Preprint typeset in JHEP style - PAPER VERSION

June 2005

TASI Lectures on Solitons

Instantons, Monopoles, Vortices and Kinks

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Recommended Books and Resources These lectures cover aspects of solitons with focus on applications to the quantum dynamics of supersymmetric gauge theories and string theory. The lectures consist of four sections, each dealing with a different soliton. We start with instantons and work down in co-dimension to monopoles, vortices and, eventually, domain walls. Emphasis is placed on the moduli space of solitons and, in particular, on the web of connections that links solitons of different types. The D-brane realization of the ADHM and Nahm construction for instantons and monopoles is reviewed, together with related constructions for vortices and domain walls. Each lecture ends with a series of vignettes detailing the roles solitons play in the quantum dynamics of supersymmetric gauge theories in various dimensions. This includes applications to the AdS/CFT correspondence, little string theory, S-duality, cosmic strings, and the quantitative correspondence between 2d sigma models and 4d gauge theories.

Contents

0. Introduction

170 years ago, a Scotsman on horseback watched a wave travelling down Edinburgh's Union canal. He was so impressed that he followed the wave for several miles, described the day of observation as the happiest of his life, and later attempted to recreate the experience in his own garden. The man's name was John Scott Russell and he is generally credited as the first person to develop an unhealthy obsession with the "singular and beautiful phenomenon" that we now call a soliton.

Russell was ahead of his time. The features of stability and persistence that so impressed him were not appreciated by his contemporaries, with Airy arguing that the "great primary wave" was neither great nor primary[1]. It wasn't until the following century that solitons were understood to play an important role in areas ranging from engineering to biology, from condensed matter to cosmology.

The purpose of these lectures is to explore the properties of solitons in gauge theories. There are four leading characters: the instanton, the monopole, the vortex, and the domain wall (also known as the kink). Most reviews of solitons start with kinks and work their way up to the more complicated instantons. Here we're going to do things backwards and follow the natural path: instantons are great and primary, other solitons follow. A major theme of these lectures is to flesh out this claim by describing the web of inter-relationships connecting our four solitonic characters.

Each lecture will follow a similar pattern. We start by deriving the soliton equations and examining the basic features of the simplest solution. We then move on to discuss the interactions of multiple solitons, phrased in terms of the moduli space. For each type of soliton, D-brane techniques are employed to gain a better understanding of the relevant geometry. Along the way, we shall discuss various issues including fermionic zero modes, dyonic excitations and non-commutative solitons. We shall also see the earlier solitons reappearing in surprising places, often nestling within the worldvolume of a larger soliton, with interesting consequences. Each lecture concludes with a few brief descriptions of the roles solitons play in supersymmetric gauge theories in various dimensions.

These notes are aimed at advanced graduate students who have some previous awareness of solitons. The basics will be covered, but only very briefly. A useful primer on solitons can be found in most modern field theory textbooks (see for example [1]). More

[1]More background on Russell and his wave can be found at http://www.ma.hw.ac.uk/~chris/scott_russell.html and http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Russell_Scott.html.

details are contained in the recent book by Manton and Sutcliffe [2]. There are also a number of good reviews dedicated to solitons of a particular type and these will be mentioned at the beginning of the relevant lecture. Other background material that will be required for certain sections includes a basic knowledge of the structure of supersymmetric gauge theories and D-brane dynamics. Good reviews of these subjects can be found in [3, 4, 5].

1. Instantons

30 years after the discovery of Yang-Mills instantons [6], they continue to fascinate both physicists and mathematicians alike. They have lead to new insights into a wide range of phenomena, from the structure of the Yang-Mills vacuum [7, 8, 9] to the classification of four-manifolds [10]. One of the most powerful uses of instantons in recent years is in the analysis of supersymmetric gauge dynamics where they play a key role in unravelling the plexus of entangled dualities that relates different theories. The purpose of this lecture is to review the classical properties of instantons, ending with some applications to the quantum dynamics of supersymmetric gauge theories.

There exist many good reviews on the subject of instantons. The canonical reference for basics of the subject remains the beautiful lecture by Coleman [11]. More recent applications to supersymmetric theories are covered in detail in reviews by Shifman and Vainshtein [12] and by Dorey, Hollowood, Khoze and Mattis [13]. This latter review describes the ADHM construction of instantons and overlaps with the current lecture.

1.1 The Basics

The starting point for our journey is four-dimensional, pure $SU(N)$ Yang-Mills theory with action[2]

$$S = \frac{1}{2e^2}\int d^4x \; \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} \tag{1.1}$$

Motivated by the semi-classical evaluation of the path integral, we search for finite action solutions to the Euclidean equations of motion,

$$\mathcal{D}_\mu F^{\mu\nu} = 0 \tag{1.2}$$

which, in the imaginary time formulation of the theory, have the interpretation of mediating quantum mechanical tunnelling events.

The requirement of finite action means that the potential A_μ must become pure gauge as we head towards the boundary $r \to \infty$ of spatial $\mathbf{R}^4$,

$$A_\mu \to i g^{-1}\,\partial_\mu g \tag{1.3}$$

[2]Conventions: We pick Hemitian generators T^m with Killing form $\mathrm{Tr}\,T^mT^n = \frac{1}{2}\delta^{mn}$. We write $A_\mu = A_\mu^m T^m$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. Adjoint covariant derivatives are $\mathcal{D}_\mu X = \partial_\mu X - i[A_\mu, X]$. In this section alone we work with Euclidean signature and indices will wander from top to bottom with impunity; in the following sections we will return to Minkowski space with signature $(+,-,-,-)$.

with $g(x) = e^{iT(x)} \in SU(N)$. In this way, any finite action configuration provides a map from $\partial\mathbf{R}^4 \cong \mathbf{S}^3_\infty$ into the group $SU(N)$. As is well known, such maps are classified by homotopy theory. Two maps are said to lie in the same homotopy class if they can be continuously deformed into each other, with different classes labelled by the third homotopy group,

$$\Pi_3(SU(N)) \cong \mathbf{Z} \tag{1.4}$$

The integer $k \in \mathbf{Z}$ counts how many times the group wraps itself around spatial $\mathbf{S}^3_\infty$ and is known as the Pontryagin number, or second Chern class. We will sometimes speak simply of the "charge" k of the instanton. It is measured by the surface integral

$$k = \frac{1}{24\pi^2}\int_{\mathbf{S}^3_\infty} d^3S_\mu \; \mathrm{Tr}\,(\partial_\nu g)g^{-1}\,(\partial_\rho g)g^{-1}\,(\partial_\sigma g)g^{-1}\,\epsilon^{\mu\nu\rho\sigma} \tag{1.5}$$

The charge k splits the space of field configurations into different sectors. Viewing $\mathbf{R}^4$ as a foliation of concentric $\mathbf{S}^3$'s, the homotopy classification tells us that we cannot transform a configuration with non-trivial winding $k \neq 0$ at infinity into one with trivial winding on an interior $\mathbf{S}^3$ while remaining in the pure gauge ansatz (1.3). Yet, at the origin, obviously the gauge field must be single valued, independent of the direction from which we approach. To reconcile these two facts, a configuration with $k \neq 0$ cannot remain in the pure gauge form (1.3) throughout all of $\mathbf{R}^4$: it must have non-zero action.

An Example: $SU(2)$

The simplest case to discuss is the gauge group $SU(2)$ since, as a manifold, $SU(2) \cong \mathbf{S}^3$ and it's almost possible to visualize the fact that $\Pi_3(\mathbf{S}^3) \cong \mathbf{Z}$. (Ok, maybe $\mathbf{S}^3$ is a bit of a stretch, but it is possible to visualize $\Pi_1(\mathbf{S}^1) \cong \mathbf{Z}$ and $\Pi_2(\mathbf{S}^2) \cong \mathbf{Z}$ and it's not the greatest leap to accept that, in general, $\Pi_n(\mathbf{S}^n) \cong \mathbf{Z}$). Examples of maps in the different sectors are

- $g^{(0)} = 1$, the identity map has winding $k = 0$
- $g^{(1)} = (x_4 + ix_i\sigma^i)/r$ has winding number $k = 1$. Here $i = 1, 2, 3$, and the σ^i are the Pauli matrices
- $g^{(k)} = [g^{(1)}]^k$ has winding number k.

To create a non-trivial configuration in $SU(N)$, we could try to embed the maps above into a suitable $SU(2)$ subgroup, say the upper left-hand corner of the $N \times N$ matrix. It's not obvious that if we do this they continue to be a maps with non-trivial winding since one could envisage that they now have space to slip off. However, it turns out that this doesn't happen and the above maps retain their winding number when embedded in higher rank gauge groups.

1.1.1 The Instanton Equations

We have learnt that the space of configurations splits into different sectors, labelled by their winding $k \in \mathbf{Z}$ at infinity. The next question we want to ask is whether solutions actually exist for different k. Obviously for $k = 0$ the usual vacuum $A_\mu = 0$ (or gauge transformations thereof) is a solution. But what about higher winding with $k \neq 0$? The first step to constructing solutions is to derive a new set of equations that the instantons will obey, equations that are first order rather than second order as in (1.2). The trick for doing this is usually referred to as the Bogomoln'yi bound [14] although, in the case of instantons, it was actually introduced in the original paper [6]. From the above considerations, we have seen that any configuration with $k \neq 0$ must have some non-zero action. The Bogomoln'yi bound quantifies this. We rewrite the action by completing the square,

$$\begin{aligned} S_{\rm inst} &= \frac{1}{2e^2}\int d^4x\ {\rm Tr}\, F_{\mu\nu}F^{\mu\nu} \\ &= \frac{1}{4e^2}\int d^4x\ {\rm Tr}\,(F_{\mu\nu} \mp {}^\star F^{\mu\nu})^2 \pm 2{\rm Tr}\ F_{\mu\nu}{}^\star F^{\mu\nu} \\ &\geq \pm\frac{1}{2e^2}\int d^4x\ \partial_\mu\left(A_\nu F_{\rho\sigma} + \tfrac{2i}{3}A_\nu A_\rho A_\sigma\right)\epsilon^{\mu\nu\rho\sigma} \end{aligned} \tag{1.6}$$

where the dual field strength is defined as ${}^\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ and, in the final line, we've used the fact that $F_{\mu\nu}{}^\star F^{\mu\nu}$ can be expressed as a total derivative. The final expression is a surface term which measures some property of the field configuration on the boundary $\mathbf{S}^3_\infty$. Inserting the asymptotic form $A_\nu \to ig^{-1}\partial_\nu g$ into the above expression and comparing with (1.5), we learn that the action of the instanton in a topological sector k is bounded by

$$S_{\rm inst} \geq \frac{8\pi^2}{e^2}\,|k| \tag{1.7}$$

with equality if and only if

$$\begin{aligned} F_{\mu\nu} &= {}^\star F_{\mu\nu} \qquad\qquad (k > 0) \\ F_{\mu\nu} &= -{}^\star F_{\mu\nu} \qquad\qquad (k < 0) \end{aligned}$$

Since parity maps $k \to -k$, we can focus on the self-dual equations $F = {}^\star F$. The Bogomoln'yi argument (which we shall see several more times in later sections) says that a solution to the self-duality equations must necessarily solve the full equations of motion since it minimizes the action in a given topological sector. In fact, in the case of instantons, it's trivial to see that this is the case since we have

$$\mathcal{D}_\mu F^{\mu\nu} = \mathcal{D}_\mu {}^\star F^{\mu\nu} = 0 \tag{1.8}$$

by the Bianchi identity.

1.1.2 Collective Coordinates

So we now know the equations we should be solving to minimize the action. But do solutions exist? The answer, of course, is yes! Let's start by giving an example, before we move on to examine some of its properties, deferring discussion of the general solutions to the next subsection.

The simplest solution is the $k = 1$ instanton in $SU(2)$ gauge theory. In singular gauge, the connection is given by

$$A_\mu = \frac{\rho^2 (x-X)_\nu}{(x-X)^2((x-X)^2+\rho^2)}\, \bar{\eta}^i_{\mu\nu}\, (g\sigma^i g^{-1}) \tag{1.9}$$

The σ^i, $i = 1, 2, 3$ are the Pauli matrices and carry the $su(2)$ Lie algebra indices of A_μ. The $\bar{\eta}^i$ are three 4×4 anti-self-dual 't Hooft matrices which intertwine the group structure of the index i with the spacetime structure of the indices μ, ν. They are given by

$$\bar{\eta}^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \bar{\eta}^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad , \quad \bar{\eta}^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{1.10}$$

It's a useful exercise to compute the field strength to see how it inherits its self-duality from the anti-self-duality of the $\bar{\eta}$ matrices. To build an anti-self-dual field strength, we need to simply exchange the $\bar{\eta}$ matrices in (1.9) for their self-dual counterparts,

$$\eta^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \eta^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad , \quad \eta^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{1.11}$$

For our immediate purposes, the most important feature of the solution (1.9) is that it is not unique: it contains a number of parameters. In the context of solitons, these are known as *collective coordinates*. The solution (1.9) has eight such parameters. They are of three different types:

i) 4 translations X_μ: The instanton is an object localized in $\mathbf{R}^4$, centered around the point $x_\mu = X_\mu$.

ii) 1 scale size ρ: The interpretation of ρ as the size of the instanton can be seen by rescaling x and X in the above solution to demote ρ to an overall constant.

iii) 3 global gauge transformations $g \in SU(2)$: This determines how the instanton is embedded in the gauge group.

At this point it's worth making several comments about the solution and its collective coordinates.

- For the $k = 1$ instanton, each of the collective coordinates described above is a Goldstone mode, arising because the instanton configuration breaks a symmetry of the Lagrangian (1.1). In the case of X_μ and g it is clear that the symmetry is translational invariance and $SU(2)$ gauge invariance respectively. The parameter ρ arises from broken conformal invariance. It's rather common that all the collective coordinates of a single soliton are Goldstone modes. It's not true for higher k.

- The apparent singularity at $x_\mu = X_\mu$ is merely a gauge artifact (hence the name "singular gauge"). A plot of a gauge invariant quantity, such as the action density, reveals a smooth solution. The exception is when the instanton shrinks to zero size $\rho \to 0$. This singular configuration is known as the small instanton. Despite its singular nature, it plays an important role in computing the contribution to correlation functions in supersymmetric theories. The small instanton lies at finite distance in the space of classical field configurations (in a way which will be made precise in Section 1.2).

- You may be surprised that we are counting the gauge modes g as physical parameters of the solution. The key point is that they arise from the *global* part of the gauge symmetry, meaning transformations that don't die off asymptotically. These are physical symmetries of the system rather than redundancies. In the early days of studying instantons the 3 gauge modes weren't included, but it soon became apparent that many of the nicer mathematical properties of instantons (for example, hyperKählerity of the moduli space) require us to include them, as do certain physical properties (for example, dyonic instantons in five dimensions)

The $SU(2)$ solution (1.9) has 8 collective coordinates. What about $SU(N)$ solutions? Of course, we should keep the $4 + 1$ translational and scale parameters but we would expect more orientation parameters telling us how the instanton sits in the larger

$SU(N)$ gauge group. How many? Suppose we embed the above $SU(2)$ solution in the upper left-hand corner of an $N \times N$ matrix. We can then rotate this into other embeddings by acting with $SU(N)$, modulo the stabilizer which leaves the configuration untouched. We have

$$SU(N)/S[U(N-2) \times U(2)] \tag{1.12}$$

where the $U(N-2)$ hits the lower-right-hand corner and doesn't see our solution, while the $U(2)$ is included in the denominator since it acts like g in the original solution (1.9) and we don't want to overcount. Finally, the notation $S[U(p) \times U(q)]$ means that we lose the overall central $U(1) \subset U(p) \times U(q)$. The coset space above has dimension $4N - 8$. So, within the ansatz (1.9) embedded in $SU(N)$, we see that the $k = 1$ solution has $4N$ collective coordinates. In fact, it turns out that this is all of them and the solution (1.9), suitably embedded, is the most general $k = 1$ solution in an $SU(N)$ gauge group. But what about solutions with higher k? To discuss this, it's useful to introduce the idea of the moduli space.

1.2 The Moduli Space

We now come to one of the most important concepts of these lectures: the *moduli space*. This is defined to be the space of all solutions to $F = {}^\star F$, modulo gauge transformations, in a given winding sector k and gauge group $SU(N)$. Let's denote this space as $\mathcal{I}_{k,N}$. We will define similar moduli spaces for the other solitons and much of these lectures will be devoted to understanding the different roles these moduli spaces play and the relationships between them.

Coordinates on $\mathcal{I}_{k,N}$ are given by the collective coordinates of the solution. We've seen above that the $k = 1$ solution has $4N$ collective coordinates or, in other words, $\dim(\mathcal{I}_{1,N}) = 4N$. For higher k, the number of collective coordinates can be determined by index theorem techniques. I won't give all the details, but will instead simply tell you the answer.

$$\dim(\mathcal{I}_{k,N}) = 4kN \tag{1.13}$$

This has a very simple interpretation. The charge k instanton can be thought of as k charge 1 instantons, each with its own position, scale, and gauge orientation. When the instantons are well separated, the solution does indeed look like this. But when instantons start to overlap, the interpretation of the collective coordinates can become more subtle.

Strictly speaking, the index theorem which tells us the result (1.13) doesn't count the number of collective coordinates, but rather related quantities known as *zero modes*. It works as follows. Suppose we have a solution A_μ satisfying $F = {}^\star F$. Then we can perturb this solution $A_\mu \rightarrow A_\mu + \delta A_\mu$ and ask how many other solutions are nearby. We require the perturbation δA_μ to satisfy the linearized self-duality equations,

$$\mathcal{D}_\mu \delta A_\nu - \mathcal{D}_\nu \delta A_\mu = \epsilon_{\mu\nu\rho\sigma} \mathcal{D}^\rho \delta A^\sigma \tag{1.14}$$

where the covariant derivative $\mathcal{D}_\mu$ is evaluated on the background solution. Solutions to (1.14) are called zero modes. The idea of zero modes is that if we have a general solution $A_\mu = A_\mu(x_\mu, X^\alpha)$, where X^α denote all the collective coordinates, then for each collective coordinate we can define the zero mode $\delta_\alpha A_\mu = \partial A_\mu / \partial X^\alpha$ which will satisfy (1.14). In general however, it is not guaranteed that any zero mode can be successfully integrated to give a corresponding collective coordinate. But it will turn out that all the solitons discussed in these lectures do have this property (at least this is true for bosonic collective coordinates; there is a subtlety with the Grassmannian collective coordinates arising from fermions which we'll come to shortly).

Of course, any local gauge transformation will also solve the linearized equations (1.14) so we require a suitable gauge fixing condition. We'll write each zero mode to include an infinitesimal gauge transformation Ω_α,

$$\delta_\alpha A_\mu = \frac{\partial A_\mu}{\partial X^\alpha} + \mathcal{D}_\mu \Omega_\alpha \tag{1.15}$$

and choose Ω_α so that $\delta_\alpha A_\mu$ is orthogonal to any other gauge transformation, meaning

$$\int d^4x \ \mathrm{Tr}\,(\delta_\alpha A_\mu)\, \mathcal{D}_\mu \eta = 0 \qquad \forall\ \eta \tag{1.16}$$

which, integrating by parts, gives us our gauge fixing condition

$$\mathcal{D}_\mu\, (\delta_\alpha A_\mu) = 0 \tag{1.17}$$

This gauge fixing condition does not eliminate the collective coordinates arising from global gauge transformations which, on an operational level, gives perhaps the clearest reason why we must include them. The Atiyah-Singer index theorem counts the number of solutions to (1.14) and (1.17) and gives the answer (1.13).

So what does the most general solution, with its $4kN$ parameters, look like? The general explicit form of the solution is not known. However, there are rather clever ansätze which give rise to various subsets of the solutions. Details can be found in the original literature [15, 16] but, for now, we head in a different, and ultimately more important, direction and study the geometry of the moduli space.

1.2.1 The Moduli Space Metric

A priori, it is not obvious that $\mathcal{I}_{k,N}$ is a manifold. In fact, it does turn out to be a smooth space apart from certain localized singularities corresponding to small instantons at $\rho \to 0$ where the field configuration itself also becomes singular.

The moduli space $\mathcal{I}_{k,N}$ inherits a natural metric from the field theory, defined by the overlap of zero modes. In the coordinates X^α, $\alpha = 1, \ldots, 4kN$, the metric is given by

$$g_{\alpha\beta} = \frac{1}{2e^2} \int d^4x \ \mathrm{Tr}\ (\delta_\alpha A_\mu)\ (\delta_\beta A_\mu) \tag{1.18}$$

It's hard to overstate the importance of this metric. It distills the information contained in the solutions to $F = {}^\star F$ into a more manageable geometric form. It turns out that for many applications, everything we need to know about the instantons is contained in the metric $g_{\alpha\beta}$, and this remains true of similar metrics that we will define for other solitons. Moreover, it is often much simpler to determine the metric (1.18) than it is to determine the explicit solutions.

The metric has a few rather special properties. Firstly, it inherits certain isometries from the symmetries of the field theory. For example, both the $SO(4)$ rotation symmetry of spacetime and the $SU(N)$ gauge action will descend to give corresponding isometries of the metric $g_{\alpha\beta}$ on $\mathcal{I}_{k,N}$.

Another important property of the metric (1.18) is that it is *hyperKähler*, meaning that the manifold has reduced holonomy $Sp(kN) \subset SO(4kN)$. Heuristically, this means that the manifold admits something akin to a quaternionic structure[3]. More precisely, a hyperKähler manifold admits three complex structures J^i, $i = 1, 2, 3$ which obey the relation

$$J^i\, J^j = -\delta^{ij} + \epsilon^{ijk}\, J^k \tag{1.19}$$

The simplest example of a hyperKähler manifold is $\mathbf{R}^4$, viewed as the quaternions. The three complex structures can be taken to be the anti-self-dual 't Hooft matrices $\bar{\eta}^i$ that we defined in (1.10), each of which gives a different complex pairing of $\mathbf{R}^4$. For example, from $\bar{\eta}^3$ we get $z^1 = x^1 + ix^2$ and $z^2 = x^3 - ix^4$.

[3]Warning: there is also something called a quaternionic manifold which arises in $\mathcal{N} = 2$ supergravity theories [17] and is different from a hyperKähler manifold. For a discussion on the relationship see [18].

The instanton moduli space $\mathcal{I}_{k,N}$ inherits its complex structures J^i from those of $\mathbf{R}^4$. To see this, note if δA_μ is a zero mode, then we may immediately write down three other zero modes $\bar{\eta}^i_{\nu\mu}\,\delta A_\mu$, each of which satisfy the equations (1.14) and (1.17). It must be possible to express these three new zero modes as a linear combination of the original ones, allowing us to define three matrices J^i,

$$\bar{\eta}^i_{\mu\nu}\ \delta_\beta A_\nu = (J^i)^\alpha{}_\beta\ [\delta_\alpha A_\mu] \tag{1.20}$$

These matrices J^i then descend to three complex structures on the moduli space $\mathcal{I}_{k,N}$ itself which are given by

$$(J^i)^\alpha{}_\beta = g^{\alpha\gamma}\ \int d^4x\ \bar{\eta}^i_{\mu\nu}\ \mathrm{Tr}\ \delta_\beta A_\mu\ \delta_\gamma A_\nu \tag{1.21}$$

So far we have shown only that J^i define almost complex structures. To prove hyperKählerity, one must also show integrability which, after some gymnastics, is possible using the formulae above. A more detailed discussion of the geometry of the moduli space in this language can be found in [19, 20] and more generally in [21, 22]. For physicists the simplest proof of hyperKählerity follows from supersymmetry as we shall review in section 1.3.

It will prove useful to return briefly to discuss the isometries. In Kähler and hyperKähler manifolds, it's often important to state whether isometries are compatible with the complex structure J. If the complex structure doesn't change as we move along the isometry, so that the Lie derivative $\mathcal{L}_k J = 0$, with k the Killing vector, then the isometry is said to be *holomorphic*. In the instanton moduli space $\mathcal{I}_{k,N}$, the $SU(N)$ gauge group action is tri-holomorphic, meaning it preserves all three complex structures. Of the $SO(4) \cong SU(2)_L \times SU(2)_R$ rotational symmetry, one half, $SU(2)_L$, is tri-holomorphic, while the three complex structures are rotated under the remaining $SU(2)_R$ symmetry.

1.2.2 An Example: A Single Instanton in $SU(2)$

In the following subsection we shall show how to derive metrics on $\mathcal{I}_{k,N}$ using the powerful ADHM technique. But first, to get a flavor for the ideas, let's take a more pedestrian route for the simplest case of a $k = 1$ instanton in $SU(2)$. As we saw above, there are three types of collective coordinates.

i) The four translational modes are $\delta_{(\nu)} A_\mu = \partial A_\mu / \partial X^\nu + \mathcal{D}_\mu \Omega_\nu$ where Ω_ν must be chosen to satisfy (1.17). Using the fact that $\partial/\partial X^\nu = -\partial/\partial x^\nu$, it is simple to see that the correct choice of gauge is $\Omega_\nu = A_\nu$, so that the zero mode is simply given by $\delta_\nu A_\mu = F_{\mu\nu}$, which satisfies the gauge fixing condition by virtue of the original equations of motion (1.2). Computing the overlap of these translational zero modes then gives

$$\int d^4x \ \mathrm{Tr}\ (\delta_{(\nu)} A_\mu \, \delta_{(\rho)} A_\mu) = S_{\rm inst}\ \delta_{\nu\rho} \tag{1.22}$$

ii) One can check that the scale zero mode $\delta A_\mu = \partial A_\mu / \partial\rho$ already satisfies the gauge fixing condition (1.17) when the solution is taken in singular gauge (1.9). The overlap integral in this case is simple to perform, yielding

$$\int d^4x \ \mathrm{Tr}\ (\delta A_\mu \, \delta A_\mu) = 2 S_{\rm inst} \tag{1.23}$$

iii) Finally, we have the gauge orientations. These are simply of the form $\delta A_\mu = \mathcal{D}_\mu \Lambda$, but where Λ does not vanish at infinity, so that it corresponds to a global gauge transformation. In singular gauge it can be checked that the three $SU(2)$ rotations $\Lambda^i = [(x-X)^2/((x-X)^2+\rho^2)]\sigma^i$ satisfy the gauge fixing constraint. These give rise to an $SU(2) \cong \mathbf{S}^3$ component of the moduli space with radius given by the norm of any one mode, say, Λ^3

$$\int d^4x \ \mathrm{Tr}\ (\delta A_\mu \, \delta A_\mu) = 2 S_{\rm inst}\, \rho^2 \tag{1.24}$$

Note that, unlike the others, this component of the metric depends on the collective coordinate ρ, growing as ρ^2. This dependence means that the $\mathbf{S}^3$ arising from $SU(2)$ gauge rotations combines with the $\mathbf{R}^+$ from scale transformations to form the space $\mathbf{R}^4$. However, there is a discrete subtlety. Fields in the adjoint representation are left invariant under the center $Z_2 \subset SU(2)$, meaning that the gauge rotations give rise to $\mathbf{S}^3/\mathbf{Z}_2$ rather than $\mathbf{S}^3$. Putting all this together, we learn that the moduli space of a single instanton is

$$\mathcal{I}_{1,2} \cong \mathbf{R}^4 \times \mathbf{R}^4/\mathbf{Z}_2 \tag{1.25}$$

where the first factor corresponds to the position of the instanton, and the second factor determines its scale size and $SU(2)$ orientation. The normalization of the flat metrics on the two $\mathbf{R}^4$ factors is given by (1.22) and (1.23). In this case, the hyperKähler structure on $\mathcal{I}_{1,2}$ comes simply by viewing each $\mathbf{R}^4 \cong \mathbb{H}$, the quaternions. As is clear from our derivation, the singularity at the origin of the orbifold $\mathbf{R}^4/\mathbf{Z}_2$ corresponds to the small instanton $\rho \to 0$.

1.3 Fermi Zero Modes

So far we've only concentrated on the pure Yang-Mills theory (1.1). It is natural to wonder about the possibility of other fields in the theory: could they also have non-trivial solutions in the background of an instanton, leading to further collective coordinates? It turns out that this doesn't happen for bosonic fields (although they do have an important impact if they gain a vacuum expectation value as we shall review in later sections). Importantly, the fermions do contribute zero modes.

Consider a single Weyl fermion λ transforming in the adjoint representation of $SU(N)$, with kinetic term $i\mathrm{Tr}\,\bar{\lambda}\,\bar{\not{\mathcal{D}}}\lambda$. In Euclidean space, we treat λ and $\bar{\lambda}$ as independent variables, a fact which leads to difficulties in defining a real action. (For the purposes of this lecture, we simply ignore the issue - a summary of the problem and its resolutions can be found in [13]). The equations of motion are

$$\bar{\not{\mathcal{D}}}\lambda \equiv \bar{\sigma}^\mu \mathcal{D}_\mu \lambda = 0 \quad , \qquad \not{\mathcal{D}}\bar{\lambda} \equiv \sigma^\mu \mathcal{D}_\mu \bar{\lambda} = 0 \tag{1.26}$$

where $\not{\mathcal{D}} = \sigma^\mu \mathcal{D}_\mu$ and the 2×2 matrices are $\sigma^\mu = (\sigma^i, -i1_2)$. In the background of an instanton $F = {}^\star F$, only λ picks up zero modes. $\bar{\lambda}$ has none. This situation is reversed in the background of an anti-instanton $F = -{}^\star F$. To see that $\bar{\lambda}$ has no zero modes in the background of an instanton, we look at

$$\bar{\not{\mathcal{D}}}\,\not{\mathcal{D}} = \bar{\sigma}^\mu \sigma^\nu D_\mu \mathcal{D}_\nu = \mathcal{D}^2\,1_2 + F^{\mu\nu}\bar{\eta}^i_{\mu\nu}\sigma^i \tag{1.27}$$

where $\bar{\eta}^i$ are the anti-self-dual 't Hooft matrices defined in (1.10). But a self-dual matrix $F_{\mu\nu}$ contracted with an anti-self-dual matrix $\bar{\eta}_{\mu\nu}$ vanishes, leaving us with $\bar{\not{\mathcal{D}}}\,\not{\mathcal{D}} = \mathcal{D}^2$. And the positive definite operator $\mathcal{D}^2$ has no zero modes. In contrast, if we try to repeat the calculation for λ, we find

$$\not{\mathcal{D}}\,\bar{\not{\mathcal{D}}} = \mathcal{D}^2\,1_2 + F^{\mu\nu}\eta^i_{\mu\nu}\sigma^i \tag{1.28}$$

where η^i are the self-dual 't Hooft matrices (1.11). Since we cannot express the operator $\not{\mathcal{D}}\,\bar{\not{\mathcal{D}}}$ as a total square, there's a chance that it has zero modes. The index theorem tells us that each Weyl fermion λ picks up $4kN$ zero modes in the background of a charge k instanton. There are corresponding Grassmann collective coordinates, which we shall denote as χ, associated to the most general solution for the gauge field and fermions. But these Grassmann collective coordinates occasionally have subtle properties. The quick way to understand this is in terms of supersymmetry. And often the quick way to understand the full power of supersymmetry is to think in higher dimensions.

1.3.1 Dimension Hopping

It will prove useful to take a quick break in order to make a few simple remarks about instantons in higher dimensions. So far we've concentrated on solutions to the self-duality equations in four-dimensional theories, which are objects localized in Euclidean spacetime. However, it is a simple matter to embed the solutions in higher dimensions simply by insisting that all fields are independent of the new coordinates. For example, in $d = 4 + 1$ dimensional theories one can set $\partial_0 = A_0 = 0$, with the spatial part of the gauge field satisfying $F = {}^\star F$. Such configurations have finite energy and the interpretation of particle like solitons. We shall describe some of their properties when we come to applications. Similarly, in $d = 5 + 1$, the instantons are string like objects, while in $d = 9 + 1$, instantons are five-branes. While this isn't a particularly deep insight, it's a useful trick to keep in mind when considering the fermionic zero modes of the soliton in supersymmetric theories as we shall discuss shortly.

When solitons have a finite dimensional worldvolume, we can promote the collective coordinates to fields which depend on the worldvolume directions. These correspond to massless excitations living on the solitons. For example, allowing the translational modes to vary along the instanton string simply corresponds to waves propagating along the string. Again, this simple observation will become rather powerful when viewed in the context of supersymmetric theories.

A note on terminology: Originally the term "instanton" referred to solutions to the self-dual Yang-Mills equations $F = {}^\star F$. (At least this was true once Physical Review lifted its censorship of the term!). However, when working with theories in spacetime dimensions other than four, people often refer to the relevant finite action configuration as an instanton. For example, kinks in quantum mechanics are called instantons. Usually this doesn't lead to any ambiguity but in this review we'll consider a variety of solitons in a variety of dimensions. I'll try to keep the phrase "instanton" to refer to (anti)-self-dual Yang-Mills instantons.

1.3.2 Instantons and Supersymmetry

Instantons share an intimate relationship with supersymmetry. Let's consider an instanton in a $d = 3 + 1$ supersymmetric theory which could be either $\mathcal{N} = 1$, $\mathcal{N} = 2$ or $\mathcal{N} = 4$ super Yang-Mills. The supersymmetry transformation for any adjoint Weyl fermion takes the form

$$\delta\lambda = F^{\mu\nu}\sigma_\mu\bar{\sigma}_\nu\epsilon \quad , \quad \delta\bar{\lambda} = F^{\mu\nu}\bar{\sigma}_\mu\sigma_\nu\bar{\epsilon} \tag{1.29}$$

where, again, we treat the infinitesimal supersymmetry parameters ϵ and $\bar{\epsilon}$ as independent. But we've seen above that in the background of a self-dual solution $F = {}^\star F$

the combination $F^{\mu\nu}\bar{\sigma}_\mu\sigma_\nu = 0$. This means that the instanton is annihilated by half of the supersymmetry transformations $\bar{\epsilon}$, while the other half, ϵ, turn on the fermions λ. We say that the supersymmetries arising from ϵ are broken by the soliton, while those arising from $\bar{\epsilon}$ are preserved. Configurations in supersymmetric theories which are annihilated by some fraction of the supersymmetries are known as BPS states (although the term Witten-Olive state would be more appropriate [23]).

Both the broken and preserved supersymmetries play an important role for solitons. The broken ones are the simplest to describe, for they generate fermion zero modes $\lambda = F^{\mu\nu}\sigma_\mu\bar{\sigma}_\nu\epsilon$. These "Goldstino" modes are a subset of the $4kN$ fermion zero modes that exist for each Weyl fermion λ. Further modes can also be generated by acting on the instanton with superconformal transformations.

The unbroken supersymmetries $\bar{\epsilon}$ play a more important role: they descend to a supersymmetry on the soliton worldvolume, pairing up bosonic collective coordinates X with Grassmannian collective coordinates χ. There's nothing surprising here. It's simply the statement that if a symmetry is preserved in a vacuum (where, in this case, the "vacuum" is the soliton itself) then all excitations above the vacuum fall into representations of this symmetry. However, since supersymmetry in $d = 0 + 0$ dimensions is a little subtle, and the concept of "excitations above the vacuum" in $d = 0 + 0$ dimensions even more so, this is one of the places where it will pay to lift the instantons to higher dimensional objects. For example, instantons in theories with 8 supercharges (equivalent to $\mathcal{N} = 2$ in four dimensions) can be lifted to instanton strings in six dimensions, which is the maximum dimension in which Yang-Mills theory with eight supercharges exists. Similarly, instantons in theories with 16 supercharges (equivalent to $\mathcal{N} = 4$ in four dimensions) can be lifted to instanton five-branes in ten dimensions. Instantons in $\mathcal{N} = 1$ theories are stuck in their four-dimensional world.

Considering Yang-Mills instantons as solitons in higher dimensions allows us to see this relationship between bosonic and fermionic collective coordinates. Consider exciting a long-wavelength mode of the soliton in which a bosonic collective coordinate X depends on the worldvolume coordinate of the instanton s, so $X = X(s)$. Then if we hit this configuration with the unbroken supersymmetry $\bar{\epsilon}$, it will no longer annihilate the configuration, but will turn on a fermionic mode proportional to $\partial_s X$. Similarly, any fermionic excitation will be related to a bosonic excitation.

The observation that the unbroken supersymmetries descend to supersymmetries on the worldvolume of the soliton saves us a lot of work in analyzing fermionic zero modes: if we understand the bosonic collective coordinates and the preserved supersymmetry, then the fermionic modes pretty much come for free. This includes some rather subtle interaction terms.

For example, consider instanton five-branes in ten-dimensional super Yang-Mills. The worldvolume theory must preserve 8 of the 16 supercharges. The only such theory in $5+1$ dimensions is a sigma-model on a hyperKähler target space [24] which, for instantons, is the manifold $\mathcal{I}_{k,N}$. The Lagrangian is

$$\mathcal{L} = g_{\alpha\beta}\partial X^\alpha \partial X^\beta + i\bar{\chi}^\alpha D_{\alpha\beta}\chi^\beta + \tfrac{1}{4}R_{\alpha\beta\gamma\delta}\bar{\chi}^\alpha\chi^\beta\bar{\chi}^\gamma\chi^\delta \tag{1.30}$$

where ∂ denotes derivatives along the soliton worldvolume and the covariant derivative is $D_{\alpha\beta} = g_{\alpha\beta}\partial + \Gamma^\gamma_{\alpha\beta}(\partial X_\gamma)$. This is the slick proof that the instanton moduli space metric must be hyperKähler: it is dictated by the 8 preserved supercharges.

The final four-fermi term couples the fermionic collective coordinates to the Riemann tensor. Suppose we now want to go back down to instantons in four dimensional $\mathcal{N}=4$ super Yang-Mills. We can simply dimensionally reduce the above action. Since there are no longer worldvolume directions for the instantons, the first two terms vanish, but we're left with the term

$$S_{\rm inst} = \tfrac{1}{4}R_{\alpha\beta\gamma\delta}\bar{\chi}^\alpha\chi^\beta\bar{\chi}^\gamma\chi^\delta \tag{1.31}$$

This term reflects the point we made earlier: zero modes cannot necessarily be lifted to collective coordinates. Here we see this phenomenon for fermionic zero modes. Although each such mode doesn't change the action of the instanton, if we turn on four Grassmannian collective coordinates at the same time then the action does increase! One can derive this term without recourse to supersymmetry but it's a bit of a pain [25]. The term is very important in applications of instantons.

Instantons in four-dimensional $\mathcal{N}=2$ theories can be lifted to instanton strings in six dimensions. The worldvolume theory must preserve half of the 8 supercharges. There are two such super-algebras in two dimensions, a non-chiral $(2,2)$ theory and a chiral $(0,4)$ theory, where the two entries correspond to left and right moving fermions respectively. By analyzing the fermionic zero modes one can show that the instanton string preserves $(0,4)$ supersymmetry. The corresponding sigma-model doesn't contain the term (1.31). (Basically because the $\bar{\chi}$ zero modes are missing). However, similar terms can be generated if we also consider fermions in the fundamental representation.

Finally, instantons in $\mathcal{N}=1$ super Yang-Mills preserve $(0,2)$ supersymmetry on their worldvolume.

In the following sections, we shall pay scant attention to the fermionic zero modes, simply stating the fraction of supersymmetry that is preserved in different theories. In many cases this is sufficient to fix the fermions completely: the beauty of supersymmetry is that we rarely have to talk about fermions!

1.4 The ADHM Construction

In this section we describe a powerful method to solve the self-dual Yang-Mills equations $F = {}^\star F$ due to Atiyah, Drinfeld, Hitchin and Manin and known as the ADHM construction [26]. This will also give us a new way to understand the moduli space $\mathcal{I}_{k,N}$ and its metric. The natural place to view the ADHM construction is twistor space. But, for a physicist, the simplest place to view the ADHM construction is type II string theory [27, 28, 29]. We'll do things the simple way.

The brane construction is another place where it's useful to consider Yang-Mills instantons embedded as solitons in a $p+1$ dimensional theory with $p \geq 3$. With this in mind, let's consider a configuration of N Dp-branes, with k D$(p-4)$-branes in type II string theory (Type IIB for p odd; type IIA for p even). A typical configuration is drawn in figure 1. We place all N Dp-branes on top of each other so that, at low energies, their worldvolume dynamics is described by

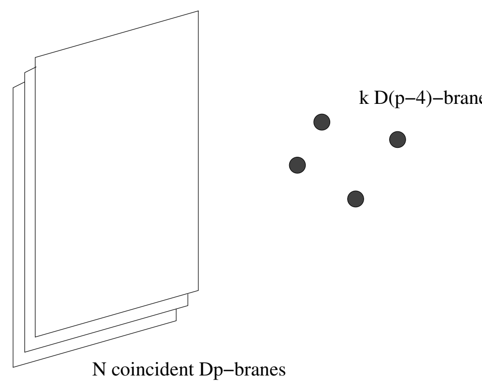

Figure 1: Dp-branes as instantons.

$$d = p+1 \; U(N) \text{ Super Yang-Mills with 16 Supercharges}$$

For example, if $p=3$ we have the familiar $\mathcal{N}=4$ theory in $d=3+1$ dimensions. The worldvolume theory of the Dp-branes also includes couplings to the various RR-fields in the bulk. This includes the term

$$\mathrm{Tr} \int_{Dp} d^{p+1}x \;\; C_{p-3} \wedge F \wedge F \qquad (1.32)$$

where F is the $U(N)$ gauge field, and C_{p-3} is the RR-form that couples to D$(p-4)$-branes. The importance of this term lies in the fact that it relates instantons on the Dp-branes to D$(p-4)$ branes. To see this, note that an instanton with non-zero $F \wedge F$ gives rise to a source $(8\pi^2/e^2) \int d^{p-3}x \; C_{p-3}$ for the RR-form. This is the same source induced by a D$(p-4)$-brane. If you're careful in comparing the factors of 2 and π and such like, it's not hard to show that the instanton has precisely the mass and charge of the D$(p-4)$-brane [3, 5]. They are the same object! We have the important result that

$$\text{Instanton in D}p\text{-Brane} \;\equiv\; \text{D}(p-4)\text{-Brane} \tag{1.33}$$

The strategy to derive the ADHM construction from branes is to view this whole story from the perspective of the D$(p-4)$-branes [27, 28, 29]. For definiteness, let's revert back to $p=3$, so that we're considering D-instantons interacting with $D3$-branes. This means that we have to write down the $d=0+0$ dimensional theory on the D-instantons. Since supersymmetric theories in no dimensions may not be very familiar, it will help to keep in mind that the whole thing can be lifted to higher p.

Suppose firstly that we don't have the D3-branes. The theory on the D-instantons in flat space is simply the dimensional reduction of $d=3+1$ $\mathcal{N}=4$ $U(k)$ super Yang-Mills to zero dimensions. We will focus on the bosonic sector, with the fermions dictated by supersymmetry as explained in the previous section. We have 10 scalar fields, each of which is a $k\times k$ Hermitian matrix. For later convenience, we split them into two batches:

$$(X^\mu, \hat{X}^m) \qquad \mu=1,2,3,4; \quad m=5,\ldots,10 \tag{1.34}$$

where we've put hats on directions transverse to the D3-brane. We'll use the index notation $(X^\mu)^\alpha{}_\beta$ to denote the fact that each of these is a $k\times k$ matrix. Note that this is a slight abuse of notation since, in the previous section, $\alpha=1,\ldots,4k$ rather than $1,\ldots,k$ here. We'll also introduce the complex notation

$$Z = X_1 + iX_2 \quad , \quad W = X_3 - iX_4 \tag{1.35}$$

When X_μ and $\hat{X}_m$ are all mutually commuting, their $10k$ eigenvalues have the interpretation of the positions of the k D-instantons in flat ten-dimensional space.

