7 Neutrinos

No one would accuse a neutrino of being gregarious. They interact less than a first year undergraduate mathematics student forced to sit next to their theoretical physics professor at a matriculation dinner (to give a weirdly specific yet shudderingly memorable analogy).

For example, in the time it takes you to read this sentence, around 100 trillion neutrinos will have passed through your body. Most of them came from the Sun, but a significant minority have a cosmic origin, and have been streaming through the universe, uninterrupted since the first few seconds after the Big Bang. Moreover, in contrast to photons, the number of neutrinos hitting you doesn't change appreciably as day turns into night. The neutrinos from the Sun will happily pass right through the Earth and out the other side. This is vividly demonstrated in the picture of the Sun at night shown in Figure 19.

There are two reasons why neutrinos are so intangible. The first is that they are the only particle to interact solely through the weak force. And, as we've seen, the weak force is weak. The second reason is that their mass is much much smaller than any other fermion which means that on the rare occasion that they do interact, they don't deliver much of a punch. The purpose of this section is to describe some properties of neutrinos in more detail.

7.1 Neutrino Masses

There is much that we don't know about neutrino masses. But we do know that the masses are not zero.

At the moment, we have no direct measurement of the mass of each neutrino. But we do have some precious information. First, we know that one neutrino must have a mass greater than

$$m_\nu \gtrsim 0.05 \text{ eV} . \tag{7.1}$$

Second, constraints from cosmology give us an upper bound on the sum of all neutrino masses. This comes from the imprint that neutrinos in the early universe leave on the cosmic microwave background radiation and on subsequent structure formation of galaxies (in particular, baryon acoustic oscillations – you can read more about this in the lecture notes on Cosmology.). This bound is

$$\sum_\nu m_\nu \lesssim 0.25 \text{ eV} . \tag{7.2}$$

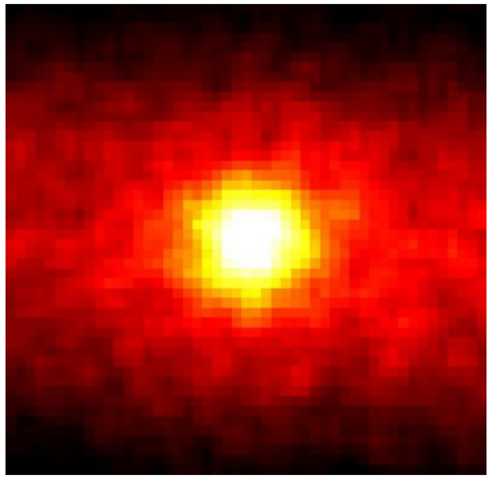

Figure 19. The Sun at night. This is a picture, taken by Super-Kamiokande, shows the neutrino flux coming from the Sun. The picture was taken at night, with the neutrinos passing through the Earth before hitting the detector.

In addition, we have information about the mass *differences* between neutrinos. We denote the mass of the neutrinos as m_1, m_2 and m_3. Much like for quarks, the mass eigenstates do not correspond to the flavour eigenstates ν_e, ν_μ and ν_τ and we will explain the relation more in the next section. We know that the mass splitting between two of the states is comparable to the overall mass of neutrinos,

$$|m_3^2 - m_2^2| = 2.5 \times 10^{-3} \text{ eV}^2 \ . \tag{7.3}$$

(We've taken the magnitude on the difference on the left-hand side to hide the fact that we don't actually know which if m_3 and m_2 is heavier: we will describe this ambiguity further below. Then there is a much smaller mass splitting between of order

$$m_2^2 - m_1^2 \approx 7.4 \times 10^{-5} \text{ eV}^2 \ . \tag{7.4}$$

There are still a number of possibilities consistent with these bounds. It may even be, for example, that one neutrino is massless while others have mass ~ 0.1 eV or so. Still, our ignorance notwithstanding, a rough summary of the masses of all fermions is shown in Figure 20.

In the rest of this section, we will describe the basics of neutrino masses. We will learn how they can get a mass in the Standard Model and its extensions, and how we are able to determine the structure of masses described above.

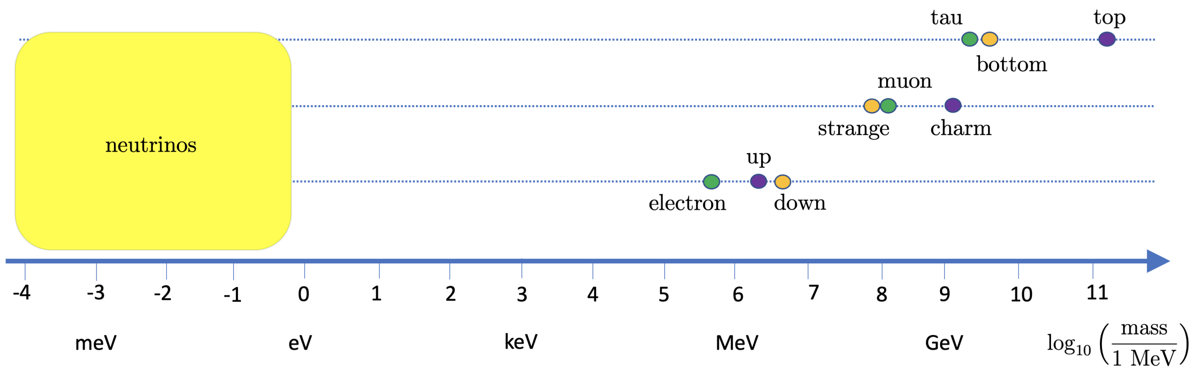

Figure 20. Fermion masses, arranged by generation. The charged leptons are green, the $-1/3$ quarks are orange, and the charge $+2/3$ quarks are purple. The neutrinos are way off to the left.

7.1.1 Dirac vs Majorana Masses

Even with our limited knowledge, it's clear that neutrinos aren't like the other particles. There is six orders of magnitude separating the mass of the top quark from the mass of the electron. Then there is a gap of another six order of magnitude before we get to the neutrinos. The first question we should ask is: why?

We don't have a definitive answer to this question. But we do have a plausible answer. In what follows, I will sketch what appear to be the most reasonable ways in which neutrinos can get a mass. They are not the only ways: if you're willing to add new fields to the Standard Model, and then try to hide them from experiments, then you can cook up other possibilities. Ultimately, experiment must be our guide to figure out which is right.

