hat{\mathbf{p}}^2/2m$. Galilean boosts of velocity $\mathbf{v}$ act on $\mathcal{H}$ through a a time-independent unitary operator $\hat{U}(\mathbf{v})$ such that

$$\hat{U}^\dagger(\mathbf{v})\hat{x}(t)\hat{U}(\mathbf{v}) = \hat{x}(t) + \mathbf{v}t \ .$$

i) Show that $\hat{U}(\mathbf{v}_1)\hat{U}(\mathbf{v}_2) = \hat{U}(\mathbf{v}_1 + \mathbf{v}_2)$ and $\hat{U}^\dagger(\mathbf{v})\hat{\mathbf{p}}\hat{U}(\mathbf{v}) = \hat{\mathbf{p}} + m\mathbf{v}$. Express $\hat{U}(\mathbf{v})$ in terms of $\hat{\mathbf{x}}$, $\hat{\mathbf{p}}$, and $\mathbf{v}$.

ii) Let $\hat{T}(\mathbf{a}) = e^{-i\mathbf{a}\cdot\hat{\mathbf{p}}/\hbar}$ be the translation operator. Evaluate $\hat{T}^\dagger(\mathbf{a})\hat{U}^\dagger(\mathbf{v})\hat{T}(\mathbf{a})\hat{U}(\mathbf{v})$. Is this compatible with classical expectations for a corresponding sequence of boosts and translations?

6. For a spin-$\frac{1}{2}$ particle, the spin operator is $\hat{\mathbf{S}} = \hbar\boldsymbol{\sigma}/2$.

i) Using the commutation and anti-commutation relations, but without matrix multiplication, explain why the Pauli matrices obey

$$\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k \ .$$

Hence that for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$

$$(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma}) = (\mathbf{a}\cdot\mathbf{b})\mathbb{1} + i(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma}.$$

ii) Hence show that

$$e^{-i\boldsymbol{\alpha}\cdot\hat{\mathbf{S}}/\hbar} = \cos\left(\frac{\alpha}{2}\right)\mathbb{1} - i\sin\left(\frac{\alpha}{2}\right)\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{\sigma}.$$

iii) A spin-$\frac{1}{2}$ particle in a magnetic field $\mathbf{B}$ has Hamiltonian $H = -\mu\mathbf{B}\cdot\hat{\mathbf{S}}$, where μ is constant. If the particle's spin is initially prepared to be in some state $|\chi\rangle$, show that the probability the spin is found to be in an orthogonal state $|\chi'\rangle$ at a time t later is

$$|\langle\chi'|\hat{\mathbf{B}}\cdot\boldsymbol{\sigma}|\chi\rangle|^2\sin^2\omega t,$$

where ω is a frequency you should specify.

iv) Obtain the Heisenberg equation of motion for $\hat{\mathbf{S}}(t)$ with this Hamiltonian.

v) Show that the spin operator in the Heisenberg picture is

$$\hat{\mathbf{S}}(t) = \cos(2\omega t)\hat{\mathbf{S}} + (1-\cos(2\omega t))\hat{\mathbf{B}}(\hat{\mathbf{B}}\cdot\hat{\mathbf{S}}) - \sin(2\omega t)\hat{\mathbf{B}}\times\hat{\mathbf{S}} \ .$$

7. A spin-$\frac{1}{2}$ particle interacts with a time-varying magnetic field such that

$$H = -\gamma\mathbf{B}(t)\cdot\hat{\mathbf{S}} \quad \text{with} \quad \mathbf{B}(t) = B_0\hat{\mathbf{z}} + b(\hat{\mathbf{x}}\cos\omega_1 t + \hat{\mathbf{y}}\sin\omega_1 t) \ .$$

Let $|\psi(t)\rangle$ be the state of the particle at time t and let $\hat{U}(\omega_1 t\hat{\mathbf{z}})$ be the rotation operator around the z-axis by the time-dependent angle $\omega_1 t$.

