Numerical Relativity's arduous path to glory

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Numerical Relativity and Fundamental Fields
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Overview

- Introduction
- The Ancient World: The Birth of Numerical Relativity
- From the Dark ages to the Renaissance
- Towards the Holy Grail
- The gold rush years

1. Introduction, Motivation

Task: Solve this!

It's simple but it isn't easy…

How do we get the metric?

- The metric must obey $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$
- Ricci tensor, Einstein tensor, matter tensor

$R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$

$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R^{\mu}{}_{\mu}$ "Trace reverse Ricci"

$T_{\alpha\beta}$ "Matter"

Λ "Cosmological constant"

- Solutions: Easy! Take metric $g_{\alpha\beta}$
- Calculate $\Rightarrow G_{\alpha\beta}$
- Use that for $\Rightarrow T_{\alpha\beta}$

- Physically meaningful solutions: That's the hard part!

Solving Einstein's Eqs.: The toolbox

- **Analytic solutions**
  - Symmetry assumptions

  Schwarzschild, Kerr, FLRW, Vaidya, Tangherlini, Myers-Perry, ...

- **Perturbation theory**
  - Assume solution is close to a known "background" $g^{(0)}_{\alpha\beta}$
  - Expand $g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \dots \quad \Rightarrow$ linear system

  Regge-Wheeler-Zerilli-Moncrief, Teukolsky, QNMs, EOB, ...

- **Post-Newtonian theory**
  - Assume small velocities $\Rightarrow$ Expansion in $\frac{v}{c}$
  - N^{th} order expressions for GWs, momenta, orbits, ...

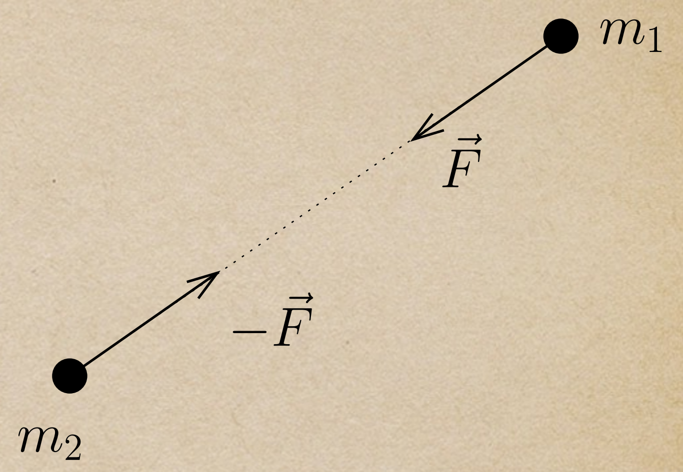
  Blanchet, Buonanno, Damour, Kidder, Schäfer, Will, ...

- **Post-Minkowskian Theory** (Weak gravity but arbitrary v)
- **Numerical Relativity**

The Newtonian 2-body problem

- Eqs. of motion

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = \vec{F} = -G\frac{m_1 m_2}{r^2}\hat{\vec{r}} = -m_2\frac{d^2\vec{r}_2}{dt^2}$$



- Solution: Kepler ellipses, parabolic, hyperolic

$$r = \frac{r_0}{1+\epsilon\cos\theta}$$

- What is the equivalent in GR?
  - No point particles in GR $\rightarrow$ Black holes!
  - Systems typically are dissipative $\rightarrow$ Gravitational waves

- The Holy Grail of numerical relativity: Inspiral of BH binary

- History: e.g. US CQG 1411.3997

Challenges in GR

- Covariance of the Einstein equations
  - Space and time on equal footing: How to evolve? Time?
  - Well posedness? Suitability for numerical methods?
- Meaning of the solutions; cf. Schwarzschild solution or GWs
  - Gauge invariant diagnostics
  - Definition of observables
- No a-priori spacetime “stage”. Coordinates are evolved.
- Singularities
- Computational costs: 3D effect
- Numerical stability

The ancient world:
The Birth of Numerical Relativity

Foundations and the first steps

- The Cauchy problem of the Einstein equations locally has a unique solution   Choquet-Bruhat   Acta Math. 1952
- Characteristic formulation   Bondi, Sachs   Proc.Roy.Soc. 1962
- Canonical 3+1 or ADM formulation of the Einstein equations

  Arnowitt, Deser, Misner (1962)   gr-qc/0405109
- First numerical relativity simulations: $\lesssim 100$ time steps

  Hahn & Lindquist   Ann.Phys. 1964
- 1D Gravitational collapse   May & White   PR 1966


AMD


Y Choquet-Bruhat


H Bondi


R K Sachs

The 1970s

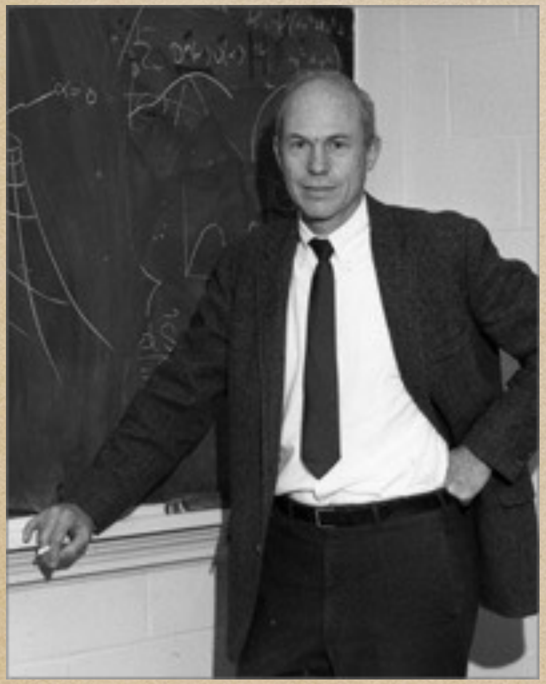
- Reinvestigation initiated by B DeWitt

  → PhD theses by A Cavez (1971), L Smarr (1975), K R Eppley (1975)

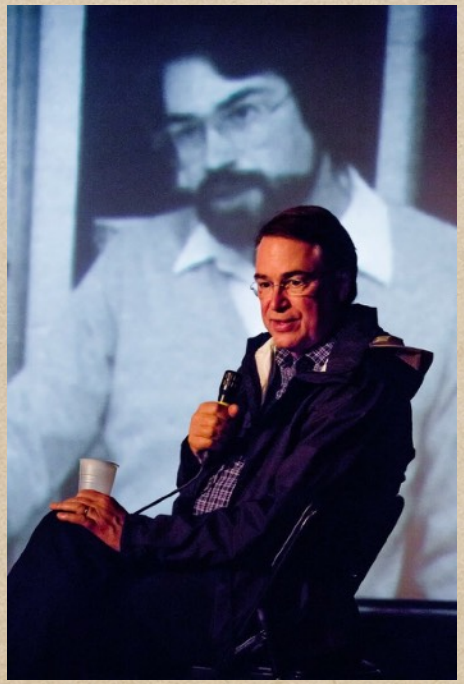
- 300 x Flops relative to Hahn & Lindquist