The most obvious way to give neutrinos a mass is to add a right-handed neutrino ν_R to the Standard Model. Indeed, we already included this in Section 5 when describing the fields of the Standard Model, although we also raised a question mark about its existence. *If* we include a right-handed neutrino that is uncharged under the Standard Model gauge group, then it can participate in a Yukawa coupling. Restricting to a single generation for now, the lepton Yukawas are then (5.74),

$$\mathcal{L}_{\text{Yuk}} = -y^e\,\bar{L}_L H e_R - y^\nu\,\bar{L}_L \tilde{H} \nu_R + \text{h.c.}\ . \tag{7.5}$$

When the Higgs condenses, the neutrino gets a mass just like all other fermions, given by

$$m = \frac{y^\nu}{\sqrt{2}} v \ . \tag{7.6}$$

We refer to this as a *Dirac mass*.

There's nothing wrong with this explanation for neutrino masses. But it does raise a question of why the dimensionless Yukawa coupling is $y^\nu \sim 10^{-12}$. Of course, as we've repeatedly seen, we don't understand the values of any of the Yukawa couplings so perhaps this is just one more mystery to add to the list. Nonetheless, it's such a wildly small number that it feels like it's crying out for some explanation. And the good news is that there is a very natural explanation at hand.

Moreover, this explanation doesn't require us to do anything than follow our original philosophy when constructing the Standard Model. That is, given all the fields at our disposal, we should write down all possible relevant and marginal terms consistent with Lorentz invariance and gauge symmetry. And the addition of the right-handed neutrino allows for something new. This is the term

$$\mathcal{L}_{\text{Maj}} = \frac{1}{2} M \nu_R \nu_R + \text{h.c.} \ . \tag{7.7}$$

Here $M \in \mathbb{C}$. This is called a *Majorana mass*.

Suppose that we have both the Dirac mass m, as in (7.6), and the Majorana mass M, as in (7.7). What is the physical mass of the neutrinos? To answer this, we write the combined mass term as

$$\mathcal{L}_{\text{mass}} = \frac{1}{2} (\bar{\nu}_L, \nu_R) \begin{pmatrix} 0 & m \\ m & M \end{pmatrix} \begin{pmatrix} \bar{\nu}_L \\ \nu_R \end{pmatrix} + \text{h.c.} \ . \tag{7.8}$$

The physical masses are the eigenvalues of this matrix. We have

$$\text{mass} = \frac{1}{2} \left| M \pm \sqrt{M^2 - 4m^2} \right| \ . \tag{7.9}$$

What does this buy us? We know that the neutrinos has a mass in the eV range. One possibility is that both m and M are in this ballpark. But there's an alternative option, which is that the Majorana mass M is very large. If we take $M \gg m$, then the two masses above become

$$\text{mass} \approx M \quad \text{and} \quad \text{mass} \approx \frac{m^2}{M} \ . \tag{7.10}$$

The particle with mass $\approx M$ is mostly the right-handed neutrino, while the particle with mass $\approx m^2/M$ is approximately the left-handed neutrino. And, crucially, it's quite possible for the latter of these to be light, even if the Yukawa couplings are the same order of magnitude as those for electrons.

For example, if $y^\nu \approx 1$ (like the extraordinarily heavy top quark) then a Majorana mass of order 10^{13} GeV or so will get us in the ballpark of the observed masses. This is getting close to the realm of grand unified theories. Obviously, for smaller Yukawa couplings, the corresponding Majorana mass should be smaller. This suggests, somewhat counterintuitively, that the smallness of the neutrino mass might be because the right-handed neutrino gets a very large mass. This is known as the *seesaw mechanism*.

7.1.2 The Dimension 5 Operator

There's something a little unsettling about the seesaw mechanism. We introduced a right-handed neutrino to give both left- and right-handed particles a mass. But then we saw that the physical mass of one of these states M was extremely large, way beyond current experiments. Which suggests that it should be possible the describe the resulting physics without invoking it in the first place!

And, indeed there is. But it does require us to go beyond our original philosophy when constructing the Standard Model. We originally set ourselves the task of writing down all relevant and marginal terms consistent with Lorentz and gauge symmetries. We can incorporate neutrino masses without a right-handed neutrino if we also allow ourselves to include irrelevant operators.

As usual, operators in quantum field theory are classified by their dimension. Those with dimension $\Delta < 4$ are relevant, and those with dimension $\Delta = 4$ are (classically) marginal. There are an infinite number of irrelevant operators, but their importance can still be judged by how irrelevant they are. And, among them, there is a unique operator with dimension $\Delta = 5$. This is

$$\mathcal{L}_5 = \frac{\lambda}{M}(\bar{L}_L\tilde{H})(\bar{L}_L\tilde{H}) + \text{h.c.}\ . \tag{7.11}$$

This is sometimes called the *Weinberg operator* although Weinberg has so many things named after him in the Standard Model that I'm not sure it's helpful terminology. It has dimension 5 because it contains two fermions (each of dimension 3/2) and two scalars (each of dimension 1). Here λ is a dimensionless coupling and M is a mass scale. If we integrate out the massive right-handed neutrino, then we generate the coupling (7.11) with M the Majorana mass and $\lambda = (y^\nu)^2$. However, the operator (7.11) may be generated by something else that isn't associated to a right-handed neutrino.

We see that (7.11) captures the spirit of the seesaw mechanism: when the Higgs gets a vev v, the left-handed neutrino ν_L gets a Majorana mass $\sim \lambda v^2/M$. This retains the irony in which detecting a very small Majorana mass points towards physics at a very high energy scale.

7.1.3 Neutrinoless Double Beta Decay

Above, we've seen that there are two ways that a neutrino can get a mass: either a bog standard Dirac mass (7.6), or a Majorana mass (7.7) which, if large, is captured in the dimension 5 operator (7.11).

There is one important difference between these: the Majorana mass violates lepton number at tree level. This means that it might be possible to detect the neutrino Majorana mass by observing a process which explicitly violates lepton number.

You can't have a process that changes lepton number by just one because (in the absence of any other fermion getting involved) that would also violate $(-1)^F$ which is part of the Lorentz group. So, in searching for signals of lepton number violation, we are looking for processes that change L by two. The most clear cut process of this type is something called *neutrinoless double beta decay*, sometime referred to rather elliptically as $0\nu\beta\beta$.

Recall that beta decay is the process $n \to p + e^- + \bar{\nu}^e$. This increases the atomic number of an element by one. Double beta decay is what it sounds like: we have $2n \to 2p + 2e^- + 2\bar{\nu}^e$, increasing the atomic number of an element by two.

Double beta decay occurs, albeit rarely. It's most easy to observe in elements for which the normal single beta decay is forbidden. For example, ^{76}Ge (with atomic number 32) can't decay through single beta decay to ^{76}As (with atomic number 33) because the germanium nucleus is lighter than the arsenic nucleus. However, it is possible for germanium to decay to ^{76}Se (with atomic number 34) which happens to have a lighter nucleus. The decay process is

$$^{76}\mathrm{Ge} \to {}^{76}\mathrm{Se} + 2e^- + 2\bar{\nu}^e \ . \tag{7.12}$$

This decay has been observed with lifetime of around 10^{21} years. (That was a *very* long experiment.)