What effect does the presence of the $D3$-branes have? The answer is well known. Firstly, they reduce the supersymmetry on the lower dimensional brane by half, to eight supercharges (equivalent to $\mathcal{N}=2$ in $d=3+1$). The decomposition (1.34) reflects this, with the $\hat{X}_m$ lying in a vector multiplet and the X_μ forming an adjoint hypermultiplet. The new fields which reduce the supersymmetry are N hypermultiplets, arising from quantizing strings stretched between the Dp-branes and D$(p-4)$-branes. Each hypermultiplet carries an $\alpha=1,\ldots k$ index, corresponding to the D$(p-4)$-brane on which the string ends, and an $a=1,\ldots,N$ index corresponding to the Dp-brane on which the other end of the string sits.. Again we ignore fermions. The two complex scalars in each hypermultiplet are denoted

$$\psi^\alpha{}_a \quad , \quad \tilde{\psi}^a{}_\alpha \tag{1.36}$$

where the index structure reflects the fact that ψ transforms in the $\mathbf{k}$ of the $U(k)$ gauge symmetry, and the $\bar{\mathbf{N}}$ of a $SU(N)$ flavor symmetry. In contrast $\tilde{\psi}$ transforms in the $(\bar{\mathbf{k}}, \mathbf{N})$ of $U(k) \times SU(N)$. (One may wonder about the difference between a gauge and flavor symmetry in zero dimensions; again the reader is invited to lift the configuration to higher dimensions where such nasty questions evaporate. But the basic point will be that we treat configurations related by $U(k)$ transformations as physically equivalent). These hypermultiplets can be thought of as the dimensional reduction of $\mathcal{N} = 2$ hypermultiplets in $d = 3 + 1$ dimensions which, in turn, are composed of two chiral multiplets ψ and $\tilde{\psi}$.

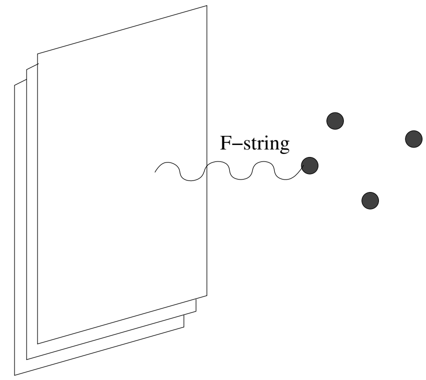

Figure 2: F-strings give rise to hypermultiplets.

The scalar potential for these fields is fixed by supersymmetry (Actually, supersymmetry in $d = 0 + 0$ dimensions is rather weak; at this stage we should lift up to, say $p = 7$, where so we can figure out the familiar $\mathcal{N} = 2$ theory on the D$(p-3)$=D3-branes, and then dimensionally reduce back down to zero dimensions). We have

$$V = \frac{1}{g^2} \sum_{m,n=5}^{10} [\hat{X}_m, \hat{X}_n]^2 + \sum_{m=5}^{10} \sum_{\mu=1}^{4} [\hat{X}_m, X_\mu]^2 + \sum_{a=1}^{N} (\psi^{a\dagger} \hat{X}_m^2 \psi_a + \tilde{\psi}^a \hat{X}_m^2 \tilde{\psi}_a^\dagger) \tag{1.37}$$

$$+ g^2 \,\mathrm{Tr}\, (\sum_{a=1}^{N} \psi_a \psi^{a\dagger} - \tilde{\psi}_a^\dagger \tilde{\psi}^a + [Z, Z^\dagger] + [W, W^\dagger])^2 + g^2 \,\mathrm{Tr}\, | \sum_{a=1}^{N} \psi_a \tilde{\psi}^a + [Z, W]|^2$$

The terms in the second line are usually referred to as D-terms and F-terms respectively (although, as we shall shall review shortly, they are actually on the same footing in theories with eight supercharges). Each is a $k \times k$ matrix. The third term in the first line ensures that the hypermultiplets get a mass if the $\hat{X}_m$ get a vacuum expectation value. This reflects the fact that, as is clear from the picture, the Dp-D$(p-4)$ strings become stretched if the branes are separated in the $\hat{X}^m$, $m = 5, \ldots, 10$ directions. In contrast, there is no mass for the hypermultiplets if the D$(p-4)$ branes are separated in the X_μ, $\mu = 1, 2, 3, 4$ directions. Finally, note that we've included an auxiliary coupling constant g^2 in (1.37). Strictly speaking we should take the limit $g^2 \to \infty$.

We are interested in the ground states of the D-instantons, determined by the solutions to $V = 0$. There are two possibilities

1. The second line vanishes if $\psi = \tilde{\psi} = 0$ and X_μ are diagonal. The first two terms vanish if $\hat{X}_m$ are also diagonal. The eigenvalues of X_μ and $\hat{X}_m$ tell us

where the k D-instantons are placed in flat space. They are unaffected by the existence of the D3-branes whose presence is only felt at the one-loop level when the hypermultiplets are integrated out. This is known as the "*Coulomb branch*", a name inherited from the structure of gauge symmetry breaking: $U(k) \to U(1)^k$. (The name is, of course, more appropriate in dimensions higher than zero where particles charged under $U(1)^k$ experience a Coulomb interaction).

2. The first line vanishes if $\hat{X}_m = 0$, $m = 5, \ldots, 10$. This corresponds to the D$(p-4)$ branes lying on top of the Dp-branes. The remaining fields ψ, $\tilde{\psi}$, Z and W are constrained by the second line in (1.37). Since these solutions allow $\psi, \tilde{\psi} \neq 0$ we will generically have the $U(k)$ gauge group broken completely, giving the name "*Higgs branch*" to this class of solutions. More precisely, the Higgs branch is defined to be the space of solutions

$$\mathcal{M}_{\text{Higgs}} \cong \{\hat{X}_m = 0, V = 0\}/U(k) \tag{1.38}$$

where we divide out by $U(k)$ gauge transformations. The Higgs branch describes the D$(p-4)$ branes nestling inside the larger Dp-branes. But this is exactly where they appear as instantons. So we might expect that the Higgs branch knows something about this. Let's start by computing its dimension. We have $4kN$ real degrees of freedom in ψ and $\tilde{\psi}$ and a further $4k^2$ in Z and W. The D-term imposes k^2 real constraints, while the F-term imposes k^2 complex constraints. Finally we lose a further k^2 degrees of freedom when dividing by $U(k)$ gauge transformations. Adding, subtracting, we have

$$\dim(\mathcal{M}_{\text{Higgs}}) = 4kN \tag{1.39}$$

which should look familiar (1.13). The first claim of the ADHM construction is that we have an isomorphism between manifolds,

$$\mathcal{M}_{\text{Higgs}} \cong \mathcal{I}_{k,N} \tag{1.40}$$

1.4.1 The Metric on the Higgs Branch

To summarize, the D-brane construction has lead us to identify the instanton moduli space $\mathcal{I}_{k,N}$ with the Higgs branch of a gauge theory with 8 supercharges (equivalent to $\mathcal{N} = 2$ in $d = 3 + 1$). The field content of this gauge theory is

$$\begin{aligned} U(k) \text{ Gauge Theory} &+ \text{Adjoint Hypermultiplet } Z, W \\ &+ N \text{ Fundamental Hypermultiplets } \psi_a, \tilde{\psi}^a \end{aligned} \tag{1.41}$$

This auxiliary $U(k)$ gauge theory defines its own metric on the Higgs branch. This metric arises in the following manner: we start with the flat metric on $\mathbf{R}^{4k(N+k)}$, parameterized by ψ, $\tilde{\psi}$, Z and W. Schematically,

$$ds^2 = |d\psi|^2 + |d\tilde{\psi}|^2 + |dZ|^2 + |dW|^2 \tag{1.42}$$

This metric looks somewhat more natural if we consider higher dimensional D-branes where it arises from the canonical kinetic terms for the hypermultiplets. We now pull back this metric to the hypersurface $V = 0$, and subsequently quotient by the $U(k)$ gauge symmetry, meaning that we only consider tangent vectors to $V = 0$ that are orthogonal to the $U(k)$ action. This procedure defines a metric on $\mathcal{M}_{\rm Higgs}$. The second important result of the ADHM construction is that this metric coincides with the one defined in terms of solitons in (1.18).

I haven't included a proof of the equivalence between the metrics here, although it's not too hard to show (for example, using Maciocia's hyperKähler potential [22] as reviewed in [13]). However, we will take time to show that the isometries of the metrics defined in these two different ways coincide. From the perspective of the auxiliary $U(k)$ gauge theory, all isometries appear as flavor symmetries. We have the $SU(N)$ flavor symmetry rotating the hypermultiplets; this is identified with the $SU(N)$ gauge symmetry in four dimensions. The theory also contains an $SU(2)_R$ R-symmetry, in which $(\psi, \tilde{\psi}^\dagger)$ and $(Z, W^\dagger)$ both transform as doublets (this will become more apparent in the following section in equation (1.44)). This coincides with the $SU(2)_R \subset SO(4)$ rotational symmetry in four dimensions. Finally, there exists an independent $SU(2)_L$ symmetry rotating just the X_μ.

The method described above for constructing hyperKähler metrics is an example of a technique known as the hyperKähler quotient [30]. As we have seen, it arises naturally in gauge theories with 8 supercharges. The D- and F-terms of the potential (1.37) give what are called the triplet of "moment-maps" for the $U(k)$ action.

1.4.2 Constructing the Solutions

As presented so far, the ADHM construction relates the moduli space of instantons $\mathcal{I}_{k,N}$ to the Higgs branch of an auxiliary gauge theory. In fact, we've omitted the most impressive part of the story: the construction can also be used to give solutions to the self-duality equations. What's more, it's really very easy! Just a question of multiplying a few matrices together. Let's see how it works.

Firstly, we need to rewrite the vacuum conditions in a more symmetric fashion. Define

$$\omega_a = \begin{pmatrix} \psi^{\alpha}{}_{a} \\ \tilde{\psi}^{\dagger\alpha}{}_{a} \end{pmatrix} \tag{1.43}$$

Then the real D-term and complex F-term which lie in the second line of (1.37) and define the Higgs branch can be combined in to the triplet of constraints,

$$\sum_{a=1}^{N} \omega_a^\dagger \sigma^i \, \omega_a - i[X_\mu, X_\nu]\bar{\eta}^i_{\mu\nu} = 0 \tag{1.44}$$

where σ^i are, as usual, the Pauli matrices and $\bar{\eta}^i$ the 't Hooft matrices (1.10). These give three $k \times k$ matrix equations. The magic of the ADHM construction is that for each solution to the algebraic equations (1.44), we can build a solution to the set of non-linear partial differential equations $F = {}^\star F$. Moreover, solutions to (1.44) related by $U(k)$ gauge transformations give rise to the same field configuration in four dimensions. Let's see how this remarkable result is achieved.

The first step is to build the $(N + 2k) \times 2k$ matrix Δ,

$$\Delta = \begin{pmatrix} \omega^T \\ X_\mu \sigma^\mu \end{pmatrix} + \begin{pmatrix} 0 \\ x_\mu \sigma^\mu \end{pmatrix} \tag{1.45}$$

where $\sigma_\mu = (\sigma^i, -i\mathbf{1}_2)$. These have the important property that $\sigma_{[\mu}\bar{\sigma}_{\nu]}$ is self-dual, while $\bar{\sigma}_{[\mu}\sigma_{\nu]}$ is anti-self-dual, facts that we also used in Section 1.3 when discussing fermions. In the second matrix we've re-introduced the spacetime coordinate x_μ which, here, is to be thought of as multiplying the $k \times k$ unit matrix. Before proceeding, we need a quick lemma:

Lemma: $\Delta^\dagger \Delta = f^{-1} \otimes 1_2$

where f is a $k \times k$ matrix, and 1_2 is the unit 2×2 matrix. In other words, $\Delta^\dagger\Delta$ factorizes and is invertible.

Proof: Expanding out, we have (suppressing the various indices)

$$\Delta^\dagger \Delta = \omega^\dagger \omega + X^\dagger X + (X^\dagger x + x^\dagger X) + x^\dagger x 1_k \tag{1.46}$$

Since the factorization happens for all $x \equiv x_\mu \sigma^\mu$, we can look at three terms separately. The last is $x^\dagger x = x_\mu \bar{\sigma}^\mu x_\nu \sigma^\nu = x^2 \, 1_2$. So that works. For the term linear in x, we simply need the fact that $X_\mu = X_\mu^\dagger$ to see that it works. What's more tricky is the term that doesn't depend on x. This is where the triplet of D-terms (1.44) comes in. Let's write the relevant term from (1.46) with all the indices, including an $m, n = 1, 2$ index to denote the two components we introduced in (1.43). We require

$$\begin{aligned} & \omega^{\dagger\alpha}_{ma}\,\omega_{a\beta n} + (X_\mu)^\alpha{}_\gamma (X_\nu)^\gamma{}_\beta\,\bar\sigma^{\mu\,mp}\sigma^\nu{}_{pn} \sim \delta^m_n \\ \Leftrightarrow \qquad & \mathrm{tr}_2\,\sigma^i\left[\omega\omega^\dagger + X^\dagger X\right] = 0 \qquad i = 1,2,3 \\ \Leftrightarrow \qquad & \omega^\dagger\sigma^i\omega + X_\mu X_\nu \bar\sigma^\mu\sigma^i\sigma^\nu = 0 \end{aligned} \tag{1.47}$$

But, using the identity $\bar\sigma^\mu\sigma^i\sigma^\nu = 2i\bar\eta^i_{\mu\nu}$, we see that this last condition is implied by the vanishing of the D-terms (1.44). This concludes our proof of the lemma. □

The rest is now plain sailing. Consider the matrix Δ as defining $2k$ linearly independent vectors in $\mathbf{C}^{N+2k}$. We define U to be the $(N+2k)\times N$ matrix containing the N normalized, orthogonal vectors. i.e

$$\Delta^\dagger U = 0 \quad , \quad U^\dagger U = 1_N \tag{1.48}$$

Then the potential for a charge k instanton in $SU(N)$ gauge theory is given by

$$A_\mu = iU^\dagger\,\partial_\mu U \tag{1.49}$$

Note firstly that if U were an $N \times N$ matrix, this would be pure gauge. But it's not, and it's not. Note also that A_μ is left unchanged by auxiliary $U(k)$ gauge transformations.

We need to show that A_μ so defined gives rise to a self-dual field strength with winding number k. We'll do the former, but the latter isn't hard either: it just requires more matrix multiplication. To help us in this, it will be useful to construct the projection operator $P = UU^\dagger$ and notice that this can also be written as $P = 1 - \Delta f \Delta^\dagger$. To see that these expressions indeed coincide, we can check that $PU = U$ and $P\Delta = 0$ for both. Now we're almost there:

$$\begin{aligned} F_{\mu\nu} &= \partial_{[\mu}A_{\nu]} - iA_{[\mu}A_{\nu]} \\ &= \partial_{[\mu}\, iU^\dagger\partial_{\nu]}U + iU^\dagger(\partial_{[\mu}U)U^\dagger(\partial_{\nu]}U) \\ &= i(\partial_{[\mu}U^\dagger)(\partial_{\nu]}U) - i(\partial_{[\mu}U^\dagger)UU^\dagger(\partial_{\nu]}U) \\ &= i(\partial_{[\mu}U^\dagger)(1 - UU^\dagger)(\partial_{\nu]}U) \\ &= i(\partial_{[\mu}U^\dagger)\Delta\, f\,\Delta^\dagger(\partial_{\nu]}U) \\ &= iU^\dagger(\partial_{[\mu}\Delta)\, f\,(\partial_{\nu]}U) \\ &= iU^\dagger\sigma_{[\mu}f\bar\sigma_{\nu]}U \end{aligned}$$

At this point we use our lemma. Because $\Delta^\dagger\Delta$ factorizes, we may commute f past σ_μ. And that's it! We can then write

$$F_{\mu\nu} = iU^\dagger f\sigma_{[\mu}\bar{\sigma}_{\nu]}U = {}^\star F_{\mu\nu} \tag{1.50}$$

since, as we mentioned above, $\sigma_{\mu\nu} = \sigma_{[\mu}\bar{\sigma}_{\nu]}$ is self-dual. Nice huh! What's harder to show is that the ADHM construction gives all solutions to the self-dualily equations. Counting parameters, we see that we have the right number and it turns out that we can indeed get all solutions in this manner.

The construction described above was first described in ADHM's original paper, which weighs in at a whopping 2 pages. Elaborations and extensions to include, among other things, $SO(N)$ and $Sp(N)$ gauge groups, fermionic zero modes, supersymmetry and constrained instantons, can be found in [31, 32, 33, 34].

An Example: The Single $SU(2)$ Instanton Revisited

Let's see how to re-derive the $k = 1$ $SU(2)$ solution (1.9) from the ADHM method. We'll set $X_\mu = 0$ to get a solution centered around the origin. We then have the 4×2 matrix

$$\Delta = \begin{pmatrix} \omega^T \\ x_\mu\sigma^\mu \end{pmatrix} \tag{1.51}$$

where the D-term constraints (1.44) tell us that $\omega^\dagger_{am}(\sigma^i)^m{}_n\omega^n_a = 0$. We can use our $SU(2)$ flavor rotation, acting on the indices $a, b = 1, 2$, to choose the solution

$$\omega^{\dagger a}{}_m\omega^m{}_b = \rho^2\delta^a{}_b \tag{1.52}$$

in which case the matrix Δ becomes $\Delta^T = (\rho 1_2, x_\mu\sigma^\mu)$. Then solving for the normalized zero eigenvectors $\Delta^\dagger U = 0$, and $U^\dagger U = 1$, we have

$$U = \begin{pmatrix} \sqrt{x^2/(x^2+\rho^2)}\, 1_2 \\ -\sqrt{\rho^2/x^2(x^2+\rho^2)}\, x_\mu\bar{\sigma}^\mu \end{pmatrix} \tag{1.53}$$

From which we calculate

$$A_\mu = iU^\dagger\partial_\mu U = \frac{\rho^2 x_\nu}{x^2(x^2+\rho^2)}\,\bar{\eta}^i_{\mu\nu}\,\sigma^i \tag{1.54}$$

which is indeed the solution (1.9) as promised.

1.4.3 Non-Commutative Instantons

There's an interesting deformation of the ADHM construction arising from studying instantons on a non-commutative space, defined by

$$[x_\mu, x_\nu] = i\theta_{\mu\nu} \tag{1.55}$$

The most simple realization of this deformation arises by considering functions on the space $\mathbf{R}^4_\theta$, with multiplication given by the $\star$-product

$$f(x) \star g(x) = \exp\left(\frac{i}{2}\theta_{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial x^\nu}\right) f(y)g(x)\Big|_{x=y} \tag{1.56}$$

so that we indeed recover the commutator $x_\mu \star x_\nu - x_\nu \star x_\mu = i\theta_{\mu\nu}$. To define gauge theories on such a non-commutative space, one must extend the gauge symmetry from $SU(N)$ to $U(N)$. When studying instantons, it is also useful to decompose the non-commutivity parameter into self-dual and anti-self-dual pieces:

$$\theta_{\mu\nu} = \xi^i\,\eta^i_{\mu\nu} + \zeta^i\,\bar{\eta}^i_{\mu\nu} \tag{1.57}$$

where η^i and $\bar{\eta}^i$ are defined in (1.11) and (1.10) respectively. At the level of solutions, both ξ and ζ affect the configuration. However, at the level of the moduli space, we shall see that the self-dual instantons $F = {}^\star F$ are only affected by the anti-self-dual part of the non-commutivity, namely ζ^i. (A similar statement holds for $F = -{}^\star F$ solutions and ξ). This change to the moduli space appears in a beautifully simple fashion in the ADHM construction: we need only add a constant term to the right hand-side of the constraints (1.44), which now read

$$\sum_{a=1}^{N} \omega_a^\dagger \sigma^i\,\omega_a - i[X_\mu, X_\nu]\bar{\eta}^i_{\mu\nu} = \zeta^i\,1_k \tag{1.58}$$

From the perspective of the auxiliary $U(k)$ gauge theory, the ζ^i are Fayet-Iliopoulous (FI) parameters.

The observation that the FI parameters ζ^i appearing in the D-term give the correct deformation for non-commutative instantons is due to Nekrasov and Schwarz [35]. To see how this works, we can repeat the calculation above, now in non-commutative space. The key point in constructing the solutions is once again the requirement that we have the factorization

$$\Delta^\dagger \star \Delta = f^{-1}\,1_2 \tag{1.59}$$

The one small difference from the previous derivation is that in the expansion (1.46), the $\star$-product means we have

$$x^\dagger \star x = x^2\, 1_2 - \zeta^i \sigma^i \tag{1.60}$$

Notice that only the anti-self-dual part contributes. This extra term combines with the constant terms (1.47) to give the necessary factorization if the D-term with FI parameters (1.58) is satisfied. It is simple to check that the rest of the derivation proceeds as before, with $\star$-products in the place of the usual commutative multiplication.

The addition of the FI parameters in (1.58) have an important effect on the moduli space $\mathcal{I}_{k,N}$: they resolve the small instanton singularities. From the ADHM perspective, these arise when $\psi = \tilde{\psi} = 0$, where the $U(k)$ gauge symmetry does not act freely. The FI parameters remove these points from the moduli space, $U(k)$ acts freely everywhere on the Higgs branch, and the deformed instanton moduli space $\mathcal{I}_{k,N}$ is smooth. This resolution of the instanton moduli space was considered by Nakajima some years before the relationship to non-commutivity was known [36]. A related fact is that non-commutative instantons occur even for $U(1)$ gauge theories. Previously such solutions were always singular, but the addition of the FI parameter stabilizes them at a fixed size of order $\sqrt{\theta}$. Reviews of instantons and other solitons on non-commutative spaces can be found in [37, 38].

1.4.4 Examples of Instanton Moduli Spaces

A Single Instanton

Consider a single $k = 1$ instanton in a $U(N)$ gauge theory, with non-commutivity turned on. Let us choose $\theta_{\mu\nu} = \zeta \bar{\eta}^3_{\mu\nu}$. Then the ADHM gauge theory consists of a $U(1)$ gauge theory with N charged hypermultiplets, and a decoupled neutral hypermultiplet parameterizing the center of the instanton. The D-term constraints read

$$\sum_{a=1}^N |\psi_a|^2 - |\tilde{\psi}_a|^2 = \zeta \quad , \quad \sum_{a=1}^N \tilde{\psi}_a \psi_a = 0 \tag{1.61}$$

To get the moduli space we must also divide out by the $U(1)$ action $\psi_a \to e^{i\alpha}\psi_a$ and $\tilde{\psi}_a \to e^{-i\alpha}\tilde{\psi}_a$. To see what the resulting space is, first consider setting $\tilde{\psi}_a = 0$. Then we have the space

$$\sum_{a=1}^N |\psi_a|^2 = \zeta \tag{1.62}$$

which is simply $\mathbf{S}^{2N-1}$. Dividing out by the $U(1)$ action then gives us the complex projective space $\mathbb{CP}^{N-1}$ with size (or Kähler class) ζ. Now let's add the $\tilde{\psi}$ back. We can turn them on but the F-term insists that they lie orthogonal to ψ, thus defining the co-tangent bundle of $\mathbb{CP}^{N-1}$, denoted $T^\star\mathbb{CP}^{N-1}$. Including the decoupled $\mathbf{R}^4$, we have [39]

$$\mathcal{I}_{1,N} \cong \mathbf{R}^4 \times T^\star\mathbb{CP}^{N-1} \tag{1.63}$$

where the size of the zero section $\mathbb{CP}^{N-1}$ is ζ. As $\zeta \to 0$, this cycle lying in the center of the space shrinks and $\mathcal{I}_{1,N}$ becomes singular at this point.

For a single instanton in $U(2)$, the relative moduli space is $T^\star\mathbf{S}^2$. This is the smooth resolution of the A_1 singularity $\mathbf{C}^2/\mathbf{Z}_2$ which we found to be the moduli space in the absence of non-commutivity. It inherits a well-known hyperKähler metric known as the Eguchi-Hanson metric [40],

$$ds^2_{EH} = \left(1 - 4\zeta^2/\rho^4\right)^{-1} d\rho^2 + \frac{\rho^2}{4}\left(\sigma_1^2 + \sigma_2^2 + \left(1 - 4\zeta^2/\rho^4\right)\sigma_3^2\right) \tag{1.64}$$

Here the σ_i are the three left-invariant $SU(2)$ one-forms which, in terms of polar angles $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 2\pi$, take the form

$$\begin{aligned}
\sigma_1 &= -\sin\psi\ d\theta + \cos\psi\sin\theta\ d\phi \\
\sigma_2 &= \cos\psi\ d\theta + \sin\psi\sin\theta\ d\phi \\
\sigma_3 &= d\psi + \cos\theta\ d\phi
\end{aligned} \tag{1.65}$$

As $\rho \to \infty$, this metric tends towards the cone over $\mathbf{S}^3/\mathbf{Z}_2$. However, as we approach the origin, the scale size is truncated at $\rho^2 = 2\zeta$, where the apparent singularity is merely due to the choice of coordinates and hides the zero section $\mathbf{S}^2$.

Two $U(1)$ Instantons

Before resolving by a non-commutative deformation, there is no topology to support a $U(1)$ instanton. However, it is perhaps better to think of the $U(1)$ theory as admitting small, singular, instantons with moduli space given by the symmetric product $\mathrm{Sym}^k(\mathbf{C}^2)$, describing the positions of k points. Upon the addition of a non-commutivity parameter, smooth $U(1)$ instantons exist with moduli space given by a resolution of $\mathrm{Sym}^k(\mathbf{C}^2)$. To my knowledge, no explicit metric is known for $k \geq 3$ $U(1)$ instantons, but in the case of two $U(1)$ instantons, the metric is something rather familiar, since $\mathrm{Sym}^2\mathbf{C}^2 \cong \mathbf{C}^2 \times \mathbf{C}^2/Z_2$ and we have already met the resolution of this space above. It is

$$\mathcal{I}_{k=2,N=1} \cong \mathbf{R}^4 \times T^\star\mathbf{S}^2 \tag{1.66}$$

endowed with the Eguchi-Hanson metric (1.64) where ρ now has the interpretation of the separation of two instantons rather than the scale size of one. This can be checked explicitly by computing the metric on the ADHM Higgs branch using the hyperKähler quotient technique [41]. Scattering of these instantons was studied in [42]. So, in this particular case we have $\mathcal{I}_{1,2} \cong \mathcal{I}_{2,1}$. We shouldn't get carried away though as this equivalence doesn't hold for higher k and N (for example, the isometries of the two spaces are different).

1.5 Applications

Until now we've focussed exclusively on classical aspects of the instanton configurations. But, what we're really interested in is the role they play in various quantum field theories. Here we sketch two examples which reveal the importance of instantons in different dimensions.

1.5.1 Instantons and the AdS/CFT Correspondence

We start by considering instantons where they were meant to be: in four dimensional gauge theories. In a semi-classical regime, instantons give rise to non-perturbative contributions to correlation functions and there exists a host of results in the literature, including exact results in both $\mathcal{N}=1$ [43, 44] and $\mathcal{N}=2$ [45, 34, 37] supersymmetric gauge theories. Here we describe the role instantons play in $\mathcal{N}=4$ super Yang-Mills and, in particular, their relationship to the AdS/CFT correspondence [47]. Instantons were first considered in this context in [48, 49]. Below we provide only a sketchy description of the material covered in the paper of Dorey et al [50]. Full details can be found in that paper or in the review [13].

In any instanton computation, there's a number of things we need to calculate [7]. The first is to count the zero modes of the instanton to determine both the bosonic collective coordinates X and their fermionic counterparts χ. We've described this in detail above. The next step is to perform the leading order Gaussian integral over all modes in the path integral. The massive (i.e. non-zero) modes around the background of the instanton leads to the usual determinant operators which we'll denote as $\det \Delta_B$ for the bosons, and $\det \Delta_F$ for the fermions. These are to be evaluated on the background of the instanton solution. However, zero modes must be treated separately. The integration over the associated collective coordinates is left unperformed, at the price of introducing a Jacobian arising from the transformation between field variables and collective coordinates. For the bosonic fields, the Jacobian is simply $J_B = \sqrt{\det g_{\alpha\beta}}$, where $g_{\alpha\beta}$ is the metric on the instanton moduli space defined in (1.18). This is the role played by the instanton moduli space metric in four dimensions: it appears in the measure when performing the path integral. A related factor J_F occurs for fermionic zero modes. The final ingredient in an instanton calculation is the action $S_{\rm inst}$ which includes both the constant piece $8\pi k/g^2$, together with terms quartic in the fermions (1.31). The end result is summarized in the instanton measure

$$d\mu_{\rm inst} = d^{n_B}X\, d^{n_F}\chi\ J_B J_F\ \frac{{\rm det}\Delta_F}{{\rm det}^{1/2}\Delta_B}\, e^{-S_{\rm inst}} \tag{1.67}$$

where there are $n_B = 4kN$ bosonic and n_F fermionic collective coordinates. In supersymmetric theories in four dimensions, the determinants famously cancel [7] and we're left only with the challenge of evaluating the Jacobians and the action. In this section, we'll sketch how to calculate these objects for $\mathcal{N}=4$ super Yang-Mills.

As is well known, in the limit of strong 't Hooft coupling, $\mathcal{N}=4$ super Yang-Mills is dual to type IIB supergravity on $AdS_5\times \mathbf{S}^5$. An astonishing fact, which we shall now show, is that we can see this geometry even at weak 't Hooft coupling by studying the $d=0+0$ ADHM gauge theory describing instantons. Essentially, in the large N limit, the instantons live in $AdS_5\times \mathbf{S}^5$. At first glance this looks rather unlikely! We've seen that if the instantons live anywhere it is in $\mathcal{I}_{k,N}$, a $4kN$ dimensional space that doesn't look anything like $AdS_5\times \mathbf{S}^5$. So how does it work?

While the calculation can be performed for an arbitrary number of k instantons, here we'll just stick with a single instanton as a probe of the geometry. To see the AdS_5 part is pretty easy and, in fact, we can do it even for an instanton in $SU(2)$ gauge theory. The trick is to integrate over the orientation modes of the instanton, leaving us with a five-dimensional space parameterized by X_μ and ρ. The rationale for doing this is that if we want to compute gauge invariant correlation functions, the $SU(N)$ orientation modes will only give an overall normalization. We calculated the metric for a single instanton in equations (1.22)-(1.24), giving us $J_B\sim\rho^3$ (where we've dropped some numerical factors and factors of e^2). So integrating over the $SU(2)$ orientation to pick up an overall volume factor, we get the bosonic measure for the instanton to be

$$d\mu_{\rm inst}\sim\rho^3\, d^4X d\rho \tag{1.68}$$

We want to interpret this measure as a five-dimensional space in which the instanton moves, which means thinking of it in the form $d\mu=\sqrt{G}\,d^4X\,d\rho$ where G is the metric on the five-dimensional space. It would be nice if it was the metric on AdS_5. But it's not! In the appropriate coordinates, the AdS_5 metric is,

$$ds^2_{AdS} = \frac{R^2}{\rho^2}(d^4X + d\rho^2) \tag{1.69}$$

giving rise to a measure $d\mu_{AdS} = (R/\rho)^5 d^4X d\rho$. However, we haven't finished with the instanton yet since we still have to consider the fermionic zero modes. The fermions are crucial for quantum conformal invariance so we may suspect that their zero modes are equally crucial in revealing the AdS structure, and this is indeed the case. A single $k = 1$ instanton in the $\mathcal{N} = 4$ $SU(2)$ gauge theory has 16 fermionic zero modes. 8 of these, which we'll denote as ξ are from broken supersymmetry while the remaining 8, which we'll call ζ arise from broken superconformal transformations. Explicitly each of the four Weyl fermions λ of the theory has a profile,

$$\lambda = \sigma^{\mu\nu} F_{\mu\nu}(\xi - \sigma^\rho \zeta\,(x_\rho - X_\rho)) \tag{1.70}$$

One can compute the overlap of these fermionic zero modes in the same way as we did for bosons. Suppressing indices, we have

$$\int d^4x\ \frac{\partial\lambda}{\partial\xi}\frac{\partial\lambda}{\partial\xi} = \frac{32\pi^2}{e^2} \quad , \quad \int d^4x\ \frac{\partial\lambda}{\partial\zeta}\frac{\partial\lambda}{\partial\zeta} = \frac{64\pi^2\rho^2}{e^2} \tag{1.71}$$

So, recalling that Grassmannian integration is more like differentiation, the fermionic Jacobian is $J_F \sim 1/\rho^8$. Combining this with the bosonic contribution above, the final instanton measure is

$$d\mu_{\rm inst} = \left(\frac{1}{\rho^5}\, d^4X d\rho\right)\, d^8\xi d^8\zeta = d\mu_{Ads}\, d^8\xi d^8\zeta \tag{1.72}$$

So the bosonic part does now look like AdS_5. The presence of the 16 Grassmannian variables reflects the fact that the instanton only contributes to a 16 fermion correlation function. The counterpart in the AdS/CFT correspondence is that D-instantons contribute to R^4 terms and their 16 fermion superpartners and one can match the supergravity and gauge theory correlators exactly.

So we see how to get AdS_5 for $SU(2)$ gauge theory. For $SU(N)$, one has $4N - 8$ further orientation modes and $8N - 16$ further fermi zero modes. The factors of ρ cancel in their Jacobians, leaving the AdS_5 interpretation intact. But there's a problem with these extra fermionic zero modes since we must saturate them in the path integral in some way even though we still want to compute a 16 fermionic correlator. This is achieved by the four-fermi term in the instanton action (1.31). However, when looked at in the right way, in the large N limit these extra fermionic zero modes will generate the $\mathbf{S}^5$ for us. I'll now sketch how this occurs.

The important step in reforming these fermionic zero modes is to introduce auxiliary variables $\hat{X}$ which allows us to split up the four-fermi term (1.31) into terms quadratic in the fermions. To get the index structure right, it turns out that we need six such auxiliary fields, let's call them $\hat{X}^m$, with $m = 1, \ldots, 6$. In fact we've met these guys before: they're the scalar fields in the vector multiplet of the ADHM gauge theory. To see that they give rise to the promised four fermi term, let's look at how they appear in the ADHM Lagrangian. There's already a term quadratic in $\hat{X}$ in (1.37), and another couples this to the surplus fermionic collective coordinates χ so that, schematically,

$$\mathcal{L}_{\hat{X}} \sim \hat{X}^2 \omega^\dagger \omega + \bar{\chi} \hat{X} \chi \tag{1.73}$$

where, as we saw in Section 1.4, the field ω contains the scale and orientation collective coordinates, with $\omega^\dagger \omega \sim \rho^2$. Integrating out $\hat{X}$ in the ADHM Lagrangian does indeed result in a four-fermi term which is identified with (1.31). However, now we perform a famous trick: we integrate out the variables we thought we were interested in, namely the χ fields, and focus on the ones we thought were unimportant, the $\hat{X}$'s. After dealing correctly with all the indices we've been dropping, we find that this results in the contribution to the measure

$$d\mu_{\text{auxiliary}} = d^6\hat{X}\ (\hat{X}^m \hat{X}^m)^{2N-4} \exp\left(-2\rho^2 \hat{X}^m \hat{X}^m\right) \tag{1.74}$$

In the large N limit, the integration over the radial variable $|\hat{X}|$ may be performed using the saddle-point approximation evaluated at $|\hat{X}| = \rho$. The resulting powers of ρ are precisely those mentioned above that are needed to cancel the powers of ρ appearing in the bosonic Jacobian. Meanwhile, the integration over the angular coordinates in $\hat{X}^m$ has been left untouched. The final result for the instanton measure becomes

$$d\mu_{\text{inst}} = \left(\frac{1}{\rho^5}\, d^4X\, d\rho\, d^5\hat{\Omega}\right)\ d^8\xi d^8\zeta \tag{1.75}$$

And the instanton indeed appears as if it's moving in $AdS_5 \times \mathbf{S}^5$ as promised.

The above discussion is a little glib. The invariant meaning of the measure alone is not clear: the real meaning is that when integrated against correlators, it gives results in agreement with gravity calculations in $\text{AdS}_5 \times \mathbf{S}^5$. This, and several further results, were shown in [50]. Calculations of this type were later performed for instantons in other four-dimensional gauge theories, both conformal and otherwise [51, 52, 53, 54, 55]. Curiously, there appears to be an unresolved problem with performing the calculation for instantons in non-commutative gauge theories.

1.5.2 Instanton Particles and the $(2,0)$ Theory

There exists a rather special superconformal quantum field theory in six dimensions known as the $(2,0)$ theory. It is the theory with 16 supercharges which lives on N M5-branes in M-theory and it has some intriguing and poorly understood properties. Not least of these is the fact that it appears to have N^3 degrees of freedom. While it's not clear what these degrees of freedom are, or even if it makes sense to talk about "degrees of freedom" in a strongly coupled theory, the N^3 behavior is seen when computing the free energy $F \sim N^3 T^6$ [56], or anomalies whose leading coefficient also scales as N^3 [57].

If the $(2,0)$ theory is compactified on a circle of radius R, it descends to $U(N)$ $d = 4+1$ super Yang-Mills with 16 supercharges, which can be thought of as living on D4-branes in Type IIA string theory. The gauge coupling e^2, which has dimension of length in five dimensions, is given by

$$e^2 = 8\pi^2 R \tag{1.76}$$

As in any theory compactified on a spatial circle, we expect to find Kaluza-Klein modes, corresponding to momentum modes around the circle with mass $M_{\rm KK} = 1/R$. Comparison with the gauge coupling constant (1.76) gives a strong hint what these particles should be, since

$$M_{\rm kk} = M_{\rm inst} \tag{1.77}$$

and, as we discussed in section 1.3.1, instantons are particle-like objects in $d = 4+1$ dimensions. The observation that instantons are Kaluza-Klein modes is clear from the IIA perspective: the instantons in the D4-brane theory are D0-branes which are known to be the Kaluza-Klein modes for the lift to M-theory.

The upshot of this analysis is a remarkable conjecture: the maximally supersymmetric $U(N)$ Yang-Mills theory in five dimensions is really a six-dimensional theory in disguise, with the size of the hidden dimension given by $R \sim e^2$ [58, 59, 60]. As $e^2 \to \infty$, the instantons become light. Usually, as solitons become light, they also become large floppy objects, losing their interpretation as particle excitations of the theory. But this isn't necessarily true for instantons because, as we've seen, their scale size is arbitrary and, in particular, independent of the gauge coupling.

Of course, the five-dimensional theory is non-renormalizable and we can only study questions that do not require the introduction of new UV degrees of freedom. With this caveat, let's see how we can test the conjecture using instantons. If they're really Kaluza-Klein modes, they should exhibit Kaluza-Klein-eqsue behavior which includes a characteristic spectrum of threshold bound state of particles with k units of momentum going around the circle. This means that if the five-dimensional theory contains the information about its six dimensional origin, it should exhibit a threshold bound state of k instantons for each k. But this is something we can test in the semi-classical regime by solving the low-energy dynamics of k interacting instantons. As we have seen, this is given by supersymmetric quantum mechanics on $\mathcal{I}_{k,N}$, with the Lagrangian given by (1.30) where $\partial = \partial_t$ in this equation.

Let's review how to solve the ground states of $d = 0+1$ dimensional supersymmetric sigma models of the form (1.30). As explained by Witten, a beautiful connection to de Rahm cohomology emerges after quantization [61]. Canonical quantization of the fermions leads to operators satisfying the algebra

$$\{\chi_\alpha, \chi_\beta\} = \{\bar{\chi}_\alpha, \bar{\chi}_\beta\} = 0 \quad \text{and} \quad \{\chi_\alpha, \bar{\chi}_\beta\} = g_{\alpha\beta} \tag{1.78}$$

which tells us that we may regard $\bar{\chi}_\alpha$ and χ_β as creation and annihilation operators respectively. The states of the theory are described by wavefunctions $\varphi(X)$ over the moduli space $\mathcal{I}_{k,N}$, acted upon by some number p of fermion creation operators. We write $\varphi_{\alpha_1,\ldots,\alpha_p}(X) \equiv \bar{\chi}_{\alpha_1} \ldots \bar{\chi}_{\alpha_p} \varphi(X)$. By the Grassmann nature of the fermions, these states are anti-symmetric in their p indices, ensuring that the tower stops when $p = \dim(\mathcal{I}_{k,N})$. In this manner, the states can be identified with the space of all p-forms on $\mathcal{I}_{k,N}$.

The Hamiltonian of the theory has a similarly natural geometric interpretation. One can check that the Hamiltonian arising from (1.30) can be written as

$$H = QQ^\dagger + Q^\dagger Q \tag{1.79}$$

where Q is the supercharge which takes the form $Q = -i\bar{\chi}_\alpha p_\alpha$ and $Q^\dagger = -i\chi_\alpha p_\alpha$, and p_α is the momentum conjugate to X^α. Studying the action of Q on the states above, we find that $Q = d$, the exterior derivative on forms, while $Q^\dagger = d^\dagger$, the adjoint operator. We can therefore write the Hamiltonian as the Laplacian acting on all p-forms,

$$H = dd^\dagger + d^\dagger d \tag{1.80}$$

We learn that the space of ground states $H = 0$ coincide with the harmonic forms on the target space.

There are two subtleties in applying this analysis to instantons. The first is that the instanton moduli space $\mathcal{I}_{k,N}$ is singular. At these points, corresponding to small instantons, new UV degrees of freedom are needed. Presumably this reflects the non-renormalizability of the five-dimensional gauge theory. However, as we have seen, one can resolve the singularity by turning on non-commutivity. The interpretation of the instantons as KK modes only survives if there is a similar non-commutative deformation of the $(2,0)$ theory which appears to be the case.

The second subtlety is an infra-red effect: the instanton moduli space is non-compact. For compact target spaces, the ground states of the sigma-model coincide with the space of harmonic forms or, in other words, the cohomology. For non-compact target spaces such as $\mathcal{I}_{k,N}$, we have the further requirement that any putative ground state wavefunction must be normalizable and we need to study cohomology with compact support. With this in mind, the relationship between the five-dimensional theory and the six-dimensional $(2,0)$ theory therefore translates into the conjecture

There is a unique normalizable harmonic form on $\mathcal{I}_{k,N}$ for each k and N

Note that even for a single instanton, this is non-trivial. As we have seen above, after resolving the small instanton singularity, the moduli space for a $k = 1$ instanton in $U(N)$ theory is $T^\star(\mathbf{CP}^{N-1})$, which has Euler character $\chi = N$. Yet, there should be only a single groundstate. Indeed, it can be shown explicitly that of these N putative ground states, only a single one has sufficiently compact support to provide an L^2 normalizable wavefunction [62]. For an arbitrary number of k instantons in $U(N)$ gauge theory, there is an index theorem argument that this unique bound state exists [63].