- ADM equations, 2D code, Misner (1960) initial data

  → single BHs, head-on collisions



B DeWitt



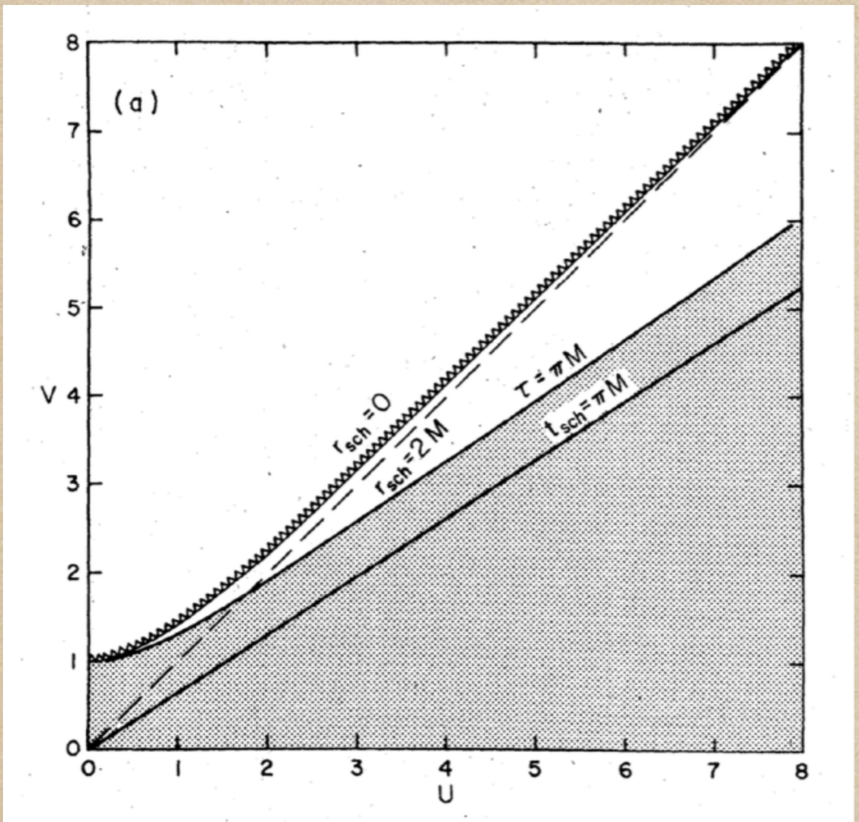
L Smarr

J W York Jr.