Ordinary double beta decay preserves lepton number. But if the neutrino has a Majorana mass, so lepton number is violated, then there is another option: this is neutrinoless double beta decay

$$^{76}\mathrm{Ge} \to {}^{76}\mathrm{Se} + 2e^- \ . \tag{7.13}$$

Despite many ongoing searches, no such decay process has been observed, either in germanium or the dozen or so other elements that exhibit ordinary double beta decay. Current bounds put the effective half-life of elements due to double beta decay at $> 10^{25}$ years or so. These put bounds on the mass of a neutrino coming from a dimension 5 operator of $m_\nu \lesssim 0.3$ eV.

7.1.4 The PMNS Matrix

The fact that we have three generations of fermions means that, as for quarks, there is a misalignment between the mass and flavour eigenstates of leptons. As we saw in Section 5, we label the three generations of leptons as (5.20),

$$L_L^i = \begin{pmatrix} \nu_L^i \\ e_L^i \end{pmatrix} = \left\{ \begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix} , \begin{pmatrix} \nu_L^\mu \\ \mu_L \end{pmatrix} , \begin{pmatrix} \nu_L^\tau \\ \tau_L \end{pmatrix} \right\} . \tag{7.14}$$

These left-handed lepton appear in the charged currents that couple to the W bosons (5.88). If we omit the quarks terms, and focus only on the leptons, we have

$$J_\mu^+ = \bar{\nu}_L^i \bar{\sigma}_\mu e_L^i \quad \text{and} \quad J_\mu^- = \bar{e}_L^i \bar{\sigma}_\mu \nu_L^i . \tag{7.15}$$

As with the quarks, the leptons that appear here are *before* we diagonalise the mass matrices. In other words, the leptons that appear here are in the *flavour* basis.

If, however, we choose to work in the mass basis, which means that the mass terms are diagonal then, as with the quarks, we get a 3×3 unitary mixing matrix U appearing in the charged current which becomes

$$J_\mu^+ = \bar{\nu}_L^i \bar{\sigma}_\mu U_{ij}^\dagger e_L^j \quad \text{and} \quad J_\mu^- = \bar{e}_L^i \bar{\sigma}_\mu U_{ij} \nu_L^j . \tag{7.16}$$

This matrix U is known as the PMNS matrix, named after Pontecorvo, Maki, Nakagawa, and Sakata or simply the neutrino mixing matrix.

We learn that there are two natural bases that we can use: the mass basis in which the masses are diagonal, or the flavour basis in which the coupling the W bosons are diagonal. And these differ from each other. Correspondingly, there are two different linear combinations of fields.

What we usually refer to as the "electron neutrino", "muon neutrino", and "tau neutrino" are fields in the flavour basis. For example, beta decay happens by $n \to p + e^- + \bar{\nu}_e$ and that neutrino $\bar{\nu}^e$ is the one that couples to the W boson and electron, so it is $\bar{\nu}_e$ in the flavour eigenbasis. Which means that the neutrino that is emitted is *not* in a mass eigenstate!

It's useful to introduce some new notation to highlight what's going on. We will refer to the left-handed neutrinos in the flavour basis as ν_e and ν_μ and ν_τ. And we will refer to the neutrinos in the mass basis simply as ν_1 and ν_2 and ν_3. Each of these is a left-handed Weyl fermion, but we've suppressed the subscript L. The ν_i in (7.16) are in the mass basis and we see that these are related to the flavour basis by the PMNS matrix,

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{7.17}$$

The PMNS matrix is to leptons what the CKM matrix is to quarks. Just as for the CKM matrix, we have no way to determine the values of U from first principle. Instead, we must measure these from experiment. The magnitude of each component is now known reasonably accurately: these are

$$\begin{pmatrix} |U_{e1}| & |U_{e2}| & |U_{e3}| \\ |U_{\mu 1}| & |U_{\mu 2}| & |U_{\mu 3}| \\ |U_{\tau 1}| & |U_{\tau 2}| & |U_{\tau 3}| \end{pmatrix} \approx \begin{pmatrix} 0.8 & 0.5 & 0.1 \\ 0.3 & 0.5 & 0.7 \\ 0.4 & 0.6 & 0.6 \end{pmatrix} . \tag{7.18}$$

Some values are known fairly well; others less well. There are, for example, error bars of ± 0.1 on $U_{\tau 2}$.

The first thing to note is that the PMNS matrix is strikingly different from the CKM matrix describing the mixing of quarks[10]. In the quark sector, the CKM matrix was close to being the unit matrix, with just small off-diagonal elements. This meant that there was close alignment between the masses and the weak force. But we see no such thing in the neutrino sector. The mixing is pretty much as big as it can be! The lepton sector is really nothing like the quark sector. We do not have an explanation for the structure of the PMNS matrix. Indeed, its form came as a surprise to theorists. Surely it is telling us something important. It's just we don't yet know what!

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[10] Recall that $\begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} \approx \begin{pmatrix} 0.97 & 0.22 & 0.004 \\ 0.22 & 0.97 & 0.04 \\ 0.009 & 0.04 & 0.999 \end{pmatrix}$. Note also that the indices are of the CKM matrix and PMNS matrix are in the opposite order. For $V_{\rm CKM}$, the different rows are labelled by the up-type quarks, which is the first component of Q_L. For $U_{\rm PMNS}$, the rows are labelled by the charged lepton, which is the second component of L_L.

7.1.5 CP Violation in the Lepton Sector

As with the CKM matrix, CP violation is captured by the complex phases of the PMNS matrix. Here we must distinguish between neutrinos getting a purely Dirac mass and neutrinos getting a Majorana mass.

In the case where there are three right-handed neutrinos and each species of neutrino gets a Dirac mass, then the story is the same as for the CKM matrix: the neutrino mixing matrix has just a single phase.

But the counting is different if we have a Majorana mass. For this exercise, we will ignore the (unknown) mass of the right-handed neutrino and assume that the neutrino mass comes from the dimension 5 operator (7.11). With three generations, this takes the form

$$\mathcal{L}_5 = \frac{C_{ij}}{M}(\bar{L}^i_L\tilde{H})(\bar{L}^j_L\tilde{H}) \ . \tag{7.19}$$

Here C_{ij} is a complex *symmetric* 3×3 matrix, which means that it has 6 complex parameter or 12 real parameters. This means that in C_{ij} and the electron Yukawa y^e_{ij}, there are a total of $12+18=30$ real parameters. And we can eliminate some of these through $U(3)^2$ rotations acting on L^i_L and e^i_R. This leaves us with

$$30 - 2\times 9 = 12 \tag{7.20}$$

physical parameters. That's two more than for the quark sector. Note that, in contrast to the quark sector, there's no overall $U(1)$ that leaves the parameters untouched: that's because of the Majorana mass.