i) Define $|\chi(t)\rangle$ via $|\psi(t)\rangle = \hat{U}(\omega_1 t\hat{\mathbf{z}})|\chi(t)\rangle$. Show that $|\chi(t)\rangle$ satisfies the time-dependent Schrödinger equation with Hamiltonian $\hat{H}_{\text{eff}} = -\mu\mathbf{B}_{\text{eff}} \cdot \hat{\mathbf{S}}$, where $\mathbf{B}_{\text{eff}}$ is a time-independent magnetic field that you should specify.

ii) Hence show that

$$\langle\hat{\mathbf{S}}\rangle_{\psi(t)} = R(\omega_1 t\hat{\mathbf{z}})R(-\mu t\mathbf{B}_{\text{eff}})\langle\hat{\mathbf{S}}\rangle_{\psi(0)},$$

where $R(\boldsymbol{\alpha})$ is a rotation matrix in $\mathbb{R}^3$.

iii) Sketch the motion of $\langle\hat{\mathbf{S}}\rangle_{\psi(t)}$ over time.

8. Consider a 2d isotropic harmonic oscillator of frequency ω.

i) Construct combinations of the raising/lowering operators for the 2d oscillator that satisfy the $so(3)$ algebra.

ii) Show how all oscillator states fit into representations of $so(3)$.

9. Let $\hat{\mathbf{J}} = (\hat{J}_1, \hat{J}_2, \hat{J}_3)$ and $|j\,m\rangle$ denote the standard angular momentum operators and states, so that, using units in which $\hbar = 1$,

$$\hat{\mathbf{J}}^2\,|j\,m\rangle = j(j+1)|j\,m\rangle, \qquad \hat{J}_3|j\,m\rangle = m|j\,m\rangle.$$

Show that $\hat{U}(\theta) = \exp(-i\theta\hat{J}_2)$ is unitary and define

$$\hat{J}_i(\theta) = \hat{U}(\theta)\,\hat{J}_i\,\hat{U}(\theta)^{-1} \;\text{ for }\; i = 1, 3\,.$$

Using the commutation relations for angular momentum show that

$$\frac{d^2\hat{J}_i(\theta)}{d\theta^2} + \hat{J}_i(\theta) \;=\; 0 \;\text{ for }\; i = 1, 3\,.$$

Hence show that

$$\hat{J}_1(\theta) = \hat{J}_1\cos\theta - \hat{J}_3\sin\theta\,, \qquad \hat{J}_3(\theta) = \hat{J}_1\sin\theta + \hat{J}_3\cos\theta\,.$$

Deduce that $\hat{U}(\frac{\pi}{2})\,|j\,m\rangle$ are eigenstates of $\hat{J}_1$.

For $j = \frac{1}{2}$, use the Pauli representation of operators and states to show that

$$\hat{U}(\theta)|\!\uparrow\rangle \;=\; \cos\tfrac{1}{2}\theta\,|\!\uparrow\rangle \;+\; \sin\tfrac{1}{2}\theta\,|\!\downarrow\rangle\,, \quad \hat{U}(\theta)|\!\downarrow\rangle \;=\; -\sin\tfrac{1}{2}\theta\,|\!\uparrow\rangle \;+\; \cos\tfrac{1}{2}\theta\,|\!\downarrow\rangle$$

where $|\!\uparrow\rangle = |\frac{1}{2}\;\frac{1}{2}\rangle$ and $|\!\downarrow\rangle = |\frac{1}{2}\;-\frac{1}{2}\rangle$. Verify in this representation that $\hat{U}(\frac{\pi}{2})|\!\uparrow\rangle$ and $\hat{U}(\frac{\pi}{2})|\!\downarrow\rangle$ are eigenstates of J_1.

10. Show that for two spin-$\frac{1}{2}$ particles the composite state

$$\frac{1}{\sqrt{2}}(\,|\!\uparrow\rangle|\!\downarrow\rangle \;-\; |\!\downarrow\rangle|\!\uparrow\rangle\,)$$

is unchanged by a transformation $|\!\uparrow\rangle \mapsto \hat{U}(\theta)|\!\uparrow\rangle$ and $|\!\downarrow\rangle \mapsto \hat{U}(\theta)|\!\downarrow\rangle$ applied to all one-particle states. How does this relate to the angular momentum properties of the two-particle state?