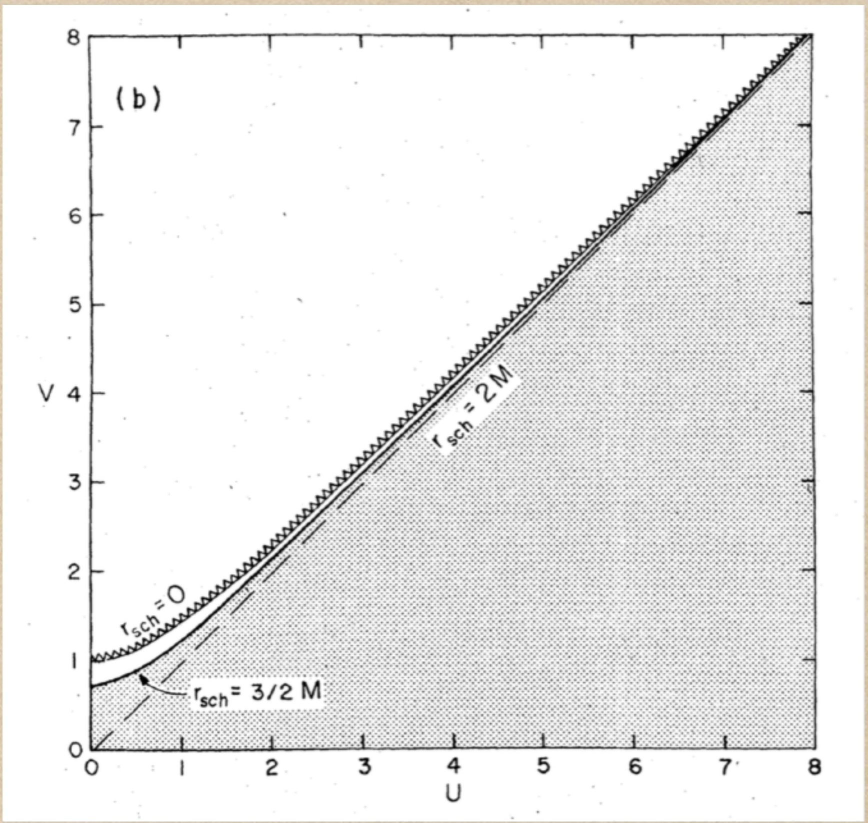
The 1970s

- Singularity avoiding slicing Smarr & York PRD 1978

  Schwarzschild-Kruskal evolved with



geodesic slicing



maximal slicing

- York formulation of the 3+1 equations

  York 1979 in "Sources of Gravitational Radiation" Ed. L Smarr

The 3+1 decomposition

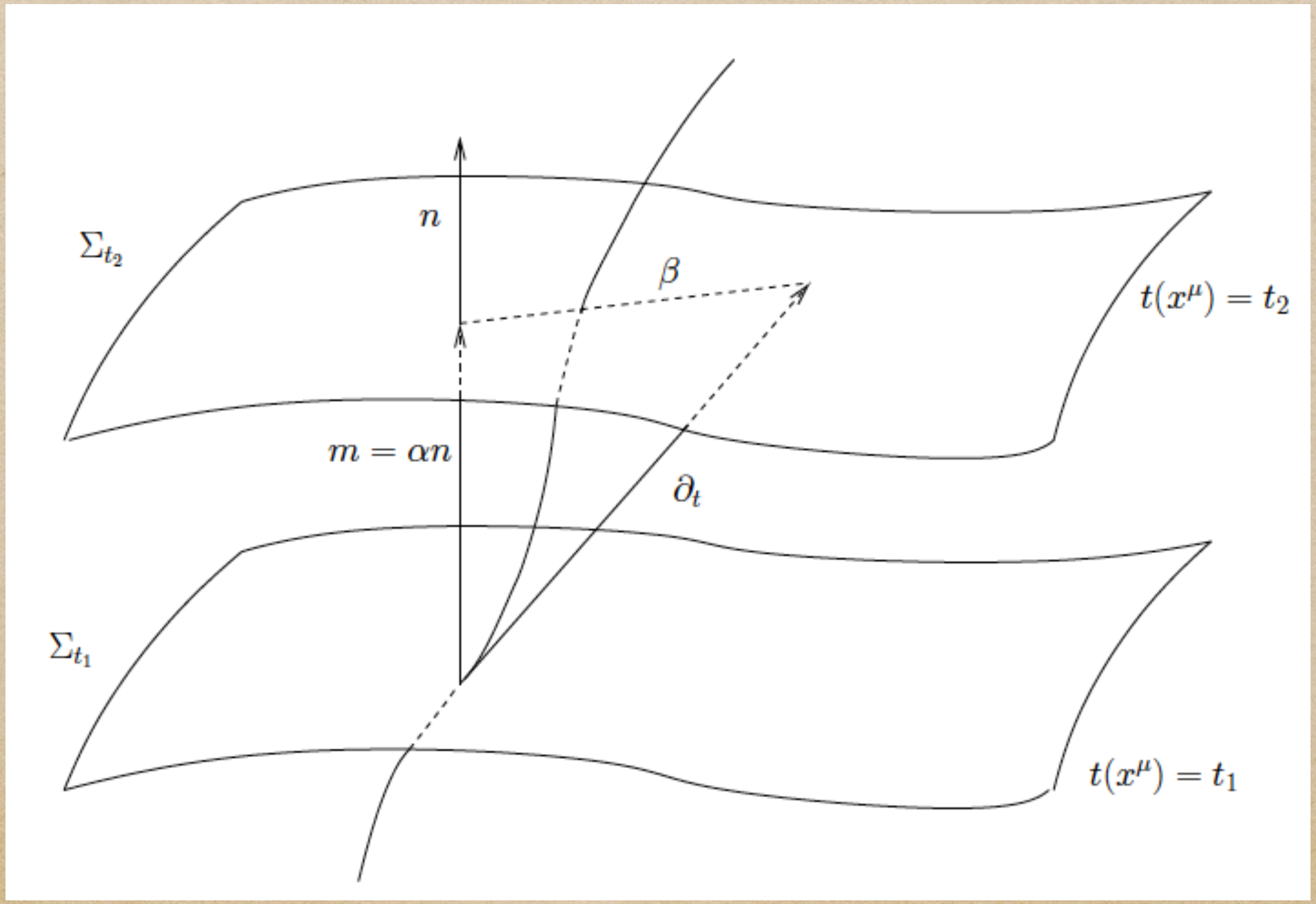
- Let $(\mathcal{M}, g)$ be a Spacetime

  = Manifold with metric of signature $-+++$

- Assume $\exists$ smooth $t : \mathcal{M} \mapsto \mathbb{R}$

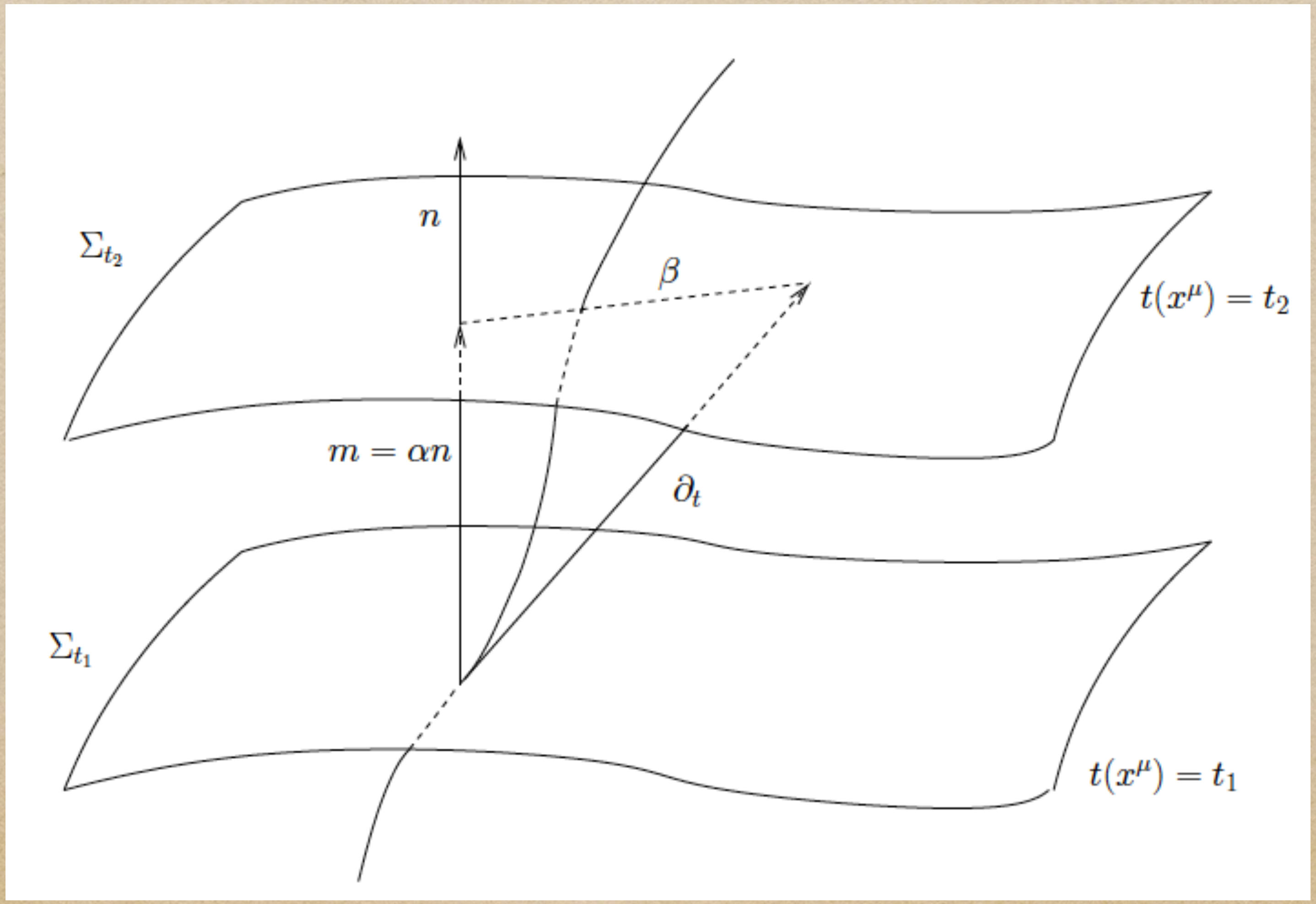
  with timelike gradient $\mathbf{d}t \neq 0$ and level surfaces

$$\forall_{t_1 \in \mathbb{R}} \;\; \Sigma_{t_1} = \{p \in \mathcal{M} \;\; : \;\; t(p) = t_1\} \; , \;\; \Sigma_{t_1} \cap \Sigma_{t_2} = \emptyset \;\Leftrightarrow\; t_1 \neq t_2$$