As for quarks, we can also see how this decomposes into real mixing angles and complex phases. A $U(3)$ matrix has 3 real parameters and 6 complex phases, so the lepton sector with Majorana masses has

$$(6+9) - 2\times 3 = 9 \text{ real parameters} \tag{7.21}$$

and

$$(6+9) - 2\times 6 = 3 \text{ complex phases} \ . \tag{7.22}$$

We see that the total number of real parameters is the same as for the quarks: it decomposes into 6 masses for electrons and neutrinos, together with three angles which live inside the PMNS matrix. In contrast, with a Majorana mass there are two more complex phases lurking inside the PMNS matrix. The usual way to parameterise these is by embellishing the CKM matrix structure (6.31) with two additional phases,

$$\begin{aligned}
U_{\rm PMNS} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & -s_{23} & c_{12} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha_1} & 0 \\ 0 & 0 & e^{i\alpha_2} \end{pmatrix} \\
&= \begin{pmatrix} c_{12}\,c_{13} & s_{12}\,c_{13} & s_{13}e^{-i\delta} \\ -s_{12}\,c_{23} - c_{12}\,s_{23}\,s_{13}e^{i\delta} & c_{12}\,c_{23} - s_{12}\,s_{23}\,s_{13}e^{i\delta} & s_{23}\,c_{13} \\ s_{12}\,s_{23} - c_{12}\,c_{23}\,s_{13}e^{i\delta} & -c_{12}\,s_{23} - s_{12}\,c_{23}\,s_{13}e^{i\delta} & c_{23}\,c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha_1} & 0 \\ 0 & 0 & e^{i\alpha_2} \end{pmatrix} .
\end{aligned}$$

While the real angles θ_{ij} are measured with some precision, as shown in (7.18), the complex phases $e^{i\delta}$ and (if they exist) $e^{i\alpha_1}$ and $e^{i\alpha_2}$ remain unknown for neutrinos. This means that we don't currently know if CP violation is possible in the lepton sector of the Standard Model. We note, however, that because none of the mixing angles θ_{ij} are particularly small, there is the possibility that CP violation in the lepton sector is significantly larger than in the quark sector. Future experiments should decide this.

7.2 Neutrino Oscillations

So far we have described the different ways in which neutrinos can get a mass. But we haven't yet explained how we know that they have mass. After all, it's not like we can simply collect a bunch of neutrinos in a jar and weigh it. Instead, our information comes in a less direct manner.

We have met the key piece of physics already: the mass eigenstates of the neutrinos are misaligned with the flavour eigenstates. The two are related through the PMNS matrix (7.17).

Neutrinos are always created or observed in flavour eigenstates. For example, in beta decay we have

$$n \longrightarrow p + e^- + \bar{\nu}_e \tag{7.23}$$

and it's definitely an electron neutrino that is emitted. Relatedly, we can detect an electron neutrino through a neutrino capture process, $\nu^e + n \longrightarrow p + e^-$. For example, the earliest neutrino detection experiments used tanks filled with dry-cleaning fluid which was rich in chlorine and looked for electron neutrinos through the process

$$\nu_e + {}^{37}\mathrm{Cl} \longrightarrow {}^{37}\mathrm{Ar} + e^- \ . \tag{7.24}$$

Again, it's necessarily an electron neutrino that induces this process, not a neutrino of any other type.

However, as we have seen, the electron neutrino ν^e is *not* a mass eigenstate. In the language of quantum mechanics, this means that it's not an energy eigenstate. But we know from our first courses on quantum mechanics what happens when systems are placed in states that are not energy eigenstates: the state you sit in varies with time. And so it is with neutrinos: the flavour of neutrino oscillates over time.

Before we put some mathematical meat on these ideas, it's worth pointing out that neutrino mixing comes with a slightly different change of perspective compared to the entirely analogous quark mixing that we met in Section 6. When we talk about quarks, we usually think of meson as energy eigenstates. The mixing then manifests itself as interactions allowing, say, a strange quark to decay to a up quark.

In contrast, in the world of leptons we can be confident that we have a particular flavour of neutrino to hand. The mixing then manifests itself as this flavour evolving, coherently, to a superposition of other flavours over time.

7.2.1 Oscillations with Two Generations

To see the basic physics, it's useful to restrict ourselves to the situation with just two flavours of neutrino. We'll take these to be the electron and muon neutrinos, related to mass eigenstates by the rotation matrix

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \ . \tag{7.25}$$

If the neutrinos have Majorana masses then there can be an additional complex phase in these relations. This will not affect neutrino oscillations and we won't consider it here.

We can think of the neutrinos as a 2-level system in quantum mechanics. Suppose that we start with an electron neutrino. Written in terms of energy eigenstates, this is

$$|\nu_e\rangle = \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle \ . \tag{7.26}$$

The neutrino ν_e is emitted with some energy E but, as we've seen, $|\nu_e\rangle$ isn't an energy eigenstate so we should view this as the average energy, $E = \cos^2\theta\, E_1 + \sin^2\theta\, E_2$, where

E_1 and E_2 are the energies of the states $|\nu_1\rangle$ and $|\nu_2\rangle$. Now, as we evolve in time, each of the energy eigenstates picks up a different phase,

$$\begin{aligned}|\nu_e(t)\rangle &= e^{-iE_1t}\cos\theta|\nu_1\rangle + e^{-iE_2t}\sin\theta|\nu_2\rangle \\ &= e^{-iE_1t}\big(\cos\theta|\nu_1\rangle + e^{-i\Delta E\, t}\sin\theta|\nu_2\rangle\big)\end{aligned} \tag{7.27}$$

where $\Delta E = E_1 - E_2$ is the energy difference between the states. Now we can convert back to the flavour eigenstates to get

$$|\nu_e(t)\rangle = e^{-iE_1t}\Big(\big(\cos^2\theta + e^{-i\Delta E\, t}\sin^2\theta\big)|\nu_e\rangle - \cos\theta\,\sin\theta\big(1 - e^{-i\Delta E\, t}\big)|\nu_\mu\rangle\Big)\ . \tag{7.28}$$

This is a standard result in quantum mechanics, entirely analogous to, say, Rabi oscillations in atomic physics. We see that, as time evolves, we have a probability of the electron neutrino ν_e to convert to a muon neutrino ν_μ,

$$P(\nu_e \to \nu_\mu) = \sin^2(2\theta)\,\sin^2\left(\frac{\Delta E\, t}{2}\right)\ . \tag{7.29}$$