So much for the ground states. What about the N^3 degrees of freedom. Is it possible to see this from the five-dimensional gauge theory? Unfortunately, so far, no one has managed this. Five dimensional gauge theories become strongly coupled in the ultra-violet where their non-renormalizability becomes an issue and we have to introduce new degrees of freedom. This occurs at an energy scale $E \sim 1/e^2N$, where the 't Hooft coupling becomes strong. This is parametrically lower than the KK scale $E \sim 1/R \sim 1/e^2$. Supergravity calculations reveal that the N^3 degrees of freedom should also become apparent at the lower scale $E \sim 1/e^2N$ [64]. This suggests that perhaps the true degrees of freedom of the theory are "fractional instantons", each with mass $M_{\rm inst}/N$. Let me end this section with some rampant speculation along these lines. It seems possible that the $4kN$ moduli of the instanton may rearrange themselves into the positions of kN objects, each living in $\mathbf{R}^4$ and each, presumably, carrying the requisite mass $1/e^2N$. We shall see a similar phenomenon occurring for vortices in Section 3.8.2. If this speculation is true, it would also explain why a naive quantization of the instanton leads to a continuous spectrum, rather strange behavior for a single particle: it's because the instanton is really a multi-particle state. However, to make sense of this idea we would need to understand why the fractional instantons are confined to lie within the instanton yet, at the same time, are also able to wander freely as evinced by the $4kN$ moduli. Which, let's face it, is odd! A possible explanation for this strange behavior may lie in the issues of non-normalizability of non-abelian modes discussed above, and related issues described in [65].

While it's not entirely clear what a fractional instanton means on $\mathbf{R}^4$, one can make rigorous sense of the idea when the theory is compactified on a further $\mathbf{S}^1$ with a Wilson line [66, 67]. Moreover, there's evidence from string dualities [68, 39] that the moduli space of instantons on compact spaces $\mathbf{M} = \mathbf{T}^4$ or $K3$ has the same topology as the symmetric product $\mathrm{Sym}^{kN}(\mathbf{M})$, suggesting an interpretation in terms of kN entities (strictly speaking, one needs to resolve these spaces into an object known as the Hilbert scheme of points over $\mathbf{M}$).

2. Monopoles

The tale of magnetic monopoles is well known. They are postulated particles with long-range, radial, magnetic field B_i, $i = 1, 2, 3$,

$$B_i = \frac{g\,\hat{r}_i}{4\pi r^2} \tag{2.1}$$

where g is the magnetic charge. Monopoles have never been observed and one of Maxwell's equations, $\nabla \cdot B = 0$, insists they never will be. Yet they have been a recurrent theme in high energy particle physics for the past 30 years! Why?

The study of monopoles began with Dirac [69] who showed how one could formulate a theory of monopoles consistent with a gauge potential A_μ. The requirement that the electron doesn't see the inevitable singularities in A_μ leads to the famed quantization condition

$$eg = 2\pi n \qquad n \in \mathbf{Z} \tag{2.2}$$

However, the key step in the rehabilitation of magnetic monopoles was the observation of 't Hooft [70] and Polyakov [71] that monopoles naturally occur in non-abelian gauge theories, making them a robust prediction of grand unified theories based on semi-simple groups. In this lecture we'll review the formalism of 't Hooft-Polyakov monopoles in $SU(N)$ gauge groups, including the properties of the solutions and the D-brane realization of the Nahm construction. At the end we'll cover several applications to quantum gauge theories in various dimensions.

There are a number of nice reviews on monopoles in the literature. Aspects of the classical solutions are dealt with by Sutcliffe [72] and Shnir [73]; the mathematics of monopole scattering can be found in the book by Atiyah and Hitchin [74]; the application to S-duality of quantum field theories is covered in the review by Harvey [75]. A comprehensive review of magnetic monopoles by Weinberg and Yi will appear shortly [76].

2.1 The Basics

To find monopoles, we first need to change our theory from that of Lecture 1. We add a single real scalar field $\phi \equiv \phi^a{}_b$, transforming in the adjoint representation of $SU(N)$. The action now reads

$$S = \mathrm{Tr} \int d^4x\; \frac{1}{2e^2} F_{\mu\nu}F^{\mu\nu} + \frac{1}{e^2}(\mathcal{D}_\mu \phi)^2 \tag{2.3}$$

where we're back in Minkowski signature $(+,-,-,-)$. The spacetime index runs over $\mu = 0,1,2,3$ and we'll also use the purely spatial index $i = 1,2,3$. Actions of this type occur naturally as a subsector of $\mathcal{N}=4$ and $\mathcal{N}=2$ super Yang-Mills theories. There is no potential for ϕ so, classically, we are free to chose the vacuum expectation value (vev) as we see fit. Gauge inequivalent choices correspond to different ground states of the theory. By use of a suitable gauge transformation, we may set

$$\langle\phi\rangle = \mathrm{diag}(\phi_1,\ldots,\phi_N) = \vec{\phi}\cdot\vec{H} \tag{2.4}$$

where the fact we're working in $SU(N)$ means that $\sum_{a=1}^{N}\phi_a = 0$. We've also introduced the notation of the root vector $\vec{\phi}$, with $\vec{H}$ a basis for the $(N-1)$-dimensional Cartan subalgebra of $su(N)$. If you're not familiar with roots of Lie algebras and the Cartn-Weyl basis then you can simply think of $\vec{H}$ as the set of N matrices, each with a single entry 1 along the diagonal. (This is actually the Cartan subalgebra for $u(N)$ rather than $su(N)$ but this will take care of itself if we remember that $\sum_a \phi_a = 0$). Under the requirement that $\phi_a \neq \phi_b$ for $a \neq b$ the gauge symmetry breaks to the maximal torus,

$$SU(N) \to U(1)^{N-1} \tag{2.5}$$

The spectrum of the theory consists of $(N-1)$ massless photons and scalars, together with $\frac{1}{2}N(N-1)$ massive W-bosons with mass $M_W^2 = (\phi_a - \phi_b)^2$. In the following we will use the Weyl symmetry to order $\phi_a < \phi_{a+1}$.

In the previous lecture, instantons arose from the possibility of winding field configurations non-trivially around the $\mathbf{S}^3_\infty$ infinity of Euclidean spacetime. Today we're interested in particle-like solitons, localized in space rather than spacetime. These objects are supported by the vev (2.4) twisting along its gauge orbit as we circumvent the spatial boundary $\mathbf{S}^2_\infty$. If we denote the two coordinates on $\mathbf{S}^2_\infty$ as θ and φ, then solitons are supported by configurations with $\langle\phi\rangle = \langle\phi(\theta,\varphi)\rangle$. Let's classify the possible windings. A vev of the form (2.4) is one point in a space of gauge equivalent vacua, given by $SU(N)/U(1)^{N-1}$ where the stabilizing group in the denominator is the unbroken symmetry group (2.5) which leaves (2.4) untouched. We're therefore left to consider maps: $\mathbf{S}^2_\infty \to SU(N)/U(1)^{N-1}$, characterized by

$$\Pi_2\left(SU(N)/U(1)^{N-1}\right) \cong \Pi_1\left(U(1)^{N-1}\right) \cong \mathbf{Z}^{N-1} \tag{2.6}$$

This classification suggests that we should be looking for $(N-1)$ different types of topological objects. As we shall see, these objects are monopoles carrying magnetic charge in each of the $(N-1)$ unbroken abelian gauge fields (2.5).

Why is winding of the scalar field ϕ at infinity associated with magnetic charge? To see the precise connection is actually a little tricky — details can be found in [70, 71] and in [77] for $SU(N)$ monopoles — but there is a simple heuristic argument to see why the two are related. The important point is that if a configuration is to have finite energy, the scalar kinetic term $\mathcal{D}_\mu\phi$ must decay at least as fast as $1/r^2$ as we approach the boundary $r \to \infty$. But if $\langle\phi\rangle$ varies asymptotically as we move around $\mathbf{S}^2_\infty$, we have $\partial\phi \sim 1/r$. To cancel the resulting infrared divergence we must turn on a corresponding gauge potential $A_\theta \sim 1/r$, leading to a magnetic field of the form $B \sim 1/r^2$.

Physically, we would expect any long range magnetic field to propagate through the massless $U(1)$ photons. This is indeed the case. If $\mathcal{D}_i\phi \to 0$ as $r \to \infty$ then $[\mathcal{D}_i, \mathcal{D}_j]\phi = -i[F_{ij}, \phi] \to 0$ as $r \to \infty$. Combining these two facts, we learn that the non-abelian magnetic field carried by the soliton is of the form,

$$B_i = \vec{g}\cdot\vec{H}(\theta,\varphi)\;\frac{\hat{r}_i}{4\pi r^2} \tag{2.7}$$

Here the notation $\vec{H}(\theta,\varphi)$ reminds us that the unbroken Cartan subalgebra twists within the $su(N)$ Lie algebra as we move around the $\mathbf{S}^2_\infty$.

2.1.1 Dirac Quantization Condition

The allowed magnetic charge vectors $\vec{g}$ may be determined by studying the winding of the scalar field ϕ around $\mathbf{S}^2_\infty$. However, since the winding is related to the magnetic charge, and the latter is a characteristic of the long range behavior of the monopole, it's somewhat easier to neglect the non-abelian structure completely and study just the $U(1)$ fields. The equivalence between the two methods is reflected in the equality between first and second homotopy groups in (2.6).

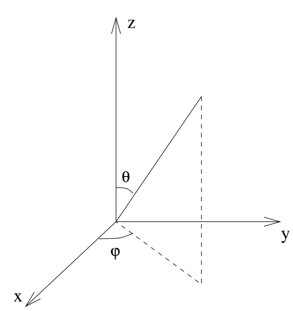

Figure 3:

For this purpose, it is notationally simpler to work in unitary, or singular, gauge in which the vev $\langle\phi\rangle = \vec{\phi}\cdot\vec{H}$ is fixed to be constant at infinity. This necessarily re-introduces Dirac string-like singularities for any single-valued gauge potential, but allows us to globally write the magnetic field in diagonal form,

$$B_i = \mathrm{diag}(g_1,\ldots,g_N)\;\frac{\hat{r}_i}{4\pi r^2} \tag{2.8}$$

where $\sum_{a=1}^N g_a = 0$ since the magnetic field lies in $su(N)$ rather than $u(N)$.

What values of g_a are allowed? A variant of Dirac's original argument, due to Wu and Yang [78], derives the magnetic field (2.8) from two gauge potentials defined respectively on the northern and southern hemispheres of $\mathbf{S}^2_\infty$:

$$\begin{aligned} A^N_\varphi &= \frac{1-\cos\theta}{4\pi r\sin\theta}\,\vec{g}\cdot\vec{H} \\ A^S_\varphi &= -\frac{1+\cos\theta}{4\pi r\sin\theta}\,\vec{g}\cdot\vec{H} \end{aligned} \tag{2.9}$$

where A^N goes bad at the south pole $\theta=\pi$, while A^S sucks at the north pole $\theta=0$. To define a consistent field strength we require that on the overlap $\theta\neq 0,\pi$, the two differ by a gauge transformation which, indeed, they do:

$$A^N_i = U(\partial_i + A^S_i)U^{-1} \tag{2.10}$$

with $U(\theta,\varphi) = \exp(-i\vec{g}\cdot\vec{H}\varphi/2\pi)$. Notice that as we've written it, this relationship only holds in unitary gauge where $\vec{H}$ doesn't depend on θ or φ, requiring that we work in singular gauge. The final requirement is that our gauge transformation is single valued, so $U(\varphi)=U(\varphi+2\pi)$ or, in other words, $\exp(i\,\vec{g}\cdot\vec{H})=1$. This requirement is simply solved by

$$g_a \in 2\pi\mathbf{Z} \tag{2.11}$$

This is the Dirac quantization condition (2.2) in units in which the electric charge $e=1$, a convention which arises from scaling the coupling outside the action in (2.3). In fact, in our theory the W-bosons have charge 2 under any $U(1)$ while matter in the fundamental representation would have charge 1.

There's another notation for the magnetic charge vector $\vec{g}$ that will prove useful. We write

$$\vec{g} = 2\pi\sum_{a=1}^{N-1} n_a\,\vec{\alpha}_a \tag{2.12}$$

where $n_a\in\mathbf{Z}$ by the Dirac quantization condition[4] and $\vec{\alpha}_a$ are the simple roots of $su(N)$. The choice of simple roots is determined by defining $\vec{\phi}$ to lie in a positive Weyl chamber. What this means in practice, with our chosen ordering $\phi_a<\phi_{a+1}$, is that we can write each root as an N-vector, with

$$\begin{aligned} \vec{\alpha}_1 &= (1,-1,0,\ldots,0) \\ \vec{\alpha}_2 &= (0,1,-1,\ldots,0) \end{aligned} \tag{2.13}$$

[4]For monopoles in a general gauge group, the Dirac quantization condition becomes $\vec{g}=4\pi\sum_a n_a\vec{\alpha}^\star_a$ where $\vec{\alpha}^\star_a$ are simple co-roots.

through to

$$\vec{\alpha}_{N-1} = (0, 0, \ldots, 1, -1) \tag{2.14}$$

Then translating between two different notations for the magnetic charge vector we have

$$\begin{aligned}\vec{g} &= \mathrm{diag}(g_1, \ldots, g_N) \\ &= 2\pi \,\mathrm{diag}(n_1\,,\, n_2 - n_1, \ldots, n_{N-1} - n_{N-2}\,,\, -n_{N-1})\end{aligned} \tag{2.15}$$

The advantage of working with the integers n_a, $a = 1, \ldots, N-1$ will become apparent shortly.

2.1.2 The Monopole Equations

As in Lecture 1, we've learnt that the space of field configurations decomposes into different topological sectors, this time labelled by the vector $\vec{g}$ or, equivalently, the $N-1$ integers n_a. We're now presented with the challenge of finding solutions in the non-trivial sectors. We can again employ a Bogomoln'yi bound argument (this time actually due to Bogomoln'yi [14]) to derive first order equations for the monopoles. We first set $\partial_0 = A_0 = 0$, so we are looking for time independent configurations with vanishing electric field. Then the energy functional of the theory gives us the mass of a magnetic monopole,

$$\begin{aligned}M_{\rm mono} &= \mathrm{Tr}\int d^3x\ \frac{1}{e^2}B_i^2 + \frac{1}{e^2}(\mathcal{D}_i\phi)^2 \\ &= \mathrm{Tr}\int d^3x\ \frac{1}{e^2}\left(B_i \mp \mathcal{D}_i\phi\right)^2 \pm \frac{2}{e^2}\,B_i\mathcal{D}_i\phi \\ &\geq \frac{2}{e^2}\int d^3x\ \partial_i\ \mathrm{Tr}(B_i\phi)\end{aligned} \tag{2.16}$$

where we've used the Bianchi identity $\mathcal{D}_iB_i = 0$ when integrating by parts to get the final line. As in the case of instantons, we've succeeded in bounding the energy by a surface term which measures a topological charge. Comparing with the expressions above we have

$$M_{\rm mono} \geq \frac{|\vec{g}\cdot\vec{\phi}|}{e^2} = \frac{2\pi}{e^2}\sum_{a=1}^{N-1} n_a\phi_a \tag{2.17}$$

with equality if and only if the monopole equations (often called the Bogomoln'yi equations) are obeyed,

$$\begin{aligned}B_i &= \mathcal{D}_i\phi \qquad\qquad \text{if } \vec{g}\cdot\vec{\phi} > 0 \\ B_i &= -\mathcal{D}_i\phi \qquad\qquad \text{if } \vec{g}\cdot\vec{\phi} < 0\end{aligned} \tag{2.18}$$

For the rest of this lecture we'll work with $\vec{g} \cdot \vec{\phi} > 0$ and the first of these equations. Our path will be the same as in lecture 1: we'll first examine the simplest solution to these equations and then study its properties before moving on to the most general solutions. So first:

2.1.3 Solutions and Collective Coordinates

The original magnetic monopole described by 't Hooft and Polyakov occurs in $SU(2)$ theory broken to $U(1)$. We have $SU(2)/U(1) \cong \mathbf{S}^2$ and $\Pi_2(\mathbf{S}^2) \cong \mathbf{Z}$. Here we'll describe the simplest such monopole with charge one. To better reveal the topology supporting this monopole (as well as to demonstrate explicitly that the solution is smooth) we'll momentarily revert back to a gauge where the vev winds asymptotically. The solution to the monopole equation (2.18) was found by Prasad and Sommerfield [79]

$$\begin{aligned} \phi &= \frac{\hat{r}_i \sigma^i}{r}\,(vr\coth(vr) - 1) \\ A_k &= -\epsilon_{ikj}\,\frac{\hat{r}^j \sigma^i}{r}\left(1 - \frac{vr}{\sinh vr}\right) \end{aligned} \tag{2.19}$$

This solution asymptotes to $\langle\phi\rangle = v\sigma^i \hat{r}^i$, where σ^i are the Pauli matrices (i.e. comparing notation with (2.4) in, say, the $\hat{r}^3$ direction, we have $v = -\phi_1 = \phi_2$). The $SU(2)$ solution presented above has 4 collective coordinates, although none of them are written explicitly. Most obviously, there are the three center of mass coordinates. As with instantons, there is a further collective coordinate arising from acting on the soliton with the unbroken gauge symmetry which, in this case, is simply $U(1)$.

For monopoles in $SU(N)$ we can always generate solutions by embedding the configuration (2.19) above into a suitable $SU(2)$ subgroup. Note however that, unlike the situation for instantons, we can't rotate from one $SU(2)$ embedding to another since the $SU(N)$ gauge symmetry is not preserved in the vacuum. Each $SU(2)$ embedding will give rise to a different monopole with different properties — for example, they will have magnetic charges under different $U(1)$ factors.

Of the many inequivalent embeddings of $SU(2)$ into $SU(N)$, there are $(N-1)$ special ones. These have generators given in the Cartan-Weyl basis by $\vec{\alpha} \cdot \vec{H}$ and $E_{\pm\vec{\alpha}}$ where $\vec{\alpha}$ is one of the simple roots (2.13). In a less sophisticated language, these are simply the $(N-1)$ contiguous 2×2 blocks which lie along the diagonal of an $N \times N$ matrix. Embedding the monopole in the a^{th} such block gives rise to the magnetic charge $\vec{g} = \vec{\alpha}_a$.

2.2 The Moduli Space

For a monopole with magnetic charge $\vec{g}$, we want to know how many collective coordinates are contained within the most general solution. The answer was given by E.

Weinberg [80]. There are subtleties that don't occur in the instanton calculation, and a variant of the Atiyah-Singer index theorem due to Callias is required [81]. But the result is very simple. Define the moduli space of monopoles with magnetic charge $\vec{g}$ to be $\mathcal{M}_{\vec{g}}$. Then the number of collective coordinates is

$$\dim(\mathcal{M}_{\vec{g}}) = 4\sum_{a=1}^{N-1} n_a \tag{2.20}$$

The interpretation of this is as follows. There exist $(N-1)$ "elementary" monopoles, each associated to a simple root $\vec{\alpha}_a$, carrying magnetic charge under exactly one of the $(N-1)$ surviving $U(1)$ factors of (2.5). Each of these elementary monopoles has 4 collective coordinates. A monopole with general charge $\vec{g}$ can be decomposed into $\sum_a n_a$ elementary monopoles, described by three position coordinates and a phase arising from $U(1)$ gauge rotations.

Figure 4:

You should be surprised by the existence of this large class of solutions since it implies that monopoles can be placed at arbitrary separation and feel no force. But this doesn't happen for electrons! Any objects carrying the same charge, whether electric or magnetic, repel. So what's special about monopoles? The point is that monopoles also experience a second long range force due to the massless components of the scalar field ϕ. This gives rise to an attraction between the monopoles that precisely cancels the electromagnetic repulsion [82]. Such cancellation of forces only occurs when there is no potential for ϕ as in (2.3).

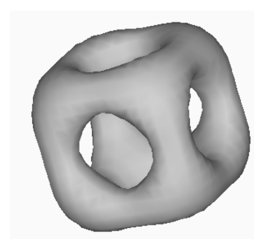

Figure 5:

The interpretation of the collective coordinates as positions of particle-like objects holds only when the monopoles are more widely separated than their core size. As the monopoles approach, weird things happen! Two monopoles form a torus. Three monopoles form a tetrahedron, seemingly splitting into four lumps of energy as seen in figure 4. Four monopoles form a cube as in figure 5. (Both of these figures are taken from [83]). We see that monopoles really lose their individual identities as the approach and merge into each other. Higher monopoles form platonic solids, or buckyball like objects.

2.2.1 The Moduli Space Metric

The metric on $\mathcal{M}_{\vec{g}}$ is defined in a similar fashion to that on the instanton moduli space $\mathcal{I}_{k,N}$. To be more precise, it's defined in an identical fashion. Literally! The key point is that the monopole equations $B = \mathcal{D}\phi$ and the instanton equations $F = {}^\star F$ are really the same: the difference between the two lies in the boundary conditions. To see this, consider instantons with $\partial_4 = 0$ and endow the component of the gauge field $A_4 \equiv \phi$ with a vev $\langle\phi\rangle$. We end up with the monopole equations. So using the notation $\delta\phi = \delta A_4$, we can reuse the linearized self-dual equations (1.14) and the gauge fixing condition (1.17) from the Lecture 1 to define the monopole zero modes. The metric on the monopole moduli space $\mathcal{M}_{\vec{g}}$ is again given by the overlap of zero modes,

$$g_{\alpha\beta} = \frac{1}{e^2}\,\mathrm{Tr}\int d^3x\;(\delta_\alpha A_i\,\delta_\beta A_i + \delta_\alpha\phi\,\delta_\beta\phi) \tag{2.21}$$

The metric on the monopole moduli space has the following properties:

- The metric is hyperKähler .
- The metric enjoys an $SO(3) \times U(1)^{N-1}$ isometry. The former descends from physical rotations of the monopoles in space. The latter arise from the unbroken gauge group. The $U(1)^{N-1}$ isometries are tri-holomorphic, while the $SO(3)$ isometry rotates the three complex structures.
- The metric is smooth. There are no singular points analogous to the small instanton singularities of $\mathcal{I}_{k,N}$ because, as we have seen, the scale of the monopole isn't a collective coordinate. It is fixed to be $L_{\rm mono} \sim 1/M_W$, the Compton wavelength of the W-bosons.
- Since the metrics on $\mathcal{I}_{k,N}$ and $\mathcal{M}_{\vec{g}}$ arise from the same equations, merely endowed with different boundary conditions, one might wonder if we can interpolate between them. In fact we can. In the study of instantons on $\mathbf{R}^3\times\mathbf{S}^1$, with a non-zero Wilson line around the $\mathbf{S}^1$, the $4N$ collective coordinates of the instanton gain the interpretation of the positions of N "fractional instantons" [66, 67]. These are often referred to as calorons and are identified as the monopoles discussed above. By taking the radius of the circle to zero, and some calorons to infinity, we can interpolate between the metrics on $\mathcal{M}_{\vec{g}}$ and $\mathcal{I}_{k,N}$ [62].

2.2.2 The Physical Interpretation of the Metric

For particles such as monopoles in $d = 3+1$ dimensions, the metric on the moduli space has a beautiful physical interpretation first described by Manton [84]. Suppose that the monopoles move slowly through space. We approximate the motion by assuming that the field configurations remain close to the static solutions, but endow the collective coordinates X^α with time dependence: $X^\alpha \to X^\alpha(t)$. If monopoles collide at very high energies this approximation will not be valid. As the monopoles hit they will spew out massive W-bosons and, on occasion, even monopole-anti-monopole pairs. The resulting field configurations will look nothing like the static monopole solutions. Even for very low-energy scattering it's not completely clear that the approximation is valid since the theory doesn't have a mass gap and the monopoles can emit very soft photons. Nevertheless, there is much evidence that this procedure, known as the *moduli space approximation*, does capture the true physics of monopole scattering at low energies. The time dependence of the fields is

$$A_\mu = A_\mu(X^\alpha(t)) \quad , \quad \phi = \phi(X^\alpha(t)) \tag{2.22}$$

which reduces the dynamics of an infinite number of field theory degrees of freedom to a finite number of collective coordinates. We must still satisfy Gauss' law,

$$\mathcal{D}_i E_i - i[\phi, \mathcal{D}_0\phi] = 0 \tag{2.23}$$

which can be achieved by setting $A_0 = \Omega_\alpha \dot{X}^\alpha$, where the Ω_α are the extra gauge rotations that we introduced in (1.15) to ensure that the zero modes satisfy the background gauge fixing condition. This means that the time dependence of the fields is given in terms of the zero modes,

$$\begin{aligned} E_i = F_{0i} &= \delta_\alpha A_i \; \dot{X}^\alpha \\ \mathcal{D}_0\phi &= \delta_\alpha \phi \; \dot{X}^\alpha \end{aligned} \tag{2.24}$$

Plugging this into the action (2.3) we find

$$\begin{aligned} S &= \mathrm{Tr} \int d^4x \; \frac{1}{e^2} \left(E_i^2 + B_i^2 + (\mathcal{D}_0\phi)^2 + (\mathcal{D}_i\phi)^2 \right) \\ &= \int dt \; \left(M_{\rm mono} + \tfrac{1}{2} g_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta \right) \end{aligned} \tag{2.25}$$

The upshot of this is analysis is that the low-energy dynamics of monopoles is given by the $d = 0+1$ sigma model on the monopole moduli space. The equations of motion following from (2.25) are simply the geodesic equations for the metric $g_{\alpha\beta}$. We learn that the moduli space metric captures the velocity-dependent forces felt by monopoles, such that low-energy scattering is given by geodesic motion.

In fact, this logic can be reversed. In certain circumstances it's possible to figure out the trajectories followed by several moving monopoles. From this one can construct a metric on the configuration space of monopoles such that the geodesics reconstruct the known motion. This metric agrees with that defined above in a very different way. This procedure has been carried out for a number of examples [85, 86, 87].

2.2.3 Examples of Monopole Moduli Spaces

Let's now give a few examples of monopole moduli spaces. We start with the simple case of a single monopole where the metric may be explicitly computed.

One Monopole

Consider the $\vec{g} = \vec{\alpha}_1$ monopole, which is nothing more that the charge one $SU(2)$ solution we saw previously (2.19). In this case we can compute the metric directly. We have two different types of collective coordinates:

i) The three translational modes. The linearized monopole equation and gauge fixing equation are solved by $\delta_{(i)}A_j = -F_{ij}$ and $\delta_{(i)}\phi = -\mathcal{D}_i\phi$, so that the overlap of zero modes is

$$\mathrm{Tr}\ \frac{1}{e^2}\int d^3x\ (\delta_{(i)}A_k\,\delta_{(j)}A_k + \delta_{(i)}\phi\,\delta_{(j)}\phi) = M_{\rm mono}\ \delta_{ij} \tag{2.26}$$

ii) The gauge mode arises from transformation $U = \exp(i\phi\chi/v)$, where the normalization has been chosen so that the collective coordinate χ has periodicity 2π. This gauge transformation leaves ϕ untouched while the transformation on the gauge field is $\delta A_i = (\mathcal{D}_i\phi)/v$.

Putting these two together, we find that single monopole moduli space is

$$\mathcal{M}_{\vec{\alpha}} \cong \mathbf{R}^3 \times \mathbf{S}^1 \tag{2.27}$$

with metric

$$ds^2 = M_{\rm mono}\ \left(dX^i dX^i + \frac{1}{v^2}\,d\chi^2\right) \tag{2.28}$$

where $M_{\rm mono} = 4\pi v/e^2$ in the notation used in the solution (2.19).

Two Monopoles

Two monopoles in $SU(2)$ have magnetic charge $\vec{g} = 2\alpha_1$. The direct approach to compute the metric that we have just described becomes impossible since the most general analytic solution for the two monopole configuration is not available. Nonetheless, Atiyah and Hitchin were able to determine the two monopole moduli space using symmetry considerations alone, most notably the constraints imposed by hyperKählerity [74, 88]. It is

$$\mathcal{M}_{2\vec{\alpha}} \cong \mathbf{R}^3 \times \frac{\mathbf{S}^1 \times \mathcal{M}_{AH}}{\mathbf{Z}_2} \tag{2.29}$$

where $\mathbf{R}^3$ describes the center of mass of the pair of monopoles, while $\mathbf{S}^1$ determines the overall phase $0 \leq \chi \leq 2\pi$. The four-dimensional hyperKähler space $\mathcal{M}_{AH}$ is the famous Atiyah-Hitchin manifold. Its metric can be written as

$$ds^2 = f(r)^2 dr^2 + a(r)^2 \sigma_1^2 + b(r)^2 \sigma_2^2 + c(r)^2 \sigma_3^2 \tag{2.30}$$

Here the radial coordinate r measures the separation between the monopoles in units of the monopole mass. The σ_i are the three left-invariant $SU(2)$ one-forms which, in terms of polar angles $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 2\pi$, take the form

$$\begin{aligned} \sigma_1 &= -\sin\psi \; d\theta + \cos\psi \sin\theta \; d\phi \\ \sigma_2 &= \cos\psi \; d\theta + \sin\psi \sin\theta \; d\phi \\ \sigma_3 &= d\psi + \cos\theta \; d\phi \end{aligned} \tag{2.31}$$

For far separated monopoles, θ and ϕ determine the angular separation while ψ is the relative phase. The $\mathbf{Z}_2$ quotient in (2.29) acts as

$$\mathbf{Z}_2 : \chi \rightarrow \chi + \pi \quad , \quad \psi \rightarrow \psi + \pi \tag{2.32}$$

The hyperKähler condition can be shown to relate the four functions f, a, b and c through the differential equation

$$\frac{2bc}{f} \frac{da}{dr} = (b - c)^2 - a^2 \tag{2.33}$$

together with two further equations obtained by cyclically permuting a, b and c. The solutions can be obtained in terms of elliptic integrals but it will prove more illuminating to present the asymptotic expansion of these functions. Choosing coordinates such that $f(r) = -b(r)/r$, we have

$$\begin{aligned} a^2 &= r^2 \left(1 - \frac{2}{r}\right) - 8r^3 e^{-r} + \ldots \\ b^2 &= r^2 \left(1 - \frac{2}{r}\right) + 8r^3 e^{-r} + \ldots \\ c^2 &= 4 \left(1 - \frac{2}{r}\right)^{-1} + \ldots \end{aligned} \tag{2.34}$$

If we suppress the exponential corrections, the metric describes the velocity dependant forces between two monopoles interacting through their long range fields. In fact, this asymptotic metric can be derived by treating the monopoles as point particles and considering their Liénard-Wiechert potentials. Note that in this limit there is an isometry associated to the relative phase ψ. However, the minus sign before the $2/r$ terms means that the metric is singular. The exponential corrections to the metric resolve this singularity and contain the information about the behavior of the monopoles as their non-abelian cores overlap.

The Atiyah-Hitchin metric appears in several places in string theory and supersymmetric gauge theories, including the M-theory lift of the type IIA O6-plane [89], the solution of the quantum dynamics of 3d gauge theories [90], in intersecting brane configurations [91], the heterotic string compactified on ALE spaces [92, 93] and NS5-branes on orientifold 8-planes [94]. In each of these places, there is often a relationship to magnetic monopoles underlying the appearance of this metric.

For higher charge monopoles of the same type $\vec{g} = n\vec{\alpha}$, the leading order terms in the asymptotic expansion of the metric, associated with the long-range fields of the monopoles, have been computed. The result is known as the Gibbons-Manton metric [86]. The full metric on the monopole moduli space remains an open problem.

Two Monopoles of Different Types

As we have seen, higher rank gauge groups $SU(N)$ for $N \geq 3$ admit monopoles of different types. If a $\vec{g} = \vec{\alpha}_a$ monopole and a $\vec{g} = \vec{\alpha}_b$ monopole live in entirely different places in the gauge group, so that $\vec{\alpha}_a \cdot \vec{\alpha}_b = 0$, then they don't see each other and their moduli space is simply the product $(\mathbf{R}^3 \times \mathbf{S}^1)^2$. However, if they live in neighboring subgroups so that $\vec{\alpha}_a \cdot \vec{\alpha}_b = -1$, then they do interact non-trivially.

The metric on the moduli space of two neighboring monopoles, sometimes referred to as the $(1,1)$ monopole, was first computed by Connell [95]. But he chose not to publish. It was rediscovered some years later by two groups when the connection with electro-magnetic duality made the study of monopoles more pressing [96, 97]. It is simplest to describe if the two monopoles have the same mass, so $\vec{\phi} \cdot \vec{\alpha}_a = \vec{\phi} \cdot \vec{\alpha}_b$. The moduli space is then

$$\mathcal{M}_{\vec{\alpha}_1+\vec{\alpha}_2} \cong \mathbf{R}^3 \times \frac{\mathbf{S}^1 \times \mathcal{M}_{TN}}{\mathbf{Z}_2} \tag{2.35}$$

where the interpretation of the $\mathbf{R}^3$ factor and $\mathbf{S}^1$ factor are the same as before. The relative moduli space is the Taub-NUT manifold, which has metric

$$ds^2 = \left(1 + \frac{2}{r}\right)(dr^2 + r^2(\sigma_1^2 + \sigma_2^2)) + \left(1 + \frac{2}{r}\right)^{-1} \sigma_3^2 \tag{2.36}$$

The $+2/r$ in the metric, rather than the $-2/r$ of Atiyah-Hitchin, means that the metric is smooth. The apparent singularity at $r = 0$ is merely a coordinate artifact, as you can check by transforming to the variables $R = \sqrt{r}$. Once again, the $1/r$ terms capture the long range interactions of the monopoles, with the minus sign traced to the fact that each sees the other with opposite magnetic charge (essentially because $\vec{\alpha}_1 \cdot \vec{\alpha}_2 = -1$). There are no exponential corrections to this metric. The non-abelian cores of the two monopoles do not interact.

The exact moduli space metric for a string of neighboring monopoles, $\vec{g} = \sum_a \vec{\alpha}_a$ has been determined. Known as the Lee-Weinberg-Yi metric, it is a higher dimensional generalization of the Taub-NUT metric [87]. It is smooth and has no exponential corrections.

2.3 Dyons

Consider the one-monopole moduli space $\mathbf{R}^3 \times \mathbf{S}^1$. Motion in $\mathbf{R}^3$ is obvious. But what does motion along the $\mathbf{S}^1$ correspond to?

We can answer this by returning to our specific $SU(2)$ solution (2.19). We determined that the zero mode for the $U(1)$ action is $\delta A_i = \mathcal{D}_i\phi$ and $\delta\phi = 0$. Translating to the time dependence of the fields (2.24), we find

$$E_i = \frac{(\mathcal{D}_i\phi)}{v}\,\dot{\chi} = \frac{B_i}{v}\;e^2\dot{\chi} \tag{2.37}$$

We see that motion along the $\mathbf{S}^1$ induces an electric field for the monopole, proportional to its magnetic field. In the unbroken $U(1)$, this gives rise to a long range electric field,

$$\mathrm{Tr}(E_i\phi) = \frac{qve^2\hat{r}_i}{2\pi r^2} \tag{2.38}$$

where, comparing with the normalization above, the electric charge q is given by

$$q = \frac{2\pi\dot{\chi}}{ve^2} \tag{2.39}$$

Note that motion in $\mathbf{R}^3$ also gives rise to an electric field, but this is the dual to the familiar statement that a moving electric charge produces a magnetic field. Motion in $\mathbf{S}^1$, on the other hand, only has the effect of producing an electric field [98].

A particle with both electric and magnetic charges is called a *dyon*, a term first coined by Schwinger [99]. Since we have understood this property from the perspective of the monopole worldline, can we return to our original theory (2.3) and find the corresponding solution there? The answer is yes. We relax the condition $E_i = 0$ when completing the Bogomoln'yi square in (2.16) and write

$$\begin{aligned} M_{\rm dyon} = \mathrm{Tr}\int d^3x\;\frac{1}{e^2}\,(E_i - \cos\alpha\,\mathcal{D}_i\phi)^2 + \frac{1}{e^2}\,(B_i - \sin\alpha\,\mathcal{D}_i\phi)^2 \\ +\frac{2}{e^2}\mathrm{Tr}\int d^3x\;\partial_i\,(\cos\alpha\,E_i\phi + \sin\alpha\,B_i\phi) \end{aligned} \tag{2.40}$$

which holds for all α. We write the long range magnetic field as $E_i = \vec{q}\cdot\vec{H}\, e^2\hat{r}^i/4\pi r^2$. Then by adjusting α to make the bound as tight as possible, we have

$$M_{\rm dyon} \geq \sqrt{\left(\vec{q}\cdot\vec{\phi}\right)^2 + \left(\frac{\vec{g}\cdot\vec{\phi}}{e^2}\right)^2} \tag{2.41}$$

and, given a solution to the monopole, it is easy to find a corresponding solution for the dyon for which this bound is saturated, with the fields satisfying

$$B_i = \sin\alpha\,\mathcal{D}_i\phi \quad \text{and} \quad E_i = \cos\alpha\,\mathcal{D}_i\phi \tag{2.42}$$

This method of finding solutions in the worldvolume theory of a soliton, and subsequently finding corresponding solutions in the parent 4d theory, will be something we'll see several more times in later sections.

I have two further comments on dyons.

- We could add a theta term $\theta F\wedge F$ to the 4d theory. Careful calculation of the electric Noether charges shows that this induces an electric charge $\vec{q} = \theta\vec{g}/2\pi$ on the monopole. In the presence of the theta term, monopoles become dyons. This is known as the Witten effect [100].

- Both the dyons arising from (2.42), and those arising from the Witten effect, have $\vec{q}\sim\vec{g}$. One can create dyons whose electric charge vector is not parallel to the magnetic charge by turning on a vev for a second, adjoint scalar field [101, 102]. These states are 1/4-BPS in $\mathcal{N}=4$ super Yang-Mills and correspond to (p,q)-string webs stretched between D3-branes. From the field theory perspective, the dynamics of these dyons is described by motion on the monopole moduli space with a potential induced by the second scalar vev [103, 104, 105].

2.4 Fermi Zero Modes

As with instantons, when the theory includes fermions they may be turned on in the background of the monopole without raising the energy of the configuration. A Dirac fermion λ in the adjoint representation satisfies

$$i\gamma^\mu\mathcal{D}_\mu\lambda - i[\phi,\lambda] = 0 \tag{2.43}$$

Each such fermion carried $4\sum_a n_a$ zero modes.

Rather than describing this in detail, we can instead resort again to supersymmetry. In $\mathcal{N} = 4$ super Yang-Mills, the monopoles preserve one-half the supersymmetry, corresponding to $\mathcal{N} = (4,4)$ on the monopole worldvolume. While, monopoles in $\mathcal{N} = 2$ supersymmetric theories preserve $\mathcal{N} = (0,4)$ on their worldvolume. Monopoles in $\mathcal{N} = 1$ theories are not BPS; they preserve no supersymmetry on their worldvolume.

There is also an interesting story with fermions in the fundamental representation, leading to the phenomenon of solitons carrying fractional fermion number [106]. A nice description of this can be found in [75].

2.5 Nahm's Equations

In the previous section we saw that the ADHM construction gave a powerful method for understanding instantons, and that it was useful to view this from the perspective of D-branes in string theory. You'll be pleased to learn that there exists a related method for studying monopoles. It's known as the Nahm construction [107]. It was further developed for arbitrary classical gauge group in [108], while the presentation in terms of D-branes was given by Diaconescu [109].

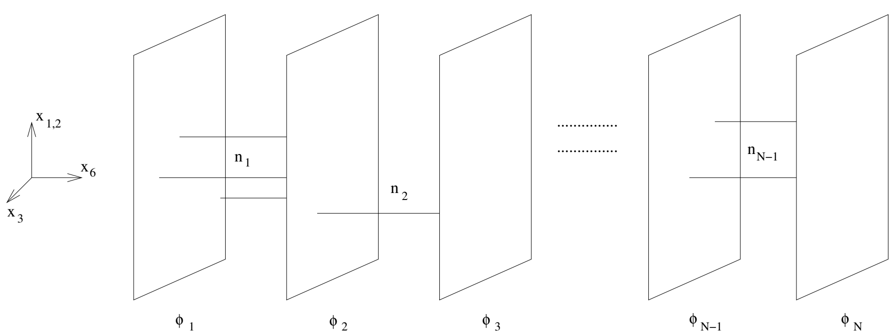

Figure 6: The D-brane set-up for monopoles of charge $\vec{g} = \sum_a n_a \vec{\alpha}_a$.

We start with $\mathcal{N} = 4$ $U(N)$ super Yang-Mills, realized on the worldvolume of D3-branes. To reflect the vev $\langle \phi \rangle = \text{diag}(\phi_1 \ldots, \phi_N)$, we separate the D3-branes in a transverse direction, say the x^6 direction. The a^{th} D3-brane is placed at position $x_6 = \phi_a$.

As is well known, the W-bosons correspond to fundamental strings stretched between the D3-branes. The monopoles are their magnetic duals, the D-strings. At this point our notation for the magnetic charge vector $\vec{g} = \sum_a n_a \vec{\alpha}_a$ becomes more visual. This monopole in sector $\vec{g}$ is depicted by stretching n_a D-strings between the a^{th} and $(a+1)^{\text{th}}$ D3-branes.

Our task now is to repeat the analysis of lecture 1 that led to the ADHM construction: we must read off the theory on the D1-branes, which we expect give us a new perspective on the dynamics of magnetic monopoles. From the picture it looks like the dynamics of the D-strings will be governed by something like a $\prod_a U(n_a)$ gauge theory, with each group living on the interval $\phi_a \leq x_6 \leq \phi_{a+1}$. And this is essentially correct. But what are the relevant equations dictating the dynamics? And what happens at the boundaries?

To get some insight into this, let's start by considering n infinite D-strings, with worldvolume x_0, x_6, and with D3-brane impurities inserted at particular points $x_6 = \phi_a$, as shown below.

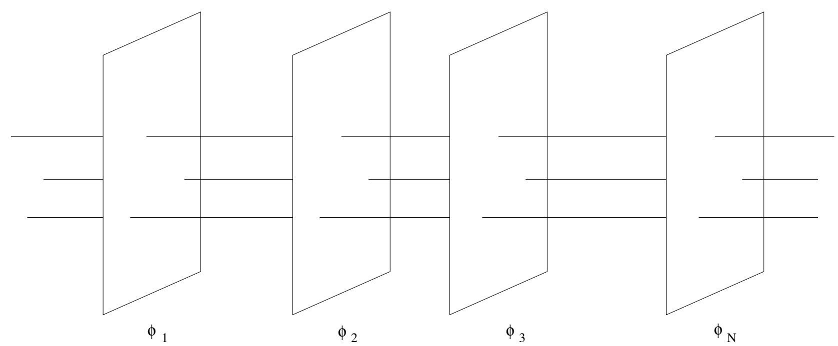

Figure 7: The D3-branes give rise to impurities on the worldvolume of the D1-branes.

The theory on the D-strings is a $d = 1+1$ $U(n)$ gauge theory with 16 supercharges (known as $\mathcal{N} = (8,8)$). Each D3-brane impurity donates a hypermultiplet to the theory, breaking supersymmetry by half to $\mathcal{N} = (4,4)$. As in lecture 1, we write the hypermultiplets as

$$\omega_a = \begin{pmatrix} \psi_a \\ \tilde{\psi}_a^\dagger \end{pmatrix} \qquad a = 1, \ldots, N \tag{2.44}$$

where ψ_a transforms in the $\mathbf{n}$ of $U(n)$, while $\tilde{\psi}_a$ transforms in the $\bar{\mathbf{n}}$. The coupling of these impurities (or defects as they're also known) is uniquely determined by supersymmetry, and again occurs in a triplet of D-terms (or, equivalently, a D-term and an F-term). In lecture 1, I unapologetically quoted the D-term and F-term arising in the ADHM construction (equation (1.44)) since they can be found in any supersymmetry text book. However, now we have an impurity theory which is a little less familiar. Nonetheless, I'm still going to quote the result, but this time I'll apologize. We could derive this interaction by examining the supersymmetry in more detail, but it's easier to simply tell you the answer and then give a couple of remarks to try and convince you that it's right. It turns out that the (admittedly rather strange) triplet of D-terms occurring in the Lagrangian is

$$\mathrm{Tr}\left(\frac{\partial X^i}{\partial x^6} - i[A_6, X^i] - \frac{i}{2}\epsilon_{ijk}[X^j, X^k] + \sum_{a=1}^{N} \omega_a^\dagger \sigma^i \omega_a \; \delta(x^6 - \phi_a)\right)^2 \tag{2.45}$$

In the ground state of the D-strings, this term must vanish. Some motivating remarks:

- The configuration shown in figure 7 arises from T-dualizing the D0-D4 system. This viewpoint makes it clear that A_6 is the right bosonic field to partner X^i in a hypermultiplet.
- Set $\partial_6 = 0$. Then, relabelling $A_6 = X^4$, this term is almost the same as the triplet of D-terms appearing in the ADHM construction. The only difference is the appearance of the delta-functions.
- We know that D-strings can end on D3-branes. The delta-function sources in the D-term are what allow this to happen. For example, consider a single $n = 1$ D-string, so that all commutators above vanish. We choose $\tilde{\psi} = 0$, to find the triplet of D-terms

$$\partial_6 X^1 = 0 \quad , \quad \partial_6 X^2 = 0 \quad , \quad \partial_6 X^3 = |\psi|^2 \delta(0) \tag{2.46}$$

which allows the D-string profile to take the necessary step (function) to split on the D3-brane as shown below.