The 3+1 decomposition

- 1-Form: $\mathbf{d}t$; vector: $\dfrac{\partial}{\partial t} =: \partial_t \quad \Rightarrow \quad \langle \mathbf{d}t, \partial_t \rangle = 1$
- Timelike normal $n_\mu := -\alpha(\mathbf{d}t)_\mu$
- Spatial projector $\perp^\alpha{}_\mu = \delta^\alpha{}_\mu + n^\alpha n_\mu$
- Adapted coordinates $(t,\, x^i)$, x^i label points inside Σ_t



(D-1)+1 decomposition of the metric

- In adapted coordinates, we write the spacetime metric

$$g_{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^2 + \beta_m \beta^m & \beta_j \\ \hline \beta_i & \gamma_{ij} \end{array}\right)$$

$$\Leftrightarrow \qquad g^{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^{-2} & \alpha^{-2}\beta^j \\ \hline \alpha^{-2}\beta^i & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{array}\right)$$

$$\Leftrightarrow \qquad ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

- Gauge variables: Lapse α, shift β^i
- For tensors tangent in all components to Σ_t we lower indices with γ_{ij} : $S^i{}_{jk} = \gamma_{jm} S^{im}{}_k$, etc.
- Details e.g. in Gourgoulhon gr-qc/0703035

Decomposition of the Einstein eqs.

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

$$\Leftrightarrow \quad R_{\alpha\beta} = 8\pi \left(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T \right) + \Lambda g_{\alpha\beta}$$

- Energy momentum tensor

$$\rho := T_{\mu\nu} n^\mu n^\nu \,,$$

$$j_\alpha := -\perp^\mu{}_\alpha T_{\mu\nu} n^\nu \,,$$

$$S_{\alpha\beta} := \perp T_{\alpha\beta} \,, \quad S = \gamma^{\mu\nu} S_{\mu\nu} \,,$$

$$T_{\alpha\beta} = S_{\alpha\beta} + n_\alpha j_\beta + n_\beta j_\alpha + \rho n_\alpha n_\beta \,, \quad T = S - \rho \,.$$

- Lie derivative

$$\mathcal{L}_{\boldsymbol{m}} K_{ij} = \partial_t K_{ij} - \beta^m \partial_m K_{ij} - K_{mj} \partial_i \beta^m - K_{im} \partial_j \beta^m$$
$$\mathcal{L}_{\boldsymbol{m}} \gamma_{ij} = \partial_t \gamma_{ij} - \beta^m \partial_m \gamma_{ij} - \gamma_{mj} \partial_i \beta^m - \gamma_{im} \partial_j \beta^m$$

The ADM version of the Einstein eqs.

- Introduction of the extrinsic curvature:

$$\mathcal{L}_{\boldsymbol{m}}\gamma_{ij} = -2\alpha K_{ij}$$

- $\perp^{\mu}{}_{\alpha}\perp^{\nu}_{\beta}$ projection

$$\mathcal{L}_{\boldsymbol{m}}K_{ij} = -D_iD_j\alpha + \alpha(\mathcal{R}_{ij} + KK_{ij} - 2K_{im}K^{m}{}_{j}) + 8\pi\alpha\left(\frac{S-\rho}{D-2}\gamma_{ij} - S_{ij}\right) - \frac{2}{D-2}\Lambda\gamma_{ij}$$

"Evolution equations"

- $n^{\mu}n^{\nu}$ projection

$$\mathcal{R} + K^2 - K^{mn}K_{mn} = 2\Lambda + 16\pi\rho$$

"Hamiltonian constraint"