The fact that this probability depends on sine functions is telling us that the change of flavour is an oscillation, in the sense that it goes back and forth. At this point, we need an expression for the energy difference ΔE. For each of the mass eigenstates, we have the usual relativistic dispersion relation

$$E_i = \sqrt{\mathbf{p}_i^2 + m_i^2} \approx |\mathbf{p}_i| + \frac{m_i^2}{2|\mathbf{p}_i|} \tag{7.30}$$

where, in the second equality, we've used the fact that our neutrinos are ultra-relativistic with $|\mathbf{p}| \gg m$. We can think of the neutrinos as sitting in momentum eigenstates, so that $\mathbf{p}_1 = \mathbf{p}_2$. Further, we can replace the $\mathbf{p}$ in the denominator with the original energy E, giving

$$\Delta E = \frac{\Delta m^2}{2E} \tag{7.31}$$

with $\Delta m^2 = m_1^2 - m_2^2$. There's one final flourish: the neutrinos are travelling at very close to the speed of light and so, in time t, travel a distance $L = t$ (because, of course, $c = 1$). We can then write the probability for an electron neutrino to convert into a muon neutrino, depending on the distance it travels

$$P(\nu_e \to \nu_\mu) = \sin^2(2\theta)\,\sin^2\left(\frac{\Delta m^2}{4E}L\right)\ . \tag{7.32}$$

We can put some numbers in this to figure out what kind of length scales L we need to see neutrino oscillations. First, we should put factors of $\hbar$ and c back into the formula. On dimensional grounds, we should have

$$P(\nu_e \to \nu_\mu) = \sin^2(2\theta)\,\sin^2\left(\frac{\Delta m^2 c^4}{4E\hbar c}L\right)\ . \tag{7.33}$$

We have $\hbar = 6.5 \times 10^{-16}$ eV s. For mass differences Δmc^2 of order an eV (which, as we will see, is a little on the high side) and neutrino energies E measured in GeV (which, as we shall see, is also a little on the high side), the argument of the sine function is of order 1 for

$$L \sim 4\hbar c \times \frac{\text{GeV}}{(\text{eV})^2} \sim 1 \text{ km}\ . \tag{7.34}$$

That's a remarkably human length scale to emerge from fundamental physics! It sets the kind of scale over which neutrino experiments should take place. We will see examples below. Putting in the numbers, the probability is often written a

$$P(\nu_e \to \nu_\mu) \approx \sin^2(2\theta)\,\sin^2\left(1.27 \times \frac{\Delta m^2}{(\text{eV})^2}\frac{(\text{GeV})}{E}\frac{L}{(\text{km})}\right)\ . \tag{7.35}$$

This formula contains two fundamental parameters: the mixing angle θ and the difference in masses Δm^2. To see oscillations, both need to be non-zero. The formula also contains two parameters that can vary from one experiment to another: the energy E of the beam and the length travelled L. In principle, by varying E and L, and seeing how one kind of neutrino morphs into another, we can determine the mixing angle θ and mass difference Δm^2. As you can see from the formula above, to see oscillations it is best to tune $E/L \sim \Delta m^2$.

Oscillations with Three Flavours

Repeating this calculation with three species of neutrinos gives the probability for oscillation from one flavour species α to another β in terms of the PMNS matrix U,

$$P(\nu_\alpha \to \nu_\beta) = \left|U_{\alpha 1}U^\star_{\beta 1} + U_{\alpha 2}U^\star_{\beta 2}\,e^{-i\Delta m^2_{21}L/2E} + U_{\alpha 3}U^\star_{\beta 3}\,e^{-i\Delta m^2_{31}L/2E}\right|^2\ . \tag{7.36}$$

If we take a limit in which $\Delta m^2_{21}L \ll E$, then we have

$$P(\nu_\alpha \to \nu_\beta) = \left|U_{\alpha 1}U^\star_{\beta 1} + U_{\alpha 2}U^\star_{\beta 2} + U_{\alpha 3}U^\star_{\beta 3}\,e^{-i\Delta m^2_{31}L/2E}\right|^2\ . \tag{7.37}$$

But, because U is unitary, we have $U_{\alpha 1}U^\star_{\beta 1} + U_{\alpha 2}U^\star_{\beta 2} + U_{\alpha 3}U^\star_{\beta 3} = \delta_{\alpha\beta}$. For $\alpha \neq \beta$, we then have

$$P(\nu_\alpha \to \nu_\beta) = \left|U_{\alpha 3}U^\star_{\beta 3}\right|^2\left|-1 + e^{i\Delta m^2_{31}L/2E}\right|^2\ . \tag{7.38}$$

This reproduces our two flavour result (7.35).

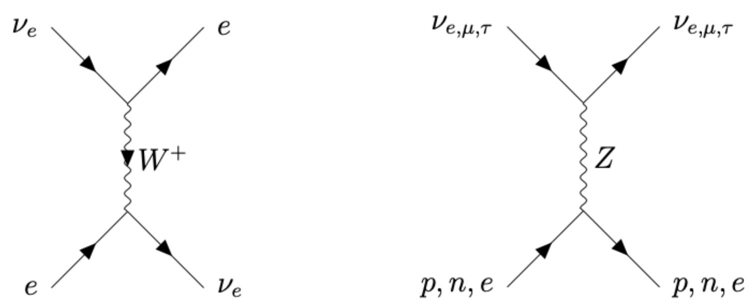

Figure 21. The scattering of electron neutrinos through a charged current, and any kind of neutrino through a neutral current.

7.2.2 Oscillations in Matter

There is a variation on the neutrino oscillation calculation that arises when neutrinos propagate through matter. This is both important and surprising.

The result is important because one source of neutrinos is the Sun, and the neutrinos that are created in the centre of the Sun have a way to travel before they emerge into empty space. And we would like to understand what happens to them on that journey. In addition, it is quite possible to detect neutrinos at night, after they have passed through the Earth and, again, we would like to understand if this last part of the journey has any noticeable effect.

The result is surprising because neutrinos are famously not impeded by things that sit in their way. Most happily pass straight through the Earth without being scattered. And yet, as we will see, the fact that they move in a density of matter does affect the oscillations. (There is also a second reason why the result is surprising which is to do with the orders of magnitude of energy involved and we will highlight this below.)

The effect that we care about arises from the elastic, forward scattering of neutrinos off a background of matter. This means that the neutrinos exchange neither energy nor momentum with the background matter. This process arises through the Feynman diagrams shown in Figure 21. All three types of neutrino can scatter off protons, neutrons and electrons through the exchange of a Z boson, while the electron neutrino can additionally scatter off electrons through the exchange of a W boson.

