Quantum Mechanics: Example Sheet 2

1. A quantum system has Hamiltonian $\hat{H}$ with normalised eigenstates χ_n and corresponding energies E_n $(n = 1, 2, 3, \ldots)$. A linear operator $\hat{Q}$ associated with the quantity Q is defined by its action on these states:

$$\hat{Q}\chi_1 = \chi_2 \ , \qquad \hat{Q}\chi_2 = \chi_1 \ , \qquad \hat{Q}\chi_n = 0 \quad n > 2 \ .$$

Show that $\hat{Q}$ has eigenvalues ± 1 (in addition to zero) and find the corresponding normalised eigenstates $\chi_\pm$, in terms of energy eigenstates. Calculate $\langle \hat{H} \rangle$ in each of the states $\chi_\pm$.

A measurement of Q is made at time zero, and the result $+1$ is obtained. The system is then left undisturbed for a time t, at which instant another measurement of Q is made. What is the probability that the result will again be $+1$? Show that the probability is zero if the measurement is made when a time $T = \pi\hbar/(E_2 - E_1)$ has elapsed (assume $E_2 - E_1 > 0$).

2. In the previous example, suppose that an experimenter makes n successive measurements of Q at regular time intervals T/n. If the result $+1$ is obtained for one measurement, show that the amplitude for the next measurement to give $+1$ is

$$A_n = 1 - \frac{iT(E_1 + E_2)}{2\hbar n} + \mathcal{O}\left(\frac{1}{n^2}\right) \ .$$

The probability that all n measurements give the result $+1$ is then $P_n = (|A_n|^2)^n$. Show that

$$\lim_{n \to \infty} P_n = 1 \ .$$

Interpreting $\chi_\pm$ *as the 'not-boiling' and 'boiling' states of a two-state 'quantum pot', this shows that a watched quantum pot never boils (also called the Quantum Zeno Paradox).*

3. Write down the Hamiltonian H for a harmonic oscillator of mass m and frequency ω. Express $\langle H \rangle$ in terms of $\langle \hat{x} \rangle$, $\langle \hat{p} \rangle$, Δx and Δp, all defined for some normalised state ψ. Use the Uncertainty Relation to deduce that $E \geq \frac{1}{2}\hbar\omega$ for any energy eigenvalue E.

4. The energy levels of the harmonic oscillator are $E_n = (n+\frac{1}{2})\hbar\omega$ for $n = 0, 1, 2, \ldots$ and the corresponding stationary state wavefunctions are

$$\chi_n(x) \;=\; h_n(y)e^{-y^2/2} \qquad \text{where} \qquad y = (m\omega/\hbar)^{1/2}x$$

and h_n is a polynomial of degree n with $h_n(-y) = (-1)^n h_n(y)$. Using *only* the orthogonality relations

$$(\chi_m, \chi_n) = \delta_{mn}\,,$$

determine χ_2 and χ_3 up to an overall constant in each case.

Give an expression for the quantum state of the oscillator $\psi(x,t)$ if the initial state is $\psi(x,0) = \sum_{n=0}^{\infty} c_n \chi_n(x)$, where c_n are complex constants. Deduce that

$$|\,\psi(\,x,\;2p\pi/\omega\,)\,|^2 = |\,\psi(\,-x,\;(2q{+}1)\pi/\omega\,)\,|^2$$

for any integers p, $q \geq 0$. Comment on this result, considering the particular case in which $\psi(x,0)$ is sharply peaked around position $x = a$.

5. A particle of mass m is in a one-dimensional infinite square well (a potential box) with $U = 0$ for $0 < x < a$ and $U = \infty$ otherwise. The normalised wavefunction for the particle at time $t = 0$ is

$$\psi(x,0) = Cx(a-x)\;.$$

(i) Determine the real constant C.

(ii) By expanding $\psi(x,0)$ as a linear combination of energy eigenfunctions (found in Example **1** above), obtain an expression for $\psi(x,t)$, the wavefunction at time t.

(iii) A measurement of the energy is made at time $t > 0$. Show that the probability that this yields the result $E_n = \hbar^2\pi^2 n^2/2ma^2$ is $960/\pi^6 n^6$ if n is odd, and zero if n is even. Why should the result for n even be expected? Which value of the energy is most likely, and why is its probability so close to unity?

6. Let $\hat{H}$ be a Hamiltonian and $\chi(x)$ any normalised eigenstate with energy E. Show that, for any operator $\hat{A}$,

$$\langle\,[\hat{H},\hat{A}]\,\rangle_\chi \;=\; 0\,.$$

For a particle in one dimension, let $\hat{H} = \hat{T} + \hat{U}$ where $\hat{T} = \hat{p}^2/2m$ is the kinetic energy and $U(\hat{x})$ is any (real) potential. By setting $\hat{A} = \hat{x}$ in the result above and using the canonical commutation relation between position and momentum, show that $\langle\hat{p}\rangle_\psi = 0$.

Now assume further that $U(\hat{x}) = k\hat{x}^n$ (with k and n constants). By taking $\hat{A} = \hat{x}\hat{p}$, show that

$$\langle\hat{T}\rangle_\chi = \frac{n}{n+2}\,E \qquad \text{and} \qquad \langle\hat{U}\rangle_\chi = \frac{2}{n+2}\,E\,.$$

7. Suppose Q is an observable that does not depend explicitly on time. Show that

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \langle \hat{Q} \rangle_\psi \;=\; \langle\, [\hat{Q}, \hat{H}]\, \rangle_\psi$$

where $\psi(\mathbf{x}, t)$ obeys the Schrödinger Equation. Apply this to the position and momentum of a particle in three dimensions, with Hamiltonian

$$\hat{H} \;=\; \frac{1}{2m}\hat{\mathbf{p}}^2 + U(\hat{\mathbf{x}}) \;,$$

by calculating the commutator of $\hat{H}$ with each component of $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$. Compare the results with the classical equations of motion.

8. Let $\hat{A}$ and $\hat{B}$ be hermitian operators. Show that $i[\hat{A}, \hat{B}]$ is hermitian.

Given a normalised state ψ, consider $\|\, (\hat{A} + i\lambda\hat{B})\psi \,\|^2$ with λ a real variable and deduce that

$$\langle \hat{A}^2 \rangle \langle \hat{B}^2 \rangle \;\geq\; \tfrac{1}{4}\, |\, \langle\, i[\hat{A}, \hat{B}]\, \rangle\, |^2 \,,$$

with all expectation values taken in the state ψ. Hence derive the generalised uncertainty relation:

$$\Delta A\, \Delta B \;\geq\; \tfrac{1}{2}\, |\, \langle\, [\hat{A}, \hat{B}]\, \rangle\, | \;.$$