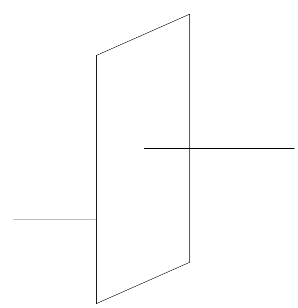

Figure 8:

If that wasn't enough motivation, one can find the full supersymmetry analysis in the original papers [110, 111] and, in most detail, in [112]. Accepting (2.45) we can make progress in understanding monopole dynamics by studying the limit in which several D-string segments, including the semi-infinite end segments, move off to infinity, leaving us back with the picture of figure 6.

The upshot of this is that the dynamics of the $\vec{g} = \sum_a n_a \vec{\alpha}_a$ monopoles are described as follows: In the interval $\phi_a \leq x_6 \leq \phi_{a+1}$, we have a $U(n_a)$ gauge theory, with three adjoint scalars X^i, $i = 1, 2, 3$ satisfying

$$\frac{dX^i}{dx^6} - i[A_6, X^i] - \tfrac{1}{2}\epsilon^{ijk}[X^i, X^j] = 0 \tag{2.47}$$

These are Nahm's equations. The boundary conditions imposed at the end of the interval depend on the number of monopoles in the neighbouring segment. (Set $n_0 = n_N = 0$ in what follows)

$\underline{n_a = n_{a+1}}$: The $U(n_a)$ gauge symmetry is extended to the interval $\phi_a \leq x^6 \leq \phi_{a+2}$ and an impurity is added to the right-hand-side of Nahm's equations

$$\frac{dX^i}{dx^6} - i[A_6, X^i] - \tfrac{1}{2}\epsilon^{ijk}[X^i, X^j] = \omega^\dagger_{a+1}\sigma^i\omega_{a+1}\ \delta(x^6 - \phi_{a+1}) \tag{2.48}$$

This, of course, follows immediately from (2.45).

$\underline{n_a = n_{a+1} - 1}$: In this case, $X^i \to (X_i)_-$, a set of three constant $n_a \times n_a$ matrices as $x^6 \to (\phi_{a+1})_-$. To the right of the impurity, the X^i are $(n_a + 1) \times (n_a + 1)$ matrices. They are required to satisfy the boundary condition

$$X^i \to \begin{pmatrix} y^i & a^{i\dagger} \\ a^i & (X^i)_- \end{pmatrix} \qquad \text{as } x_6 \to (\phi_{a+1})_+ \tag{2.49}$$

where $y^i \in \mathbf{R}$ and each a^i is a complex n_a-vector. One can derive this boundary condition without too much trouble by starting with (2.48) and taking $|\omega| \to \infty$ to remove one of the monopoles [113].

$\underline{n_a \leq n_{a+1} - 2}$ Once again $X^i \to (X^i)_-$ as $x_6 \to (\phi_{a+1})_-$ but, from the other side, the matrices X_μ now have a simple pole at the boundary,

$$X^i \to \begin{pmatrix} J^i/s + Y^i & \mathcal{O}(s^\gamma) \\ \mathcal{O}(s^\gamma) & (X^i)_- \end{pmatrix} \qquad \text{as } x_6 \to (\phi_{a+1})_+ \tag{2.50}$$

Here $s = (x^6 - \phi_{a+1})$ is the distance to the impurity. The matrices J^i are the irreducible $(n_{a+1} - n_a) \times (n_{a+1} - n_a)$ representation of $su(2)$, and Y^i are now constant $(n_{a+1} - n_a) \times (n_{a+1} - n_a)$ matrices. Note that the simple pole structure is compatible with Nahm's equations, with both the derivative and the commutator term going like $1/s^2$. Finally, $\gamma = \frac{1}{2}(n_{a+1} - n_a - 1)$, so the off-diagonal terms vanish as we approach the boundary. The boundary condition (2.50) can also be derived from (2.49) by removing a monopole to infinity [113].

When $n_a > n_{a+1}$, the obvious parity flipped version of the above conditions holds.

2.5.1 Constructing the Solutions

Just as in the case of ADHM construction, Nahm's equations capture information about both the monopole solutions and the monopole moduli space. The space of solutions to Nahm's equations (2.47), subject to the boundary conditions detailed above, is isomorphic to the monopole moduli space $\mathcal{M}_{\vec{g}}$. The phases of each monopole arise from the gauge field A_6, while X^i carry the information about the positions of the monopoles. Moreover, there is a natural metric on the solutions to Nahm's equations which coincides with the metric on the monopole moduli space. I don't know if anyone has calculated the Atiyah-Hitchin metric using Nahm data, but a derivation of the Lee-Weinberg-Yi metric was given in [114].

Given a solution to Nahm's equations, one can explicitly construct the corresponding solution to the monopole equation. The procedure is analogous to the construction of instantons in 1.4.2, although its a little harder in practice as its not entirely algebraic. We now explain how to do this. The first step is to build a Dirac-like operator from the solution to (2.47). In the segment $\phi_a \leq x^6 \leq \phi_{a+1}$, we construct the Dirac operator

$$\Delta = \frac{d}{dx^6} - iA_6 - i(X^i + r^i)\sigma^i \tag{2.51}$$

where we've reintroduced the spatial coordinates r^i into the game. We then look for normalizable zero modes U which are solutions to the equation

$$\Delta U = 0 \tag{2.52}$$

One can show that there are N such solutions, and so we consider U as a $2n_a \times N$-dimensional matrix. Note that the dimension of U jumps as we move from one interval to the next. We want to appropriately normalize U, and to do so choose to integrate over all intervals, so that

$$\int_{\phi_1}^{\phi_N} dx^6 \; U^\dagger U = \mathbf{1}_N \tag{2.53}$$

Once we've figured out the expression for U, a Higgs field ϕ and a gauge field A_i which satisfy the monopole equation are given by,

$$\phi = \int_{\phi_1}^{\phi_N} dx^6 \; x^6 \, U^\dagger U \quad , \quad A_i = \int_{\phi_1}^{\phi_N} dx^6 \; U^\dagger \partial_6 U \tag{2.54}$$

The similarity between this procedure and that described in section 1.4.2 for instantons should be apparent.

In fact, there's a slight complication that I've brushed under the rug. The above construction only really holds when $n_a \neq n_{a+1}$. If we're in a situation where $n_a = n_{a+1}$ for some a, then we have to take the hypermultiplets ω_a into account, since their value affects the monopole solution. This isn't too hard — it just amounts to adding some extra discrete pieces to the Dirac operator Δ. Details can be found in [108].

A string theory derivation of the construction part of the Nahm construction was recently given in [115].

An Example: The Single $SU(2)$ Monopole Revisited

It's very cute to see the single $n = 1$ solution (2.19) for the $SU(2)$ monopole drop out of this construction. This is especially true since the Nahm data is trivial in this case: $X^i = A_6 = 0$!

To see how this arises, we look for solutions to

$$\Delta U = \left(\frac{d}{dx^6} - r^i \sigma^i \right) U = 0 \tag{2.55}$$

where $U = U(x^6)$ is a 2×2 matrix. This is trivially solved by

$$U = \sqrt{\frac{r}{\sinh(2vr)}} \left(\cosh(rx^6)\, \mathbf{1}_2 + \sinh(rx^6)\, \hat{r}^i \sigma^i \right) \tag{2.56}$$

which has been designed to satisfy the normalizability condition $\int_{-v}^{+v} U^\dagger U \; dx^6 = \mathbf{1}_2$. Armed with this, we can easily reproduce the monopole solution (2.19). For example, the Higgs field is given by

$$\phi = \int_{-v}^{+v} dx^6 \; x^6 U^\dagger U = \frac{\hat{r}^i \sigma^i}{r} \left(vr \coth(vr) - 1 \right) \tag{2.57}$$

as promised. And the gauge field A_i drops out just as easily. See — told you it was cute! Monopole solutions with charge of the type $(1, 1, \ldots, 1)$ were constructed using this method in [116].

2.6 What Became of Instantons

In the last lecture we saw that pure Yang-Mills theory contains instanton solutions. Now we've added a scalar field, where have they gone?! The key point to note is that the theory was conformal before ϕ gained its vev. As we saw in Lecture 1, this led to a collective coordinate ρ, the scale size of the instanton. Now with $\langle \phi \rangle \neq 0$ we have introduced a mass scale into the game and no longer expect ρ to correspond to an exact collective coordinate. This turns out to be true: in the presence of a non-zero vev $\langle \phi \rangle$, the instanton minimizes its action by shrinking to zero size $\rho \to 0$. Although, strictly speaking, no instanton exists in the theory with $\langle \phi \rangle \neq 0$, they still play a crucial role. For example, the famed Seiberg-Witten solution can be thought of as summing these small instanton corrections.

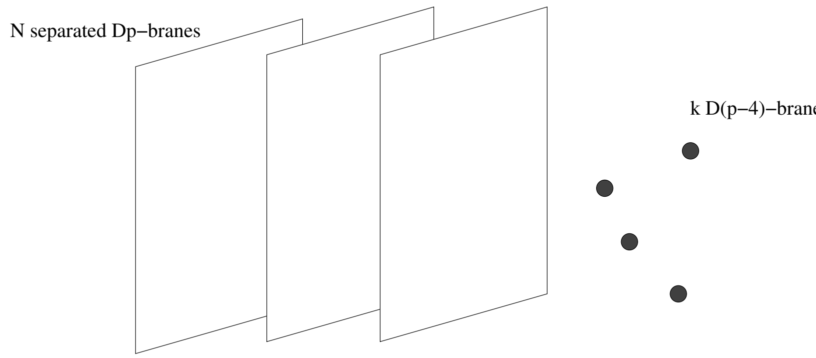

Figure 9: Separating the Dp-branes gives rise to a mass for the hypermultiplets

How can we see this behavior from the perspective of the worldvolume theory? We can return to the D-brane set-up, now with the Dp-branes separated in one direction, say x_6, to mimic the vev $\langle\phi\rangle$. Each Dp-D$(p-4)$ string is now stretched by a different amount, reflecting the fact that each hypermultiplet has a different mass. The potential on the worldvolume theory of the D-instantons is now

$$V = \frac{1}{g^2}\sum_{m,n=5}^{10}[X_m, X_n]^2 + \sum_{m,\mu}[X_m, X_\mu]^2 + \sum_{a=1}^{N}\psi^{a\dagger}(X_m - \phi^m{}_a)^2\psi_a + \tilde{\psi}^a(X_m - \phi^m{}_a)^2\tilde{\psi}^\dagger_a$$
$$+g^2\,\mathrm{Tr}\,(\sum_{a=1}^{N}\psi_a\psi^{a\dagger} - \tilde{\psi}^\dagger_a\tilde{\psi}^a + [Z,Z^\dagger] + [W,W^\dagger])^2 + g^2\,\mathrm{Tr}\,|\sum_{a=1}^{N}\psi_a\tilde{\psi}^a + [Z,W]|^2$$

We've actually introduced more new parameters here than we need, since the D3-branes can be separated in 6 different dimensions, so we have the corresponding positions $\phi^m{}_a$, $m = 4,\ldots,9$ and $a = 1,\ldots N$. Since we have been dealing with just a single scalar field ϕ in this section, we will set $\phi^m_i = 0$ except for $m = 6$ (I know...why 6?!). The parameters $\phi^6{}_a = \phi_a$ are the components of the vev (2.4).

We can now re-examine the vacuum condition for the Higgs branch. If we wish to try to turn on ψ and $\tilde{\psi}$, we must first set $X_m = \phi_a$, for some a. Then the all ψ_b and $\tilde{\psi}^b$ must vanish except for $b = a$. But, taking the trace of the D- and F-term conditions tells us that even ψ_a and $\tilde{\psi}_a$ vanish. We have lost our Higgs branch completely. The interpretation is that the instantons have shrunk to zero size. Note that in the case of non-commutativity, the instantons don't vanish but are pushed to the $U(1)$ instantons with, schematically, $|\psi|^2 \sim \zeta$.

Although the instantons shrink to zero size, there's still important information to be gleaned from the potential above. One can continue to think of the instanton moduli space $\mathcal{I}_{k,N} \cong \mathcal{M}_{\rm Higgs}$ as before, but now with a potential over it. This potential arises after integrating out the X_m and it is not hard to show that it is of a very specific form: it is the length-squared of a triholomorphic Killing vector on $\mathcal{I}_{k,N}$ associated with the $SU(N)$ isometry.

This potential on $\mathcal{I}_{k,N}$ can be derived directly within field theory without recourse to D-branes or the ADHM construction [117]. This is the route we follow here. The question we want to ask is: given an instanton solution, how does the presence of the ϕ vev affect its action? This gives the potential on the instanton moduli space which is simply

$$V = \int d^4x \ {\rm Tr}\,(\mathcal{D}_\mu\phi)^2 \tag{2.58}$$

where $\mathcal{D}_\mu$ is evaluated on the background instanton solution. We are allowed to vary ϕ so it minimizes the potential so that, for each solution to the instanton equations, we want to find ϕ such that

$$\mathcal{D}^2\phi = 0 \tag{2.59}$$

with the boundary condition that $\phi \rightarrow \langle\phi\rangle$. But we've seen an equation of this form, evaluated on the instanton background, before. When we were discussing the instanton zero modes in section 1.2, we saw that the zero modes arising from the overall $SU(N)$ gauge orientation were of the form $\delta A_\mu = \mathcal{D}_\mu\Lambda$, where Λ tends to a constant at infinity and satisfies the gauge fixing condition $\mathcal{D}_\mu\delta A_\mu = 0$. This means that we can re-write the potential in terms of the overlap of zero modes

$$V = \int d^4x \ {\rm Tr}\ \delta A_\mu \delta A_\mu \tag{2.60}$$

for the particular zero mode $\delta A_\mu = \mathcal{D}_\mu\phi$ associated to the gauge orientation of the instanton. We can give a nicer geometrical interpretation to this. Consider the action of the Cartan subalgebra $\vec{H}$ on $\mathcal{I}_{k,N}$ and denote the corresponding Killing vector as $\vec{k} = \vec{k}^\alpha\partial_\alpha$. Then, since ϕ generates the transformation $\vec{\phi}\cdot\vec{H}$, we can express our zero mode in terms of the basis $\delta A_\mu = (\vec{\phi}\cdot\vec{k}^\alpha)\,\delta_\alpha A_\mu$. Putting this in our potential and performing the integral over the zero modes, we have the final expression

$$V = g_{\alpha\beta}\,(\vec{\phi}\cdot\vec{k}^\alpha)\,(\vec{\phi}\cdot\vec{k}^\beta) \tag{2.61}$$

The potential vanishes at the fixed points of the $U(1)^{N-1}$ action. This is the small instanton singularity (or related points on the blown-up cycles in the resolved instanton moduli space). Potentials of the form (2.61) were first discussed by Alvarez-Gaume and Freedman who showed that, for tri-holomorphic Killing vectors k, they are the unique form allowed in a sigma-model preserving eight supercharges [118].

The concept of a potential on the instanton moduli space $\mathcal{I}_{k,N}$ is the modern way of viewing what used to known as the "constrained instanton", that is an approximate instanton-like solution to the theory with $\langle\phi\rangle \neq 0$ [119]. These potentials play an important role in Nekrasov's first-principles computation of the Seiberg-Witten prepotential [46]. Another application occurs in the five-dimensional theory, where instantons are particles. Here the motion on the moduli space may avoid the fate of falling to the zeroes of (2.61) by spinning around the potential like a motorcyclist on the wall of death. These solutions of the low-energy dynamics are dyonic instantons which carry electric charge in five dimensions [117, 120, 121].

2.7 Applications

Time now for the interesting applications, examining the role that monopoles play in the quantum dynamics of supersymmetric gauge theories in various dimensions. We'll look at monopoles in 3, 4, 5 and 6 dimensions in turn.

2.7.1 Monopoles in Three Dimensions

In $d = 2+1$ dimensions, monopoles are finite action solutions to the Euclidean equations of motion and the role they play is the same as that of instantons in $d = 3+1$ dimensions: in a semi-classical evaluation of the path-integral, one must sum over these monopole configurations. In 1975, Polyakov famously showed how a gas of these monopoles leads to linear confinement in non-supersymmetric Georgi-Glashow model [122] (that is, an $SU(2)$ gauge theory broken to $U(1)$ by an adjoint scalar field).

In supersymmetric theories, monopoles give rise to somewhat different physics. The key point is that they now have fermionic zero modes, ensuring that they can only contribute to correlation functions with a suitable number of fermionic insertions to soak up the integrals over the Grassmannian collective coordinates. In $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theories[5] in $d = 2 + 1$ dimensions, instantons generate superpotentials, lifting moduli spaces of vacua [123]. In $\mathcal{N} = 8$ theories, instantons contribute to particular 8 fermi correlation functions which have a beautiful interpretation in terms of membrane scattering in M-theory [124, 125]. In this section, I'd like to describe one of the nicest applications of monopoles in three dimensions which occurs in theories with $\mathcal{N} = 4$ supersymmetry, or 8 supercharges.

[5] A (foot)note on nomenclature. In any dimension, the number of supersymmetries $\mathcal{N}$ counts the number of supersymmetry generators in units of the minimal spinor. In $d = 2+1$ the minimal Majorana spinor has 2 real components. This is in contrast to $d = 3+1$ dimensions where the minimal Majorana (or equivalently Weyl) spinor has 4 real components. This leads to the unfortunate fact that $\mathcal{N} = 1$ in $d = 3+1$ is equivalent to $\mathcal{N} = 2$ in $d = 2+1$. It's annoying. The invariant way to count is in terms of supercharges. Four supercharges means $\mathcal{N} = 1$ in four dimensions or $\mathcal{N} = 2$ in three dimensions.

We'll consider $\mathcal{N} = 4$ $SU(2)$ super Yang-Mills. The superpartners of the gauge field include 3 adjoint scalar fields, ϕ^α, $\alpha = 1, 2, 3$ and 2 adjoint Dirac fermions. When the scalars gain an expectation value $\langle \phi^\alpha \rangle \neq 0$, the gauge group is broken $SU(2) \to U(1)$ and the surviving, massless, bosonic fields are 3 scalars and a photon. However, in $d = 2 + 1$ dimensions, the photon has only a single polarization and can be exchanged in favor of another scalar σ. We achieve this by a duality transformation:

$$F_{ij} = \frac{e^2}{2\pi} \epsilon_{ijk} \partial^k \sigma \tag{2.62}$$

where we have chosen normalization so that the scalar σ is periodic: $\sigma = \sigma + 2\pi$. Since supersymmetry protects these four scalars against becoming massive, the most general low-energy effective action we can write down is the sigma-model

$$L_{\mathrm{low-energy}} = \frac{1}{2e^2}\, g_{\alpha\beta}\, \partial_i \phi^\alpha\, \partial^i \phi^\beta \tag{2.63}$$

where $\phi^\alpha = (\phi^1, \phi^2, \phi^3, \sigma)$. Remarkably, as shown by Seiberg and Witten, the metric $g_{\alpha\beta}$ can be determined uniquely [90]. It turns out to be an old friend: it is the Atiyah-Hitchin metric (2.30)! The dictionary is $\phi^i = e^2 r^i$ and $\sigma = \psi$. Comparing with the functions a, b and c listed in (2.34), the leading constant term comes from tree level in our 3d gauge theory, and the $1/r$ terms arise from a one-loop correction. Most interesting is the e^{-r} term in (2.34). This comes from a semi-classical monopole computation in $d = 2 + 1$ which can be computed exactly [128]. So we find monopoles arising in two very different ways: firstly as an instanton-like configuration in the 3d theory, and secondly in an auxiliary role as the description of the low-energy dynamics. The underlying reason for this was explained by Hanany and Witten [91], and we shall see a related perspective in section 2.7.4.

So the low-energy dynamics of $\mathcal{N} = 4$ $SU(2)$ gauge theory is dictated by the two monopole moduli space. It can also be shown that the low-energy dynamics of the $\mathcal{N} = 4$ $SU(N)$ gauge theory in $d = 2 + 1$ is governed by a sigma-model on the moduli space of N magnetic monopoles in an $SU(2)$ gauge group [126]. There are 3d quiver gauge theories related to monopoles in higher rank, simply laced (i.e. ADE) gauge groups [91, 127] but, to my knowledge, there is no such correspondence for monopoles in non-simply laced groups.

2.7.2 Monopoles and Duality

Perhaps the most important application of monopoles is the role they play in uncovering the web of dualities relating various theories. Most famous is the S-duality of $\mathcal{N} = 4$ super Yang-Mills in four dimensions. The idea is that we can re write the gauge theory treating magnetic monopoles as elementary particles rather than solitons [129]. The following is a lightening review of this large subject. Many more details can be found in [75].

The conjecture of S-duality states that we may re-express the theory, treating monopoles as the fundamental objects, at the price of inverting the coupling $e \to 4\pi/e$. Since this is a strong-weak coupling duality, we need to have some control over the strong coupling behavior of the theory to test the conjecture. The window on this regime is provided by the BPS states [23], whose mass is not renormalized in the maximally supersymmetric $\mathcal{N} = 4$ theory which, among other reasons, makes it a likely place to look for S-duality [130]. In fact, this theory exhibits a more general $SL(2,\mathbf{Z})$ group of duality transformations which acts on the complexified coupling $\tau = \theta/2\pi + 4\pi i/e^2$ by

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d} \qquad \text{with } a,b,c,d \in \mathbf{Z} \;\text{ and } ad - bc = 1 \tag{2.64}$$

A transformation of this type mixes up what we mean by electric and magnetic charges. Let's work in the $SU(2)$ gauge theory for simplicity so that electric and magnetic charges in the unbroken $U(1)$ are each specified by an integer (n_e, n_m). Then under the $SL(2,\mathbf{Z})$ transformation,

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \longrightarrow \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix} \tag{2.65}$$

The conjecture of S-duality has an important prediction that can be tested semi-classically: the spectrum must form multiplets under the $SL(2,Z)$ transformation above. In particular, if S-duality holds, the existence of the W-boson state $(n_e, n_m) = (1,0)$ implies the existence of a slew of further states with quantum numbers $(n_e, n_m) = (a, c)$ where a and c are relatively prime. The states with magnetic charge $n_m = c = 1$ are the dyons that we described in Section 2.3 and can be shown to exist in the quantum spectrum. But we have to work much harder to find the states with magnetic charge $n_m = c > 1$. To do so we must examine the low-energy dynamics of n_m monopoles, described by supersymmetric quantum mechanics on the monopole moduli space. Bound states saturating the Bogomoln'yi bound correspond to ground states of the quantum mechanics. But, as we described in section 1.5.2, this questions translates into the more geometrical search for normalizable harmonic forms on the monopole moduli space.

In the $n_m = 2$ monopole sector, the bound states were explicitly demonstrated to exist by Sen [131]. S-duality predicts the existence of a tower of dyon states with charges $(n_e, 2)$ for all n_e odd which translates into the requirement that there is a unique harmonic form ω on the Atiyah-Hitchin manifold. The electric charge still comes from motion in the $\mathbf{S}^1$ factor of the monopole moduli space (2.29), but the need for only odd charges n_e to exist requires that the form ω is odd under the $\mathbf{Z}_2$ action (2.32). Uniqueness requires that ω is either self-dual or anti-self-dual. In fact, it is the latter. The ansatz,

$$\omega = F(r)(d\sigma_1 - \frac{fa}{bc}dr \wedge \sigma_1) \tag{2.66}$$

is harmonic provided that $F(r)$ satisfies

$$\frac{dF}{dr} = -\frac{fa}{bc}F \tag{2.67}$$

One can show that this form is normalizable, well behaved at the center of the moduli space and, moreover, unique. Historically, the existence of this form was one the compelling pieces of evidence in favor of S-duality, leading ultimately to an understanding of strong coupling behavior of many supersymmetric field theories and string theories.

The discussion above is for $\mathcal{N} = 4$ theories. In $\mathcal{N} = 2$ theories, the bound state described above does not exist (a study of the $\mathcal{N} = (0,4)$ supersymmetric quantum mechanics reveals that the Hilbert space is identified with holomorphic forms and ω is not holomorphic). Nevertheless, there exists a somewhat more subtle duality between electrically and magnetically charged states, captured by the Seiberg-Witten solution [132]. Once again, there is a semi-classical test of these ideas along the lines described above [133]. There is also an interesting story in this system regarding quantum corrections to the monopole mass [134].

2.7.3 Monopole Strings and the $(2,0)$ Theory

We've seen that the moduli space of a single monopole is $\mathcal{M} \cong \mathbf{R}^3 \times \mathbf{S}^1$ with metric,

$$ds^2 = M_{\rm mono}\left(dX^i dX^i + \frac{1}{v^2}\,d\chi^2\right) \tag{2.68}$$

where $\chi \in [0, 2\pi)$. It looks as if, at low-energies, the monopole is moving in a higher dimensional space. Is there any situation where we can actually interpret this motion in the $\mathbf{S}^1$ as motion in an extra, hidden dimension of space?

One problem with interpreting internal degrees of freedom, such as χ, in terms of an extra dimension is that there is no guarantee that motion in these directions will be Lorentz covariant. For example, Einstein's speed limit tells us that the motion of the monopole in $\mathbf{R}^3$ is bounded by the speed of light: i.e. $\dot{X} \leq 1$. But is there a similar bound on $\dot{\chi}$? This is an question which goes beyond the moduli space approximation, which keeps only the lowest velocities, but is easily answered since we know the exact spectrum of the dyons. The energy of a relativistically moving dyon is $E^2 = M^2_{\rm dyon} + p_i p_i$, where p_i is the momentum conjugate to the center of mass X_i. Using the mass formula (2.41), we have the full Hamiltonian

$$H_{\rm dyon} = \sqrt{M^2_{\rm mono} + v^2 p_\chi^2 + p_i p_i} \tag{2.69}$$

where $p_\chi = 2q$ is the momentum conjugate to χ. This gives rise to the Lagrangian,

$$L_{\rm dyon} = -M_{\rm mono}\sqrt{1 - \dot{\chi}^2/v^2 - \dot{X}^i\dot{X}^i} \tag{2.70}$$

which, at second order in velocities, agrees with the motion on the moduli space (2.68). So, surprisingly, the internal direction χ does appear in a Lorentz covariant manner in this Lagrangian and is therefore a candidate for an extra, hidden, dimension.

However, looking more closely, our hopes are dashed. From (2.70) (or, indeed, from (2.68)), we see that the radius of the extra dimension is proportional to $1/v$. But the width of the monopole core is also $1/v$. This makes it a little hard to convincingly argue that the monopole can happily move in this putative extra dimension since there's no way the dimension can be parametrically larger than the monopole itself. It appears that χ is stuck in the auxiliary role of endowing monopoles with electric charge, rather than being promoted to a physical dimension of space.

Things change somewhat if we consider the monopole as a string-like object in a $d = 4 + 1$ dimensional gauge theory. Now the low-energy effective action for a single monopole is simply the action (2.70) lifted to the two dimensional worldsheet of the string, yielding the familiar Nambu-Goto action

$$S_{\rm string} = -T_{\rm mono}\int d^2y\ \sqrt{1 - (\partial\chi)^2/v^2 - (\partial X^i)^2} \tag{2.71}$$

where ∂ denotes derivatives with respect to both worldsheet coordinates, σ and τ. We've rewritten $M_{\rm mono} = T_{\rm mono} = 4\pi v/e^2$ to stress the fact that it is a tension, with dimension 2 (recall that e^2 has dimension -1 in $d = 4 + 1$). As it stands, we're in no better shape. The size of the circle is still $1/v$, the same as the width of the monopole string. However, now we have a two dimensional worldsheet we may apply T-duality. This means exchanging momentum modes around $\mathbf{S}^1$ for winding modes so that

$$\partial\chi = {}^\star\partial\tilde{\chi} \tag{2.72}$$

We need to be careful with the normalization. A careful study reveals that,

$$\frac{1}{4\pi}\int d^2y\, R^2\,(\partial\chi)^2 \;\rightarrow\; \frac{1}{4\pi}\int d^2y\,\frac{1}{R^2}\,(\partial\tilde{\chi})^2 \tag{2.73}$$

where, up to that important factor of 4π, R is the radius of the circle measured in string units. Comparing with our normalization, we have $R^2 = 8\pi^2/ve^2$, and the dual Lagrangian is

$$S_{\rm string} = -T_{\rm mono}\int d^2y\,\sqrt{1-(e^2/8\pi^2)^2(\partial\tilde{\chi})^2-(\partial X^i)^2} \tag{2.74}$$

We see that the physical radius of this dual circle is now $e^2/8\pi^2$. This can be arbitrarily large and, in particular, much larger than the width of the monopole string. It's a prime candidate to be interpreted as a real, honest, extra dimension. In fact, in the maximally supersymmetric Yang-Mills theory in five dimensions, it is known that this extra dimension is real. It is precisely the hidden circle that takes us up to the six-dimensional $(2,0)$ theory that we discussed in section 1.5.2. The monopole even tells us that the instantons must be the Kaluza-Klein modes since the inverse radius of the dual circle is exactly $M_{\rm inst}$. Once again, we see that solitons allow us to probe important features of the quantum physics where myopic perturbation theory fails. Note that the derivation above does rely on supersymmetry since, for the Hamiltonian (2.69) to be exact, we need the masses of the dyons to saturate the Bogomoln'yi bound (2.41).

2.7.4 D-Branes in Little String Theory

Little string theories are strongly interacting string theories without gravity in $d=5+1$ dimensions. For a review see [136]. The maximally supersymmetric variety can be thought of as the decoupled theory living on NS5-branes. They come in two flavors: the type iia little string theory is a $(2,0)$ supersymmetric theory which reduces at low-energies to the conformal field theory discussed in sections 1.5.2 and 2.7.3. In contrast, the type iib little string has $(1,1)$ non-chiral supersymmetry and reduces at low-energies to $d=5+1$ Yang-Mills theory. When this theory sits on the Coulomb branch it admits monopole solutions which, in six dimensions, are membranes. Let's discuss some of the properties of these monopoles in the $SU(2)$ theory.

The low-energy dynamics of a single monopole is the $d=2+1$ dimensional sigma model with target space $\mathbf{R}^3\times\mathbf{S}^1$ and metric (2.68). But, as we already discussed, in $d=2+1$ we can exchange the periodic scalar χ for a $U(1)$ gauge field living on the monopole. Taking care of the normalization, we find

$$F_{mn} = \frac{8\pi^2}{e^2}\epsilon_{mnp}\,\partial^p\chi \tag{2.75}$$

with $m, n = 0, 1, 2$ denoting the worldvolume dimensions of the monopole 2-brane. The low-energy dynamics of this brane can therefore be written as

$$S_{\text{brane}} = \int d^3x \; \frac{1}{2} T_{\text{mono}} \left((\partial_m X^i)^2 + \frac{1}{v^2} (\partial_m \chi)^2 \right) \tag{2.76}$$

$$= \int d^3x \; \frac{1}{2g^2} \left((\partial_m \varphi^i)^2 + \frac{1}{2} F_{mn} F^{mn} \right) \tag{2.77}$$

where $g^2 = 4\pi^2 T_{\text{mono}}/v^2$ is fixed by the duality (2.75) and insisting that the scalar has canonical kinetic term dictates $\varphi^i = (8\pi^2/e^2)\, X^i = T_{\text{inst}} X^i$. This normalization will prove important. Including the fermions, we therefore find the low-energy dynamics of a monopole membrane to be free $U(1)$ gauge theory with 8 supercharges (called $\mathcal{N} = 4$ in three dimensions), containing a photon and three real scalars.

Six dimensional gauge theories also contain instanton strings. These are the "little strings" of little string theory. We will now show that strings can end on the monopole 2-brane. This is simplest to see from the worldvolume perspective in terms of the original variable χ. Defining the complex coordinate on the membrane worldvolume $z = x^4 + ix^5$, we have the BPS "BIon" spike [137, 138] solution of the theory (2.76)

$$X^1 + \frac{i}{v} \chi = \frac{1}{v} \log(vz) \tag{2.78}$$

Plotting the value of the transverse position X^1 as a function of $|z|$, we see that this solution indeed has the profile of a string ending on the monopole 2-brane. Since χ winds once as we circumvent the origin $z = 0$, after the duality transformation we see that this string sources a radial electric field. In other words, the end of the string is charged under the $U(1)$ gauge field on the brane (2.75). We have found a D-brane in the six-dimensional little string theory.

Having found the string solution from the perspective of the monopole worldvolume theory, we can ask whether we can find a solution in the full $d = 5 + 1$ dimensional theory. In fact, as far as I know, no one has done this. But it is possible to write down the first order equations that this solution must solve [139]. They are the dimensional reduction of equations found in [140] and read

$$\begin{gathered} F_{23} + F_{45} = \mathcal{D}_1 \phi \quad , \quad F_{35} = -F_{42} \quad , \quad F_{34} = -F_{25} \\ F_{31} = \mathcal{D}_2 \phi \quad , \quad F_{12} = \mathcal{D}_3 \phi \quad , \quad F_{51} = \mathcal{D}_4 \phi \quad , \quad F_{14} = \mathcal{D}_5 \phi \end{gathered} \tag{2.79}$$

Notice that among the solutions to these equations are instanton strings stretched in the x^1 directions, and monopole 2-branes with spatial worldvolume (x^4, x^5). It would be interesting to find an explicit solution describing the instanton string ending on the monopole brane.

We find ourselves in a rather familiar situation. We have string-like objects which can terminate on D-brane objects, where their end is electrically charged. Yet all this is within the context of a gauge theory, with no reference to string theory or gravity. Let's remind ourselves about some further properties of D-branes in string theory to see if the analogy can be pushed further. For example, there are two methods to understand the dynamics of D-branes in string theory, using either closed or open strings. The first method — the closed string description — uses the supergravity solution for D-branes to compute their scattering. In contrast, in the second method — the open string description — the back-reaction on the bulk is ignored. Instead the strings stretched between two branes are integrated in, giving rise to new, light fields of the worldvolume theory as the branes approach. In flat space, this enhances $U(1)^n$ worldvolume gauge symmetry to $U(n)$ [141]. The quantum effects from these non-abelian fields capture the scattering of the D-branes. The equivalence of these two methods is assured by open-closed string duality, where the diagram drawn in figure 10 can be interpreted as tree-level closed string or one-loop open string exchange. Generically the two methods have different regimes of validity.

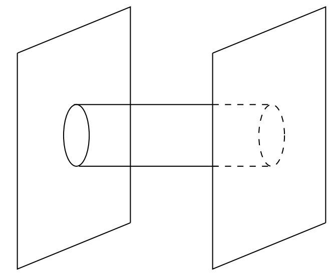

Figure 10:

Is there an analogous treatment for our monopole D-branes? The analogy of the supergravity description is simply the Manton moduli space approximation described in section 2.2. What about the open string description? Can we integrate in the light states arising from instanton strings stretched between two D-branes? They have charge $(+1, -1)$ under the two branes and, by the normalization described above, mass $T_{\rm inst}|X_1^i - X_2^i| = |\varphi_1^i - \varphi_2^i|$. Let's make the simplest assumption that quantization of these strings gives rise to W-bosons, enhancing the worldvolume symmetry of n branes to $U(n)$. Do the quantum effects of these open strings mimic the classical scattering of monopoles? Of course they do! This is precisely the calculation we described in section 2.7.1: the Coulomb branch of the $U(n)$ $\mathcal{N} = 4$ super Yang-Mills in $d = 2 + 1$ dimensions is the n monopole moduli space.

The above discussion is not really new. It is nothing more than the "Hanany-Witten" story [91], with attention focussed on the NS5-brane worldvolume rather than the usual 10-dimensional perspective. Nevertheless, it's interesting that one can formulate the story without reference to 10-dimensional string theory. In particular, if we interpret our results in terms of open-closed string duality summarized in figure 10, it strongly suggests that the bulk six-dimensional Yang-Mills fields can be thought of as quantized loops of instanton strings.

To finish, let me confess that, as one might expect, the closed and open string descriptions have different regimes of validity. The bulk calculation is valid in the full quantum theory only if we can ignore higher derivative corrections to the six-dimensional action. These scale as $e^{2n}\partial^{2n}$. Since the size of the monopole is $\partial \sim v^{-1}$, we have the requirement $v^2e^2 \ll 1$ for the "closed string" description to be valid. What about the open string description? We integrate in an object of energy $E = T_{\rm inst}\Delta X$, where ΔX is the separation between branes. We do not want to include higher excitations of the string which scale as v. So we have $E \ll v$. At the same time, we want $\Delta X > 1/v$, the width of the branes, in order to make sense of the discussion. These two requirements tell us that $v^2e^2 \gg 1$. The reason that the two calculations yield the same result, despite their different regimes of validity, is due to a non-renormalization theorem, which essentially boils down the restrictions imposed by the hyperKähler nature of the metric.

3. Vortices

In this lecture, we're going to discuss vortices. The motivation for studying vortices should be obvious: they are one of the most ubiquitous objects in physics. On table-tops, vortices appear as magnetic flux tubes in superconductors and fractionally charged quasi-excitations in quantum Hall fluids. In the sky, vortices in the guise of cosmic strings have been one one of the most enduring themes in cosmology research. With new gravitational wave detectors coming on line, there is hope that we may be able to see the distinctive signatures of these strings as the twist and whip. Finally, and more formally, vortices play a crucial role in determining the phases of low-dimensional quantum systems: from the phase-slip of superconducting wires, to the physics of strings propagating on Calabi-Yau manifolds, the vortex is key.

As we shall see in detail below, in four dimensional theories vortices are string like objects, carrying magnetic flux threaded through their core. They are the semi-classical cousins of the more elusive QCD flux tubes. In what follows we will primarily be interested in the dynamics of infinitely long, parallel vortex strings and the long-wavelength modes they support. There are a number of reviews on the dynamics of vortices in four dimensions, mostly in the context of cosmic strings [142, 143, 144].

3.1 The Basics

In order for our theory to support vortices, we must add a further field to our Lagrangian. In fact we must make two deformations

- We increase the gauge group from $SU(N)$ to $U(N)$. We could have done this before now, but as we have considered only fields in the adjoint representation the central $U(1)$ would have simply decoupled.
- We add matter in the fundamental representation of $U(N)$. We'll add N_f scalar fields q_i, $i = 1\ldots, N_f$.

The action that we'll work with throughout this lecture is

$$S = \int d^4x \ \mathrm{Tr}\left(\frac{1}{2e^2}F^{\mu\nu}F_{\mu\nu} + \frac{1}{e^2}(\mathcal{D}_\mu\phi)^2\right) + \sum_{i=1}^{N_f}|\mathcal{D}_\mu q_i|^2$$
$$- \sum_{i=1}^{N_f} q_i^\dagger \phi^2 q_i - \frac{e^2}{4}\,\mathrm{Tr}\,(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2\, 1_N)^2 \qquad (3.1)$$

The potential is of the type admitting a completion to $\mathcal{N}=1$ or $\mathcal{N}=2$ supersymmetry. In this context, the final term is called the D-term. Note that everything in the bracket of the D-term is an $N \times N$ matrix. Note also that the couplings in front of the potential are not arbitrary: they have been tuned to critical values.

We've included a new parameter, v^2, in the potential. Obviously this will induce a vev for q. In the context of supersymmetric gauge theories, this parameter is known as a Fayet-Iliopoulos term.

We are interested in ground states of the theory with vanishing potential. For $N_f < N$, one can't set the D-term to zero since the first term is, at most, rank N_f, while the v^2 term is rank N. In the context of supersymmetric theories, this leads to spontaneous supersymmetry breaking. In what follows we'll only consider $N_f \geq N$. In fact, for the first half of this section we'll restrict ourselves to the simplest case:

$$N_f = N \tag{3.2}$$

With this choice, we can view q as an $N \times N$ matrix $q^a{}_i$, where a is the color index and i the flavor index. Up to gauge transformations, there is a unique ground state of the theory,

$$\phi = 0 \quad , \quad q^a{}_i = v\delta^a{}_i \tag{3.3}$$

Studying small fluctuations around this vacuum, we find that all gauge fields and scalars are massive, and all have the same mass $M^2 = e^2v^2$. The fact that all masses are equal is a consequence of tuning the coefficients of the potential.

The theory has a $U(N)_G \times SU(N)_F$ gauge and flavor symmetry. On the quark fields q this acts as

$$q \to UqV^\dagger \qquad U \in U(N)_G, \quad V \in SU(N)_F \tag{3.4}$$

The vacuum expectation value (3.3) is preserved only for transformations of the form $U = V$, meaning that we have the pattern of spontaneous symmetry breaking

$$U(N)_G \times SU(N)_F \to SU(N)_{\rm diag} \tag{3.5}$$

This is known as the color-flavor-locked phase in the high-density QCD literature [145].

When $N = 1$, our theory is the well-studied abelian Higgs model, which has been known for many years to support vortex strings [146, 147]. These vortex strings also exist in the non-abelian theory and enjoy rather rich properties, as we shall now see. Let's choose the strings to lie in the x^3 direction. To support such objects, the scalar fields q must wind around $\mathbf{S}^1_\infty$ at spatial infinity in the (x^1, x^2) plane, transverse to the string. As we're used to by now, such winding is characterized by the homotopy group, this time

$$\Pi_1\left(U(N) \times SU(N)/SU(N)_{\rm diag}\right) \cong \mathbf{Z} \tag{3.6}$$

which means that we can expect vortex strings supported by a single winding number $k \in \mathbf{Z}$. To see that this winding of the scalar is associated with magnetic flux, we use the same trick as for monopoles. Finiteness of the quark kinetic term requires that $\mathcal{D}q \sim 1/r^2$ as $r \to \infty$. But a winding around $\mathbf{S}^1_\infty$ necessarily means that $\partial q \sim 1/r$. To cancel this, we must turn on $A \to i\partial q\, q^{-1}$ asymptotically. The winding of the scalar at infinity is determined by an integer k, defined by

$$2\pi k = \text{Tr} \oint_{\mathbf{S}^1_\infty} i\partial_\theta q\; q^{-1} = \text{Tr} \oint_{\mathbf{S}^1_\infty} A_\theta = \text{Tr} \int dx^1 dx^2\; B_3 \tag{3.7}$$

This time however, in contrast to the case of magnetic monopoles, there is no long range magnetic flux. Physically this is because the theory has a mass gap, ensuring any excitations die exponentially. The result, as we shall see, is that the magnetic flux is confined in the center of the vortex string.

