Mathematical Biology: Example Sheet 1

David Tong, January 2025

1. The population $N(t)$ of a certain insect is modelled by the ODE

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - p(N)$$

where $p(N) = BN/(A+N)$ with $A, B > 0$.

a) Give suggestions as to the meaning of the terms in this equation.

b) Show by rescaling that the dynamics depends only on the two parameters $\alpha = A/K, \beta = B/rK$ [Hint: focus on simplifying the logistic terms first].

c) Investigate how many *positive* steady states there are, i.e. fixed points with $N > 0$. Sketch the (α, β) plane, dividing it into regions where there are zero, one and two positive steady states.

d) What is the number of *stable* solutions, including the fixed point at $N = 0$, in each region? [Hint: investigating $N = 0$ stability will be enough to deduce the rest.]

2. A variant of the Hutchinson-Wright equation is

$$\frac{dx(t)}{dt} = \alpha\left[x(t-T) - x(t)^2\right],$$

where $\alpha, T > 0$. Give a brief interpretation of what this might represent in terms of population dynamics.

Show that the constant solution with $x(t) = 1$ is stable for all $\alpha, T > 0$.

3. The population density $n(a,t)$ of individuals of age a at time t satisfies

$$\frac{\partial n(a,t)}{\partial t} + \frac{\partial n(a,t)}{\partial a} = -\mu(a)n(a,t), \quad \text{with} \quad n(0,t) = \int_0^\infty b(a)n(a,t)da,$$

where $\mu(a)$ is the age-dependent death rate and $b(a)$ is the birth rate per individual of age a.

Using the standard similarity solution $n(a,t) = \tilde{n}(a)e^{rt}$ for each of the examples below, give: (i) the mean number of offspring; (ii) the population growth rate r (solve where possible otherwise give an implicit expression); (iii) the value of the birth rate parameter B (defined below) for which there is neither growth nor decay and sketch the age-profile of the population in this case.

a) The birth rate $b(a)$ is a constant B for $a_1 < a < a_2$ and zero otherwise. The death rate $\mu(a)$ is a constant d for $a > a_2$ and zero otherwise.

b) Individuals give birth only at age a^*: $b(a) = B\,\delta(a - a^*)$. The death rate $\mu(a)$ is a constant d for all ages.

c) The birth rate $b(a)$ is a constant B for all ages. All individuals die at age A. [Hint: in this extreme case, rather than using $\mu(a)$, just reformulate the standard approach slightly.]

4. A simple model of two competing populations eating the same food takes the form

$$\dot{N}_1 = r_1 N_1 \left(1 - \frac{N_1}{K_1} - b_1 \frac{N_2}{K_2}\right),$$
$$\dot{N}_2 = r_2 N_2 \left(1 - \frac{N_2}{K_2} - b_2 \frac{N_1}{K_1}\right).$$

Rescale the equations to simplify them, and show that the solutions depend only on $\rho = r_2/r_1, b_1$ and b_2.

Now assume that $\rho, b_1, b_2 > 0$. Find all the physically relevant fixed points and determine their stability. Give conditions on the coefficients such that there is a stable state of *coexistence*, with $N_1, N_2 > 0$.

5. Consider the 'harvesting' model

$$\dot{u} = u(1 - v) - \epsilon u^2 - f,$$
$$\dot{v} = -\alpha v(1 - u),$$

with constants $\alpha > 0, f > 0$ and $0 < \epsilon < 1/2$.

Find all the biologically relevant fixed points of this system, and investigate their stability, distinguishing between different ranges of f:

a) $0 < f < \epsilon$,

b) $\epsilon < f < 1 - \epsilon$,

c) $1 - \epsilon < f < 1/(4\epsilon)$,

d) $1/(4\epsilon) < f$.

In each case, sketch trajectories in the $u - v$ phase-plane, and discuss what would happen to the predator and prey populations in practice.

Note that for this model something odd happens at $u = 0$. Comment on this, and discuss how the model might be improved in this respect.

6. A fungal disease is introduced into an isolated population of frogs. Without disease, the population size x would obey the (normalised) logistic equation $\dot{x} = x(1 - x)$. The disease causes death at rate d and there is no recovery. The disease transmission rate is β and in addition, offspring of infected frogs are also infected from birth.

a) Briefly explain why the population sizes of the uninfected x and infected y frogs now satisfy

$$\begin{aligned} \dot{x} &= x\left[1 - x - (1 + \beta)y\right], \\ \dot{y} &= y\left[(1 - d) - (1 - \beta)x - y\right]. \end{aligned}$$

b) The population starts at the disease-free population size ($x = 1$) and a small number of infected frogs are introduced. Show that the disease will successfully invade iff $\beta > d$.

c) By finding all the equilibria in $x, y \geq 0$ and considering their stability, find the long term outcome for the frog population. Specify d as a function of β at any boundaries.

d) Plot the long-term steady *total* population size as a function of d for fixed β, and note that an intermediate mortality rate is actually the most harmful for overall population numbers. Explain why this is the case.

7. Consider an infectious disease in some population where immunity wanes with time (and ignore births and deaths). This can be captured by an SIRS model of an infectious disease: start with the SIR model from lectures, but recovered individuals (R) can lose their immunity and become susceptible again at a rate γR. Using the fact that the total population size N remains constant, reduce the system of equations to two, for S and I (the populations of susceptibles and infecteds respectively).

Give an expression for the basic reproduction ratio R_0 and show that when $R_0 > 1$ the system has a stable fixed point where both $S > 0$ and $I > 0$.

Find the nullclines and sketch trajectories in the $S - I$ plane. What happens in the long term?