The neutral currents give the same contribution to all flavours of neutrinos while, for oscillations, we care about differences in neutrino energies. For this reason, we look only at the contribution from charged currents. We've already seen in Section 5 that, at low energies, this is captured by the 4-fermion current-current interaction (5.92) which, in the present context, we view as contribution to the Hamiltonian

$$\Delta H = 2\sqrt{2} G_F \, J_{+\,\mu} \, J_-^{\mu} \; . \tag{7.39}$$

Here, $G_F \approx 10^{-5}$ GeV^{-2} the Fermi coupling. The currents $J_\mu^\pm$ were given in (5.88) and include the term

$$\begin{aligned} J_{+\,\mu} \, J_-^{\mu} &= (\bar{\nu}_L \bar{\sigma}_\mu e_L) \, (\bar{e}_L \bar{\sigma}^\mu \nu_L) + \ldots \\ &= (\bar{e}_L \bar{\sigma}_\mu e_L) \, (\bar{\nu}_L \bar{\sigma}^\mu \nu_L) + \ldots \end{aligned} \tag{7.40}$$

where, in the second line, we've done a Fierz shuffle to reorder the fermions. In the presence of matter, the $\mu = 0$ component of the vector $\bar{e}_L \bar{\sigma}^\mu e_L$ gets an expectation value

$$\langle \bar{e}_L \bar{\sigma}^\mu e_L \rangle = n \delta^{\mu 0} \tag{7.41}$$

where n is the background (number) density of electrons. This expectation value breaks Lorentz invariance, as a background density of matter must. It also breaks both CP and CPT as the background is made of normal matter, not anti-matter. (Recall that the CPT theorem is a statement about Lorentz invariant theories only.) The upshot is that we get a contribution to the Hamiltonian governing neutrinos that takes the form

$$\Delta H = V \, \bar{\nu}_L \bar{\sigma}^0 \nu_L \quad \text{where} \quad V = 2\sqrt{2} G_F n \; . \tag{7.42}$$

At this point, we see the next surprise. The extra term in the Hamiltonian H_c is quadratic in neutrinos and so, in that sense, looks like an additional contribution to the neutrino mass. The mass density of matter in the Sun is about $\rho \approx 1$ g cm^{-3} which gives $V \approx 10^{-12}$ eV. In the centre of the Earth, the density is an order of magnitude larger and, correspondingly, $V \approx 10^{-13}$ eV. Both of these are tiny compared to typical neutrino masses of 10^{-3} eV which naively suggests that this effect can't possibly be important for neutrino propagation.

But that intuition is wrong. And it's wrong because of the different index structure. That extra factor of $\bar{\sigma}^0$ in (7.42) makes all the difference: it is telling us that the background matter couples to neutrinos much like a background gauge field of the form $V^\mu = (V, 0, 0, 0)$. This means that the dispersion relation for neutrinos now takes the form

$$(p_\mu - V_\mu)(p^\mu - V^\mu) = m^2 \quad \Longrightarrow \quad (E - V)^2 = m^2 + \mathbf{p}^2 \; . \tag{7.43}$$

We're in a ultra-relativistic regime, with $E, p \gg m \gg V$, so we expand and drop the V^2 term to get the

$$E \approx p + \frac{m^2 + 2EV}{2p^2} + \dots \; . \tag{7.44}$$

We see that the relevant comparison is not m vs V but, instead, m^2 vs EV. And for energies in the MeV range, these can be comparable.

Our next task is to understand how this affects the oscillations. Recall that, in the vacuum, the neutrino Hamiltonian was diagonal in the mass basis. But now we've added an extra term that is diagonal in the flavour basis, contributing only to the electron neutrino. This means that we have some more matrix diagonalisation ahead of us.

To keep things simple, we'll stick to just two flavours of neutrino which we take to be ν_e and ν_μ. We'll again reduce things to a two-state quantum system. In the flavour basis, the vacuum Hamiltonian is given by

$$H = U \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} U^\dagger \quad \text{with} \quad U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \; . \tag{7.45}$$

We use the result (7.31) that gives the energy difference in terms of the mass difference, $E_2 - E_1 = \Delta m^2/2E$, to write

$$H = \frac{1}{2}(E_1 + E_2)\mathbb{1} + \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \; . \tag{7.46}$$

The overall energy contribution $\frac{1}{2}(E_1 + E_2)\mathbb{1}$ is unimportant for our needs and we drop it in what follows. This is the vacuum Hamiltonian. Now we want to include the effects of matter which, as we have seen, give a new contribution

$$H + \Delta H = \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} \; . \tag{7.47}$$

We need to extract the new eigenvalues and eigenvectors of this matrix. If we call these eigenvalues λ_1 and λ_2 then the effective mass splitting in the presence of matter is $\Delta m_m^2 = 2E(\lambda_2 - \lambda_1)$. A short calculation shows that

$$\Delta m_m^2 = \sqrt{(\Delta m^2 \cos 2\theta - 2EV)^2 + (\Delta m^2 \sin 2\theta)^2} \; . \tag{7.48}$$

Meanwhile, we also want to know the effective mixing angle θ_m. This comes from computing the eigenvectors of the new Hamiltonian which take the form $(\cos\theta_m, -\sin\theta_m)$ and $(\sin\theta_m, \cos\theta_m)$. The result is most simply expressed using a double angle formula as

$$\tan 2\theta_m = \frac{\sin 2\theta}{\cos 2\theta - 2EV/\Delta m^2} \ . \tag{7.49}$$

The probability for oscillation from one species to the other is then given by our previous expression (7.33) with Δm^2 and θ replaced by Δm_m^2 and θ_m. This probability is maximised when

$$\cos 2\theta = \frac{2EV}{\Delta m^2} \quad \Longrightarrow \quad \theta_m = \frac{\pi}{4} \ . \tag{7.50}$$

For anti-neutrinos, we replace V with $-V$ in the expressions above. This means that when mixing is maximal for neutrinos, with $\cos 2\theta = 2EV/\Delta m^2$, it is not maximal for anti-neutrinos.

Briefly, the MSW Effect

You might think that it's rather unlikely that we will hit the resonance condition (7.50) for maximal mixing. However, as neutrinos propagate outwards from the centre of the Sun, they experience a changing matter density. This means that we should think of the parameter V in our 2-state quantum system as being time-dependent. It may well be that, at some point on its journey, a given neutrino experiences a point where the effective mixing is maximal. In this way, large mixing can be generated even though the fundamental mixing angles may be small. This is known as the *MSW effect*.