The Lagrangian of equation (3.1) is very special, and far from the only theory admitting vortex solutions. Indeed, the vortex zoo is well populated with different objects, many exhibiting curious properties. Particularly interesting examples include Alice strings [148, 149], and vortices in Chern-Simons theories [150]. In this lecture we shall stick with the vortices arising from (3.1) since, as we shall see, they are closely related to the instantons and monopoles described in the previous lectures. To my knowledge, the properties of non-abelian vortices in this model were studied only quite recently in [151] (a related model, sharing similar properties, appeared at the same time [152]).

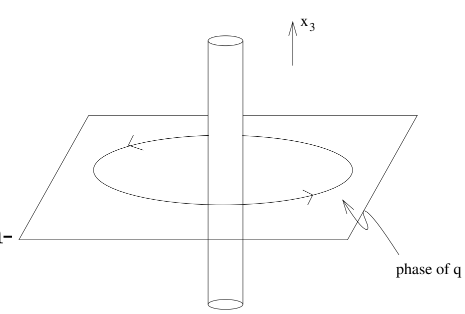

Figure 11:

3.2 The Vortex Equations

To derive the vortex equations we once again perform the Bogomoln'yi completing the square trick (due, once again, to Bogomoln'yi [14]). We look for static strings in the x^3 direction, so make the ansatz $\partial_0 = \partial_3 = 0$ and $A_0 = A_3 = 0$. We also set $\phi = 0$. In fact ϕ will not play a role for the remainder of this lecture, although it will be resurrected in the following lecture. The tension (energy per unit length) of the string is

$$T_{\rm vortex} = \int dx^1 dx^2 \;{\rm Tr}\, \left(\frac{1}{e^2}B_3^2 + \frac{e^2}{4}(\sum_{i=1}^N q_i q_i^\dagger - v^2\, 1_N)^2\right) + \sum_{i=1}^N |\mathcal{D}_1 q_i|^2 + |\mathcal{D}_2 q_i|^2$$

$$= \int dx^1 dx^2 \;\frac{1}{e^2}{\rm Tr}\, \left(B_3 \mp \frac{e^2}{2}(\sum_{i=1}^N q_i q_i^\dagger - v^2\, 1_N)\right)^2 + \sum_{i=1}^N |\mathcal{D}_1 q_i \mp i\mathcal{D}_2 q_i|^2$$

$$\mp v^2 \int dx^1 dx^2 \;{\rm Tr}\, B_3 \tag{3.8}$$

To get from the first line to the second, we need to use the fact that $[D_1, D_2] = -iB_3$, to cancel the cross terms from the two squares. Using (3.7), we find that the tension of the charge $|k|$ vortex is bounded by

$$T_{\rm vortex} \geq 2\pi v^2\, |k| \tag{3.9}$$

In what follows we focus on vortex solutions with winding $k < 0$. (These are mapped into $k > 0$ vortices by a parity transformation, so there is no loss of generality). The inequality is then saturated for configurations obeying the vortex equations

$$B_3 = \frac{e^2}{2}(\sum_i q_i q_i^\dagger - v^2\, 1_N) \quad , \quad \mathcal{D}_z q_i = 0 \tag{3.10}$$

where we've introduced the complex coordinate $z = x^1 + ix^2$ on the plane transverse to the vortex string, so $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$. If we choose $N = 1$, then the Lagrangian (3.1) reduces to the abelian-Higgs model and, until recently, attention mostly focussed on this abelian variety of the equations (3.10). However, as we shall see below, when the vortex equations are non-abelian, so each side of the first equation (3.10) is an $N \times N$ matrix, they have a much more interesting structure.

Unlike monopoles and instantons, no analytic solution to the vortex equations is known. This is true even for a single $k = 1$ vortex in the $U(1)$ theory. There's nothing sinister about this. It's just that differential equations are hard and no one has decided to call the vortex solution a special function and give it a name! However, it's not difficult to plot the solution numerically and the profile of the fields is sketched below. The energy density is localized within a core of the vortex of size $L = 1/ev$, outside of which all fields return exponentially to their vacuum.

The simplest $k = 1$ vortex in the abelian $N = 1$ theory has just two collective coordinates, corresponding to its position on the z-plane. But what are the collective coordinates of a vortex in $U(N)$? We can use the same idea we saw in the instanton

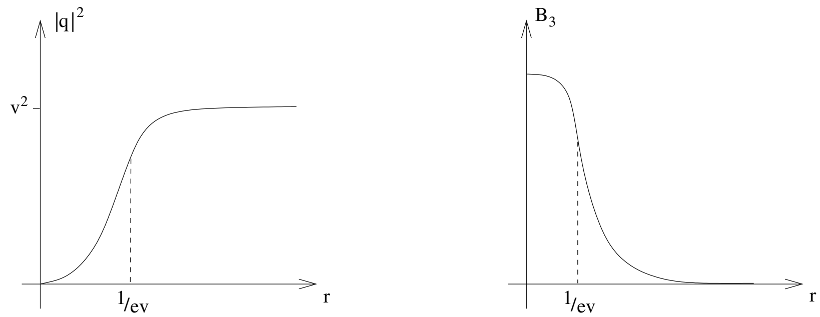

Figure 12: A sketch of the vortex profile.

lecture, and embed the abelian vortex — let's denote it $q^\star$ and $A_z^\star$ — in the $N \times N$ matrices of the non-abelian theory. We have

$$A_z = \begin{pmatrix} A_z^\star & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \quad , \qquad q = \begin{pmatrix} q^\star & & & \\ & v & & \\ & & \ddots & \\ & & & v \end{pmatrix} \tag{3.11}$$

where the columns of the q matrix carry the color charge, while the rows carry the flavor charge. We have chosen the embedding above to lie in the upper left-hand corner but this isn't unique. We can rotate into other embeddings by acting with the $SU(N)_{\rm diag}$ symmetry preserved in the vacuum. Dividing by the stabilizer, we find the internal moduli space of the single non-abelian vortex to be

$$SU(N)_{\rm diag}/S[U(N-1) \times U(1)] \cong \mathbb{CP}^{N-1} \tag{3.12}$$

The appearance of $\mathbb{CP}^{N-1}$ as the internal space of the vortex is interesting: it tells us that the low-energy dynamics of a vortex string is the much studied quantum $\mathbb{CP}^{N-1}$ sigma model. We'll see the significance of this in the following lecture. For now, let's look more closely at the moduli of the vortices.

3.3 The Moduli Space

We've seen that a single vortex has $2N$ collective coordinates: 2 translations, and $2(N-1)$ internal modes, dictating the orientation of the vortex in color and flavor space. We denote the moduli space of charge k vortices in the $U(N)$ gauge theory as $\mathcal{V}_{k,N}$. We've learnt above that

$$\mathcal{V}_{1,N} \cong \mathbf{C} \times \mathbb{CP}^{N-1} \tag{3.13}$$

What about higher k? An index theorem [154, 151] tells us that the number of collective coordinates is

$$\dim(\mathcal{V}_{k,N}) = 2kN \tag{3.14}$$

Look familiar? Remember the result for k instantons in $U(N)$ that we found in lecture 1: $\dim(\mathcal{I}_{k,N}) = 4kN$. We'll see more of this similarity between instantons and vortices in the following.

As for previous solitons, the counting (3.14) has a natural interpretation: k parallel vortex strings may be placed at arbitrary positions, each carrying $2(N-1)$ independent orientational modes. Thinking physically in terms of forces between vortices, this is a consequence of tuning the coefficient $e^2/4$ in front of the D-term in (3.1) so that the mass of the gauge bosons equals the mass of the q scalars. If this coupling is turned up, the scalar mass increases and so mediates a force with shorter range than the gauge bosons, causing the vortices to repel. (Recall the general rule: spin 0 particles give rise to attractive forces; spin 1 repulsive). This is a type II non-abelian superconductor. If the coupling decreases, the mass of the scalar decreases and the vortices attract. This is a non-abelian type I superconductor. In the following, we keep with the critically coupled case (3.1) for which the first order equations (3.10) yield solutions with vortices at arbitrary position.

3.3.1 The Moduli Space Metric

There is again a natural metric on $\mathcal{V}_{k,N}$ arising from taking the overlap of zero modes. These zero modes must solve the linearized vortex equations together with a suitable background gauge fixing condition. The linearized vortex equations read

$$\mathcal{D}_z\delta A_{\bar{z}} - \mathcal{D}_{\bar{z}}\delta A_z = \frac{ie^2}{4}(\delta q\, q^\dagger + q\,\delta q^\dagger) \quad \text{and} \quad \mathcal{D}_z\delta q = i\delta A_z q \tag{3.15}$$

where q is to be viewed as an $N\times N$ matrix in these equations. The gauge fixing condition is

$$\mathcal{D}_z\delta A_{\bar{z}} + \mathcal{D}_{\bar{z}}\delta A_z = -\frac{ie^2}{4}(\delta q\, q^\dagger - q\,\delta q^\dagger) \tag{3.16}$$

which combines with the first equation in (3.15) to give

$$\mathcal{D}_{\bar{z}}\delta A_z = -\frac{ie^2}{4}\delta q\, q^\dagger \tag{3.17}$$

Then, from the index theorem, we know that there are $2kN$ zero modes $(\delta_\alpha A_z, \delta_\alpha q)$, $\alpha,\beta = 1,\ldots,2kN$ solving these equations, providing a metric on $\mathcal{V}_{k,N}$ defined by

$$g_{\alpha\beta} = \mathrm{Tr}\int dx^1dx^2\ \frac{1}{e^2}\delta_\alpha A_a\delta_\beta A_{\bar{z}} + \frac{1}{2}\delta_\alpha q\delta_\beta q^\dagger\ +\ \text{h.c.} \tag{3.18}$$

The metric has the following properties [155, 156]

- The metric is Kähler. This follows from similar arguments to those given for hyperKählerity of the instanton moduli space, the complex structure now descending from that on the plane $\mathbf{R}^2$, together with the obvious complex structure on q.
- The metric is smooth. It has no singularities as the vortices approach each other. Strictly speaking this statement has been proven only for abelian vortices. For non-abelian vortices, we shall show this using branes in the following section.
- The metric inherits a $U(1) \times SU(N)$ holomorphic isometry from the rotational and internal symmetry of the Lagrangian.
- The metric is unknown for $k \geq 2$. The leading order, exponentially suppressed, corrections to the flat metric were computed recently [157].

3.3.2 Examples of Vortex Moduli Spaces

A Single $U(N)$ Vortex

We've already seen above that the moduli space for a single $k = 1$ vortex in $U(N)$ is

$$\mathcal{V}_{1,N} \cong \mathbf{C} \times \mathbb{CP}^{N-1} \tag{3.19}$$

where the isometry group $SU(N)$ ensures that $\mathbb{CP}^{N-1}$ is endowed with the round, Fubini-Study metric. The only question remaining is the size, or Kähler class, of the $\mathbb{CP}^{N-1}$. This can be computed either from a D-brane construction [151] or, more conventionally, from the overlap of zero modes [158]. We'll see the former in the following section. Here let's sketch the latter. The orientational zero modes of the vortex take the form

$$\delta A_z = \mathcal{D}_z \Omega \quad , \qquad \delta q = i(\Omega q - q\Omega_0) \tag{3.20}$$

where the gauge transformation asymptotes to $\Omega \to \Omega_0$, and Ω_0 is the flavor transformation. The gauge fixing condition requires

$$\mathcal{D}^2 \Omega = \frac{e^2}{2}\{\Omega, qq^\dagger\} - 2qq^\dagger \Omega_0 \tag{3.21}$$

By explicitly computing the overlap of these zero modes, it can be shown that the size of the $\mathbb{CP}^{N-1}$ is

$$r = \frac{4\pi}{e^2} \tag{3.22}$$

This important equation will play a crucial role in the correspondence between 2d sigma models and 4d gauge theories that we'll meet in the following lecture.

Two $U(1)$ Vortices

The moduli space of two vortices in a $U(1)$ gauge theory is topologically

$$\mathcal{V}_{k=2,N=1} \cong \mathbf{C} \times \mathbf{C}/\mathbf{Z}_2 \tag{3.23}$$

where the $\mathbf{Z}_2$ reflects the fact that the two solitons are indistinguishable. Note that the notation we used above actually describes more than the topology of the manifold because, topologically, $\mathbf{C}^k/\mathbf{S}_k \cong \mathbf{C}^k$ (as any polynomial will tell you). So when I write $\mathbf{C}/\mathbf{Z}_2$ in (3.23), I mean that asymptotically the space is endowed with the flat metric on $\mathbf{C}/\mathbf{Z}_2$. Of course, this can't be true closer to the origin since we know the vortex moduli space is complete. The cone must be smooth at the tip, as shown in figure 13. The metric on the cone has been computed numerically [159], no analytic form is known. The deviations from the flat, singular, metric on the cone are exponentially suppressed and parameterized by the size of the vortex $L \sim 1/ev$.

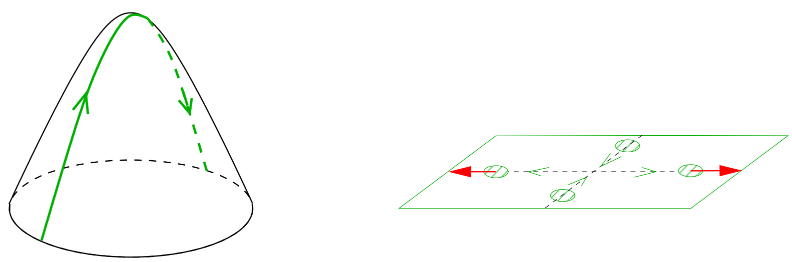

Figure 13: Right-angle scattering from the moduli space of two vortices.

Even without the exact form of the metric, we learn something very important about vortices. Consider two vortices colliding head on. This corresponds to the trajectory in moduli space that goes up and over the tip of the cone, as shown in the figure. What does this correspond to in real space? One might think that the vortices collide and rebound. But that's wrong: it would correspond to the trajectory going to the tip of the cone, and returning down the same side. Instead, the trajectory corresponds to vortices scattering at right angles [160]. The key point is that the $\mathbf{Z}_2$ action in (3.23), arising because the vortices are identical, means that the single valued coordinate on the moduli space is z^2 rather than z, the separation between the vortices. The collision sends $z^2 \rightarrow -z^2$ or $z \rightarrow iz$. This result doesn't depend on the details of the metric on the vortex moduli space, but follows simply from the fact that, near the origin, the space is smooth. Right-angle scattering of this type is characteristic of soliton collisions, occurring also for magnetic monopoles.

For $k \geq 3$ $U(1)$ vortices, the moduli space is topologically and asymptotically $\mathbf{C}^k/\mathbf{Z}_k$. The leading order exponential corrections to the flat metric on this space are known, although the full metric is not [157].

3.4 Brane Construction

For both instantons and monopoles, it was fruitful to examine the solitons from the perspective of D-branes. This allowed us to re-derive the ADHM and Nahm constructions respectively. What about for vortices? Here we present a D-brane constuction of vortices [151] that will reveal interesting information about the moduli space of solutions although, ultimately, won't be as powerful as the ADHM and Nahm constructions described in previous sections.

We use the brane set-ups of Hanany and Witten [91], consisting of D-branes suspended between a pair of NS5-branes. We work in type IIA string theory, and build the $d = 3{+}1$, $U(N)$ gauge theory[6] with $\mathcal{N} = 2$ supersymmetry. The D-brane set-up is shown in figure 14, and consists of N D4-branes with worldvolume 01236, stretched between two NS5-branes, each with worldvolume 012345, and separated in the x^6 direction. The gauge coupling e^2 is determined by the separation between the NS5-branes,

$$\frac{1}{e^2} = \frac{\Delta x^6\, l_s}{2g_s} \tag{3.24}$$

where l_s is the string length, and g_s the string coupling.

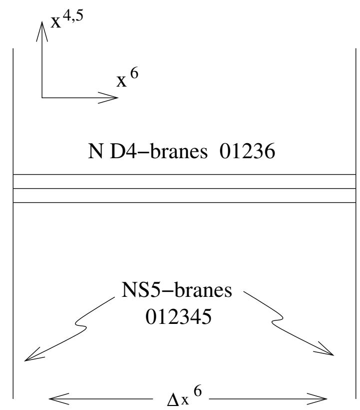

Figure 14:

The D4-branes may slide up and down between the NS5-branes in the x^4 and x^5 direction. This corresponds to turning on a vev for the complex adjoint scalar in the $\mathcal{N} = 2$ vector multiplet. Since we consider only a real adjoint scalar ϕ in our theory, we have

$$\phi_a = \left.\frac{x^4}{l_s^2}\right|_{D4_a} \tag{3.25}$$

and we'll take all D4-branes to lie coincident in the x^5 direction.

The hypermultiplets arise in the form of N D6-branes with worldvolume 0123789. The positions of the D6-branes in the $x^4 + ix^5$ directions will correspond to complex masses for the hypermultiplets. We shall consider these in the following section, but for now we set all D6-branes to lie at the origin of the x^4 and x^5 plane.

[6]In fact, for four-dimensional theories the overall $U(1)$ decouples in the brane set-up, and we have only $SU(N)$ gauge theory [161]. This doesn't affect our study of the vortex moduli space; if you're bothered by this, simply T-dualize the problem to type IIB where you can study vortices in $d = 2 + 1$ dimensions.

We also need to turn on the FI parameter v^2. This is achieved by taking the right-hand NS5-brane and pulling it out of the page in the x^9 direction. In order to remain in the ground state, the D4-branes are not allowed to tilt into the x^9 direction: this would break supersymmetry and increase their length, reflecting a corresponding increase in the ground state energy of the theory. Instead, they must split on the D6-branes. Something known as the S-rule [91, 162] tells us that only one D4-brane can end on a given D6-brane while preserving supersymmetry, ensuring that we need at least N D6-branes to find a zero-energy ground state. The final configuration is drawn in the figure 16, with the field theory dictionary given by

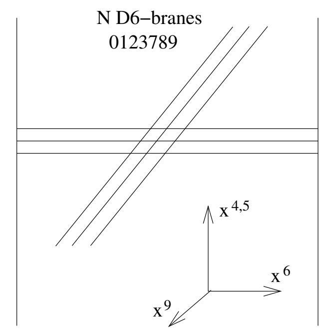

Figure 15:

$$v^2 = \frac{\Delta x^9}{(2\pi)^3 g_s l_s^3} \tag{3.26}$$

Now we've built our theory, we can look to find the vortices. We expect them to appear as other D-branes in the configuration. There is a unique BPS D-brane with the correct mass: it is a D2-brane, lying coincident with the D6-branes, with worldvolume 039, as shown in figure 16 [163]. The x^3 direction here is the direction of the vortex string.

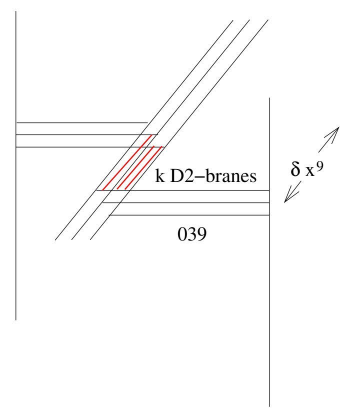

Figure 16:

The problem is: what is the worldvolume theory on the D2-branes. It's hard to read off the theory directly because of the boundary conditions where the D2-branes end on the D4-branes. But, already by inspection, we might expect that it's related to the Dp-D$(p-4)$ system described in Lecture 1 in the context of instantons. To make progress we play some brane games. Move the D6-branes to the right. As they pass the NS5-brane, the Hanany-Witten transition occurs and the right-hand D4-branes disappear [91]. We get the configuration shown in figure 17.

Let's keep the D6-branes moving. Off to infinity. Finally, we rotate our perspective a little, viewing the D-branes from a different angle, shown in figure 18. This is our final D-brane configuration and we can now read off the dynamics.

We want to determine the theory on the D2-branes in figure 18. Let's start with the easier problem in figure 19. Here the D4-branes extend to infinity in both $x^6 \to \pm\infty$ directions, and the D2-branes end on the other NS5. The theory on the D2-branes

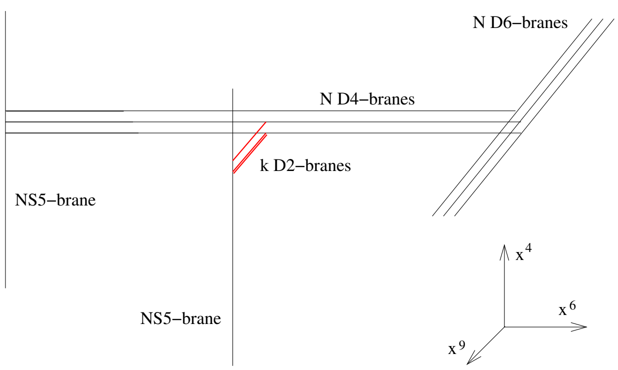

Figure 17: Moving the D6-branes

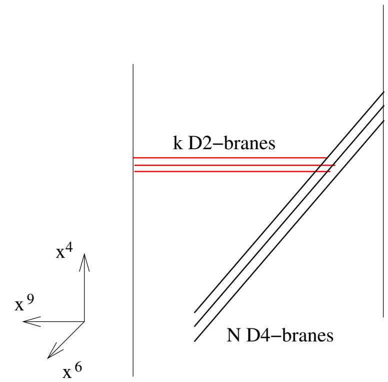

Figure 18: Rotating our viewpoint

is simple to determine: it is a $U(k)$ gauge theory with 4 real adjoint scalars, or two complex scalars

$$\sigma = X^4 + iX^5 \quad , \qquad Z = X^1 + iX^2 \tag{3.27}$$

which combine to give the $\mathcal{N} = (4,4)$ theory in $d = 1+1$. The D4-branes contribute hypermultiplets $(\psi_a, \tilde{\psi}_a)$ with $a = 1, \ldots, N$. These hypermultiplets get a mass only when the D2-branes and D4-branes are separated in the X^4 and X^5 directions. This means we have a coupling like

$$\sum_{a=1}^{N} \psi_a^\dagger \{\sigma^\dagger, \sigma\} \psi_a + \tilde{\psi}_a \{\sigma^\dagger, \sigma\} \tilde{\psi}_a^\dagger \tag{3.28}$$

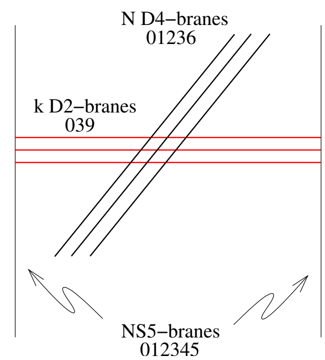

Figure 19:

But there is no such coupling between the hypermultiplets and Z. The coupling (3.28) breaks supersymmetry to $\mathcal{N} = (2,2)$. So we now understand the D2-brane theory of figure 19. However, the D2-brane theory that we're really interested in, shown in figure 18, differs from this in two ways

- The right-hand NS5-brane is moved out of the page. But we already saw in the manoeuvres around figure 16 that this induces a FI parameter on brane theory. Except this this time the FI parameter is for the D2-brane theory. It's given by

$$r = \frac{\Delta x^6}{2\pi g_s l_s} = \frac{4\pi}{e^2} \tag{3.29}$$

- We only have half of the D4-branes, not all of them. If a full D4-brane gives rise to a hypermultiplet, one might guess that half a D4-brane should give rise to half a hypermultiplet, otherwise known as a chiral multiplet. Although the argument is a little glib, it turns out that this is the correct answer [164].

We end up with the gauge theory in $d = 1+1$ dimensions with $\mathcal{N} = (2,2)$ supersymmetry

$$\begin{aligned} U(k) \text{ Gauge Theory } &+ \text{ Adjoint Chiral Multiplet } Z \\ &+ N \text{ Fundamental Chiral Multiplets } \psi_a \end{aligned}$$

This theory has a FI parameter $r = 4\pi/e^2$. Now this should be looking very familiar — it's very similar to the instanton theory we described in Lecture 1. We'll return to this shortly. For now let's keep examining our vortex theory. The potential for the various scalars is dictated by supersymmetry and is given by

$$\begin{aligned} V = &\frac{1}{g^2}\mathrm{Tr}\,|[\sigma,\sigma^\dagger]|^2 + \mathrm{Tr}\,|[\sigma,Z]|^2 + \mathrm{Tr}\,|[\sigma,Z^\dagger]|^2 + \sum_{a=1}^{N} \psi_a^\dagger\sigma^\dagger\sigma\psi_a \\ &+\frac{g^2}{2}\,\mathrm{Tr}\left(\sum_a \psi_a\psi_a^\dagger + [Z,Z^\dagger] - r\,1_k\right)^2 \end{aligned} \tag{3.30}$$

Here g^2 is an auxiliary gauge coupling which we take to infinity $g^2 \to \infty$ to restrict us to the Higgs branch, the vacuum moduli space defined by

$$\mathcal{M}_{\rm Higgs} \cong \{\sigma = 0, V = 0\}/U(k) \tag{3.31}$$

Counting the various degrees of freedom, the Higgs branch has real dimension $2kN$. From the analogy with the instanton case, it is natural to conjecture that this is the vortex moduli space [151]

$$\mathcal{V}_{k,N} \cong \mathcal{M}_{\rm Higgs} \tag{3.32}$$

While the ADHM construction has a field theoretic underpinning, I know of no field theory derivation of the above result for vortices. So what evidence do we have that the Higgs branch indeed coincides with the vortex moduli space? Because of the FI parameter, $\mathcal{M}_{\rm Higgs}$ is a smooth manifold, as is $\mathcal{V}_{k,N}$ and, obviously the dimensions work out. Both spaces have a $SU(N)\times U(1)$ isometry which, in the above construction, act upon ψ and Z respectively. Finally, in all cases we can check, the two spaces agree (as, indeed, do their Kähler classes). Let's look at some examples.

3.4.1 Examples of Vortex Moduli Spaces Revisited

One Vortex in $U(N)$

The gauge theory for a single $k = 1$ vortex in $U(N)$ is a $U(1)$ gauge theory. The adjoint scalar Z decouples, parameterizing the complex plane $\mathbf{C}$, leaving us with the N charged scalars satisfying

$$\sum_{a=1}^{N} |\psi_a|^2 = r \tag{3.33}$$

modulo the $U(1)$ action $\psi_a \to e^{i\alpha}\psi_a$. This gives us the moduli space

$$\mathcal{V}_{1,N} \cong \mathbf{C} \times \mathbb{CP}^{N-1} \tag{3.34}$$

where the $\mathbb{CP}^{N-1}$ has the correct Kähler class $r = 4\pi/e^2$ in agreement with (3.22). The metric on $\mathbb{CP}^{N-1}$ is, again, the round Fubini-Study metric.

k Vortices in $U(1)$

The Higgs branch corresponding to the k vortex moduli space is

$$\{\psi\psi^\dagger + [Z, Z^\dagger] = r\, 1_k\}/U(k) \tag{3.35}$$

which is asymptotic to the cone $\mathbf{C}^k/\mathbf{Z}_k$, with the singularities resolved. This is in agreement with the vortex moduli space. *However*, the metric on $\mathcal{M}_{\rm Higgs}$ differs by power law corrections from the flat metric on the orbifold $\mathbf{C}^k/\mathbf{Z}_k$. But, as we've discussed, $\mathcal{V}_{k,N}$ differs from the flat metric by exponential corrections.

More recently, the moduli space of two vortices in $U(N)$ was studied in some detail and shown to possess interesting and non-trivial topology [165], with certain expected features of $\mathcal{V}_{2,N}$ reproduced by the Higgs branch.

In summary, it is conjectured that the vortex moduli space $\mathcal{V}_{k,N}$ is isomorphic to the Higgs branch (3.34). But, except for the case $k = 1$ where the metric is determined by the isometry, the metrics do not agree. A direct field theory proof of this correspondence remains to be found.

3.4.2 The Relationship to Instantons

As we've mentioned a few times, the vortex theory bears a striking resemblance to the ADHM instanton theory we met in Lecture 1. In fact, the gauge theoretic construction of vortex moduli space $\mathcal{V}_{k,N}$ involves exactly half the fields of the ADHM construction. Or, put another way, the vortex moduli space is half of the instanton moduli space. We can state this more precisely: $\mathcal{V}_{k,N}$ is a complex, middle dimensional submanifold of $\mathcal{I}_{k,N}$. It can be defined by looking at the action of the isometry rotating the instantons in the $x^3 - x^4$ plane. Denote the corresponding Killing vector as h. Then

$$\mathcal{V}_{k,N} \cong \mathcal{I}_{k,N}|_{h=0} \tag{3.36}$$

where $\mathcal{I}_{k,N}$ is the resolved instanton moduli space with non-commutativity parameter $\theta_{\mu\nu} = r\bar{\eta}^3_{\mu\nu}$. We'll see a physical reason for this relationship shortly.

An open question: The ADHM construction is constructive. As we have seen, it allows us to build solutions to $F = {}^\star F$ from the variables of the Higgs branch. Does a similar construction exist for vortices?

Relationships between the instanton and vortex equations have been noted in the past. In particular, a twisted reduction of instantons in $SU(2)$ Yang-Mills on $\mathbf{R}^2 \times \mathbf{S}^2$ gives rise to the $U(1)$ vortex equations [166]. While this relationship appears to share several characteristics to the correspondence described above, it differs in many important details. It don't understand the relationship between the two approaches.

3.5 Adding Flavors

Let's now look at vortices in a $U(N_c)$ gauge theory with $N_f \geq N_c$ flavors. Note that we've added subscripts to denote color and flavor. In theories with $N_c = 1$ and $N_f > 1$, these were called semi-local vortices [167, 168, 169, 170]. The name derives from the fact the theory has both a gauge (local) group and a flavor (global) group. But for us, it's not a great name as all our theories have both types of symmetries, but it's only when $N_f > N_c$ that the extra properties of "semi-local" vortices become apparent.

The Lagrangian (3.1) remains but, unlike before, the theory no longer has a mass gap in vacuum. Instead there are N_c^2 massive scalar fields and scalars, and $2N_c(N_f - N_c)$ massless scalars. At low-energies, the theory reduces to a σ-model on the Higgs branch of the gauge theory (3.1),

$$\mathcal{M}_{\rm Higgs} \cong \{\sum_{i=1}^{N_f} q_i q_i^\dagger = v^2\, 1_{N_c}\}/U(N_c) \cong G(N_c, N_f) \tag{3.37}$$

When we have an abelian $N_c = 1$ theory, this Higgs branch is the projective space $G(1, N_f) \cong \mathbb{CP}^{N_f - 1}$. For non-abelian theories, the Higgs branch is the Grassmannian $G(N_c, N_f)$, the space of $\mathbf{C}^{N_c}$ planes in $\mathbf{C}^{N_f}$. In a given vacuum, the symmetry breaking pattern is $U(N_c) \times SU(N_f) \rightarrow S[U(N_c) \times U(N_f - N_c)]$.

The first order vortex equations (3.10) still give solutions to the full Lagrangian, now with the flavor index running over values $i = 1 \ldots, N_f$. Let's denote the corresponding vortex moduli space as $\hat{\mathcal{V}}_{k,N_c,N_f}$, so our previous notation becomes $\mathcal{V}_{k,N} \cong \hat{\mathcal{V}}_{k,N,N}$. The index theorem now tells us the dimension of the vortex moduli space

$$\dim(\hat{\mathcal{V}}_{k,N_c,N_f}) = 2kN_f \tag{3.38}$$

The dimension depends only on the number of flavors, and the semi-local vortices inherit new modes. These modes are related to scaling modes of the vortex — the size of the vortex becomes a parameter, just as it was for instantons [171].

These vortices arising in the theory with extra flavors are related to other solitons, known as a sigma-model lumps. (These solitons have other names, depending on the context, sometimes referred to as "textures", "Skyrmions" or, in the context of string theory, "worldsheet instantons"). Let's see how this works. At low-energies (or, equivalently, in the strong coupling limit $e^2 \to \infty$) our gauge theory flows to the sigma-model on the Higgs branch $\mathcal{M}_{\rm Higgs} \cong G(N_c, N_f)$. In this limit our vortices descend to lumps, objects which gain their topological support once we compactify the $(x^1 - x^2)$-plane at infinity, and wrap this sphere around $\mathcal{M}_{\rm Higgs} \cong G(N_c, N_f)$ [172, 173]

$$\Pi_2(G(N_c, N_f)) \cong \mathbf{Z} \tag{3.39}$$

When $N_f = N_c$ there is no Higgs branch, the vortices have size $L = 1/ev$ and become singular as $e^2 \to \infty$. In contrast, when $N_f > N_c$, the vortices may have arbitrary size and survive the strong coupling limit. However, while the vortex moduli space is smooth, the lump moduli space has singularities, akin to the small instanton singularities we saw in Lecture 1. We see that the gauge coupling $1/e^2$ plays the same role for lumps as θ plays for Yang-Mills instantons.

The brane construction for these vortices is much like the previous section - we just need more $D6$ branes. By performing the same series of manoeuvres, we can deduce the worldvolume theory. It is again a $d = 1 + 1$ dimensional, $\mathcal{N} = (2, 2)$ theory with

$$\begin{aligned} U(k) \text{ Gauge Theory } &+ \text{ Adjoint Chiral Multiplet } Z \\ &+ N_c \text{ Fundamental Chiral Multiplets } \psi_a \\ &+ (N_f - N_c) \text{ Anti-Fundamental Chiral Multiplets } \tilde{\psi}_a \end{aligned}$$

Once more, the FI parameter is $r = 4\pi/e^2$. The D-term constraint of this theory is

$$\sum_{a=1}^{N_c} \psi_a \psi_a^\dagger - \sum_{b=1}^{N_f - N_c} \tilde{\psi}_b^\dagger \tilde{\psi}_b + [Z, Z^\dagger] = r\, 1_k \tag{3.40}$$

A few comments

- Unlike the moduli space $\mathcal{V}_{k,N}$, the presence of the $\tilde{\psi}$ means that this space doesn't collapse as we send $r \to 0$. Instead, in this limit it develops singularities at $\psi = \tilde{\psi} = 0$ where the $U(k)$ gauge group doesn't act freely. This is the manifestation of the discussion above.

- The metric inherited from the D-term (3.40) again doesn't coincide with the metric on the vortex moduli space $\hat{\mathcal{V}}_{k,N_c,N_f}$. In fact, here the discrepancy is more pronounced, since the metric on $\hat{\mathcal{V}}_{k,N_c,N_f}$ has non-normalizable modes: the directions in moduli space corresponding to the scaling the solution are suffer an infra-red logarithmic divergence [174, 171]. The vortex theory arising from branes doesn't capture this.

3.5.1 Non-Commutative Vortices

As for instantons, we can consider vortices on the non-commutative plane

$$[x^1, x^2] = i\vartheta \tag{3.41}$$

These objects were first studied in [175]. How does this affect the moduli space? In the ADHM construction for instantons, we saw that non-commutivity added a FI parameter to the D-term constraints. But, for vortices, we already have a FI parameter: $r = 4\pi/e^2$. It's not hard to show using D-branes [151], that the effect of non-commutivity is to deform,

$$r = \frac{4\pi}{e^2} + 2\pi v^2 \vartheta \tag{3.42}$$

This has some interesting consequences. Note that for $N_f = N_c$, there is a critical FI parameter $\vartheta_c = -v^2/e^2$ for which $r = 0$. At this point the vortex moduli space becomes singular. For $\vartheta < \vartheta_c$, no solutions to the D-term equations exist. Indeed, it can be shown that in this region, no solutions to the vortex equations exist either [176]. We see that the Higgs branch correctly captures the physics of the vortices.

For $N_f > N_c$, the Higgs branch makes an interesting prediction: the vortex moduli space should undergo a topology changing transition as $\vartheta \to \vartheta_c$. For example, in the case of a single $k = 1$ vortex in $U(2)$ with $N_f = 4$, this is the well-known flop transition of the conifold. To my knowledge, no one has confirmed this behavior of the vortex moduli space from field theory. Nor has anyone found a use for it!

3.6 What Became Of.......

Let's now look at what became of the other solitons we studied in the past two lectures.

3.6.1 Monopoles

Well, we've set $\phi = 0$ throughout this lecture and, as we saw, the monopoles live on the vev of ϕ. So we shouldn't be surprised if they don't exist in our theory (3.1). We'll see them reappear in the following section.

3.6.2 Instantons

These are more interesting. Firstly the vev $q \neq 0$ breaks conformal invariance, causing the instantons to collapse. This is the same behavior that we saw in Section 2.6. But recall that in the middle of the vortex string, $q \to 0$. So maybe it's possible for the instanton to live inside the vortex string, where the non-abelian gauge symmetry is restored. To see that this can indeed occur, we can look at the worldsheet of the vortex string. As we've seen, the low-energy dynamics for a single string is

$U(1)$ with N charged chiral multiplets and FI parameter $r = 4\pi/e^2$

But this falls into the class of theories we discussed in section 3.5. So if the worldsheet is Euclidean, the theory on the vortex string itself admits a vortex solution: a vortex in a vortex. The action of this vortex is [177]

$$S_{\text{vortex in vortex}} = 2\pi r = \frac{8\pi^2}{e^2} = S_{\text{inst}} \tag{3.43}$$

which is precisely the action of the Yang-Mills instanton. Such a vortex has $2N$ zero modes which include scaling modes but, as we mentioned previously, not all are normalizable.

There is also a 4d story for these instantons buried in the vortex string. This arises by completing the square in the Lagrangian in a different way to (3.8). We still set $\phi = 0$, but now allow for all fields to vary in all four dimensions [177]. We write $z = x^1 + ix^2$ and $w = x^3 - ix^4$,

$$\begin{aligned} S &= \int d^4x \; \frac{1}{2e^2}\text{Tr}\; F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^{N_f} |\mathcal{D}_\mu q_i|^2 + \frac{e^2}{4}\text{Tr}(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2\,\mathbf{1}_{N_c})^2 \\ &= \int d^4x \; \frac{1}{2e^2}\text{Tr}\;\left(F_{12} - F_{34} - \frac{e^2}{2}(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2\,\mathbf{1}_{N_c})\right)^2 \\ &\quad + \sum_{i=1}^{N_f} |\mathcal{D}_z q_i|^2 + |\mathcal{D}_\omega q_i|^2 + \frac{1}{e^2}\text{Tr}\left((F_{14} - F_{23})^2 + (F_{13} + F_{24})^2\right) \\ &\quad + \frac{1}{e^2}\text{Tr}\,F_{\mu\nu}{}^\star F^{\mu\nu} + F_{12}v^2 + F_{34}v^2 \\ &\geq \int d^4x \; \frac{1}{e^2}\text{Tr}\,F_{\mu\nu}{}^\star F^{\mu\nu} + \text{Tr}\,(F_{12}v^2 + F_{34}v^2) \end{aligned} \tag{3.44}$$

The last line includes three topological charges, corresponding to instantons, vortex strings in the $(x^1 - x^2)$ plane, and further vortex strings in the $(x^3 - x^4)$ plane. The

Bogomoln'yi equations describing these composite solutions are

$$F_{14}=F_{23}\quad,\quad F_{13}=F_{24}\quad,\quad F_{12}-F_{34}=\frac{e^2}{2}(\sum_{i=1}^{N_f}q_iq_i^\dagger-v^2\mathbf{1}_{N_c})\quad,\quad \mathcal{D}_zq_i=\mathcal{D}_wq_i=0$$

It is not known if solutions exist, but the previous argument strongly suggests that there should be solutions describing an instanton trapped inside a vortex string. Some properties of this configuration were studied in [178].

The observation that a vortex in the vortex string is a Yang-Mills instanton gives some rationale to the fact that $\mathcal{V}_{k,N}\subset\mathcal{I}_{k,N}$.

3.7 Fermi Zero Modes

In this section, I'd like to describe an important feature of fermionic zero modes on the vortex string: they are chiral. This means that a Weyl fermion in four dimensions will give rise to a purely left-moving (or right-moving) mode on the (anti-) vortex worldsheet. In fact, a similar behavior occurs for instantons and monopoles, but since this is the first lecture where the solitons are string-like in four-dimensions, it makes sense to discuss this phenomenon here.

The exact nature of the fermionic zero modes depends on the fermion content in four dimensions. Let's stick with the supersymmetric generalization of the Lagrangian (3.1). Then we have the gaugino λ, an adjoint valued Weyl fermion which is the superpartner of the gauge field. We also have fermions in the fundamental representation, χ_i with $i=1,\ldots,N$, which are the superpartners of the scalars q_i. These two fermions mix through Yukawa couplings of the form $q_i^\dagger\lambda\chi_i$, and the Dirac equations read

$$-i\,\bar{\not{\mathcal{D}}}\lambda+i\sqrt{2}\sum_{i=1}^{N}q_i\bar{\chi}_i=0\quad\text{ and }\quad -i\,\not{\mathcal{D}}\bar{\chi}_i-i\sqrt{2}q_i^\dagger\lambda=0 \tag{3.45}$$

where the Dirac operators take the form,

$$\not{\mathcal{D}}\equiv\sigma^\mu\mathcal{D}_\mu=\begin{pmatrix}\mathcal{D}_+ & \mathcal{D}_z\\ \mathcal{D}_{\bar z} & \mathcal{D}_-\end{pmatrix}\quad\text{ and }\quad\bar{\not{\mathcal{D}}}\equiv\bar{\sigma}^\mu\mathcal{D}_\mu=\begin{pmatrix}\mathcal{D}_- & -\mathcal{D}_z\\ -\mathcal{D}_{\bar z} & \mathcal{D}_+\end{pmatrix} \tag{3.46}$$

which, as we can see, nicely split into $\mathcal{D}_\pm=\mathcal{D}_0\pm\mathcal{D}_3$ and $\mathcal{D}_z=\mathcal{D}_1-i\mathcal{D}_2$ and $\mathcal{D}_{\bar z}=\mathcal{D}_1+i\mathcal{D}_2$. The bosonic fields in (3.45) are evaluated on the vortex solution which, crucially, includes $\mathcal{D}_zq_i=0$ for the vortex (or $\mathcal{D}_{\bar z}q_i=0$ for the anti-vortex). We see the importance of this if we take the first equation in (3.45) and hit it with $\not{\mathcal{D}}$, while hitting the second equation with $\bar{\not{\mathcal{D}}}$. In each equation terms of the form $\mathcal{D}_zq_i$ will appear, and subsequently vanish as we evaluate them on the vortex background. Let's do the calculation. We split up the spinors into their components λ_α and $(\chi_\alpha)_i$ with $\alpha = 1,2$ and, for now, look for zero modes that don't propagate along the string, so $\partial_+ = \partial_- = 0$. Then the Dirac equations in component form become

$$(-\mathcal{D}_z\mathcal{D}_{\bar z} + 2q_iq_i^\dagger)\lambda_1 = 0 \quad \text{and} \quad (-\mathcal{D}_{\bar z}\mathcal{D}_z + 2q_iq_i^\dagger)\lambda_2 - \sqrt{2}(\mathcal{D}_{\bar z}q_i)\bar\chi_{1i} = 0$$
$$(-\mathcal{D}_{\bar z}\mathcal{D}_z\delta^j{}_i + 2q_i^\dagger q_j)\bar\chi_{2j} = 0 \quad \text{and} \quad (-\mathcal{D}_z\mathcal{D}_{\bar z}\delta^j{}_i + 2q_iq_j^\dagger)\bar\chi_{1j} - \sqrt{2}(\mathcal{D}_zq_i^\dagger)\lambda_2 = 0$$

The key point is that the operators appearing in the first column are positive definite, ensuring that λ_1 and χ_{2i} have no zero modes. In contrast, the equations for λ_2 and $\bar\chi_{1i}$ do have zero modes, guaranteed by the index. We therefore know that any zero modes of the vortex are of the form,

$$\lambda = \begin{pmatrix} 0 \\ \lambda \end{pmatrix} \quad \text{and} \quad \bar\chi_i = \begin{pmatrix} \bar\chi_i \\ 0 \end{pmatrix} \tag{3.47}$$

If we repeat the analysis for the anti-vortex, we find that the other components turn on. To see the relationship to the chirality on the worldsheet, we now allow the zero modes to propagate along the string, so that $\lambda = \lambda(x^0, x^3)$ and $\bar\chi_i = \bar\chi_i(x^0, x^3)$. Plugging this ansatz back into the Dirac equation, now taking into account the derivatives $\mathcal{D}_\pm$ in (3.46), we find the equations of motion

$$\partial_+\lambda = 0 \quad \text{and} \quad \partial_+\bar\chi_i = 0 \tag{3.48}$$

Or, in other words, $\lambda = \lambda(x_-)$ and $\bar\chi = \bar\chi(x_-)$: both are right movers.