- $\perp^{\mu}{}_{\alpha}n^{\nu}$ projection

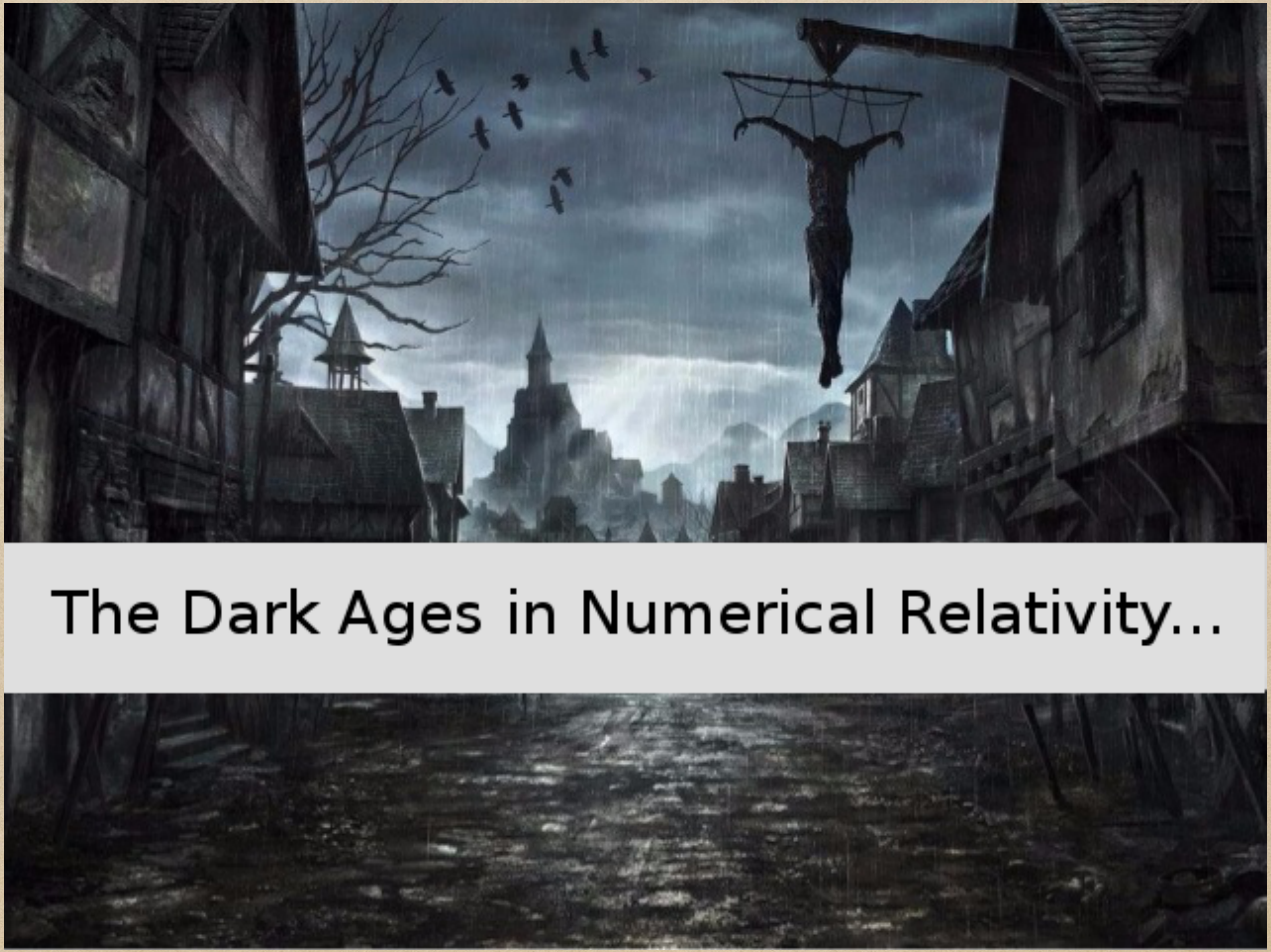
$$D_iK - D_mK^m_i = -8\pi j_i$$

"Momentum constraints"

- Backbone of numerical relativity for ~20 years

2. From the dark ages to the Renaissance

The 1980s



The 1990s

"Binary Black Hole Grand Challenge Project"

- 40 researchers in 10 institutions

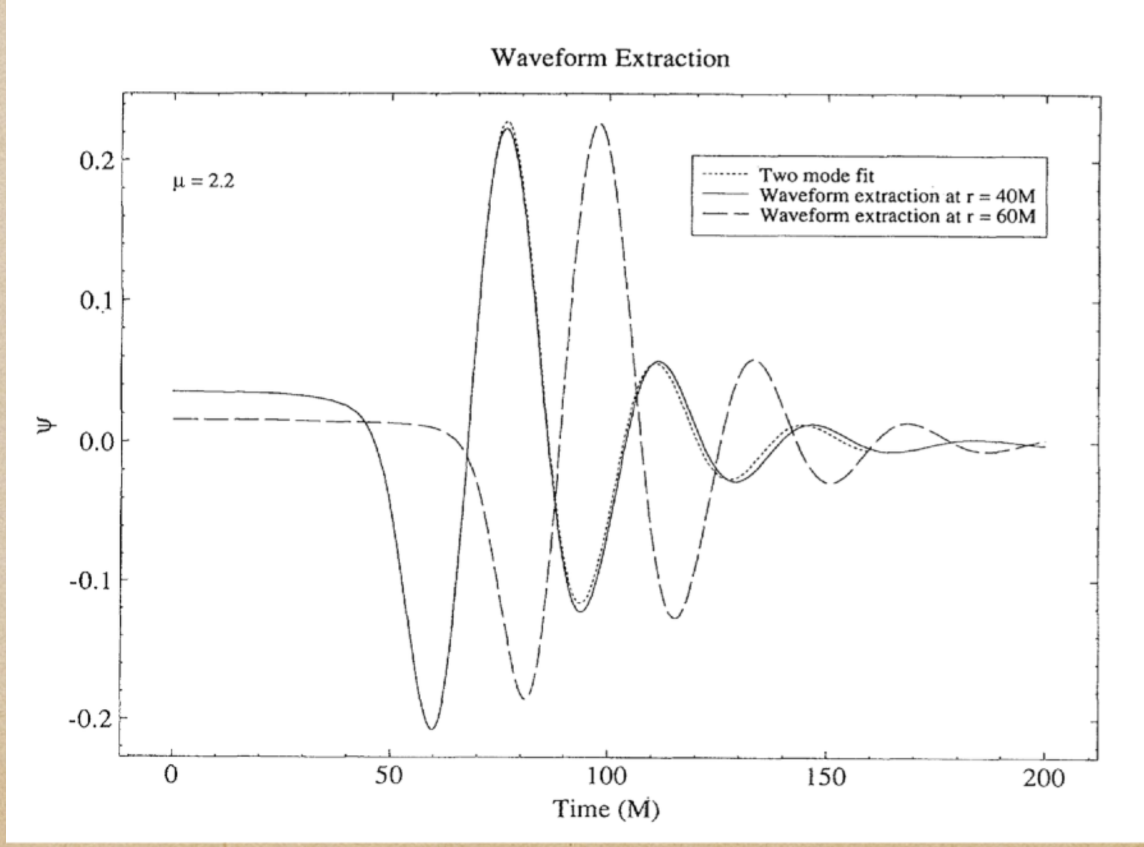
  Austin, Cornell, Illinois, North Carolina, Northwestern, Penn State, Pittsburgh, Syracuse

- Goal: Simulate BH binary inspiral, compute GW signals

- ADM formalism, axisymmetry, head-on collisions
  - $l = 2,\ l = 4$ waveforms
  - Horizon calculations
  - Unequal masses