We saw in the lectures on Topics in Quantum Mechanics that there are two limits in which it is straightforward to analyse systems with time-dependent parameters. When the time dependent is slow (in a suitable sense), we can use the adiabatic approximation. This is appropriate in the interior of the Sun. When the time dependence is fast, we can use the sudden approximation. This is appropriate when the neutrinos exit the Sun or when they enter the Earth. Both of these effects are important when understanding the observed oscillations in solar neutrinos.

7.2.3 Neutrino Detection Experiments

Nature provides two different sources of neutrinos that allow us to see oscillations. In what follows, we provide some very brief sketches of the experiments that revealed oscillations in each of these sources. In recent years, these results have been confirmed by looking at terrestrial neutrinos, created in reactors and accelerators.

Solar Neutrinos

Most neutrinos in the Sun are created in a reaction that turns hydrogen into helium,

$$4p \to {}^4\mathrm{He} + 2e^+ + 2\nu_e + 2\gamma \ . \tag{7.51}$$

This produces neutrinos with energy $E \lesssim 400$ keV. There are also further reactions, notably those involving ^{7}Be and ^{8}Be that produce significantly fewer neutrinos, but at energy up to 10 MeV. It is now thought that we have a reasonably good understanding of the neutrinos at various energy scales produced by the Sun. A number of experiments show very cleanly that what leaves the Sun is rather different from what reaches Earth.

- The first set of experiments use neutrino capture,

  $$\nu_e + n \to p + e^- \ . \tag{7.52}$$

  Clearly, this only works for electron neutrinos. This was first done in the late 1960s, useing tanks of chlorine with the reaction

  $$\nu_e + {}^{37}\mathrm{Cl} \ \longrightarrow \ {}^{37}\mathrm{Ar} + e^- \ . \tag{7.53}$$

  The resulting argon atoms were then counted and used as a proxy for the original neutrino. The incoming neutrinos require an energy of $E > 800$ keV to achieve this heat, which means that this is detecting the neutrinos produced in the rarer neutrino processes. The observed solar neutrinos are a factor of 3 smaller than expected.

  This experiment can be repeated with the chlorine replaced by gallium,

  $$\nu_e + {}^{71}\mathrm{Ga} \ \longrightarrow \ {}^{71}\mathrm{Ge} + e^- \ . \tag{7.54}$$

  Now the threshold is lower, needing only energies of $E \approx 200$ keV, meaning that many more of the Sun's neutrinos can partake. Indeed, the number of events seen is significantly higher, but still with a shortfall of about 40% compared to the theoretical prediction. This shows that the oscillations are energy-dependent, as predicted.

- It is possible to see neutrinos of any type by looking at the scattering process

  $$\nu_\alpha + e^- \to \nu_\alpha n + e^- \ . \tag{7.55}$$

  As shown in Figure 21, all neutrinos scatter by exchanging Z bosons, while the electron neutrinos have an additional contribution coming from exchanging a W boson.

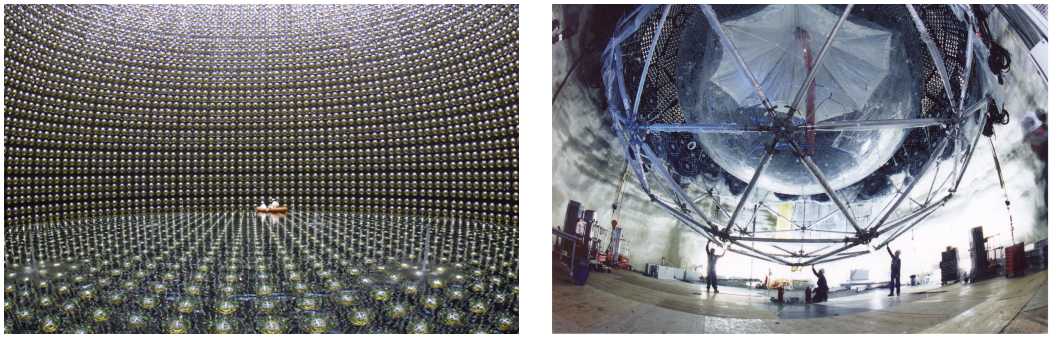

Figure 22. Neutrino detectors tend to look like the lair of a James Bond villain. On the left is a boat cleaning the Super-Kamiokande photosensors as the tank slowly fills up. On the right is the SNO tank, filled with heavy water.

Typically, the neutrinos are scattered off electrons which sit in a large tank of water and detected by the resulting Cerenkov radiation. This, for example, is how the super-Kamiokande experiment in Japan works. The neutrinos must have an energy threshold of $E \approx 8$ MeV and so, as with the chlorine experiments, is sensitive only to the rarer beryllium neutrinos. This time there is a shortfall of around 50%.

These experiments have the advantage that they reveal the direction of the incoming neutrino, and show clearly that the neutrinos are indeed coming from the Sun. In addition, the neutrinos are measured in real time which means that it's possible to detect differences between day, when the neutrinos come directly from the Sun, and night, when the neutrinos must first pass through the Earth before reaching the detector. (We will explain below why such a difference is expected.)

- The state of the art in neutrino detection is offered by the Sudbury neutrino observatory (SNO). This has a tank was filled with heavy water, D_20, where the hydrogen is replaced by deuterium D. It doesn't take much to split the deuterium nucleus apart; just 2 MeV of energy is enough. Moreover, neutrinos can knock apart a deuterium nucleus in two different ways. A weak interaction involving an intermediate W boson does the job through a neutrino capture process analogous to those that occur in chlorine or gallium,

$$\nu_e + D \;\rightarrow\; p + p + e^- \; . \tag{7.56}$$

Only electron neutrinos contribute to such processes. However, the neutrinos can also split the deuterium through a weak interaction involving a Z boson,

$$\nu + D \;\rightarrow\; n + p + \nu \tag{7.57}$$

This time there is no charged lepton created, meaning that all three kinds of neutrinos, ν_e, ν_μ and ν_τ contribute.

In addition, SNO measured neutrino scattering events of the form $\nu + e^- \rightarrow \nu + e^-$ where, again, the electron neutrinos have an additional scattering mode through the W boson. The upshot is that SNO was able to see everything – electron, muon and tau neutrinos. And once you see everything, nothing is missing. The end result agreed perfectly with theoretical expectations of the nuclear reactions inside the Sun. The electron neutrinos missed by previous experiments had transmuted into muon and tau neutrinos, incontrovertible evidence for neutrino oscillations.