In fact, the four-dimensional theory with only fundamental fermions χ_i is anomalous. Happily, so is the $\mathbb{CP}^{N-1}$ theory on the string with only right-moving fermions, suffering from the sigma-model anomaly [179]. To rectify this, one may add four dimensional Weyl fermions $\tilde\chi_i$ in the anti-fundamental representation, which provide left movers on the worldsheet. If the four-dimensional theory has $\mathcal{N} = 2$ supersymmetry, the worldsheet theory preserves $\mathcal{N} = (2,2)$ [180]. Alternatively, one may work with a chiral, non-anomalous $\mathcal{N} = 1$ theory in four-dimensions, resulting in a chiral non-anomalous $\mathcal{N} = (0,2)$ theory on the worldsheet.

3.8 Applications

Let's now turn to discussion of applications of vortices in various field theoretic contexts. We review some of the roles vortices play as finite action, instanton-like, objects in two dimensions, as particles in three dimensions, and as strings in four dimensions.

3.8.1 Vortices and Mirror Symmetry

Perhaps the most important application of vortices in string theory is in the context of the $d = 1+1$ dimensional theory on the string itself. You might protest that the string worldsheet theory doesn't involve a gauge field, so why would it contain vortices?! The trick, as described by Witten [172], is to view sigma-models in terms of an auxiliary gauge theory known as a *gauged linear sigma model*. We've already met this trick several times in these lectures: the sigma-model target space is the Higgs branch of the gauge theory. Witten showed how to construct gauge theories that have compact Calabi-Yau manifolds as their Higgs branch.

In $d = 1+1$ dimensions, vortices are finite action solutions to the Euclidean equations of motion. In other words, they play the role of instantons in the theory. As we explained Section 3.5 above, the vortices are related to worldsheet instantons wrapping the 2-cycles of the Calabi-Yau Higgs branch. It turns out that it is much easier to deal with vortices than directly with worldsheet instantons (essentially because their moduli space is free from singularities). Indeed, in a beautiful paper, Morrison and Plesser succeeded in summing the contribution of all vortices in the topological A-model on certain Calabi-Yau manifolds, showing that it agreed with the classical prepotential derived from the B-model on the mirror Calabi-Yau [181].

More recently, Hori and Vafa used vortices to give a proof of $\mathcal{N} = (2,2)$ mirror symmetry for all Calabi-Yau which can be realized as complete intersections in toric varieties [135]. Hori and Vafa work with dual variables, performing the so-called Rocek-Verlinde transformation to twisted chiral superfields [182]. They show that vortices contribute to a two fermi correlation function which, in terms of these dual variables, is cooked up by a superpotential. This superpotential then captures the relevant quantum information about the original theory. Similar methods can be used in $\mathcal{N} = (4,4)$ theories to derive the T-duality between NS5-branes and ALE spaces [183, 184, 185, 186], with the instantons providing the necessary ingredient to break translational symmetry after T-duality, leading to localized, rather than smeared, NS5-branes.

3.8.2 Swapping Vortices and Electrons

In lecture 2, we saw that it was possible to rephrase four-dimensional field theories, treating the monopoles as elementary particles instead of solitons. This trick, called electric-magnetic duality, gives key insight into the strong coupling behavior of four-dimensional field theories. In three dimensions, vortices are particle like objects and one can ask the same question: is it possible to rewrite a quantum field theory, treating the vortices as fundamental degrees of freedom?

The answer is yes. In fact, condensed matter theorists have been using this trick for a number of years (see for example [187]). Things can be put on a much more precise footing in the supersymmetric context, with the first examples given by Intriligator and Seiberg [188]. They called this phenomenon "mirror symmetry" in three dimensions as it had some connection to the mirror symmetry of Calabi-Yau manifolds described above.

Let's describe the basic idea. Following Intriligator and Seiberg, we'll work with a theory with eight supercharges (which is $\mathcal{N} = 4$ supersymmetry in three dimensions). Each gauge field comes with three real scalars and four Majorana fermions. The charged matter, which we'll refer to as "electrons", lives in a hypermultiplet, containing two complex scalars together with two Dirac fermions. The theory we start with is:

Theory A: $U(1)$ with N charged hypermultiplets

The vortices in this theory fall into the class described in Section 3.5. Each vortex has $2N$ zero modes but, as we discussed, not all of these zero modes are normalizable. The overall center of mass is, of course, normalizable (the vortex has mass $M = 2\pi v^2$) but the remaining $2(N-1)$ modes of a single vortex are logarithmically divergent.

We now wish to rewrite this theory, treating the vortices as fundamental objects. What properties must the theory have in order to mimic the behavior of the vortex? It will prove useful to think of each vortex as containing N individual "fractional vortices". We postulate that these fractional vortices suffer a logarithmic confining potential, so that any number $n < N$ have a logarithmically divergent mass, but N together form a state with finite mass. Such a system would exhibit the properties of the vortex zero modes described above: the $2N$ zero modes correspond to the positions of the N fractional vortices. They can move happily as a whole, but one pays a logarithmically divergent cost to move these objects individually. (Note: a logarithmically divergent cost isn't really that much!)

In fact, it's very easy to cook up a theory with these properties. In $d = 2+1$, an electron experiences logarithmic confinement, since its electric field goes as $E \sim 1/r$ so its energy $\int d^2x\, E^2$ suffers a logarithmic infra-red divergence. These electrons will be our "fractional vortices". We will introduce N different types of electrons and, in order to assure that only bound states of all N are gauge singlets, we introduce $N-1$ gauge fields with couplings dictated by the quiver diagram shown in the figure. Recall that quiver diagrams are read in the following way: the nodes of the quiver are gauge groups, each giving a $U(1)$ factor in this case. Meanwhile, the links denote hypermultiplets with charge $(+1,-1)$ under the gauge groups to which it is attached. Although there are N nodes in the quiver, the overall $U(1)$ decouples, leaving us with the theory

$$\text{Theory B: } U(1)^{N-1} \text{ with } N \text{ hypermultiplets}$$

This is the Seiberg-Intriligator mirror theory, capturing the same physics as Theory A. The duality also works the other way, with the electrons of Theory A mapping to the vortices of Theory B. It can be shown that the low-energy dynamics of these two theories exactly agree. This statement can be made precise at the two-derivative level. The Higgs branch of Theory A coincides with the Coulomb branch of Theory B: both are $T^\star(\mathbb{CP}^{N-1})$. Similarly, the Coulomb branch of Theory A coincides with the Higgs branch of Theory A: both are the A_{N-1} ALE space.

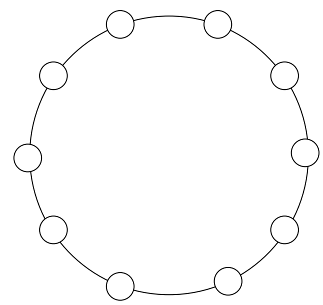

Figure 20:

There are now many mirror pairs of theories known in three dimensions. In particular, it's possible to tinker with the mirror theories so that they actually coincide at all length scales, rather than simply at low-energies [189]. Mirror pairs for non-abelian gauge theories are known, but are somewhat more complicated to due to presence of instanton corrections (which, recall, are monopoles in three dimensions) [190, 191, 192, 193, 194]. Finally, one can find mirror pairs with less supersymmetry [195, 196], including mirrors for interesting Chern-Simons theories [197, 198, 199]. Finite quantum correction to the vortex mass in $\mathcal{N}=2$ theories was described in [200]. The Chern-Simons mirrors reduce to Hori-Vafa duality under compactification to two dimensions [201].

3.8.3 Vortex Strings

In $d=3+1$ dimensions, vortices are string like objects. There is a very interesting story to be told about how we quantize vortex worldsheet theory, which is a sigma-model on $\mathcal{V}_{k,N}$. But this will have to wait for the next lecture.

Here let me mention an application of vortices in the context of cosmic strings which shows that reconnection of vortices in gauge theories is inevitable at low-energies. Reconnection of strings means that they swap partners as they intersect as shown in the figure. In general, it's a difficult problem to determine whether reconnection occurs and requires numerical study. However, at low-energies we may reliably employ the techniques of the moduli space approximation that we learnt above [202, 203, 204].

The first step is to reduce the dynamics of cosmic strings to that of particles by considering one of two spatial slices shown in the figure. The vertical slice cuts the strings to reveal a vortex-anti-vortex pair. After reconnection, this slice no longer intersects the strings, implying the annihilation of this pair. Alternatively, one can slice horizontally to reveal two vortices. Here the smoking gun for reconnection is the right-angle scattering of the vortices at (or near) the interaction point. Such 90^o degree scattering is a requirement since, as is clear from the figure, the two ends of each string are travelling in opposite directions after the collision. By varying the slicing along the string, one can reconstruct the entire dynamics of the two strings in this manner and show the inevitability of reconnection at low-energies.

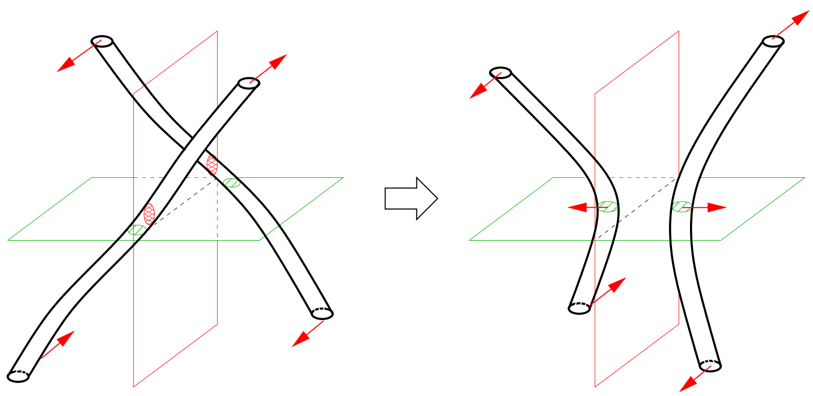

Figure 21: The reconnection cosmic strings. Slicing vertically, one sees a vortex-anti-vortex pair annihilate. Slicing horizontally, one sees two vortices scattering at right angles.

Hence, reconnection of cosmic strings requires both the annihilation of vortex-anti-vortex pairs and the right-angle scattering of two vortices. The former is expected (at least for suitably slow collisions). And we saw in Section 3.3.2 that the latter occurs for abelian vortices in the moduli space approximation. We conclude that abelian cosmic strings do reconnect at low energies. Numerical simulations reveal that these results are robust, holding for very high energy collisions [205].

For cosmic strings in non-abelian theories this result continues to hold, with strings reconnecting except for very finely tuned initial conditions [165]. However, in this case there exist mechanisms to push the strings to these finely tuned conditions, resulting in a probability for reconnection less than 1.

Recently, there has been renewed interest in the reconnection of cosmic strings, with the realization that cosmic strings may be fundamental strings, stretched across the sky

[206]. These objects differ from abelian cosmic strings as they have a reduced probability of reconnection, proportional to the string coupling g_s^2 [207, 208]. If cosmic strings are ever discovered, it may be possible to determine their probability of reconnection, giving a vital clue to their microscopic origin. The recent developments of this story have been nicely summarized in the review [144].

4. Domain Walls

So far we've considered co-dimension 4 instantons, co-dimension 3 monopoles and co-dimension 2 vortices. We now come to co-dimension 1 domain walls, or kinks as they're also known. While BPS domain walls exist in many supersymmetric theories (for example, in Wess-Zumino models [209]), there exists a special class of domain walls that live in gauge theories with 8 supercharges. They were first studied by Abraham and Townsend [210] and have rather special properties. These will be the focus of this lecture. As we shall explain below, the features of these domain walls are inherited from the other solitons we've met, most notably the monopoles.

4.1 The Basics

To find domain walls, we need to deform our theory one last time. We add masses m_i for the fundamental scalars q_i. Our Lagrangian is that of a $U(N_c)$ gauge theory, coupled to a real adjoint scalar field ϕ and N_f fundamental scalars q_i

$$S = \int d^4x \ \mathrm{Tr}\left(\frac{1}{2e^2}F^{\mu\nu}F_{\mu\nu} + \frac{1}{e^2}(\mathcal{D}_\mu\phi)^2\right) + \sum_{i=1}^{N_f}|\mathcal{D}_\mu q_i|^2 - \sum_{i=1}^{N_f} q_i^\dagger(\phi - m_i)^2 q_i - \frac{e^2}{4}\,\mathrm{Tr}\,(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2\, 1_N)^2 \tag{4.1}$$

Notice the way the masses mix with ϕ, so that the true mass of each scalar is $|\phi - m_i|$. Adding masses in this way is consistent with $\mathcal{N} = 2$ supersymmetry. We'll pick all masses to be distinct and, without loss of generality, choose

$$m_i < m_{i+1} \tag{4.2}$$

As in Lecture 3, there are vacua with $V = 0$ only if $N_f \geq N_c$. The novelty here is that, for $N_f > N_c$, we have multiple isolated vacua. Each vacuum is determined by a choice of N_c distinct elements from a set of N_f

$$\Xi = \{\xi(a) :\ \xi(a) \neq \xi(b) \text{ for } a \neq b\} \tag{4.3}$$

where $a = 1, \ldots, N_c$ runs over the color index, and $\xi(a) \in \{1, \ldots, N_f\}$. Let's set $\xi(a) < \xi(a+1)$. Then, up to a Weyl transformation, we can set the first term in the potential to vanish by

$$\phi = \mathrm{diag}(m_{\xi(1)}, \ldots, m_{\xi(N_c)}) \tag{4.4}$$

This allows us to turn on the particular components ${q^a}_i \sim {\delta^a}_{i=\xi(a)}$ without increasing the energy. To cancel the second term in the potential, we require

$$ {q^a}_i = v{\delta^a}_{i=\xi(a)} \tag{4.5} $$

The number of vacua of this type is

$$ N_{\rm vac} = \begin{pmatrix} N_f \\ N_c \end{pmatrix} = \frac{N_f!}{N_c!(N_f - N_c)!} \tag{4.6} $$

Each vacuum has a mass gap in which there are N_c^2 gauge bosons with $M_\gamma^2 = e^2v^2 + |m_{\xi(a)} - m_{\xi(b)}|^2$, and $N_c(N_f - N_c)$ quark fields with mass $M_q^2 = |m_{\xi(a)} - m_i|^2$ with $i \notin \Xi$.

Turning on the masses has explicitly broken the $SU(N_f)$ flavor symmetry to

$$ SU(N_f) \to U(1)_F^{N_f-1} \tag{4.7} $$

while the $U(N_c)$ gauge group is also broken completely in the vacuum. (Strictly speaking it is a combination of the $U(N_c)$ gauge group and $U(1)_F^{N_f-1}$ that survives in the vacuum).

4.2 Domain Wall Equations

The existence of isolated vacua implies the existence of a domain wall, a configuration that interpolates from a given vacuum Ξ_- at $x^3 \to -\infty$ to a distinct vacuum Ξ_+ at $x^3 \to +\infty$. As in each previous lecture, we can derive the first order equations satisfied by the domain wall using the Bogomoln'yi trick. We'll chose x^3 to be the direction transverse to the wall, and set $\partial_0 = \partial_1 = \partial_3 = 0$ as well as $A_0 = A_1 = A_2 = 0$. The tension of the domain wall can be written as [246]

$$ \begin{aligned} T_{\rm wall} = & \int dx^3 \ \frac{1}{e^2}{\rm Tr}\left(\mathcal{D}_3\phi + \frac{e^2}{2}(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2)\right)^2 - \mathcal{D}_3\phi\,(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2) \\ & + \sum_{i=1}^{N_f}\Big(|\mathcal{D}_3 q_i + (\phi - m_i)q_i|^2 - q_i^\dagger(\phi - m_i)\mathcal{D}_3 q_i - \mathcal{D}_3 q_i^\dagger(\phi - m_i)q_i\Big) \\ \geq & \ v^2\ [{\rm Tr}\phi]^{+\infty}_{-\infty} \end{aligned} \tag{4.8} $$

With our vacua Ξ_- and Ξ_+ at left and right infinity, we have the tension of the domain wall bounded by

$$ T_{\rm wall} \geq v^2[{\rm Tr}\phi]^{+\infty}_{-\infty} = v^2\sum_{i\in\Xi_+} m_i - v^2\sum_{i\in\Xi_-} m_i \tag{4.9} $$

and the minus signs have been chosen so that this quantity is positive (if this isn't the case we must swap left and right infinity and consider the anti-wall). The bound is saturated when the domain wall equations are satisfied,

$$\mathcal{D}_3\phi = -\frac{e^2}{2}(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2) \quad , \quad \mathcal{D}_3 q_i = -(\phi - m_i)q_i \tag{4.10}$$

Just as the monopole equations $\mathcal{D}\phi = B$ arise as the dimensional reduction of the instanton equations $F = {}^\star F$, so the domain wall equations (4.10) arise from the dimensional reduction of the vortex equations. To see this, we look for solutions to the vortex equations with $\partial_2 = 0$ and relabel $x^1 \to x^3$ and $(A_1, A_2) \to (A_3, \phi)$. Finally, the analogue of turning on the vev in going from the instanton to the monopole, is to turn on the masses m_i in going from the vortex to the domain wall. These can be thought of as a "vev" for $SU(N_f)$ the flavor symmetry.

4.2.1 An Example

The simplest theory admitting a domain wall is $U(1)$ with $N_f = 2$ scalars q_i. The domain wall equations are

$$\partial_3\phi = -\frac{e^2}{2}(|q_1|^2 + |q_2|^2 - v^2) \quad , \quad \mathcal{D}_3 q_i = -(\phi - m_i)q_i \tag{4.11}$$

We'll chose $m_2 = -m_1 = m$. The general solution to these equations is not known. The profile of the wall depends on the value of the dimensionless constant $\gamma = e^2v^2/m^2$. For $\gamma \ll 1$, the wall can be shown to have a three layer structure, in which the q_i fields decrease to zero in the outer layers, while ϕ interpolates between its two expectation values at a more leisurely pace [211]. The result is a domain wall with width $L_{\rm wall} \sim m/e^2v^2$. Outside of the wall, the fields asymptote exponentially to their vacuum values.

In the opposite limit $\gamma \gg 1$, the inner segment collapses and the two outer layers coalesce, leaving us with a domain wall of width $L_{\rm wall} \sim 1/m$. In fact, if we take the limit $e^2 \to \infty$, the first equation (4.11) becomes algebraic while the second is trivially solved. We find the profile of the domain wall to be [212]

$$q_1 = \frac{v}{A}\, e^{-m(x_3 - X) + i\theta} \quad , \quad q_2 = \frac{v}{A}\, e^{+m(x_3 - X) - i\theta} \tag{4.12}$$

where $A^2 = e^{-2m(x_3 - X)} + e^{+2m(x_3 - X)}$.

The solution (4.12) that we've found in the $e^2 \to \infty$ limit has two collective coordinates, X and θ. The former is simply the position of the domain wall in the transverse

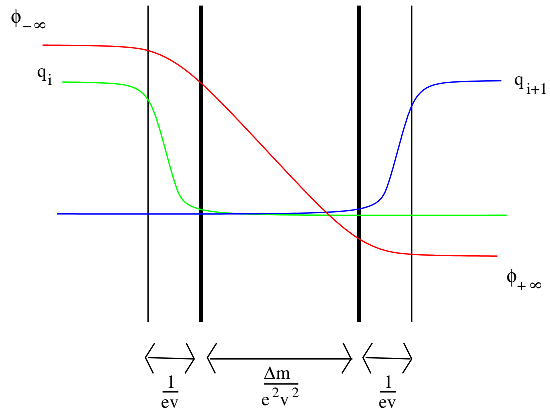

Figure 22: The three layer structure of the domain wall when $e^2v^2 \ll m^2$.

x^3 direction. The latter is also easy to see: it arises from acting on the domain wall with the $U(1)_F$ flavor symmetry of the theory [210]:

$$U(1)_F : q_1 \to e^{i\theta} q_1 \quad , \quad q_2 \to e^{-i\theta} q_2 \tag{4.13}$$

In each vacuum, this coincides with the $U(1)$ gauge symmetry. However, in the interior of the domain wall, it acts non-trivially, giving rise to a phase collective coordinate θ for the solution. It can be shown that X and θ remain the only two collective coordinates of the domain wall when we return to finite e^2 [213].

4.2.2 Classification of Domain Walls

So we see above that the simplest domain wall has two collective coordinates. What about the most general domain wall, characterized by the choice of vauca Ξ_- and Ξ_+ at left and right infinity. At first sight it appears a little daunting to classify these objects. After all, a strict classification of the topological charge requires a statement of the vacuum at left and right infinity, and the number of vacua increases exponentially with N_f. To ameliorate this sense of confusion, it will help to introduce a coarser classification of domain walls which will capture some information about the topological sector, without specifying the vacua completely. This classification, introduced in [214], will prove most useful when relating our domain walls to the other solitons we've met previously. To this end, define the N_f-vector

$$\vec{m} = (m_1, \ldots, m_{N_f}) \tag{4.14}$$

We can then write the tension of the domain wall as

$$T_{\rm wall} = v^2 \vec{g} \cdot \vec{m} \tag{4.15}$$

which defines a vector $\vec{g}$ that contains entries 0 and ± 1 only. Following the classification of monopoles in Lecture 2, let's decompose this vector as

$$\vec{g} = \sum_{i=1}^{N_f} n_i \vec{\alpha}_i \tag{4.16}$$

with $n_i \in \mathbf{Z}$ and the $\vec{\alpha}_i$ the simple roots of $su(N_f)$,

$$\begin{aligned} \vec{\alpha}_1 &= (1, -1, 0, \ldots, 0) \\ \vec{\alpha}_2 &= (0, 1, -1, \ldots, 0) \\ \vec{\alpha}_{N_f-1} &= (0, \ldots, 0, 1, -1) \end{aligned}$$

Since the vector $\vec{g}$ can only contain 0's, 1's and -1's, the integers n_i cannot be arbitrary. It's not hard to see that this restriction means that neighboring n_i's are either equal or differ by one: $n_i = n_{i+1}$ or $n_i = n_{i+1} \pm 1$.

4.3 The Moduli Space

A choice of $\vec{g}$ does not uniquely determine a choice of vacua at left and right infinity. Nevertheless, domain wall configurations which share the same $\vec{g}$ share certain characteristics, including the number of collective coordinates. The collective coordinates carried by a given domain wall was calculated in a number of situations in [215, 216, 217]. Using our classification, the index theorem tells us that there are solutions to the domain wall equations (4.10) only if $n_i \geq 0$ for all i. Then the number of collective coordinates is given by [214],

$$\dim \mathcal{W}_{\vec{g}} = 2 \sum_{i=1}^{N_f-1} n_i \tag{4.17}$$

where $\mathcal{W}_{\vec{g}}$ denotes the moduli space of any set of domain walls with charge $\vec{g}$. Again, this should be looking familiar! Recall the result for monopoles with charge $\vec{g}$ was $\dim(\mathcal{M}_{\vec{g}}) = 4\sum_a n_a$. The interpretation of the result (4.17) is, as for monopoles, that there are $N_f - 1$ elementary types of domain walls associated to the simple roots $\vec{g} = \vec{\alpha}_i$. A domain wall sector in sector $\vec{g}$ then splits up into $\sum_i n_i$ elementary domain walls, each with its own position and phase collective coordinate.

4.3.1 The Moduli Space Metric

The low-energy dynamics of multiple, parallel, domain walls is described, in the usual fashion, by a sigma-model from the domain wall worldvolume to the target space is $\mathcal{W}_{\vec{g}}$. As with other solitons, the domain walls moduli space $\mathcal{W}_{\vec{g}}$ inherits a metric from the zero modes of the solution. In notation such that $q = q^a{}_i$ is an $N_c \times N_f$ matrix, the linearized domain wall equations (4.10)

$$\begin{aligned} \mathcal{D}_3\delta\phi - i[\delta A_3, \phi] &= -\frac{e^2}{2}(\delta q\, q^\dagger + q\delta q^\dagger) \\ \mathcal{D}_3\delta q - i\delta A_3\, q &= -(\phi\delta q + \delta\phi\, q - \delta q\, m) \end{aligned} \tag{4.18}$$

where $m = \text{diag}(m_1, \ldots, m_{N_f})$ is an $N_f \times N_f$ matrix. Again, these are to be supplemented by a background gauge fixing condition,

$$\mathcal{D}_3\delta A_3 - i[\phi, \delta\phi] = i\frac{e^2}{2}(q\delta q^\dagger - \delta q\, q^\dagger) \tag{4.19}$$

and the metric on the moduli space $\mathcal{W}_{\vec{g}}$ is defined by the overlap of these zero modes,

$$g_{\alpha\beta} = \int dx^3\ \text{Tr}\left(\frac{1}{e^2}\left[\delta_\alpha A_3\, \delta_\beta A_3 + \delta_\alpha\phi\, \delta_\beta\phi\right] + \delta_\alpha q\, \delta_\beta q^\dagger + \delta_\beta q\, \delta_\alpha q^\dagger\right) \tag{4.20}$$

By this stage, the properties of the metric on the soliton moduli space should be familiar. They include.

- The metric is Kähler.
- The metric is smooth. There is no singularity as two domain walls approach each other.
- The metric inherits a $U(1)^{N-1}$ isometry from the action of the unbroken flavor symmetry (4.7) acting on the domain wall.

4.3.2 Examples of Domain Wall Moduli Spaces

Let's give some simple examples of domain wall moduli spaces.

One Domain Wall

We've seen that a single elementary domain wall $\vec{g} = \vec{\alpha}_1$ (for example, the domain wall described above in the theory with $N_c = 1$ and $N_f = 2$) has two collective coordinates: its center of mass X and a phase θ. The moduli space is

$$\mathcal{W}_\alpha \cong \mathbf{R} \times \mathbf{S}^1 \tag{4.21}$$

The metric on this space is simple to calculate. It is

$$ds^2 = (v^2\vec{m}\cdot\vec{g})\ dX^2 + v^2\, d\theta^2 \tag{4.22}$$

with the phase collective coordinate living in $\theta \in [0, 2\pi)$.

Two Domain Walls

We can't have two domain walls of the same type, say $\vec{g} = 2\vec{\alpha}_1$, since there is no choice of vacua that leads to this charge. Two elementary domain walls must necessarily be of different types, $\vec{g} = \vec{\alpha}_i + \vec{\alpha}_j$ for $i \neq j$. Let's consider $\vec{g} = \vec{\alpha}_1 + \vec{\alpha}_2$.

The moduli space is simplest to describe if the two domain walls have the same mass, so $\vec{m} \cdot \vec{\alpha}_a = \vec{m} \cdot \vec{\alpha}_b$. The moduli space is

$$\mathcal{W}_{\vec{\alpha}_1+\vec{\alpha}_2} \cong \mathbf{R} \times \frac{\mathbf{S}^1 \times \mathcal{M}_{\rm cigar}}{\mathbf{Z}_2} \tag{4.23}$$

where the interpretation of the $\mathbf{R}$ factor and $\mathbf{S}^1$ factor are the same as before. The relative moduli space has the topology and asymptotic form of a cigar, The relative

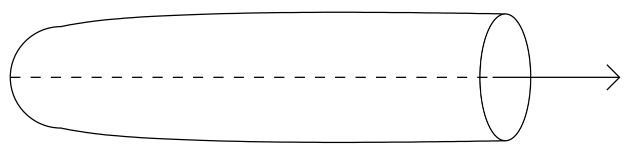

Figure 23: The relative moduli space of two domain walls is a cigar.

separation between domain walls is denoted by R. The tip of the cigar, $R = 0$, corresponds to the two domain walls sitting on top of each other. At this point the relative phase of the two domain walls degenerates, resulting in a smooth manifold. The metric on this space has been computed in the $e^2 \to \infty$ limit, although it's not particularly illuminating [212] and gives a good approximation to the metric at large finite e^2 [218]. Asymptotically, it deviates from the flat metric on the cylinder by exponentially suppressed corrections e^{-R}, as one might expect since the profile of the domain walls is exponentially localized.

4.4 Dyonic Domain Walls

You will have noticed that, rather like monopoles, the domain wall moduli space includes a phase collective coordinate $\mathbf{S}^1$ for each domain wall. For the monopole, excitations along this $\mathbf{S}^1$ give rise to dyons, objects with both magnetic and electric charges. For domain walls, excitations along this $\mathbf{S}^1$ also give rise to dyonic objects, now carrying both topological (kink) charge and flavor charge. Abraham and Townsend called these objects "Q-kinks" [210].

First order equations of motion for these dyonic domain walls may be obtained by completing the square in the Lagrangian (4.1), now looking for configurations that depend on both x^0 and x^3, allowing for a non-zero electric field F_{03}. We have

$$\begin{aligned}
T_{\rm wall} = \int dx^3\ \frac{1}{e^2}\text{Tr}\left(\cos\alpha\,\mathcal{D}_3\phi + \frac{e^2}{2}(\sum_{i=1}^{N_f}|q_iq_i^\dagger - v^2)\right)^2 - \cos\alpha\,\mathcal{D}_3\phi\,(\sum_{i=1}^{N_f}q_iq_i^\dagger - v^2) \\
+\sum_{i=1}^{N_f}\Big(|\mathcal{D}_3q_i + \cos\alpha\,(\phi - m_i)q_i|^2 - \cos\alpha\,(q_i^\dagger(\phi-m_i)\mathcal{D}_3q_i + \text{ h.c.})\Big) \\
+\frac{1}{e^2}\text{Tr}(F_{03} - \sin\alpha\,\mathcal{D}_3\phi)^2 + \frac{1}{e^2}\sin\alpha\,F_{03}\mathcal{D}_3\phi \\
+\sum_{i=1}^{N_f}\Big(|\mathcal{D}_0q_i + i\sin\alpha(\phi-m_i)q_i|^2 - \sin\alpha(iq_i^\dagger(\phi-m_i)\mathcal{D}_0q_i + \text{ h.c})\Big)
\end{aligned}$$

As usual, insisting upon the vanishing of the total squares yields the Bogomoln'yi equations. These are now to augmented with Gauss' law,

$$\mathcal{D}_3F_{03} = ie^2\sum_{i=1}^{N_f}(q_i\mathcal{D}_0q_i^\dagger - (\mathcal{D}_0q_i)q_i^\dagger) \tag{4.24}$$

Using this, we may re-write the cross terms in the energy-density to find the Bogomoln'yi bound,

$$T_{\rm wall} \geq \pm v^2[\text{Tr}\phi]^{+\infty}_{-\infty}\,\cos\alpha + (\vec{m}\cdot\vec{S})\sin\alpha \tag{4.25}$$

where $\vec{S}$ is the Noether charge associated to the surviving $U(1)^{N_f-1}$ flavor symmetry, an N_f-vector with $i^{\rm th}$ component given by

$$S_i = i(q_i\mathcal{D}_0q_i^\dagger - (\mathcal{D}_0q_i)q_i^\dagger) \tag{4.26}$$

Maximizing with respect to α results in the Bogomoln'yi bound for dyonic domain walls,

$$\mathcal{H} \geq \sqrt{v^4(\vec{m}\cdot\vec{g})^2 + (\vec{m}\cdot\vec{S})^2} \tag{4.27}$$

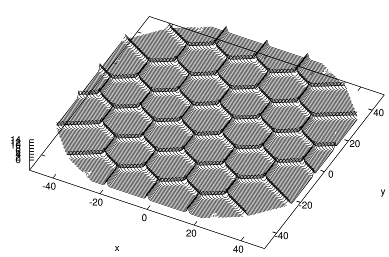

Figure 24:

This square-root form is familiar from the spectrum of dyonic monopoles that we saw in Lecture 2. More on this soon. For now, some further comments, highlighting the some similarities between dyonic domain walls and monopoles.

- There is an analog of the Witten effect. In two dimensions, where the domain walls are particle-like objects, one may add a theta term of the form θF_{01}. This induces a flavor charge on the domain wall, proportional to its topological charge, $\vec{S} \sim \vec{g}$ [219].

- One can construct dyonic domain walls with $\vec{g}$ and $\vec{S}$ not parallel if we turn on complex masses and, correspondingly, consider a complex adjoint scalar ϕ [220, 221]. The resulting 1/4 and 1/8-BPS states are the analogs of the 1/4-BPS monopoles we briefly mentioned in Lecture 2.

- The theory with complex masses also admits interesting domain wall junction configurations [222, 223]. Most notably, Eto et al. have recently found beautiful webs of domain walls, reminiscent of (p, q)5-brane webs of IIB string theory, with complicated moduli as the strands of the web shift, causing cycles to collapse and grow [224, 225]. Examples include the intricate honeycomb structure shown in figure 24 (taken from [224]).

Other aspects of these domain walls were discussed in [226, 227, 228, 229, 230]. A detailed discussed of the mass renormalization of supersymmetric kinks in $d = 1 + 1$ dimensions can be found in [231, 232].

4.5 The Ordering of Domain Walls

The cigar moduli space for two domain walls illustrates an important point: domain walls cannot pass each other. In contrast to other solitons, they must satisfy a particular ordering on the line. This is apparent in the moduli space of two domain walls since the relative separation takes values in $R \in \mathbf{R}^+$ rather than $\mathbf{R}$. The picture in spacetime shown in figure 25.

However, it's not always true that domain walls cannot pass through each other. Domain walls which live in different parts of the flavor group, so have $\vec{\alpha}_i \cdot \vec{\alpha}_j = 0$, do not interact so can happily move through each other. When these domain walls are two of many in a topological sector $\vec{g}$, an interesting pattern of interlaced walls arises, determined by which walls bump into each other, and which pass through each other. This pattern was first explored in [217]. Let's see how the ordering emerges. Start at left infinity in a particular vacuum Ξ_-. Then each elementary domain wall shifts the vacuum by increasing a single element $\xi(a) \in \Xi$ by one. The restriction that the N_c elements are distinct means that only certain domain walls can occur. This point is one that is best illustrated by a simple example:

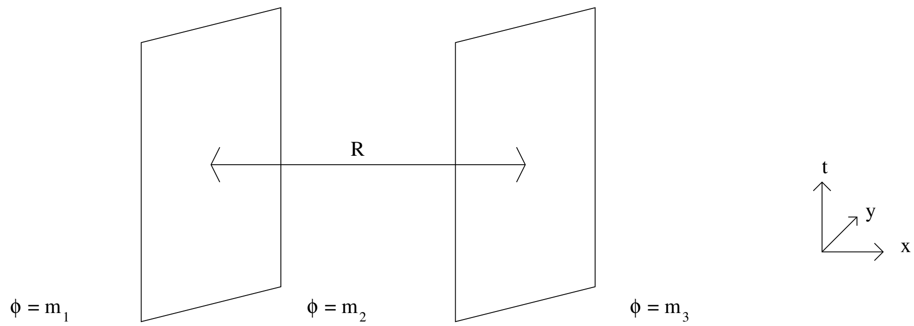

Figure 25: Two interacting domain walls cannot pass through each other. The $\vec{\alpha}_1$ domain wall is always to the left of the $\vec{\alpha}_2$ domain wall.

An Example: $N_c = 2$, $N_f = 4$

Consider the domain walls in the $U(2)$ theory with $N_f = 4$ flavors. We'll start at left infinity in the vacuum $\Xi_- = \{1,2\}$ and end at right infinity in the vacuum $\Xi_+ = \{3,4\}$. There are two different possibilities for the intermediate vacua. They are:

$$\Xi_- = \{1,2\} \longrightarrow \{1,3\} \longrightarrow \{1,4\} \longrightarrow \{2,4\} \longrightarrow \{3,4\} = \Xi_+$$
$$\Xi_- = \{1,2\} \longrightarrow \{1,3\} \longrightarrow \{2,3\} \longrightarrow \{2,4\} \longrightarrow \{3,4\} = \Xi_+$$

In terms of domain walls, these two ordering become,

$$\begin{aligned} &\vec{\alpha}_2 \longrightarrow \vec{\alpha}_3 \longrightarrow \vec{\alpha}_1 \longrightarrow \vec{\alpha}_2 \\ &\vec{\alpha}_2 \longrightarrow \vec{\alpha}_1 \longrightarrow \vec{\alpha}_3 \longrightarrow \vec{\alpha}_2 \end{aligned} \tag{4.28}$$

We see that the two $\vec{\alpha}_2$ domain walls must play bookends to the $\vec{\alpha}_1$ and $\vec{\alpha}_3$ domain walls. However, one expects that these middle two walls are able to pass through each other.

The General Ordering of Domain Walls

We may generalize the discussion above to deduce the rule for ordering of general domain walls [217]. One finds that the n_i elementary $\vec{\alpha}_i$ domain walls must be interlaced between the $\vec{\alpha}_{i-1}$ and $\vec{\alpha}_{i+1}$ domain walls. (Recall that $n_i = n_{i+1}$ or $n_i = n_{i+1} \pm 1$ so the concept of interlacing is well defined). The final pattern of domain walls is captured in figure 26, where x^3 is now plotted horizontally and the vertical position of the domain wall simply denotes its type. We shall see this vertical position take on life in the D-brane set-up we describe shortly.

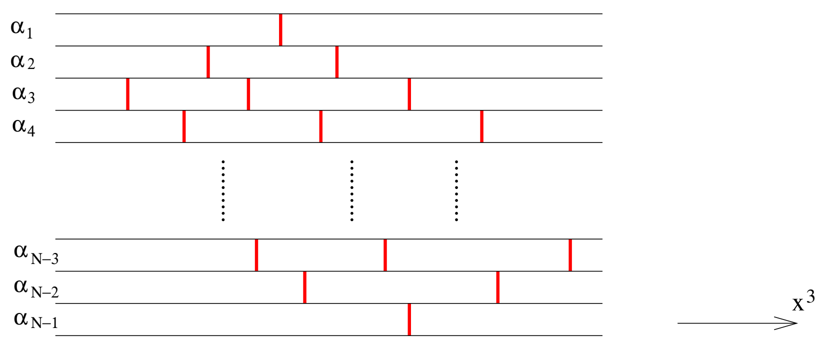

Figure 26: The ordering of many domain walls. The horizontal direction is their position, while the vertical denotes the type of domain wall.

Notice that the $\vec{\alpha}_1$ domain wall is trapped between the two $\vec{\alpha}_2$ domain walls. These in turn are trapped between the three $\vec{\alpha}_3$ domain walls. However, the relative positions of the $\vec{\alpha}_1$ and middle $\vec{\alpha}_3$ domain walls are not fixed: these objects can pass through each other.

4.6 What Became Of......

Now let's play our favorite game and ask what happened to the other solitons now that we've turned on the masses. We start with.....

4.6.1 Vortices

The vortices described in the previous lecture enjoyed zero modes arising from their embedding in $SU(N)_{\rm diag} \subset U(N_c) \times SU(N_f)$. Let go back to the situation with $N_f = N_c = N$, but with the extra terms from (4.1) added to the Lagrangian,

$$V = \frac{1}{e^2}{\rm Tr}\,(\mathcal{D}_\mu\phi)^2 + \sum_{i=1}^{N} q_i^\dagger(\phi - m_i)^2 q_i \tag{4.29}$$

As we've seen, this mass term breaks $SU(N)_{\rm diag} \to U(1)^{N-1}_{\rm diag}$, which means we can no longer rotate the orientation of the vortices within the gauge and flavor groups. We learn that the masses are likely to lift some zero modes of the vortex moduli space [233, 158, 177].

The vortex solutions that survive are those whose energy isn't increased by the extra terms in V above. Or, in other words, those vortex configurations which vanish when evaluated on V above. If we don't want the vortex to pick up extra energy from the kinetic terms $\mathcal{D}\phi^2$, we need to keep ϕ in its vacuum,

$$\phi = {\rm diag}(\phi_1, \ldots, \phi_N) \tag{4.30}$$

which means that only the components $q^a{}_i \sim \delta^a{}_i$ can turn on keeping $V = 0$.

For the single vortex $k = 1$ in $U(N)$, this means that the internal moduli space $\mathbb{CP}^{N-1}$ is lifted, leaving behind N different vortex strings, each with magnetic field in a different diagonal component of the gauge group,

$$\begin{aligned} B_3 &= \mathrm{diag}(0, \ldots, B_3^\star, \ldots, 0) \\ q &= \mathrm{diag}(v, \ldots, q^\star, \ldots, v) \end{aligned} \tag{4.31}$$

In summary, rather than having a moduli space of vortex strings, we are left with N different vortex strings, each carrying magnetic flux in a different $U(1) \subset U(N)$.

How do we see this from the perspective of the vortex worldsheet? We can re-derive the vortex theory using the brane construction of the previous lecture, but now with the D6-branes separated in the x^4 direction, providing masses m_i for the hypermultiplets q_i. After performing the relevant brane-game manipulations, we find that these translate into masses m_i for the chiral multiplets in the vortex theory. The potential for the vortex theory (3.30) is replaced by,

$$\begin{aligned} V &= \frac{1}{g^2}\mathrm{Tr}\,|[\sigma, \sigma^\dagger]|^2 + \mathrm{Tr}\,|[\sigma, Z]|^2 + \mathrm{Tr}\,|[\sigma, Z^\dagger]|^2 \\ &\quad + \sum_{a=1}^{N} \psi_a^\dagger(\sigma - m_a)^2\psi_a + \frac{g^2}{2}\,\mathrm{Tr}\left(\sum_a \psi_a\psi_a^\dagger + [Z, Z^\dagger] - r\,1_k\right)^2 \end{aligned} \tag{4.32}$$

where $r = 2\pi/e^2$ as before. The masses m_i of the four-dimensional theory have descended to masses m_a on the vortex worldsheet.

To see the implications of this, consider the theory on a single $k = 1$ vortex. The potential is simply,

$$V_{k=1} = \sum_{a=1}^{N}(\sigma - m_a)^2|\psi_a|^2 + \frac{g^2}{2}(\sum_{a=1}^{N}|\psi_a|^2 - r)^2 \tag{4.33}$$

Whereas before we could set $\sigma = 0$, leaving ψ_a to parameterize $\mathbb{CP}^{N-1}$, now the Higgs branch is lifted. We have instead N isolated vacua,

$$\sigma = m_a \quad , \quad |\psi_b|^2 = r\delta_{ab} \tag{4.34}$$

These correspond to the N different vortex strings we saw above.

A Potential on the Vortex Moduli Space

We can view the masses m_i as inducing a potential on the Higgs branch of the vortex theory after integrating out σ. This potential is equal to the length of Killing vectors on the Higgs branch associated to the $U(1)^{N-1} \subset SU(N)_{\rm diag}$ isometry. This is the same story we saw in Lecture 2.6, where the a vev for ϕ induced a potential on the instanton moduli space.