  Anninos et al

  PRL 1993, PRD 1995+

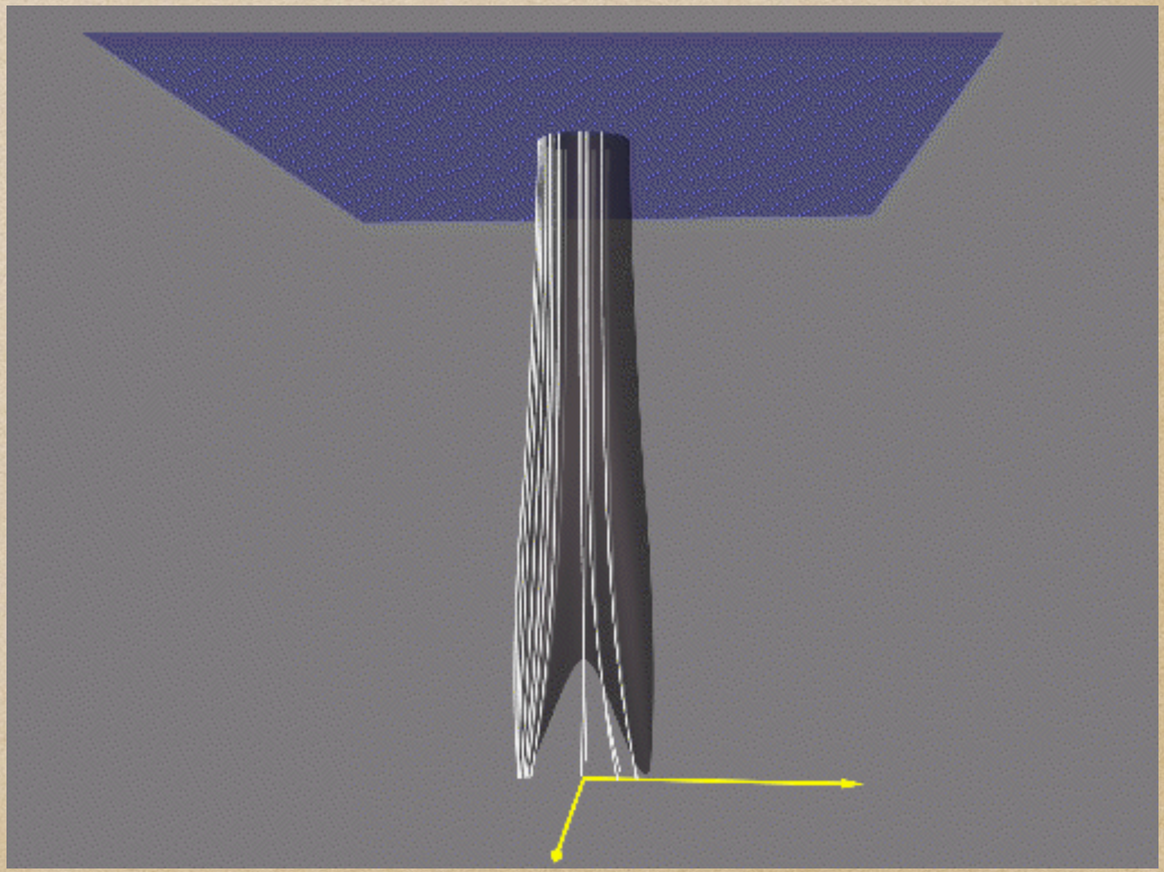


The 1990s

"Binary Black Hole Grand Challenge Project" (continued)

- First 3+1 dimensional BH simulations "G-code"
  - Single Schwarzschild BH stable up to $t \approx 50\ M$
  - Single moving BH stable up to $t \approx 60\ M$
  - Comparisons with axisymmetric simulations

Anninos et al PRD 1995



- Event horizon

  "pair of pants"

- Problem: long-term stability!

  $\rightarrow$ Well-posedness studies

Well posedness

From G Papallo, PhD thesis Cambridge, 2018

- Consider linear const.coeff. PDE system $A\partial_t u + P^i \partial_i u + Cu = 0$

Fourier trafo $\tilde{u}(t,k) = \frac{1}{\sqrt{2\pi}^n} \int u(t,x) e^{-ik_i x^i} \, d^n x$

$$\Rightarrow \partial_t \tilde{u} - i\mathcal{M}\,\tilde{u} = 0\,, \qquad \mathcal{M}(k) = A^{-1}(-P^i k_i + iC)$$

Solution $\tilde{u}(t,k) = e^{i\mathcal{M}(k)t}\,\tilde{u}(0,k)$

$$\Rightarrow u(t,x) = \frac{1}{\sqrt{2\pi}^n} \int e^{ik_i x^i} \, e^{i\mathcal{M}t}\,\tilde{u}(0,k)\, d^n k$$

- May not converge if integrand fails to decay fast with $k = \sqrt{k_i k^i}$
- ADM equations are only weakly hyperbolic
- Strong or symmetric hyperbolicity with new formulations:

BSSN, CCZ4, GHG

The post-Grand Challenge era

- Many smaller groups explored the problem independently
- Critical collapse: Collapse of spherically symmetric scalar field.

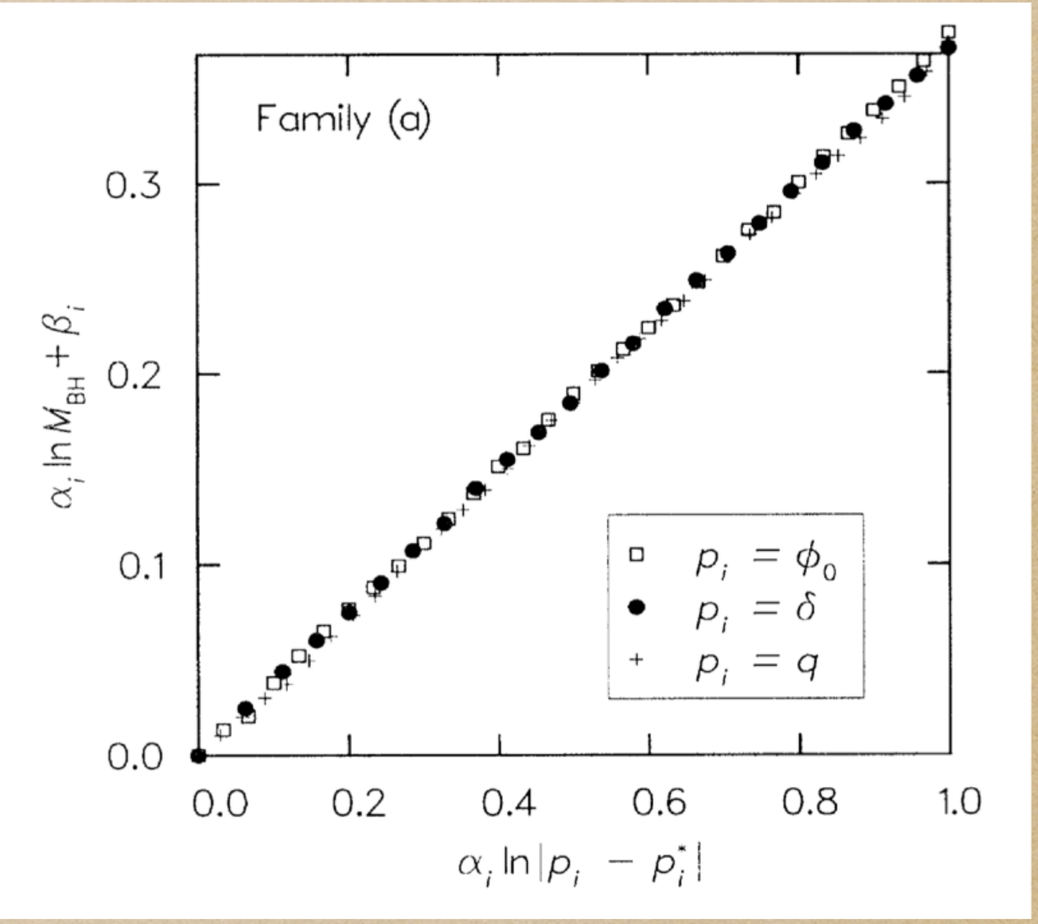
  BH formation or dispersal; Mesh refinement! Choptuik PRL 1993
- 1st Mesh refinement for 3+1 BHs

  Brügmann PRD 1996
- 1st long-term stable BH sim.