11. What is the unitary operator $\hat{U}(\alpha)$ corresponding to translation through α for a one-dimensional quantum system with position $\hat{x}$ and momentum $\hat{p}$? Calculate $[\hat{x}, \hat{U}(\alpha)]$ and show that the result is consistent with the assumption that position eigenstates obey $|x{+}\alpha\rangle = \hat{U}(\alpha)|x\rangle$. Given this assumption, express the wavefunction for $\hat{U}(\alpha)|\psi\rangle$ in terms of the wavefunction $\psi(x)$ for $|\psi\rangle$.

If the system is a one-dimensional harmonic oscillator of mass m and frequency ω, show that

$$\hat{U}(\alpha) \;=\; e^{-\frac{1}{2}\gamma^2} e^{\gamma \hat{a}^\dagger} e^{-\gamma \hat{a}} \qquad \text{where} \qquad \gamma = \alpha(m\omega/2\hbar)^{\frac{1}{2}}\ .$$

Deduce that if $\psi_n(x)$ are wavefunctions for the usual normalised states with energies $\hbar\omega(n{+}\frac{1}{2})$, then

$$\psi_0(x{-}\alpha) \;=\; e^{-\frac{1}{2}\gamma^2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}\, \gamma^n \psi_n(x)\ .$$

[Recall that $[\hat{A}, e^{\hat{B}}] = [\hat{A}, \hat{B}]e^{\hat{B}}$ and $e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}}e^{\frac{1}{2}[\hat{A},\hat{B}]}$ provided $[\hat{A}, \hat{B}]$ commutes with $\hat{A}$ and $\hat{B}$.]

12. Write down the commutation relations for the components of a vector operator $\hat{\mathbf{V}} = (\hat{V}_1, \hat{V}_2, \hat{V}_3)$ and the angular momentum operator $\hat{\mathbf{J}} = (\hat{J}_1, \hat{J}_2, \hat{J}_3)$. Use these to show that

$$\hat{\mathbf{V}}(\theta) = \exp(i\theta\mathbf{n}{\cdot}\hat{\mathbf{J}}/\hbar)\,\hat{\mathbf{V}}\,\exp(-i\theta\mathbf{n}{\cdot}\hat{\mathbf{J}}/\hbar)$$

satisfies

$$\hat{\mathbf{V}}'(\theta) = \mathbf{n}{\times}\hat{\mathbf{V}}(\theta)$$

where $\mathbf{n}$ is a unit vector and θ a real parameter. Deduce that $\mathbf{n}{\cdot}\hat{\mathbf{V}}(\theta) = \mathbf{n}{\cdot}\hat{\mathbf{V}}$ and hence that

$$\hat{\mathbf{V}}''(\theta) + \hat{\mathbf{V}}(\theta) = (\mathbf{n}{\cdot}\hat{\mathbf{V}})\,\mathbf{n}\,.$$

Solve this equation to find $\hat{\mathbf{V}}(\theta)$ in terms of $\hat{\mathbf{V}}$ and interpret your result.

13. Two identical spin-1 particles, whose centre of mass is at rest, have combined spin $\hat{\mathbf{S}}$, relative orbital angular momentum $\hat{\mathbf{L}}$ and total angular momentum $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$, with corresponding quantum numbers S, L and J. Show that $L + S$ must be even. If $J = 1$, what are the possible values of L and S?

14. Show that a particle of spin 1 cannot decay into two identical particles of spin 0. The ρ-meson has spin 1 and can decay into two spinless π-mesons, or pions, with different charges. If the intrinsic parity of any π is negative, what is the intrinsic parity of the ρ?

15. A particle X is observed to undergo the decays $X \to \rho^+ + \pi^-$ and $X \to K + K$, where K is a particle of spin 0. What is the lowest value for the spin of X that is consistent with this, and what is the corresponding intrinsic parity of X? [Assume that total angular momentum and parity are conserved is all these processes.]