In fact, just as we saw for instantons, this result can also be derived directly within the field theory itself [177]. Suppose we fix a vortex configuration (A_z, q) that solves the vortex equations before we introduce masses. We want to determine how much the new terms (4.29) lift the energy of this vortex. We minimize V by solving the equation of motion for ϕ in the background of the vortex,

$$\mathcal{D}^2\phi = \frac{e^2}{2}\sum_{i=1}^{N}\{\phi, q_i q_i^\dagger\} - 2q_i q_i^\dagger m_i \tag{4.35}$$

subject to the vev boundary condition $\phi \to {\rm diag}(m_1, \ldots, m_N)$ as $r \to \infty$. But we have seen this equation before! It is precisely the equation (3.21) that an orientational zero mode of the vortex must satisfy. This means that we can write the excess energy of the vortex in terms of the relevant orientational zero mode

$$V = \int d^2x\ \frac{2}{e^2}{\rm Tr}\,\delta A_z\,\delta A_{\bar z} + \frac{1}{2}\sum_{i=1}^{N}\delta q_i\,\delta q_i^\dagger \tag{4.36}$$

for the particular orientation zero mode $\delta A_z = \mathcal{D}_z\phi$ and $\delta q_i = i(\phi q_i - q_i m_i)$. We can give a nicer geometrical interpretation to this following the discussion in Section 2.6. Denote the Cartan subalgebra of $SU(N)_{\rm diag}$ as $\vec{H}$, and the associated Killing vectors on $\mathcal{V}_{k,N}$ as $\vec{k}_\alpha$. Then, since ϕ generates the transformation $\vec{m}\cdot\vec{H}$, we can express our zero mode in terms of the basis $\delta A_z = (\vec{m}\cdot\vec{k}^\alpha)\,\delta_\alpha A_z$ and $\delta q_i = (\vec{m}\cdot\vec{k}^\alpha)\,\delta_\alpha q_i$. Putting this in our potential and performing the integral over the zero modes, we have the final expression

$$V = g_{\alpha\beta}\,(\vec{m}\cdot\vec{k}^\alpha)\,(\vec{m}\cdot\vec{k}^\beta) \tag{4.37}$$

This potential vanishes at the fixed points of the $U(1)^{N-1}$ action. For the one-vortex moduli space $\mathbb{CP}^{N-1}$, it's not hard to see that this gives rise to the N vacuum states described above (4.34).

4.6.2 Monopoles

To see where the monopoles have gone, it's best if we first look at the vortex worldsheet theory [233]. This is now have a $d = 1+1$ dimensional theory with isolated vacua, guaranteeing the existence of domain wall, or kink, in the worldsheet. In fact, for a single $k = 1$ vortex, the theory on the worldsheet is precisely of the form (4.1) that we started with at the beginning of this lecture. (For $k > 1$, the presence of the adjoint scalar Z means that isn't precisely the same action, but is closely related). The equations describing kinks on the worldsheet are the same as (4.10),

$$\partial_3 \sigma = g^2 (\sum_{a=1}^{N} |\psi_a|^2 - r) \quad , \quad \mathcal{D}_3 \psi_a = (\sigma - m_a)\psi_a \tag{4.38}$$

where we should take the limit $g^2 \to \infty$, in which the first equation becomes algebraic. What's the interpretation of this kink on the worldsheet? We can start by examining its mass,

$$M_{\rm kink} = (\vec{m} \cdot \vec{g})\, r = \frac{2\pi}{e^2}\, (\vec{\phi} \cdot \vec{g}) = M_{\rm mono} \tag{4.39}$$

So the kink has the same mass as the monopole! In fact, it also has the same quantum numbers. To see this, recall that the different vacua on the vortex string correspond to flux tubes lying in different $U(1) \subset U(N)$ subgroups. For example, for $N = 2$, the kink must take the form shown in figure 27. So whatever the kink is, it must soak

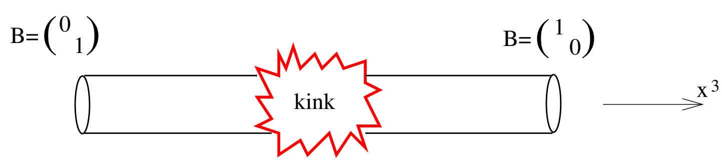

Figure 27: The kink on the vortex string.

up magnetic field $B = \text{diag}(0, 1)$ and spit out magnetic field $B = \text{diag}(1, 0)$. In other words, it is a source for the magnetic field $B = \text{diag}(1, -1)$. This is precisely the magnetic field sourced by an $SU(2)$ 't Hooft-Polyakov monopole.

What's happening here? We are dealing with a theory with a mass gap, so any magnetic monopole that lives in the bulk can't emit a long-range radial magnetic field since the photon can't propagate. We're witnessing the Meissner effect in a non-abelian superconductor. The monopole is confined, its magnetic field departing in two semi-classical flux tubes. This effect is, of course, well known and it is conjectured that a dual effect leads to the confinement of quarks in QCD. Here we have a simple, semi-classical realization in which to explore this scenario.

Can we find the monopole in the $d = 3+1$ dimensional bulk? Although no solution is known, it turns out that we can write down the Bogomoln'yi equations describing the configuration [233]. Let's go back to our action (4.1) and complete the square in a different way. We now insist only that $\partial_0 = A_0 = 0$, and write the Hamiltonian as,

$$\begin{aligned}\mathcal{H} &= \int d^3x\, \frac{1}{e^2}\mathrm{Tr}\, \left[(\mathcal{D}_1\phi + B_1)^2 + (\mathcal{D}_2 + B_2)^2 + (\mathcal{D}_3\phi + B_3 - \frac{e^2}{2}(\sum_{i=1}^{N} q_i q_i^\dagger - v^2))^2\right] \\ &\quad + \sum_{i=1}^{N} |(\mathcal{D}_1 - i\mathcal{D}_2)q_i|^2 + \sum_{i=1}^{N} |\mathcal{D}_3 q_i - (\phi - m_i)q_i|^2 + \mathrm{Tr}\, [-v^2 B_3 - \frac{2}{e^2}\partial_i(\phi B_i)] \\ &\geq \left(\int dx^3\; T_{\rm vortex}\right) + M_{\rm mono} \end{aligned} \tag{4.40}$$

where the inequality is saturated when the terms in the brackets vanish,

$$\begin{aligned} \mathcal{D}_1\phi + B_1 = 0 \quad &, \quad \mathcal{D}_1 q_i = i\mathcal{D}_2 q_i \\ \mathcal{D}_2\phi + B_2 = 0 \quad &, \quad \mathcal{D}_3 q_i = (\phi - m_i) q_i \\ \mathcal{D}_3\phi + B_3 &= \frac{e^2}{2}(\sum_{i=1}^{N} q_i q_i^\dagger - v^2) \end{aligned} \tag{4.41}$$

As you can see, these are an interesting mix of the monopole equations and the vortex equations. In fact, they also include the domain wall equations — we'll see the meaning of this when we come to discuss the applications. These equations should be thought of as the master equations for BPS solitons, reducing to the other equations in various limits. Notice moreover that these equations are over-determined, but it's simple to check that they satisfy the necessary integrability conditions to admit solutions. However, no non-trivial solutions are known analytically. (Recall that even the solution for a single vortex is not known in closed form). We expect that there exist solutions that look like figure 28.

The above discussion was for $k = 1$ and $N_f = N_c$. Extensions to $k \geq 2$ and also to $N_f \geq N_c$ also exist, although the presence of the adjoint scalar Z on the vortex worldvolume means that the kinks on the string aren't quite the same as the domain wall equations (4.10). But if we set $Z = 0$, so that the strings lie on top of each other, then the discussion of domain walls in four-dimensions carries over to kinks on the string. In fact, it's not hard to check that we've chosen our notation wisely: magnetic monopoles of charge $\vec{g}$ descend to kinks on the vortex strings with topological charge $\vec{g}$.

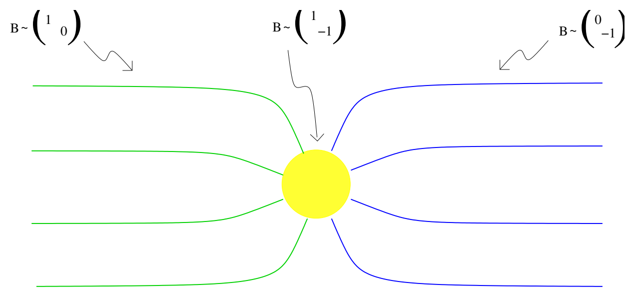

Figure 28: The confined magnetic monopole.

In summary,

Kink on the Vortex String = Confined Magnetic Monopole

The BPS confined monopole was first described in [233], but the idea that kinks on string should be interpreted as confined monopoles arose previously in [234] in the context of Z_N flux tubes. More recently, confined monopoles have been explored in several different theories [235, 236, 237, 238]. We'll devote Section 4.7 to more discussion on this topic.

4.6.3 Instantons

We now ask what became of instantons. At first glance, it doesn't look promising for the instanton! In the bulk, the FI term v^2 breaks the gauge group, causing the instanton to shrink. And the presence of the masses means that even in the center of various solitons, there's only a $U(1)$ restored, not enough to support an instanton. For example, an instanton wishing to nestle within the core of the vortex string shrinks to vanishing size and it looks as if the theory (4.1) admits only singular, small instantons.

While the above paragraph is true, it also tells us how we should change our theory to allow the instantons to return: we should consider non-generic mass parameters, so that the $SU(N_f)$ flavor symmetry isn't broken to the maximal torus, but to some non-abelian subgroup. Let's return to the example discussed in Section 4.5: $U(2)$ gauge theory with $N_f = 4$ flavors. Rather than setting all masses to be different, we chose $m_1 = m_2 = m$ and $m_3 = m_4 = -m$. In this limit, the breaking of the flavor symmetry is $SU(4) \to S[U(2) \times U(2)]$, and this has interesting consequences.

To find our instantons, we look at the domain wall which interpolates between the two vacua $\phi = m\mathbf{1}_2$ and $\phi = -m\mathbf{1}_2$. When all masses were distinct, this domain wall had 8 collective coordinates which had the interpretation of the position and phase of 4 elementary domain walls (4.28). Now that we have non-generic masses, the domain wall retains all 8 collective coordinates, but some develop a rather different interpretation: they correspond to new orientation modes in the unbroken flavor group. In this way, part of the domain wall theory becomes the $SU(2)$ chiral Lagrangian [239].

Inside the domain wall, the non-abelian gauge symmetry is restored, and the instantons may safely nestle there, finding refuge from the symmetry breaking of the bulk. One can show that, from the perspective of the domain wall worldvolume theory, they appear as Skyrmions [240]. Indeed, closer inspection reveals that the low-energy dynamics of the domain wall also includes a four derivative term necessary to stabilize the Skyrmion, and one can successfully compare the action of the instanton and Skyrmion. The relationship between instantons and Skyrmions was first noted long ago by Atiyah and Manton [241], and has been studied recently in the context of deconstruction [242, 243, 244].

4.7 The Quantum Vortex String

So far our discussion has been entirely classical. Let's now turn to the quantum theory. We have already covered all the necessary material to explain the main result. The basic idea is that $d = 1 + 1$ worldsheet theory on the vortex string captures quantum information about the $d = 3 + 1$ dimensional theory in which it's embedded. If we want certain information about the 4d theory, we can extract it using much simpler calculations in the 2d worldsheet theory.

I won't present all the calculations here, but instead simply give a flavor of the results [158, 177]. The precise relationship here holds for $\mathcal{N} = 2$ theories in $d = 3 + 1$, corresponding to $\mathcal{N} = (2, 2)$ theories on the vortex worldsheet. The first hint that the 2d theory contains some information about the 4d theory in which its embedded comes from looking at the relationship between the 2d FI parameter and the 4d gauge coupling,

$$r = \frac{4\pi}{e^2} \tag{4.42}$$

This is a statement about the classical vortex solution. Both e^2 in 4d and r in 2d run at one-loop. However, the relationship (4.42) is preserved under RG flow since the beta functions computed in 2d and 4d coincide,

$$r(\mu) = r_0 - \frac{N_c}{2\pi} \log\left(\frac{\mu_{UV}}{\mu}\right) \tag{4.43}$$

This ensures that both 4d and 2d theories hit strong coupling at the same scale $\Lambda = \mu \exp(-2\pi r/N_c)$.

Exact results about the 4d theory can be extracted using the Seiberg-Witten solution [132]. In particular, this allows us to determine the spectrum of BPS states in the theory. Similarly, the exact spectrum of the 2d theory can also be determined by computing the twisted superpotential [219, 245]. The punchline is that the spectrum of the two theories coincide. Let's see what this means. We saw in (4.39) that the classical kink mass coincides with the classical monopole mass

$$M_{\text{kink}} = M_{\text{mono}} \tag{4.44}$$

This equality is preserved at the quantum level. Let me stress the meaning of this. The left-hand side is computed in the $d = 1+1$ dimensional theory. When $(m_i - m_j) \gg \Lambda$, this theory is weakly coupled and M_{kink} receives a one-loop correction (with, obviously, two-dimensional momenta flowing in the loop). Although supersymmetry forbids higher loop corrections, there are an infinite series of worldsheet instanton contributions. The final expression for the mass of the kink schematically of the form,

$$M = M_{\text{clas}} + M_{\text{one-loop}} + \sum_{n=1}^{\infty} M_{\text{n-inst}} \tag{4.45}$$

The right-hand-side of (4.44) is computed in the $d = 3+1$ dimensional theory, which is also weakly coupled for $(m_i - m_J) \gg \Lambda$. The monopole mass M_{mono} receives corrections at one-loop (now integrating over four-dimensional momenta), followed by an infinite series of Yang-Mills instanton corrections. *And term by term these two series agree*!

The agreement of the worldsheet and Yang-Mills instanton expansions apparently has its microscopic origin in the results if the previous lecture. Recall that performing an instanton computation requires integration over the moduli space ($\mathcal{V}$ for the worldsheet instantons; $\mathcal{I}$ for Yang-Mills). Localization theorems hold when performing the integrals over $\mathcal{I}_{k,N}$ in $\mathcal{N} = 2$ super Yang-Mills, and the final answer contains contributions from only a finite number of points in $\mathcal{I}_{k,N}$ [46]. It is simple to check that all of these points lie on $\mathcal{V}_{k,N}$ which, as we have seen, is a submanifold of $\mathcal{I}_{k,N}$.

The equation (4.44) also holds in strong coupling regimes of the 2d and 4d theories where no perturbative expansion is available. Nevertheless, exact results allow the masses of BPS states to be computed and successfully compared. Moreover, the quantum correspondence between the masses of kinks and monopoles is not the only agreement between the two theories. Other results include:

- The elementary internal excitations of the string can be identified with W-bosons of the 4d theory. When in the bulk, away from the string, these W-bosons are non-BPS. But they can reduce their mass by taking refuge in the core of the vortex whereupon they regain their BPS status.

This highlights an important point: the spectrum of the 4d theory, both for monopoles and W-bosons, is calculated in the Coulomb phase, when the FI parameter $v^2 = 0$. However, the vortex string exists only in the Higgs phase $v^2 \neq 0$. What's going on? A heuristic explanation is as follows: inside the vortex, the Higgs field q dips to zero and the gauge symmetry is restored. The vortex theory captures information about the 4d theory on its Coulomb branch.

- As we saw in Sections 2.3 and 4.4, both the 4d theory and the 2d theory contain dyons. We've already seen that the spectrum of both these objects is given by the "square-root" formula (2.41) and (4.27). Again, these agree at the quantum level.

- Both theories manifest the Witten effect: adding a theta angle to the 4d theory induces an electric charge on the monopole, shifting its mass. This also induces a theta angle on the vortex worldsheet and, hence, turns the kinks into dyons.

- We have here described the theory with $N_f = N_c$. For $N_f > N_C$, the story can be repeated and again the spectrum of the vortex string coincides with the spectrum of the 4d theory in which it's embedded.

In summary, we have known for over 20 years that gauge theories in 4d share many qualitative features with sigma models in 2d, including asymptotic freedom, a dynamically generated mass gap, large N expansions, anomalies and the presences of instantons. However, the vortex string provides a quantitative relationship between the two: in this case, they share the same quantum spectrum.

4.8 The Brane Construction

In Lecture 3, we derived the brane construction for $U(N_c)$ gauge theory with N_f hypermultiplets. To add masses, one must separate the hypermultiplets in the x^4 direction. One can now see the number of vacua (4.6) since each of the N_c D4-branes must end on one of the N_f D6-branes.

To describe a domain wall, the D4-branes must start in one vacua, Ξ_- at $x_3 \rightarrow -\infty$, and interpolate to the final vacua Ξ_+ as $x^3 \rightarrow +\infty$. Viewing this integrated over all x^3, we have the picture shown in figure 29. To extract the dynamics of domain walls, we need to understand the worldvolume theory of the curved D4-brane. This isn't at all clear. Related issues have troubled previous attempts to extract domain wall

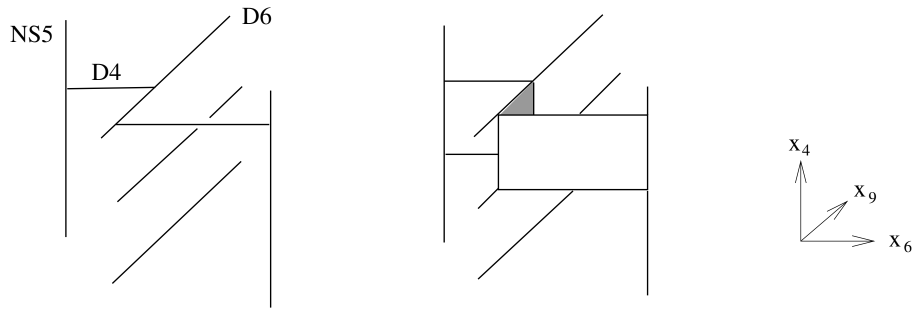

Figure 29: The D-brane configuration for an elementary $\vec{g} = \vec{\alpha}_1$ domain wall when $N_c = 1$ and $N_f = 3$.

dynamics from D-brane set-up [246, 247], although some qualitative features can be seen. However, we can make progress by studying this system in the limit $e^2 \to \infty$, so that the two NS5-branes and the N_f D6-branes lie coincident in the x^6 direction [248]. The portions of the D4-branes stretched in x^6 vanish, and we're left with D4-branes with worldvolume 01249, trapped in squares in the 49 directions where they are sandwiched between the NS5 and D6-branes. Returning to the system of domain walls in an arbitrary topological sector $\vec{g} = \sum_i n_i \vec{\alpha}_i$, we have the system drawn in figure 30.

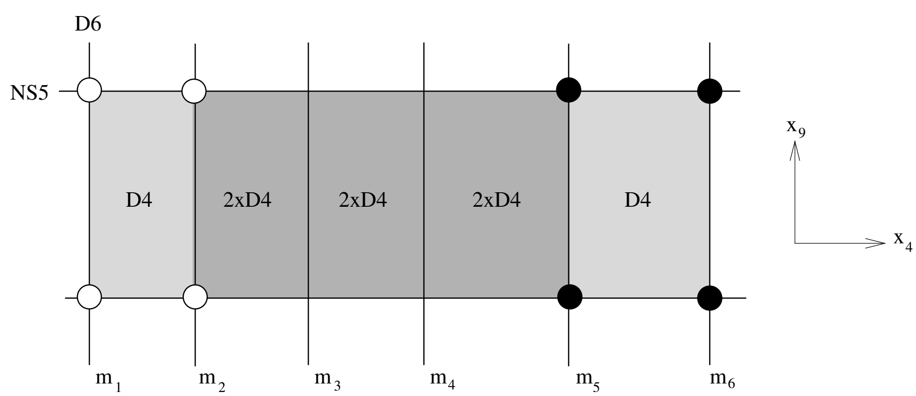

Figure 30: The D-brane configuration in the $e^2 \to \infty$ limit.

We can now read off the gauge theory living on the D4-branes. One might expect that it is of the form $\prod_i U(n_i)$. This is essentially correct. The NS5-branes project out the A_9 component of the gauge field, however the A_4 component survives and each $U(n_a)$ gauge theory lives in the interval $m_i \leq x_4 \leq m_{i+1}$. In each segment, we have A_4

and X_3, each an $n_i \times n_i$ matrix. These fields satisfy

$$\frac{dX_3}{dx^4} - i[A_4, X_3] = 0 \tag{4.46}$$

modulo $U(n_i)$ gauge transformations acting on the interval $m_i \leq x_4 \leq m_{i+1}$, and vanishing at the boundaries. These equations are kind of trivial: the interesting details lie in the boundary conditions. As in the case of monopoles, the interactions between neighbouring segments depends on the relative size of the matrices:

$\underline{n_i = n_{i+1}}$: The $U(n_i)$ gauge symmetry is extended to the interval $m_i \leq x_4 \leq m_{i+2}$ and an impurity is added to the right-hand-side of Nahm's equations, which now read

$$\frac{dX_3}{dx_4} - i[A_4, X_3] = \psi\psi^\dagger \delta(x_4 - m_{i+1}) \tag{4.47}$$

where the impurity degree of freedom ψ transforms in the fundamental representation of the $U(n_i)$ gauge group, ensuring the combination $\psi\psi^\dagger$ is a $n_i \times n_i$ matrix transforming, like X_1, in the adjoint representation. These ψ degrees of freedom are chiral multiplets which survive the NS5-brane projection.

$\underline{n_i = n_{i+1} - 1}$: In this case $X_3 \rightarrow (X_3)_-$, a $n_i \times n_i$ matrix, as $x_4 \rightarrow (m_i)_-$ from the left. To the right of m_i, X_3 is a $(n_i + 1) \times (n_i + 1)$ matrix obeying

$$X_3 \rightarrow \begin{pmatrix} y & a^\dagger \\ a & (X)_- \end{pmatrix} \qquad \text{as } x_4 \rightarrow (m_i)_+ \tag{4.48}$$

where $y_\mu \in \mathbf{R}$ and each a_μ is a complex n_i-vector. The obvious analog of this boundary condition holds when $n_i = n_{i+1} + 1$.

These boundary conditions are obviously related to the Nahm boundary conditions for monopoles that we met in Lecture 2.

4.8.1 The Ordering of Domain Walls Revisited

We now come to the important point: the ordering of domain walls. Let's see how the brane construction captures this. We can use the gauge transformations to make A_4 constant over the interval $m_i \leq x^4 \leq m_{i+1}$. Then (4.46) can be trivially integrated in each segment to give

$$X_3(x^4) = e^{iA_4 x^4} \hat{X}_3 e^{-iA_4 x^4} \tag{4.49}$$

Then the positions of the $\vec{\alpha}_i$ domain walls are given by the eigenvalues of X_3 restricted to the interval $m_i \leq x_4 \leq m_{i+1}$. Let us denote this matrix as $X_3^{(i)}$ and the eigenvalues as

$\lambda_m^{(i)}$, where $m = 1, \ldots n_i$. We have similar notation for the $\vec{\alpha}_{i+1}$ domain walls. Suppose first that $n_i = n_{i+1}$. Then the impurity (4.47) relates the two sets of eigenvalues by the jumping condition

$$X_1^{(i+1)} = X_1^{(i)} + \psi\psi^\dagger \tag{4.50}$$

We will now show that this jumping condition (4.50) correctly captures the interlacing nature of neighboring domain walls.

To see this, consider firstly the situation in which $\psi^\dagger\psi \ll \Delta\lambda_m^{(i)}$ so that the matrix $\psi\psi^\dagger$ may be treated as a small perturbation of $X_1^{(i)}$. The positivity of $\psi\psi^\dagger$ ensures that each $\lambda_m^{(i+1)} \geq \lambda_m^{(i)}$. Moreover, it is simple to show that the $\lambda_m^{(i+1)}$ increase monotonically with $\psi^\dagger\psi$. This leaves us to consider the other extreme, in which $\psi^\dagger\psi \to \infty$. It this limit ψ becomes one of the eigenvectors of $X_1^{(i+1)}$ with corresponding eigenvalue $\lambda_{n_i}^{(i+1)} = \psi^\dagger\psi$, corresponding to the limit in which the last domain wall is taken to infinity. What we want to show is that the remaining $n_i - 1$ $\vec{\alpha}_{i+1}$ domain walls are trapped between the n_i $\vec{\alpha}_i$ domain walls as depicted in figure 26. Define the $n_i \times n_i$ projection operator

$$P = 1 - \hat{\psi}\hat{\psi}^\dagger \tag{4.51}$$

where $\hat{\psi} = \psi/\sqrt{\psi^\dagger\psi}$. The positions of the remaining $(n_i - 1)$ $\vec{\alpha}_{i+1}$ domain walls are given by the (non-zero) eigenvalues of $PX_1^{(i)}P$. We must show that, given a rank n hermitian matrix X, the eigenvalues of PXP are trapped between the eigenvalues of X. This well known property of hermitian matrices is simple to show:

$$\begin{aligned}
\det(PXP - \mu) &= \det(XP - \mu) \\
&= \det(X - \mu - X\hat{\psi}\hat{\psi}^\dagger) \\
&= \det(X - \mu)\det(1 - (X - \mu)^{-1}X\hat{\psi}\hat{\psi}^\dagger)
\end{aligned}$$

Since $\hat{\psi}\hat{\psi}^\dagger$ is rank one, we can write this as

$$\begin{aligned}
\det(PXP - \mu) &= \det(X - \mu)\,[1 - \mathrm{Tr}((X - \mu)^{-1}X\hat{\psi}\hat{\psi}^\dagger)] \\
&= -\mu\,\det(X - \mu)\,\mathrm{Tr}((X - \mu)^{-1}\hat{\psi}\hat{\psi}^\dagger) \\
&= -\mu\left[\prod_{m=1}^{n}(\lambda_m - \mu)\right]\left[\sum_{m=1}^{n}\frac{|\hat{\psi}_m|^2}{\lambda_m - \mu}\right]
\end{aligned} \tag{4.52}$$

where $\hat{\psi}_m$ is the m^{th} component of the vector ψ. We learn that PXP has one zero eigenvalue while, if the eigenvalues λ_m of X are distinct, then the eigenvalues of PXP lie at the roots the function

$$R(\mu) = \sum_{m=1}^{n}\frac{|\hat{\psi}_m|^2}{\lambda_m - \mu} \tag{4.53}$$

The roots of $R(\mu)$ indeed lie between the eigenvalues λ_m. This completes the proof that the impurities (4.47) capture the correct ordering of the domain walls.

The same argument shows that the boundary condition (4.48) gives rise to the correct ordering of domain walls when $n_{i+1} = n_i + 1$, with the $\vec{\alpha}_i$ domain walls interlaced between the $\vec{\alpha}_{i+1}$ domains walls. Indeed, it is not hard to show that (4.48) arises from (4.47) in the limit that one of the domain walls is taken to infinity.

4.8.2 The Relationship to Monopoles

You will have noticed that the brane construction above is closely related to the Nahm construction we discussed in Lecture 2. In fact, just as the vortex moduli space $\mathcal{V}_{k,N}$ is related to the instanton moduli space $\mathcal{I}_{k,N}$, so the domain wall moduli space $\mathcal{W}_{\vec{g}}$ is related to the monopole moduli space $\mathcal{M}_{\vec{g}}$. The domain wall theory is roughly a subset of the monopole theory. Correspondingly, the domain wall moduli space is a complex submanifold of the monopole moduli space. To make this more precise, consider the isometry rotating the monopoles in the $x^1 - x^2$ plane (mixed with a suitable $U(1)$ gauge action). It we denote the corresponding Killing vector as h, then

$$\mathcal{W}_{\vec{g}} \cong \mathcal{M}_{\vec{g}}|_{h=0} \tag{4.54}$$

This is the analog of equation (3.36), relating the vortex and instanton moduli spaces.

Nahm's equations have appeared previously in describing domain walls in the $\mathcal{N} = 1^\star$ theory [249]. I don't know how those domain walls are related to the ones discussed here.

4.9 Applications

We've already seen one application of kinks in section 4.7, deriving a relationship between 2d sigma models and 4d gauge theories. I'll end with a couple of further interesting applications.

4.9.1 Domain Walls and the 2d Black Hole

Recall that we saw in Section 4.3.2 that the relative moduli space of a two domain walls with charge $\vec{g} = \vec{\alpha}_1 + \vec{\alpha}_2$ is the cigar shown in figure 22. Suppose we consider domain walls as strings in a $d = 2 + 1$ dimensional theory, so that the worldvolume of the domain walls is $d = 1 + 1$ dimensional. Then the low-energy dynamics of two domain walls is described by a sigma-model on the cigar.

There is a very famous conformal field theory with a cigar target space. It is known as the two-dimensional black hole [250]. It has metric,

$$ds^2_{BH} = k^2[dR^2 + \tanh^2 R \, d\theta^2] \tag{4.55}$$

The non-trivial curvature at the tip of the cigar is cancelled by a dilaton which has the profile

$$\Phi = \Phi_0 - 2\cosh R \tag{4.56}$$

So is the dynamics of the domain wall system determined by this conformal field theory? Well, not so obviously: the metric on the domain wall moduli space $\mathcal{W}_{\vec{\alpha}_1+\vec{\alpha}_2}$ does not coincide with (4.55). However, $d = 1 + 1$ dimensional theory is not conformal and the metric flows as we move towards the infra-red. There is a subtlety with the dilaton which one can evade by endowing the coordinate R with a suitable anomalous transformation under RG flow. With this caveat, it can be shown that the theory on two domain walls in $d = 2 + 1$ dimensions does indeed flow towards the conformal theory of the black hole with the identification $k = 2v^2/m$ [251].

The conformal field theory of the 2d black hole is dual to Liouville theory [252, 253]. If we deal with supersymmetric theories, this $\mathcal{N} = (2, 2)$ conformal field theory has Lagrangian

$$L_{Liouville} = \int d^4\theta \, \frac{1}{2k}|Y|^2 + \frac{\mu}{2}\int d^2\theta \, e^{-Y} + \text{h.c.} \tag{4.57}$$

and the equivalence between the two theories was proven using the techniques of mirror symmetry in [254]. In fact, one can also prove this duality by studying the dynamics of domain walls. Which is rather cute. We work with the $\mathcal{N} = 4$ (eight supercharges) $U(1)$ gauge theory in $d = 2+1$ with N_f charged hypermultiplets. As we sketched above, if we quantize the low-energy dynamics of the domain walls, we find the $\mathcal{N} = (2, 2)$ conformal theory on the cigar. However, there is an alternative way to proceed: we could choose first to integrate out some of the matter in three dimensions. Let's get rid of the charged hypermultiplets to leave a low-energy effective action for the vector multiplet. As well as the gauge field, the vector multiplet contains a triplet of real scalars $\boldsymbol{\phi}$, the first of which is identified with the ϕ we met in (4.1). The low-energy dynamics of this effective theory in $d = 2 + 1$ dimensions can be shown to be

$$L_{eff} = H(\boldsymbol{\phi}) \, \partial_\mu \boldsymbol{\phi} \cdot \partial^\mu \boldsymbol{\phi} + H^{-1}(\boldsymbol{\phi})(\partial_\mu \sigma + \boldsymbol{\omega} \cdot \partial_\mu \boldsymbol{\phi})^2 - v^4 H^{-1} \tag{4.58}$$

Here σ is the dual photon (see (2.62)) and $\nabla \times \boldsymbol{\omega} = \nabla H$, while the harmonic function H includes the corrections from integrating out the N_f hypermultiplets,

$$H(\boldsymbol{\phi}) = \frac{1}{e^2} + \sum_{i=1}^{N_f} \frac{1}{|\boldsymbol{\phi} - \mathbf{m}_i|} \tag{4.59}$$

where each triplet $\mathbf{m}_i$ is given by $\mathbf{m}_i = (m_i, 0, 0)$. We can now look for domain walls in this $d = 2+1$ effective theory. Since we want to study two domain walls, let's set $N_f = 3$. We see that the theory then has three, isolated vacua, at $\boldsymbol{\phi} = (\phi, 0, 0) = (m_i, 0, 0)$.

We now want to study the domain wall that interpolates between the two outer vacua $\phi = m_1$ and $\phi = m_3$. It's not hard to show that, in contrast to the microscopic theory (4.1), there is no domain wall solutions interpolating between these vacua. One can find a $\vec{\alpha}_1$ domain wall interpolating between $\phi = m_1$ and $\phi = m_2$. There is also a $\vec{\alpha}_2$ domain wall interpolating between $\phi = m_2$ and $\phi = m_3$. But no $\vec{\alpha}_1 + \vec{\alpha}_2$ domain wall between the two extremal vacua $\phi = m_1$ and $\phi = m_3$. The reason is essentially that only a single scalar, ϕ, changes in the domain wall profile, with equations of motion given by flow equations,

$$\partial_3 \phi = v^2 H^{-1}(\phi) \tag{4.60}$$

But since we have only a single scalar field, it must actually pass through the middle vacuum (as opposed to merely getting close) at which point the flow equations tell us $\partial_3 \phi = 0$ and it doesn't move anymore.

Although there is no solution interpolating between $\phi = m_1$ and $\phi = m_2$, one can always write down an approximate solution simply by superposing the $\vec{\alpha}_1$ and $\vec{\alpha}_2$ domain walls in such a way that they are well separated. One can then watch the evolution of this configuration under the equations of motion and, from this, extract an effective force between the domain wall [255]. For the case in hand, this calculation was performed in [251], where it was shown that the force is precisely that arising from the Liouville Lagrangian (4.57). In this way, we can use the dynamics of domain walls to derive the mirror symmetry between the cigar and Liouville theory.

4.9.2 Field Theory D-Branes

As we saw in Section 4.3.2 of this lecture, the moduli space of a single domain wall is $\mathcal{W} \cong \mathbf{R} \times \mathbf{S}^1$. This means that the theory living on the $d = 2 + 1$ dimensional worldvolume of the domain wall contains a scalar X, corresponding to fluctuations of the domain wall in the x^3 direction, together with a periodic scalar θ determining the phase of the wall. But in $d = 2 + 1$ dimensions, a periodic scalar can be dualized in favor of a photon living on the wall $4\pi v^2\partial_\mu\theta = \epsilon_{\mu\nu\rho}F^{\nu\rho}$. Thus the low-energy dynamics of the wall can alternatively be described by a free $U(1)$ gauge theory with a neutral scalar X,

$$L_{\rm wall} = \tfrac{1}{2}T_{\rm wall}\left((\partial_\mu X)^2 + \frac{1}{16\pi^2 v^4}F_{\mu\nu}F^{\mu\nu}\right) \tag{4.61}$$

This is related to the mechanism for gauge field localization described in [256].

As we have seen above, the theory also contains vortex strings. These vortex strings can end on the domain wall, where their ends are electrically charged. In other words, the domain walls are semi-classical D-branes for the vortex strings. These D-branes were first studied in [257, 211, 239]. (Semi-classical D-brane configurations in other theories have been studied in [258, 259, 260] in situations without the worldvolume gauge field). The simplest way to see that the domain wall is D-brane is using the BIon spike described in Section 2.7.4, where we described monopole as D-branes in $d = 5+1$ dimensions.

We can also see this D-brane solution from the perspective of the bulk theory. In fact, the solution obeys the equations (4.41) that we wrote down before. To see this, let's complete the square again but we should be more careful in keeping total derivatives. In a theory with multiple vacua, we have

$$\begin{aligned}\mathcal{H} = &\int d^3x\ \frac{1}{2e^2}{\rm Tr}\ \left[(\mathcal{D}_1\phi + B_1)^2 + (\mathcal{D}_2 + B_2)^2 + (\mathcal{D}_3\phi + B_3 - e^2(\sum_{i=1}^N q_i q_i^\dagger - v^2))^2\right] \\ &+ \sum_{i=1}^N |(\mathcal{D}_1 - i\mathcal{D}_2)q_i|^2 + \sum_{i=1}^N |\mathcal{D}_3 q_i - (\phi - m_i)q_i|^2 + {\rm Tr}\ [-v^2 B_3 - \frac{1}{e^2}\partial_i(\phi B_i) + v^2\partial_3\phi] \\ \geq &\left(\int dx^1 dx^2\ T_{\rm wall}\right) + \left(\int dx^3\ T_{\rm vortex}\right) + M_{\rm mono}\end{aligned} \tag{4.62}$$

and we indeed find the central charge appropriate for the domain wall. In fact these equations were first discovered in abelian theories to describe D-brane objects [211].

These equations have been solved analytically in the limit $e^2 \to \infty$ [257, 229]. Moreover, when multiple domain walls are placed in parallel along the line, one can construct solutions with many vortex strings stretched between them as figure 31, taken from [229], graphically illustrates.

Some final points on the field theoretic D-branes

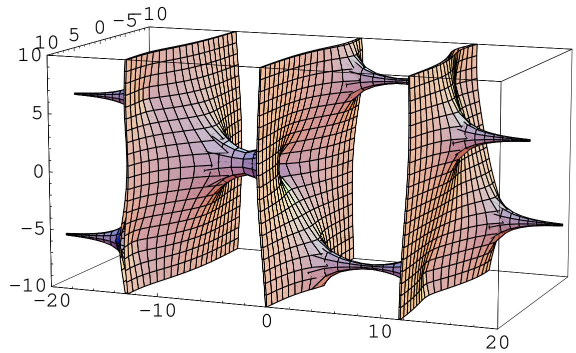

Figure 31: Plot of a field theoretic D-brane configuration [229].

- In each vacuum there are N_c different vortex strings. Not all of them can end on the bordering domain walls. There exist selection rules describing which vortex string can end on a given wall. For the $\vec{\alpha}_i$ domain wall, the string associated to q_i can end from the left, while the string associated to q_{i+1} can end from the right [214].

- For finite e^2, there is a negative binding energy when the string attaches itself to the domain wall, arising from the monopole central charge in (4.62). Known as a boojum, it was studied in this context in [214, 261]. (The name boojum was given by Mermin to a related configuration in superfluid ^{3}He [262]).

- One can develop an open string description of the domain wall dynamics, in which the motion of the walls is governed by the quantum effects of new light states that appear as the walls approach. Chern-Simons interactions on the domain wall worldvolume are responsible for stopping the walls from passing. Details can be found in [263].

Acknowledgements

These lectures were given in June 2005 at TASI, the Theoretical Advanced Study Institute held at the University of Colorado in Boulder. It was a fun week, made especially enjoyable by the sharp questions and interest of the students. I'd also like to thank the organizers, Shamit Kachru, K.T. Mahanthappa and Eva Silverstein, for inviting me to speak. I'm grateful to Jacob Bourjaily, Mohammad Edalati and Bonnie Parker for offering many comments. Finally, my thanks to Allan Adams, Nick Dorey, Greg Moore, Arvind Rajaraman, Moshe Rozali, Paul Townsend and Erick Weinberg for recent discussions on some of the issues mentioned in these notes.

References

[1] S. Weinberg, "*The quantum theory of fields. Vol. 2: Modern applications*," Cambridge University Press.

[2] N. S. Manton and P. Sutcliffe, "*Topological solitons*,", Cambridge University Press.

[3] J. Polchinski, "*Lectures on D-branes*," arXiv:hep-th/9611050; "*String Theory, Vols 1 and 2*", Cambridge University Press.

[4] A. Giveon and D. Kutasov, "*Brane dynamics and gauge theory*," Rev. Mod. Phys. **71**, 983 (1999) [arXiv:hep-th/9802067].

[5] C. V. Johnson, "D-brane primer," arXiv:hep-th/0007170; "*D-Branes*", Cambridge University Press.

[6] A. A. Belavin, A. M. Polyakov, A. S. Schwartz and Y. S. Tyupkin, "*Pseudoparticle Solutions Of The Yang-Mills Equations*," Phys. Lett. B **59**, 85 (1975).

[7] G. 't Hooft, "*Computation Of The Quantum Effects Due To A Four-Dimensional Pseudoparticle*," Phys. Rev. D **14**, 3432 (1976) [Erratum-ibid. D **18**, 2199 (1978)].

[8] R. Jackiw and C. Rebbi, "*Vacuum Periodicity In A Yang-Mills Quantum Theory*," Phys. Rev. Lett. **37**, 172 (1976).

[9] C. G. . Callan, R. F. Dashen and D. J. Gross, "*Toward A Theory Of The Strong Interactions*," Phys. Rev. D **17**, 2717 (1978).

[10] S.K. Donaldson and P.B. Kronheimer, "The Geometry of Four Manifolds", Oxford University Press (1990).

[11] S. Coleman, "*Uses of Instantons*" in Aspects of Symmetry, Cambridge University Press (1985)

[12] M. A. Shifman and A. I. Vainshtein, "*Instantons versus supersymmetry: Fifteen years later*," arXiv:hep-th/9902018.

[13] N. Dorey, T. J. Hollowood, V. V. Khoze and M. P. Mattis, "*The calculus of many instantons*," Phys. Rept. **371**, 231 (2002) [arXiv:hep-th/0206063].

[14] E. B. Bogomoln'yi, "*The Stability of Classical Solutions,*" Sov. J. Nucl. Phys, **24** (1976) 49.

[15] E. Witten, "*Some Exact Multipseudoparticle Solutions Of Classical Yang-Mills Theory*," Phys. Rev. Lett. **38**, 121 (1977).

[16] G. 't Hooft, Unpublished; R. Jackiw, C. Nohl and C. Rebbi, "*Conformal Properties Of Pseudoparticle Configurations*," Phys. Rev. D **15**, 1642 (1977).

[17] J. Bagger and E. Witten, "*Matter Couplings In N=2 Supergravity* ," Nucl. Phys. B **222**, 1 (1983).

[18] B. de Wit, M. Rocek and S. Vandoren, "*Hypermultiplets, hyperkaehler cones and quaternion-Kaehler geometry*" JHEP **0102**, 039 (2001) [arXiv:hep-th/0101161].

[19] J. A. Harvey and A. Strominger, "*String theory and the Donaldson polynomial*," Commun. Math. Phys. **151**, 221 (1993) [arXiv:hep-th/9108020].

[20] J. P. Gauntlett, "*Low-energy dynamics of N=2 supersymmetric monopoles*" Nucl. Phys. B **411**, 443 (1994) [arXiv:hep-th/9305068].

[21] M. F. Atiyah and I. M. Singer, "*Dirac Operators Coupled To Vector Potentials*," Proc. Nat. Acad. Sci. **81**, 2597 (1984).

[22] A. Maciocia, "*Metrics on the moduli spaces of instantons over Euclidean four space*," Commun. Math. Phys. **135**, 467 (1991).

[23] E. Witten and D. I. Olive, "*Supersymmetry Algebras That Include Topological Charges*," Phys. Lett. B **78**, 97 (1978).

[24] G. Sierra and P. K. Townsend, "*The Hyperkahler Supersymmetric Sigma Model In Six-Dimensions*," Phys. Lett. B **124**, 497 (1983).

[25] J. D. Blum, "*Supersymmetric quantum mechanics of monopoles in N=4 Yang-Mills theory*," Phys. Lett. B **333**, 92 (1994) [arXiv:hep-th/9401133].

[26] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Y. I. Manin, "*Construction Of Instantons*," Phys. Lett. A **65** (1978) 185.

[27] M. R. Douglas, "*Branes within branes*," arXiv:hep-th/9512077.

[28] M. R. Douglas, "*Gauge Fields and D-branes*," J. Geom. Phys. **28**, 255 (1998) [arXiv:hep-th/9604198].

[29] E. Witten, "*Sigma models and the ADHM construction of instantons*," J. Geom. Phys. **15**, 215 (1995) [arXiv:hep-th/9410052].

[30] N. J. Hitchin, A. Karlhede, U. Lindstrom and M. Rocek, "*Hyperkahler Metrics And Supersymmetry*," Commun. Math. Phys. **108**, 535 (1987).