Atmospheric Neutrinos

The story of missing neutrinos is repeated when we look elsewhere. Cosmic rays, mostly in the form of protons or helium nuclei, are constantly bombarding the Earth. When they hit the atmosphere they create a constant stream of $\pi^\pm$ pions. These pions decay to muons

$$\pi^+ \;\longrightarrow \mu^+ + \nu_\mu \quad \text{and} \quad \pi^- \;\longrightarrow \mu^- + \bar{\nu}_\mu$$

and the muons then quickly decay to electrons,

$$\mu^+ \;\longrightarrow e^+ + \nu_e + \bar{\nu}_\mu \quad \text{and} \quad \mu^- \;\longrightarrow e^- + \bar{\nu}_e + \nu_\mu$$

The resulting *atmospheric neutrinos* have significantly higher energies than solar neutrinos; often around a GeV or higher. Given the decay processes described above, each collision should result in two muon neutrinos (strictly one ν_μ, one $\bar{\nu}_\mu$) for every electron neutrino. The question is: can we find them?

The answer, given by Super-Kamiokande, is interesting and shown in Figure 23. These show plots of the neutrino flux (on the vertical axis) against the angle at which the neutrinos come into the detector (on the horizontal axis). An angle $\cos\theta = 1$, on the far right, means that the neutrinos come directly down. An angle $\cos\theta = -1$, on the far left, means that neutrinos come up, through the Earth.

The data on the left two boxes is for electron neutrinos, both for low-energy events (shown in the top box) and high-energy events (in the bottom box). The red line is the theoretical expectation; the black dots the observed flux. We see that the agreement between experiment and theory works well.

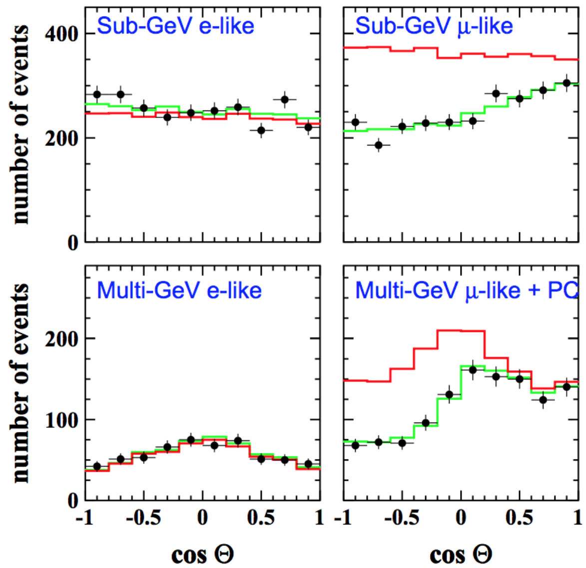

Figure 23. The observed flux of electron neutrinos (on the left) and muon neutrinos (on the right). The top boxes show low-energy neutrinos; the lower boxes high-energy neutrinos. The red line is the theoretical expectation without neutrino oscillations, and the black boxes the data.

The story is more interesting for muon neutrinos, shown in the two boxes on the right. The number of neutrinos coming straight down agrees perfectly with what we expect, but there's a clear deficit for those that come up through the Earth. Why?

For any other particle, you might think that the Earth is simply getting in the way. But neutrinos pass right through the Earth without any difficulty. (Remember the picture of the Sun at night in Figure 19.) Besides: theorists aren't stupid and had taken the presence of the Earth into account when computing the red line! Instead, the key point is that the muon neutrinos have travelled further, and so had more opportunity to convert into other neutrinos, in this case tau.

Importantly, the atmospheric neutrinos clearly show us that neutrino oscillations depend on the length L that neutrinos travel. For those neutrinos that come straight down, we have $L \approx 15$ km and no oscillations are seen. Meanwhile, for those that come up through the Earth we have $L \approx 13000$ km and ν_e is unaffected, while $\nu_\mu \to \nu_\tau$.

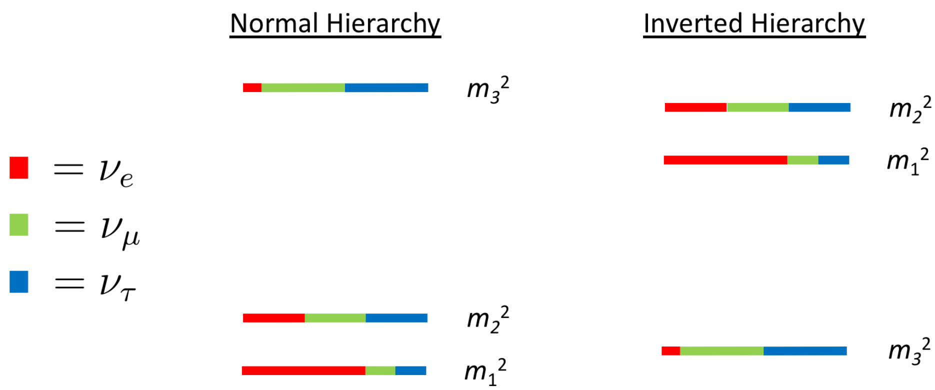

Figure 24. A colour coded description of the possible ordering of neutrino masses.

Neutrino Mass Differences

The experiments sketched above, together with similar terrestrial experiments, are how we determine the precious information about the fundamental parameters in the Standard Model. These tell us the values of the mixing angles that lie within the PMNS matrix (7.17) which, roughly speaking, translate into the following statements about the mass eigenstates: ν_1, ν_2 and ν_3

- ν_1 acts like an electron neutrino two thirds of the time, and as a muon or tau neutrino the other third.
- ν_2 acts like any one of the three neutrinos one third of the time.
- ν_3 acts like a tau neutrino 45% of the time and like a muon neutrino 45% of the time. The remaining 10%, it acts like an electron neutrino.

We also get information about mass differences. The eigenstate ν_1 is known to be lighter than ν_2 and the squares of their masses differ by

$$m_2^2 - m_1^2 \approx 7.4 \times 10^{-5}\ \text{eV}^2$$

The resulting difference in their masses is of order $\sim 10^{-2}$ eV, an order of magnitude smaller than the biggest mass. We also know the difference between the masses of ν_3 and ν_2 but, crucially, we don't yet know which one is heavier! We have

$$m_3^2 - m_2^2 = \pm\, 2.5 \times 10^{-3}\ \text{eV}^2$$

Of course, if we could measure the mass difference between m_1 and m_3.then we would be able to resolve this $\pm$ ambiguity. As it stands, we just don't know the order of the masses.

The two possibilities are shown in Figure 24. Given the pattern seen in all other fermions, one might expect that the electron neutrino ν_e would be the lightest. Since the ν_e has the biggest overlap with ν_1, this would mean that ν_1 is lightest. This is referred to as the *normal hierarchy*. But, as we've seen, very little about the neutrinos follows our expectation. So another possibility is that ν_3, which contains very little of the electron neutrino, is the lightest. This is called the *inverted hierarchy*. The latest evidence from cosmological observations of the CMB and structure formation give an improved bound on $\sum_i m_i$ and point towards the normal hierarchy.