  (characteristic code)

  Gómez et al PRL 1998
- 1st BH Grazing collision

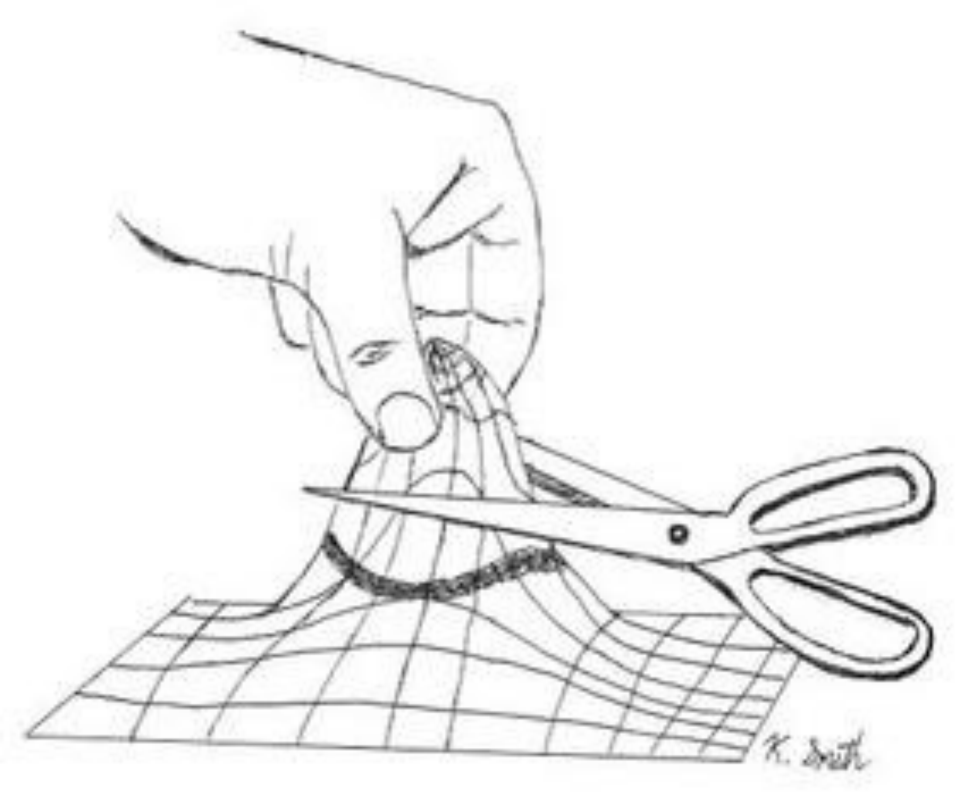
  Brandt et al PRL 2000
- Release of `Cactus 1.0`

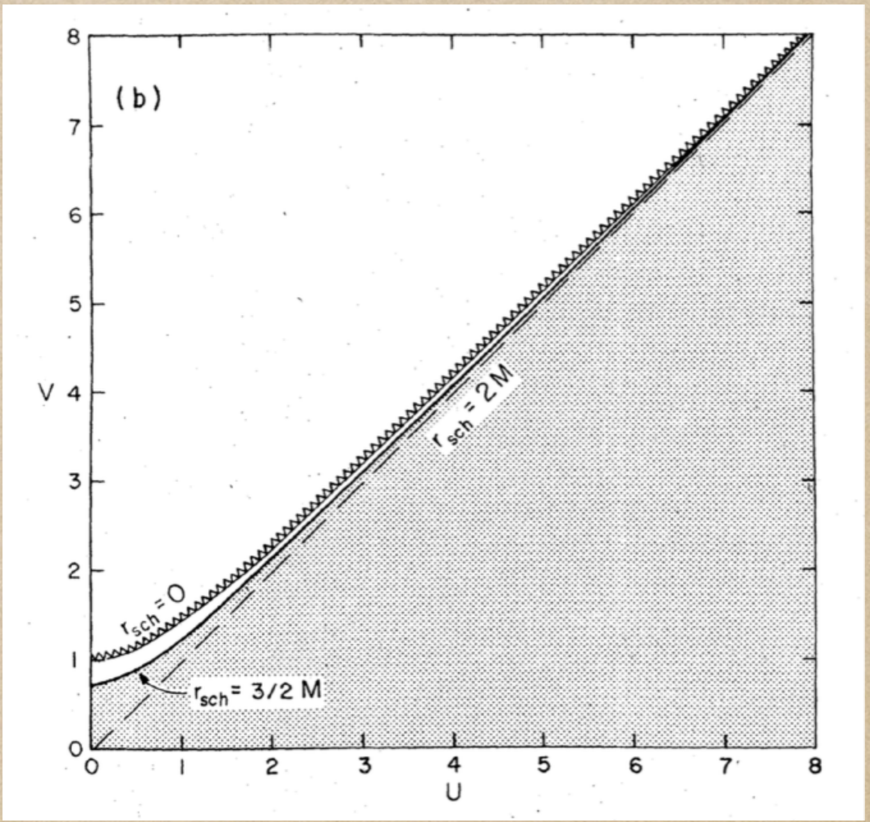


Towards the Holy Grail

Remaining problems

- Formulations: Well-posedness required
- Gauge conditions: Avoid coordinate singularities
- Physical singularities: Excision or good slicing





The generalized harmonic gauge (GHG)

- Harmonic gauge: Choose coordinates such that

$$\Box x^\alpha = \nabla^\mu \nabla_\mu x^\alpha = -g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = 0$$

- 4 dimensional Einstein eqs. in harmonic gauge:

$$R_{\alpha\beta} = -\frac{1}{2} g^{\mu\nu}\partial_\mu \partial_\nu g_{\alpha\beta} + \ldots$$

principle part of wave equation $\Rightarrow$ Manifestly hyperbolic!

- Problem: Start with a hyper surface $t = \text{const}$

Does t remain timelike?

- Goal: Generalize the harmonic gauge

Garfinkle PRD gr-qc/0110013; Pretorius CQG gr-qc/0407110;

Lindblom et al CQG gr-qc/0512093

$\rightarrow$ Source function $H^\alpha = \nabla^\mu \nabla_\mu x^\alpha = -g^{\mu\nu}\Gamma^\alpha_{\mu\nu}$

The generalized harmonic equations

- Any spacetime in any coordinates can be formulated in GH form!