[31] N. H. Christ, E. J. Weinberg and N. K. Stanton, "*General Self-Dual Yang-Mills Solutions*," Phys. Rev. D **18**, 2013 (1978).

[32] E. Corrigan and P. Goddard, "*Some Aspects Of Instantons*," DAMTP 79/18, Lectures given at Canadian Mathematical Society Meeting, Montreal, Canada, Sep 3-8, 1979

[33] H. Osborn, "*Semiclassical Functional Integrals For Selfdual Gauge Fields*," Annals Phys. **135**, 373 (1981).

[34] N. Dorey, V. V. Khoze and M. P. Mattis, "*Multi-Instanton Calculus in N=2 Supersymmetric Gauge Theory*," Phys. Rev. D **54**, 2921 (1996) [arXiv:hep-th/9603136]; "*Supersymmetry and the multi-instanton measure*," Nucl. Phys. B **513**, 681 (1998) [arXiv:hep-th/9708036].

[35] N. Nekrasov and A. Schwarz, "*Instantons on noncommutative R**4 and (2,0) superconformal six dimensional theory*," Commun. Math. Phys. **198**, 689 (1998) [arXiv:hep-th/9802068].

[36] H. Nakajima, "*Resolutions of Moduli Spaces of Ideal Instantons on* R^4", in Topology, Geometry and Field Theory, World Scientific 1994.

[37] N. A. Nekrasov, "*Trieste lectures on solitons in noncommutative gauge theories*," arXiv:hep-th/0011095.

[38] J. A. Harvey, "*Komaba lectures on noncommutative solitons and D-branes*," arXiv:hep-th/0102076.

[39] N. Seiberg and E. Witten, "*The D1/D5 system and singular CFT*," JHEP **9904**, 017 (1999) [arXiv:hep-th/9903224].

[40] T. Eguchi and A. J. Hanson, "*Asymptotically Flat Selfdual Solutions To Euclidean Gravity*," Phys. Lett. B **74**, 249 (1978).

[41] K. M. Lee, D. Tong and S. Yi, "*The moduli space of two U(1) instantons on noncommutative* R^4 *and* $R^3 \times S^1$," Phys. Rev. D **63**, 065017 (2001) [arXiv:hep-th/0008092].

[42] M. Hamanaka, Y. Imaizumi and N. Ohta, "*Moduli space and scattering of D0-branes in noncommutative super Yang-Mills theory*," Phys. Lett. B **529**, 163 (2002) [arXiv:hep-th/0112050].

[43] I. Affleck, M. Dine and N. Seiberg, "*Supersymmetry Breaking By Instantons*," Phys. Rev. Lett. **51**, 1026 (1983); "*Dynamical Supersymmetry Breaking In Supersymmetric QCD*," Nucl. Phys. B **241**, 493 (1984).

[44] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, "*Instanton Effects In Supersymmetric Theories*," Nucl. Phys. B **229**, 407 (1983); "*Exact Gell-Mann-Low Function Of Supersymmetric Yang-Mills Theories From Instanton Calculus*," Nucl. Phys. B **229**, 381 (1983); "*Supersymmetric Instanton Calculus: Gauge Theories With Matter*," Nucl. Phys. B **260**, 157 (1985) [Yad. Fiz. **42**, 1499 (1985)].

[45] D. Finnell and P. Pouliot, "*Instanton calculations versus exact results in four-dimensional SUSY gauge theories*," Nucl. Phys. B **453**, 225 (1995) [arXiv:hep-th/9503115].

[46] N. A. Nekrasov, "*Seiberg-Witten prepotential from instanton counting*," Adv. Theor. Math. Phys. **7**, 831 (2004) [arXiv:hep-th/0206161]; arXiv:hep-th/0306211.

[47] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "*Large N field theories, string theory and gravity*," Phys. Rept. **323**, 183 (2000) [arXiv:hep-th/9905111].

[48] T. Banks and M. B. Green, "*Non-perturbative effects in* $AdS_\times S^5$ *string theory and* $d = 4$ *SUSY Yang-Mills*," JHEP **9805**, 002 (1998) [arXiv:hep-th/9804170].

[49] M. Bianchi, M. B. Green, S. Kovacs and G. Rossi, "*Instantons in supersymmetric Yang-Mills and D-instantons in IIB superstring theory*," JHEP **9808**, 013 (1998) [arXiv:hep-th/9807033].

[50] N. Dorey, T. J. Hollowood, V. V. Khoze, M. P. Mattis and S. Vandoren, "*Multi instanton calculus and the AdS/CFT correspondence in N=4 superconformal field theory*," Nucl. Phys. B **552**, 88 (1999) [arXiv:hep-th/9901128].

[51] T. J. Hollowood, V. V. Khoze and M. P. Mattis, "*Summing the instanton series in* $N = 2$ *superconformal large-N QCD*," JHEP **9910**, 019 (1999) [arXiv:hep-th/9905209].

[52] T. J. Hollowood and V. V. Khoze, "*ADHM and D-instantons in orbifold AdS/CFT duality*," Nucl. Phys. B **575**, 78 (2000) [arXiv:hep-th/9908035].

[53] E. Gava, K. S. Narain and M. H. Sarmadi, "*Instantons in* $N = 2$ *Sp(N) superconformal gauge theories and the AdS/CFT correspondence*," Nucl. Phys. B **569**, 183 (2000) [arXiv:hep-th/9908125].

[54] T. J. Hollowood, "*Instantons, finite N = 2 Sp(N) theories and the AdS/CFT correspondence*," JHEP **9911**, 012 (1999) [arXiv:hep-th/9908201].

[55] M. B. Green and C. Stahn, "*D3-branes on the Coulomb branch and instantons*," JHEP **0309**, 052 (2003) [arXiv:hep-th/0308061].

[56] I. R. Klebanov and A. A. Tseytlin, "*Entropy of Near-Extremal Black p-branes*," Nucl. Phys. B **475**, 164 (1996) [arXiv:hep-th/9604089].

[57] J. A. Harvey, R. Minasian and G. W. Moore, "*Non-abelian tensor-multiplet anomalies*," JHEP **9809**, 004 (1998) [arXiv:hep-th/9808060].

[58] M. Rozali, "*Matrix theory and U-duality in seven dimensions*," Phys. Lett. B **400**, 260 (1997) [arXiv:hep-th/9702136].

[59] M. Berkooz, M. Rozali and N. Seiberg, "*Matrix description of M theory on T^4 and T^5*," Phys. Lett. B **408**, 105 (1997) [arXiv:hep-th/9704089].

[60] N. Seiberg, "*Notes on theories with 16 supercharges*," Nucl. Phys. Proc. Suppl. **67**, 158 (1998) [arXiv:hep-th/9705117].

[61] E. Witten, "*Constraints On Supersymmetry Breaking*," Nucl. Phys. B **202**, 253 (1982).

[62] K. M. Lee and P. Yi, "*Quantum spectrum of instanton solitons in five dimensional noncommutative $U(N)$ theories*," Phys. Rev. D **61**, 125015 (2000) [arXiv:hep-th/9911186].

[63] N. Dorey, T. J. Hollowood and V. V. Khoze, "*Notes on soliton bound-state problems in gauge theory and string theory*," arXiv:hep-th/0105090.

[64] N. Itzhaki, J. M. Maldacena, J. Sonnenschein and S. Yankielowicz, "*Supergravity and the large N limit of theories with sixteen supercharges*," Phys. Rev. D **58**, 046004 (1998) [arXiv:hep-th/9802042].

[65] P. Nelson and S. R. Coleman, "*What Becomes Of Global Color*," Nucl. Phys. B **237**, 1 (1984).

[66] K. M. Lee and P. Yi, "*Monopoles and instantons on partially compactified D-branes*," Phys. Rev. D **56**, 3711 (1997) [arXiv:hep-th/9702107].

[67] T. C. Kraan and P. van Baal, "*Periodic instantons with non-trivial holonomy*," Nucl. Phys. B **533**, 627 (1998) [arXiv:hep-th/9805168].

[68] R. Dijkgraaf, "*Instanton strings and hyperKaehler geometry*," Nucl. Phys. B **543**, 545 (1999) [arXiv:hep-th/9810210].

[69] P. A. M. Dirac, "*Quantised Singularities In The Electromagnetic Field,*" Proc. Roy. Soc. Lond. A **133** (1931) 60; "*The Theory Of Magnetic Poles*," Phys. Rev. **74** (1948) 817.

[70] G. 't Hooft, "*Magnetic Monopoles In Unified Gauge Theories*," Nucl. Phys. B **79**, 276 (1974).

[71] A. M. Polyakov, "*Particle Spectrum In Quantum Field Theory*," JETP Lett. **20** (1974) 194 [Pisma Zh. Eksp. Teor. Fiz. **20** (1974) 430].

[72] P. M. Sutcliffe, "*BPS monopoles*," Int. J. Mod. Phys. A **12**, 4663 (1997) [arXiv:hep-th/9707009].

[73] Y. Shnir, "*SU(N) monopoles with and without SUSY*," arXiv:hep-th/0508210; "*Magnetic Monopoles*", Springer (2005).

[74] M. F. Atiyah and N. J. Hitchin, "*The Geometry And Dynamics Of Magnetic Monopoles.*", M.B. Porter Lectures, Princeton University Press (1988)

[75] J. A. Harvey, "*Magnetic monopoles, duality, and supersymmetry*," arXiv:hep-th/9603086.

[76] E. J Weinberg and P. Yi, to appear.

[77] P. Goddard, J. Nuyts and D. I. Olive, "*Gauge Theories And Magnetic Charge*," Nucl. Phys. B **125**, 1 (1977).

[78] T. T. Wu and C. N. Yang, "*Dirac Monopole Without Strings: Monopole Harmonics*," Nucl. Phys. B **107**, 365 (1976).

[79] M. K. Prasad and C. M. Sommerfield, "*An Exact Classical Solution For The 't Hooft Monopole And The Julia-Zee Dyon*," Phys. Rev. Lett. **35**, 760 (1975).

[80] E. J. Weinberg, "*Parameter Counting For Multi Monopole Solutions*," Phys. Rev. D **20**, 936 (1979); "*Fundamental Monopoles And Multi Monopole Solutions For Arbitrary Simple Gauge Groups*," Nucl. Phys. B **167**, 500 (1980).

[81] C. Callias, "*Index Theorems On Open Spaces*," Commun. Math. Phys. **62**, 213 (1978).

[82] N. S. Manton, "*The Force Between 'T Hooft-Polyakov Monopoles*," Nucl. Phys. B **126** (1977) 525.

[83] C. J. Houghton and P. M. Sutcliffe, "*Tetrahedral and cubic monopoles*," Commun. Math. Phys. **180**, 343 (1996) [arXiv:hep-th/9601146].

[84] N. S. Manton, "*A Remark On The Scattering Of Bps Monopoles*," Phys. Lett. B **110**, 54 (1982).

[85] N. S. Manton, "*Monopole Interactions At Long Range*," Phys. Lett. B **154**, 397 (1985) [Erratum-ibid. **157B**, 475 (1985)].

[86] G. W. Gibbons and N. S. Manton, "*The Moduli space metric for well separated BPS monopoles*," Phys. Lett. B **356**, 32 (1995) [arXiv:hep-th/9506052].

[87] K. M. Lee, E. J. Weinberg and P. Yi, "*The Moduli Space of Many BPS Monopoles for Arbitrary Gauge Groups*," Phys. Rev. D **54**, 1633 (1996) [arXiv:hep-th/9602167].

[88] G. W. Gibbons and N. S. Manton, "*Classical And Quantum Dynamics Of BPS Monopoles*," Nucl. Phys. B **274**, 183 (1986).

[89] A. Sen, "*Strong coupling dynamics of branes from M-theory*," JHEP **9710**, 002 (1997) [arXiv:hep-th/9708002].

[90] N. Seiberg and E. Witten, "*Gauge dynamics and compactification to three dimensions*," arXiv:hep-th/9607163.

[91] A. Hanany and E. Witten, "*Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics*," Nucl. Phys. B **492**, 152 (1997) [arXiv:hep-th/9611230].

[92] A. Sen, "*Dynamics of multiple Kaluza-Klein monopoles in M and string theory*," Adv. Theor. Math. Phys. **1**, 115 (1998) [arXiv:hep-th/9707042].

[93] E. Witten, "*Heterotic string conformal field theory and A-D-E singularities*," JHEP **0002**, 025 (2000) [arXiv:hep-th/9909229].

[94] A. Hanany and A. Zaffaroni, "*Monopoles in string theory*," JHEP **9912**, 014 (1999) [arXiv:hep-th/9911113].

[95] S. A. Connell, "*The Dynamics of the* $SU(3)$ *Charge* $(1,1)$ *Magnetic Monopole*", University of South Australia Preprint.

[96] J. P. Gauntlett and D. A. Lowe, "*Dyons and S-Duality in N=4 Supersymmetric Gauge Theory*," Nucl. Phys. B **472**, 194 (1996) [arXiv:hep-th/9601085].

[97] K. M. Lee, E. J. Weinberg and P. Yi, "*Electromagnetic Duality and* $SU(3)$ *Monopoles*," Phys. Lett. B **376**, 97 (1996) [arXiv:hep-th/9601097].

[98] B. Julia and A. Zee, "*Poles With Both Magnetic And Electric Charges In Nonabelian Gauge Theory*," Phys. Rev. D **11**, 2227 (1975).

[99] J. S. Schwinger, "*A Magnetic Model Of Matter*," Science **165** (1969) 757.

[100] E. Witten, "*Dyons Of Charge* $e\theta/2\pi$," Phys. Lett. B **86**, 283 (1979).

[101] O. Bergman, "*Three-pronged strings and 1/4 BPS states in N = 4 super-Yang-Mills theory*," Nucl. Phys. B **525**, 104 (1998) [arXiv:hep-th/9712211].

[102] K. M. Lee and P. Yi, "*Dyons in N = 4 supersymmetric theories and three - pronged strings*," Phys. Rev. D **58**, 066005 (1998) [arXiv:hep-th/9804174].

[103] D. Tong, "*A note on 1/4-BPS states*," Phys. Lett. B **460**, 295 (1999) [arXiv:hep-th/9902005].

[104] D. Bak, C. k. Lee, K. M. Lee and P. Yi, "*Low energy dynamics for 1/4 BPS dyons*," Phys. Rev. D **61**, 025001 (2000) [arXiv:hep-th/9906119].

[105] J. P. Gauntlett, C. j. Kim, K. M. Lee and P. Yi, "*General low energy dynamics of supersymmetric monopoles*," Phys. Rev. D **63**, 065020 (2001) [arXiv:hep-th/0008031].

[106] R. Jackiw and C. Rebbi, "*Solitons With Fermion Number 1/2*," Phys. Rev. D **13**, 3398 (1976).

[107] W. Nahm, "*The Construction of all Self-Dual Multi-Monopoles by the ADHM Method*", in "Monopoles in Quantum Field Theory", eds. N. S. Craigie, P. Goddard and W. Nahm, Singapore, World Scientific, 1982.

[108] J. Hurtubise and M. K. Murray, "*On The Construction Of Monopoles For The Classical Groups*" Commun. Math. Phys. **122**, 35 (1989).

[109] D. E. Diaconescu, "*D-branes, monopoles and Nahm equations*," Nucl. Phys. B **503**, 220 (1997) [arXiv:hep-th/9608163].

[110] A. Kapustin and S. Sethi, "*The Higgs branch of impurity theories*," Adv. Theor. Math. Phys. **2**, 571 (1998) [arXiv:hep-th/9804027].

[111] D. Tsimpis, "*Nahm equations and boundary conditions*," Phys. Lett. B **433**, 287 (1998) [arXiv:hep-th/9804081].

[112] O. DeWolfe, D. Z. Freedman and H. Ooguri, "*Holography and defect conformal field theories*," Phys. Rev. D **66**, 025009 (2002) [arXiv:hep-th/0111135].

[113] X. G. Chen and E. J. Weinberg, "*ADHMN boundary conditions from removing monopoles*," Phys. Rev. D **67**, 065020 (2003) [arXiv:hep-th/0212328].

[114] M. K. Murray, "*A note on the (1, 1,..., 1) monopole metric*," J. Geom. Phys. **23**, 31 (1997) [arXiv:hep-th/9605054].

[115] K. Hashimoto and S. Terashima, "*Stringy derivation of Nahm construction of monopoles*," arXiv:hep-th/0507078.

[116] E. J. Weinberg and P. Yi, "*Explicit multimonopole solutions in SU(N) gauge theory*," Phys. Rev. D **58**, 046001 (1998) [arXiv:hep-th/9803164].

[117] N. D. Lambert and D. Tong, "*Dyonic instantons in five-dimensional gauge theories*," Phys. Lett. B **462**, 89 (1999) [arXiv:hep-th/9907014].

[118] L. Alvarez-Gaume and D. Z. Freedman, "*Potentials For The Supersymmetric Nonlinear Sigma Model*," Commun. Math. Phys. **91**, 87 (1983).

[119] I. Affleck, "*On Constrained Instantons*," Nucl. Phys. B **191**, 429 (1981).

[120] E. Eyras, P. K. Townsend and M. Zamaklar, "*The heterotic dyonic instanton*," JHEP **0105**, 046 (2001) [arXiv:hep-th/0012016].

[121] K. Peeters and M. Zamaklar, "*Motion on moduli spaces with potentials*," JHEP **0112**, 032 (2001) [arXiv:hep-th/0107164].

[122] A. M. Polyakov, "*Quark Confinement And Topology Of Gauge Groups*," Nucl. Phys. B **120**, 429 (1977).

[123] I. Affleck, J. A. Harvey and E. Witten, "*Instantons And (Super)Symmetry Breaking In (2+1)-Dimensions*," Nucl. Phys. B **206**, 413 (1982).

[124] J. Polchinski and P. Pouliot, "*Membrane scattering with M-momentum transfer*," Phys. Rev. D **56**, 6601 (1997) [arXiv:hep-th/9704029].

[125] N. Dorey, V. V. Khoze and M. P. Mattis, "*Multi-instantons, three dimensional gauge theory, and the Gauss-Bonnet-Chern theorem*" Nucl. Phys. B **502**, 94 (1997) [arXiv:hep-th/9704197].

[126] G. Chalmers and A. Hanany, "*Three dimensional gauge theories and monopoles*," Nucl. Phys. B **489**, 223 (1997) [arXiv:hep-th/9608105].

[127] D. Tong, "*Three-dimensional gauge theories and ADE monopoles*," Phys. Lett. B **448**, 33 (1999) [arXiv:hep-th/9803148].

[128] N. Dorey, V. V. Khoze, M. P. Mattis, D. Tong and S. Vandoren, "*Instantons, three-dimensional gauge theory, and the Atiyah-Hitchin manifold*," Nucl. Phys. B **502**, 59 (1997) [arXiv:hep-th/9703228].

[129] C. Montonen and D. I. Olive, "*Magnetic Monopoles As Gauge Particles?*," Phys. Lett. B **72**, 117 (1977).

[130] H. Osborn, "*Topological Charges For N=4 Supersymmetric Gauge Theories And Monopoles Of Spin 1*," Phys. Lett. B **83**, 321 (1979).

[131] A. Sen, "*Dyon - monopole bound states, selfdual harmonic forms on the multi - monopole moduli space, and SL(2,Z) invariance in string theory*," Phys. Lett. B **329**, 217 (1994) [arXiv:hep-th/9402032].

[132] N. Seiberg and E. Witten, "*Electric magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory*," Nucl. Phys. B **426**, 19 (1994) [Erratum-ibid. B **430**, 485 (1994)] [arXiv:hep-th/9407087]; "*Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD*," Nucl. Phys. B **431**, 484 (1994) [arXiv:hep-th/9408099].

[133] J. P. Gauntlett and J. A. Harvey, "*S duality and the dyon spectrum in N=2 superYang-Mills theory*," Nucl. Phys. B **463**, 287 (1996) [arXiv:hep-th/9508156].

[134] A. Rebhan, P. van Nieuwenhuizen and R. Wimmer, "*A new anomaly in the central charge of the* $N = 2$ *monopole*," Phys. Lett. B **594**, 234 (2004) [arXiv:hep-th/0401116].

[135] K. Hori and C. Vafa, "*Mirror symmetry*," arXiv:hep-th/0002222.

[136] O. Aharony, "*A brief review of little string theories*," Class. Quant. Grav. **17**, 929 (2000) [arXiv:hep-th/9911147].

[137] C. G. . Callan and J. M. Maldacena, "*Brane dynamics from the Born-Infeld action*," Nucl. Phys. B **513**, 198 (1998) [arXiv:hep-th/9708147].

[138] G. W. Gibbons, "*Born-Infeld particles and Dirichlet p-branes*," Nucl. Phys. B **514**, 603 (1998) [arXiv:hep-th/9709027].

[139] J. Gauntlett, D. Tong and P. Townsend, unpublished (2000)

[140] E. Corrigan, C. Devchand, D. B. Fairlie and J. Nuyts, "*First Order Equations For Gauge Fields In Spaces Of Dimension Greater Than Four*," Nucl. Phys. B **214**, 452 (1983).

[141] E. Witten, "*Bound states of strings and p-branes*," Nucl. Phys. B **460**, 335 (1996) [arXiv:hep-th/9510135].

[142] A. Vilenkin and E. P. S. Shellard, "*Cosmic Strings and other Topological Defects*", Cambridge University Press, 1994.

[143] M. B. Hindmarsh and T. W. B. Kibble, "*Cosmic strings*," Rept. Prog. Phys. **58**, 477 (1995) [arXiv:hep-ph/9411342].

[144] J. Polchinski, "*Introduction to cosmic F- and D-strings*," arXiv:hep-th/0412244.

[145] K. Rajagopal and F. Wilczek, "*The condensed matter physics of QCD*," arXiv:hep-ph/0011333.

[146] A. A. Abrikosov, "*On The Magnetic Properties Of Superconductors Of The Second Group*," Sov. Phys. JETP **5**, 1174 (1957) [Zh. Eksp. Teor. Fiz. **32**, 1442 (1957)].

[147] H. B. Nielsen and P. Olesen, "*Vortex-Line Models For Dual Strings*," Nucl. Phys. B **61**, 45 (1973).

[148] A. S. Schwarz, "*Field Theories With No Local Conservation Of The Electric Charge*," Nucl. Phys. B **208**, 141 (1982).

[149] M. G. Alford, K. Benson, S. R. Coleman, J. March-Russell and F. Wilczek, "*The Interactions And Excitations Of Nonabelian Vortices*," Phys. Rev. Lett. **64**, 1632 (1990) [Erratum-ibid. **65**, 668 (1990)]. Nucl. Phys. B **349**, 414 (1991).

[150] G. V. Dunne, "*Aspects of Chern-Simons theory*," arXiv:hep-th/9902115.

[151] A. Hanany and D. Tong, "*Vortices, instantons and branes*," JHEP **0307**, 037 (2003) [arXiv:hep-th/0306150].

[152] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, "*Nonabelian superconductors: Vortices and confinement in N=2 SQCD*," Nucl. Phys. B **673**, 187 (2003) [arXiv:hep-th/0307287].

[153] H. J. de Vega and F. A. Schaposnik, "*A Classical Vortex Solution Of The Abelian Higgs Model*," Phys. Rev. D **14**, 1100 (1976).

[154] E. J. Weinberg, "*Multivortex Solutions Of The Ginzburg-Landau Equations*," Phys. Rev. D **19**, 3008 (1979); "*Index Calculations For The Fermion-Vortex System*," Phys. Rev. D **24**, 2669 (1981).

[155] C. H. Taubes, "*Arbitrary N - Vortex Solutions To The First Order Landau-Ginzburg Equations*," Commun. Math. Phys. **72**, 277 (1980).

[156] T. M. Samols, "*Vortex scattering*," Commun. Math. Phys. **145**, 149 (1992).

[157] N. S. Manton and J. M. Speight, "*Asymptotic interactions of critically coupled vortices*," Commun. Math. Phys. **236**, 535 (2003) [arXiv:hep-th/0205307].

[158] M. Shifman and A. Yung, "*Non-Abelian string junctions as confined monopoles*," Phys. Rev. D **70**, 045004 (2004) [arXiv:hep-th/0403149].

[159] E. P. S. Shellard and P. J. Ruback, "*Vortex Scattering In Two-Dimensions*," Phys. Lett. B **209**, 262 (1988).

[160] P. J. Ruback, "*Vortex String Motion In The Abelian Higgs Model*," Nucl. Phys. B **296**, 669 (1988).

[161] E. Witten, "*Solutions of four-dimensional field theories via M-theory*," Nucl. Phys. B **500**, 3 (1997) [arXiv:hep-th/9703166].

[162] C. Bachas and M. B. Green, "*A classical manifestation of the Pauli exclusion principle*," JHEP **9801**, 015 (1998) [arXiv:hep-th/9712187].

[163] B. H. Lee, H. j. Lee, N. Ohta and H. S. Yang, "*Maxwell Chern-Simons solitons from type IIB string theory*," Phys. Rev. D **60**, 106003 (1999) [arXiv:hep-th/9904181].

[164] A. Hanany and K. Hori, "*Branes and $N = 2$ theories in two dimensions*," Nucl. Phys. B **513**, 119 (1998) [arXiv:hep-th/9707192].

[165] K. Hashimoto and D. Tong, "*Reconnection of non-abelian cosmic strings*," arXiv:hep-th/0506022.

[166] P. Forgacs and N. S. Manton, "*Space-Time Symmetries In Gauge Theories*," Commun. Math. Phys. **72**, 15 (1980).

[167] T. Vachaspati and A. Achucarro, "*Semilocal cosmic strings*," Phys. Rev. D **44**, 3067 (1991).

[168] A. Achucarro and T. Vachaspati, "*Semilocal and electroweak strings*," Phys. Rept. **327**, 347 (2000) [Phys. Rept. **327**, 427 (2000)] [arXiv:hep-ph/9904229].

[169] M. Hindmarsh, R. Holman, T. W. Kephart and T. Vachaspati, "*Generalized semilocal theories and higher Hopf maps*," Nucl. Phys. B **404**, 794 (1993) [arXiv:hep-th/9209088].

[170] M. Hindmarsh, "*Existence and stability of semilocal strings*," Phys. Rev. Lett. **68**, 1263 (1992).

[171] R. A. Leese and T. M. Samols, "*Interaction of semilocal vortices*," Nucl. Phys. B **396**, 639 (1993).

[172] E. Witten, "*Phases of $N = 2$ theories in two dimensions*," Nucl. Phys. B **403**, 159 (1993) [arXiv:hep-th/9301042].

[173] B. J. Schroers, "*The Spectrum of Bogomol'nyi Solitons in Gauged Linear Sigma Models*," Nucl. Phys. B **475**, 440 (1996) [arXiv:hep-th/9603101].

[174] R. S. Ward, "*Slowly Moving Lumps In The Cp**1 Model In (2+1)-Dimensions*," Phys. Lett. B **158**, 424 (1985).

[175] D. P. Jatkar, G. Mandal and S. R. Wadia, "*Nielsen-Olesen vortices in noncommutative Abelian Higgs model*," JHEP **0009**, 018 (2000) [arXiv:hep-th/0007078].

[176] D. Bak, K. M. Lee and J. H. Park, "*Noncommutative vortex solitons*," Phys. Rev. D **63**, 125010 (2001) [arXiv:hep-th/0011099].

[177] A. Hanany and D. Tong, "*Vortex strings and four-dimensional gauge dynamics*," JHEP **0404**, 066 (2004) [arXiv:hep-th/0403158].

[178] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, "*Instantons in the Higgs phase*," Phys. Rev. D **72**, 025011 (2005) [arXiv:hep-th/0412048].

[179] G. W. Moore and P. Nelson, "*The Etiology Of Sigma Model Anomalies*," Commun. Math. Phys. **100**, 83 (1985).

[180] J. D. Edelstein, C. Nunez and F. Schaposnik, "*Supersymmetry and Bogomolny equations in the Abelian Higgs model*," Phys. Lett. B **329**, 39 (1994) [arXiv:hep-th/9311055].

[181] D. R. Morrison and M. Ronen Plesser, "*Summing the instantons: Quantum cohomology and mirror symmetry in toric varieties*," Nucl. Phys. B **440**, 279 (1995) [arXiv:hep-th/9412236].

[182] M. Rocek and E. P. Verlinde, "*Duality, quotients, and currents*," Nucl. Phys. B **373**, 630 (1992) [arXiv:hep-th/9110053].

[183] R. Gregory, J. A. Harvey and G. W. Moore, "*Unwinding strings and T-duality of Kaluza-Klein and H-monopoles*," Adv. Theor. Math. Phys. **1**, 283 (1997) [arXiv:hep-th/9708086].

[184] D. Tong, "*NS5-branes, T-duality and worldsheet instantons*," JHEP **0207**, 013 (2002) [arXiv:hep-th/0204186].

[185] J. A. Harvey and S. Jensen, "*Worldsheet instanton corrections to the Kaluza-Klein monopole*," arXiv:hep-th/0507204.

[186] K. Okuyama, "*Linear sigma models of H and KK monopoles*," JHEP **0508**, 089 (2005) [arXiv:hep-th/0508097].

[187] A, Zee, "*Quantum Hall Fluids*", cond-mat/9501022.

[188] K. A. Intriligator and N. Seiberg, "*Mirror symmetry in three dimensional gauge theories*," Phys. Lett. B **387**, 513 (1996) [arXiv:hep-th/9607207].

[189] A. Kapustin and M. J. Strassler, "*On mirror symmetry in three dimensional Abelian gauge theories*," JHEP **9904**, 021 (1999) [arXiv:hep-th/9902033].

[190] J. de Boer, K. Hori, H. Ooguri and Y. Oz, "*Mirror symmetry in three-dimensional gauge theories, quivers and D-branes*," Nucl. Phys. B **493**, 101 (1997) [arXiv:hep-th/9611063].

[191] J. de Boer, K. Hori, H. Ooguri, Y. Oz and Z. Yin, "*Mirror symmetry in three-dimensional gauge theories, SL(2,Z) and D-brane moduli spaces*," Nucl. Phys. B **493**, 148 (1997) [arXiv:hep-th/9612131].

[192] B. Feng and A. Hanany, "*Mirror symmetry by O3-planes*," JHEP **0011**, 033 (2000) [arXiv:hep-th/0004092].

[193] V. Borokhov, A. Kapustin and X. k. Wu, "*Monopole operators and mirror symmetry in three dimensions*," JHEP **0212**, 044 (2002) [arXiv:hep-th/0207074].

[194] N. Ohta and D. Tomino, "*Noncommutative gauge dynamics from brane configurations with background B field*," Prog. Theor. Phys. **105**, 287 (2001) [arXiv:hep-th/0009021].

[195] O. Aharony, A. Hanany, K. A. Intriligator, N. Seiberg and M. J. Strassler, "*Aspects of $N = 2$ supersymmetric gauge theories in three dimensions*," Nucl. Phys. B **499**, 67 (1997) [arXiv:hep-th/9703110].

[196] J. de Boer, K. Hori, Y. Oz and Z. Yin, "*Branes and mirror symmetry in $N = 2$ supersymmetric gauge theories in three dimensions*," Nucl. Phys. B **502**, 107 (1997) [arXiv:hep-th/9702154].

[197] T. Kitao, K. Ohta and N. Ohta, "*Three-dimensional gauge dynamics from brane configurations with (p,q)-fivebrane*," Nucl. Phys. B **539**, 79 (1999) [arXiv:hep-th/9808111].

[198] N. Dorey and D. Tong, "*Mirror symmetry and toric geometry in three dimensional gauge theories*," JHEP **0005**, 018 (2000) [arXiv:hep-th/9911094].

[199] D. Tong, "*Dynamics of $N = 2$ supersymmetric Chern-Simons theories*," JHEP **0007**, 019 (2000) [arXiv:hep-th/0005186].

[200] A. Rebhan, P. van Nieuwenhuizen and R. Wimmer, "*Nonvanishing quantum corrections to the mass and central charge of the N=2 vortex and BPS saturation*," Nucl. Phys. B **679**, 382 (2004) [arXiv:hep-th/0307282].

[201] M. Aganagic, K. Hori, A. Karch and D. Tong, "*Mirror symmetry in 2+1 and 1+1 dimensions*," JHEP **0107**, 022 (2001) [arXiv:hep-th/0105075].

[202] E. J. Copeland and N. Turok, "*Cosmic String Interactions*," FERMILAB PUB 86-127-A

[203] E. P. S. Shellard, "*Understanding Intercommuting*," Proc. of Yale Workshop: Cosmic Strings: The Current Status (1988)

[204] A. Hanany and K. Hashimoto, "*Reconnection of colliding cosmic strings*," JHEP **0506**, 021 (2005) [arXiv:hep-th/0501031].

[205] R. A. Matzner, "*Interaction of $U(1)$ Cosmic Strings: Numerical Intercommutation*", Computers in Physics, **2**(5), 51 (1988)

[206] E. Witten, "*Cosmic Superstrings*," Phys. Lett. B **153**, 243 (1985).

[207] J. Polchinski, "*Collision Of Macroscopic Fundamental Strings*," Phys. Lett. B **209** (1988) 252.

[208] M. G. Jackson, N. T. Jones and J. Polchinski, "*Collisions of cosmic F- and D-strings*," arXiv:hep-th/0405229.

[209] S. Cecotti and C. Vafa, "*On classification of N=2 supersymmetric theories*," Commun. Math. Phys. **158**, 569 (1993) [arXiv:hep-th/9211097].

[210] E. R. Abraham and P. K. Townsend, "*Q Kinks*", Phys. Lett. B **291**, 85 (1992); '*'More On Q Kinks: A (1+1)-Dimensional Analog Of Dyons*", Phys. Lett. B **295**, 225 (1992).

[211] M. Shifman and A. Yung, "*Domain walls and flux tubes in $N = 2$ SQCD: D-brane prototypes*" Phys. Rev. D **67**, 125007 (2003) [arXiv:hep-th/0212293].

[212] D. Tong, "*The moduli space of BPS domain walls*" Phys. Rev. D **66**, 025013 (2002) [arXiv:hep-th/0202012].

[213] N. Sakai and Y. Yang, "*Moduli sapce of BPS walls in supersymmetric gauge theories*," arXiv:hep-th/0505136.

[214] N. Sakai and D. Tong, "*Monopoles, vortices, domain walls and D-branes: The rules of interaction*," JHEP **0503**, 019 (2005) [arXiv:hep-th/0501207].

[215] J. P. Gauntlett, D. Tong and P. K. Townsend, "*Multi Domain Walls in Massive Supersymmetric Sigma Models*", Phys. Rev. D **64**, 025010 (2001) [arXiv:hep-th/0012178].

[216] K. S. M. Lee, "*An index theorem for domain walls in supersymmetric gauge theories*," Phys. Rev. D **67**, 045009 (2003) [arXiv:hep-th/0211058].

[217] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, "*Non-Abelian walls in supersymmetric gauge theories*," Phys. Rev. D **70**, 125014 (2004) [arXiv:hep-th/0405194].

[218] Y. Isozumi, K. Ohashi and N. Sakai, "*Exact wall solutions in 5-dimensional SUSY QED at finite coupling*" JHEP **0311**, 060 (2003) [arXiv:hep-th/0310189].

[219] N. Dorey, "*The BPS spectra of two-dimensional supersymmetric gauge theories with twisted mass terms*" JHEP **9811**, 005 (1998) [arXiv:hep-th/9806056].

[220] K. Lee and H. U. Yee, "*New BPS objects in $N = 2$ supersymmetric gauge theories*," arXiv:hep-th/0506256.

[221] M. Eto, Y. Isozumi, M. Nitta and K. Ohashi, "*1/2, 1/4 and 1/8 BPS equations in SUSY Yang-Mills-Higgs systems: Field theoretical brane configurations*," arXiv:hep-th/0506257.

[222] J. P. Gauntlett, D. Tong and P. K. Townsend, "*Supersymmetric intersecting domain walls in massive hyper-Kaehler sigma models*" Phys. Rev. D **63**, 085001 (2001) [arXiv:hep-th/0007124].

[223] K. Kakimoto and N. Sakai, "*Domain wall junction in N = 2 supersymmetric QED in four dimensions*" Phys. Rev. D **68**, 065005 (2003) [arXiv:hep-th/0306077].

[224] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, "*Webs of walls*," arXiv:hep-th/0506135.

[225] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, "*Non-abelian webs of walls*," arXiv:hep-th/0508241.

[226] M. Arai, M. Naganuma, M. Nitta and N. Sakai, "*BPS wall in N = 2 SUSY nonlinear sigma model with Eguchi-Hanson manifold*," arXiv:hep-th/0302028; "*Manifest supersymmetry for BPS walls in N = 2 nonlinear sigma models*," Nucl. Phys. B **652**, 35 (2003) [arXiv:hep-th/0211103].

[227] A. Losev and M. Shifman, "*N = 2 sigma model with twisted mass and superpotential: Central charges and solitons*" Phys. Rev. D **68**, 045006 (2003) [arXiv:hep-th/0304003].

[228] M. Arai, E. Ivanov and J. Niederle, "*Massive nonlinear sigma models and BPS domain walls in harmonic superspace*," Nucl. Phys. B **680**, 23 (2004) [arXiv:hep-th/0312037].

[229] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, "*Construction of non-Abelian walls and their complete moduli space*," Phys. Rev. Lett. **93**, 161601 (2004); [arXiv:hep-th/0404198]. "*All exact solutions of a 1/4 BPS equation*," Phys. Rev. D **71**, 065018 (2005) [arXiv:hep-th/0405129].

[230] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, N. Sakai and Y. Tachikawa, "*Global structure of moduli space for BPS walls*," Phys. Rev. D **71**, 105009 (2005) [arXiv:hep-th/0503033].

[231] A. Rebhan, P. van Nieuwenhuizen and R. Wimmer, "*The anomaly in the central charge of the supersymmetric kink from dimensional regularization and reduction*," Nucl. Phys. B **648**, 174 (2003) [arXiv:hep-th/0207051].

[232] A. S. Goldhaber, A. Rebhan, P. van Nieuwenhuizen and R. Wimmer, "*Quantum corrections to mass and central charge of supersymmetric solitons*," Phys. Rept. **398**, 179 (2004) [arXiv:hep-th/0401152].

[233] D. Tong, "*Monopoles in the Higgs phase*" Phys. Rev. D **69**, 065003 (2004) [arXiv:hep-th/0307302].

[234] M. Hindmarsh and T. W. Kibble, "*Beads On Strings*" Phys. Rev. Lett. **55**, 2398 (1985).

[235] M. A. Kneipp, "*Z(k) string fluxes and monopole confinement in non-Abelian theories*" arXiv:hep-th/0211049.

[236] R. Auzzi, S. Bolognesi, J. Evslin and K. Konishi, "*Nonabelian monopoles and the vortices that confine them*" arXiv:hep-th/0312233.

[237] V. Markov, A. Marshakov and A. Yung, "*Non-Abelian vortices in N = 1* gauge theory*" arXiv:hep-th/0408235.

[238] R. Auzzi, S. Bolognesi and J. Evslin, "*Monopoles can be confined by 0, 1 or 2 vortices*", arXiv:hep-th/0411074.

[239] M. Shifman and A. Yung, "*Localization of non-Abelian gauge fields on domain walls at weak coupling (D-brane prototypes II)*" [arXiv:hep-th/0312257].

[240] M. Eto, M. Nitta, K. Ohashi and D. Tong, "*Skyrmions from Instantons inside Domain Walls*," arXiv:hep-th/0508130.

[241] M. F. Atiyah and N. S. Manton, "*Skyrmions From Instantons*," Phys. Lett. B **222**, 438 (1989).

[242] C. T. Hill, "*Topological solitons from deconstructed extra dimensions*," Phys. Rev. Lett. **88**, 041601 (2002) [arXiv:hep-th/0109068].

[243] C. T. Hill and C. K. Zachos, "*Dimensional deconstruction and Wess-Zumino-Witten terms*," Phys. Rev. D **71**, 046002 (2005) [arXiv:hep-th/0411157].

[244] D. T. Son and M. A. Stephanov, "*QCD and dimensional deconstruction*," Phys. Rev. D **69**, 065020 (2004) [arXiv:hep-ph/0304182].

[245] N. Dorey, T. J. Hollowood and D. Tong, "*The BPS spectra of gauge theories in two and four dimensions*" JHEP **9905**, 006 (1999) [arXiv:hep-th/9902134].

[246] N. D. Lambert and D. Tong, "*Kinky D-strings*" Nucl. Phys. B **569**, 606 (2000) [arXiv:hep-th/9907098].

[247] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta and N. Sakai, "*D-brane construction for non-Abelian walls*," Phys. Rev. D **71**, 125006 (2005) [arXiv:hep-th/0412024].

[248] A. Hanany and D. Tong, "*On monopoles and domain walls*," arXiv:hep-th/0507140.

[249] C. Bachas, J. Hoppe and B. Pioline, "*Nahm equations, $N = 1^\star$ domain walls, and D-strings in $AdS_5 \times S^5$*," JHEP **0107**, 041 (2001) [arXiv:hep-th/0007067].

[250] E. Witten, "*String Theory and Black Holes*", Phys. Rev. **D44** (1991) 314;

[251] D. Tong, "*Mirror mirror on the wall: On two-dimensional black holes and Liouville theory,*" JHEP **0304**, 031 (2003) [arXiv:hep-th/0303151].

[252] V. Fateev, A. Zamalodchikov and Al. Zamalodchikov, unpublished, as cited in [253]

[253] V. Kazakov, I. Kostov, D. Kutasov, "*A Matrix Model for the Two-Dimensional Black Hole*", Nucl. Phys. **B622** (2002) 141, hep-th/0101011.

[254] K. Hori and A. Kapustin "*Duality of the Fermionic 2d Black Hole and N=2 Liouville Theory as Mirror Symmetry*", JHEP **0108**, 045 (2001), hep-th/0104202.

[255] N. S. Manton, "*An Effective Lagrangian For Solitons*," Nucl. Phys. B **150**, 397 (1979).

[256] G. R. Dvali and M. A. Shifman, "*Domain walls in strongly coupled theories*" Phys. Lett. B **396**, 64 (1997) [Erratum-ibid. B **407**, 452 (1997)] [arXiv:hep-th/9612128].

[257] J. P. Gauntlett, R. Portugues, D. Tong and P. K. Townsend, "*D-brane solitons in supersymmetric sigma models*" Phys. Rev. D **63**, 085002 (2001) [arXiv:hep-th/0008221].

[258] S. M. Carroll and M. Trodden, "*Dirichlet topological defects*" Phys. Rev. D **57**, 5189 (1998) [arXiv:hep-th/9711099].

[259] G. Dvali and A. Vilenkin, "*Solitonic D-branes and brane annihilation,*" Phys. Rev. D **67**, 046002 (2003) [arXiv:hep-th/0209217].

[260] M. Bowick, A. De Felice and M. Trodden, "*The shapes of Dirichlet defects,*" JHEP **0310**, 067 (2003) [arXiv:hep-th/0306224].

[261] R. Auzzi, M. Shifman and A. Yung, "*Studying boojums in N = 2 theory with walls and vortices*," Phys. Rev. D **72**, 025002 (2005) [arXiv:hep-th/0504148].

[262] N. D. Mermin, "*Surface Singularities and Superflow in ^3He-A*", in Quantum Fluids and Solids, eds. S. B. Trickey, E. D. Adams and J. W. Dufty, Plnum, New York, pp 3-22.

[263] D. Tong, "*D-Branes in Field Theory*", arXiv:hep-th/0512192.