  Problem: find the corresponding H^α
- Promote the H^α to evolution variables
- Einstein equations in GH form:

$$\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} = -\,\partial_\nu g_{\mu(\alpha}\partial_{\beta)}g^{\mu\nu} - \partial_{(\alpha}H_{\beta)} + H_\mu\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\nu\alpha}\Gamma^\nu_{\mu\beta} - \frac{2}{3}\Lambda g_{\alpha\beta} - 8\pi\left(T_{\mu\nu} - \frac{1}{2}T\,g_{\alpha\beta}\right).$$

with constraints $\mathcal{C}^\alpha = H^\alpha - \Box x^\alpha = 0$

- Still has principle part of the wave equation!!! Manifestly hyperbolic

  Friedrich Comm.Math.Phys. 1985; Garfinkle gr-qc/0110013;

  Pretorius gr-qc/0407110

Initial data: Analytic data

- Schwarzschild, Kerr, Tangherlini, Myers-Perry,...

  e.g. Schwarzschild in isotropic coordinates

$$ds^2 = -\left(\frac{2r-M}{2r+M}\right)^2 dt^2 + \left(1+\frac{M}{2r}\right)^4 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

  Time symmetric initial data with n BHs:

  Brill & Lindquist PR 1963, Misner PR 1960

- Problem: Find initial data for dynamic systems
- Goals: 1) Solve constraints

  2) Realistic snapshot of physical system
- This is mostly done using the ADM 3+1 split

Initial data and conformal decomposition

- Recall: we need to satisfy the constraints

$$\mathcal{R} + K^2 - K^{mn}K_{mn} = 2\Lambda + 16\pi\rho \qquad D_i K - D_m K^m_i = -8\pi j_i$$

- Conformal metric $\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$

Lichnerowicz J.Math.Pures Appl. 1944; York PRL 1971, PRL 1972

- Conformal traceless split of the extrinsic curvature

$$K_{ij} = A_{ij} + \frac{1}{3} K \gamma_{ij} \,,$$

$$A^{ij} = \psi^{-10} \bar{A}^{ij} \quad \Leftrightarrow \quad A_{ij} = \psi^{-2} \bar{A}_{ij}$$

A Lichnerowicz

Bowen-York data

- By further splitting $\bar{A}_{ij}$ into a longitudinal and a transverse traceless part, the momentum constraints simplify substantially

  Cook LRR gr-qc/0007085

- Further assume: Vacuum, $K = 0\,, \quad \bar{\gamma}_{ij} = f_{ij}\,, \quad \lim_{r\to\infty} \psi = 0$,
  where f_{ij} is the flat metric in arbitrary coords.

  In words: Traceless E.Curv., conformal flatness, asymptotic flatness

- Then there exists an analytic solution to the momentum constraints

$$\bar{A}_{ij} = \frac{3}{2r^2}\left[P_i n_j + P_j n_i - (f_{ij} - n_i n_j)P^k n_k\right] + \frac{3}{r^3}\left(\epsilon_{kil}S^l n^k n_j + \epsilon_{kjl}S^l n^k n_i\right)\,,$$

P^k = Momentum

S^k = Spin

where r is a coordinate radius and $n^i = \frac{x^i}{r}$

Bowen & York PRD 1980

Puncture data

Brandt & Brügmann PRL gr-qc/9703066

- The Hamiltonian constraint is then given by

$$\bar{\nabla}^2\psi + \frac{1}{8}\psi^{-7}\bar{A}_{mn}\bar{A}^{mn} = 0$$

- Ansatz for conformal factor $\psi = \psi_{\rm BL} + u$

where $\psi_{\rm BL} = \sum_{a=1}^{N} \frac{m_a}{|\vec{r} - \vec{r}_a|}$ is the Brill-Lindquist conformal factor,

i.e. the solution for $\bar{A}_{ij} = 0$.

- There then exist unique $\mathcal{C}^2$ solutions u to the Hamiltonian constr.

- The Hamiltonian constraint in this form is particularly suitable for numerical solution.

  E.g. Ansorg, Brügmann & Tichy gr-qc/0404056

Beyond conformally flat initial data

- Problem: Conformally flat data limits spins to $S/M^2 \lesssim 0.928$

  Dain et al. PRD gr-qc/0201062

- Similar problems arise for large linear momenta

- Solution: Non-conformally flat initial data

  - Superpose Kerr-Schild data Lovelace et al. PRD 0805.4192

    Solve constraints with Conformal Thin Sandwich approach

    York PRL 82 (1999) 1350

  - Superpose boosted conformal Kerr BHs; attenuation functions

    Zlochower et al. PRD 1706.01980, Ruchlin et al. PRD 1410.8607

    Evolve with CCZ4 (constraint damping variant of BSSN)

    Alic et al. PRD 1106.2254, Hilditch et al. PRD 1212.2901

The BSSNOK system

- Goal: Modify ADM eqs. to get a strongly hyperbolic system

  Nakamura et al PTPS 1987, Shibata & Nakamura PRD 1995,

  Baumgarte & Shapiro gr-qc/9810065

- Use (i) conformal decomposition, (ii) trace split, (iii) aux. variables

$$\gamma := \det \gamma_{ij}\,, \qquad \chi := \gamma^{-1/3}\,, \qquad K = \gamma^{mn} K_{mn}\,,$$

$$\tilde{\gamma}_{ij} := \chi \gamma_{ij} \qquad \Leftrightarrow \qquad \tilde{\gamma}^{ij} = \frac{1}{\chi}\gamma^{ij}\,,$$

$$\tilde{A}_{ij} := \chi \left(K_{ij} - \frac{1}{3}\gamma_{ij} K \right) \qquad \Leftrightarrow \qquad K_{ij} = \frac{1}{\chi}\left(\tilde{A}_{ij} + \frac{1}{3}\tilde{\gamma}_{ij} K \right)\,,$$

$$\tilde{\Gamma}^i := \tilde{\gamma}^{mn} \tilde{\Gamma}^i_{mn}\,.$$

- Auxiliary constraints

$$\tilde{\gamma} = 1, \qquad \tilde{\gamma}^{mn}\tilde{A}_{mn} = 0, \qquad \mathcal{G}^i \equiv \tilde{\Gamma}^i - \tilde{\gamma}^{mn}\tilde{\Gamma}^i_{mn} = 0\,.$$

The BSSN equations

$$\mathcal{H} := \mathcal{R} + \frac{2}{3}K^2 - \tilde{A}^{mn}\tilde{A}_{mn} - 16\pi\rho - 2\Lambda = 0\,,$$

$$\mathcal{M}_i := \tilde{\gamma}^{mn}\tilde{D}_m\tilde{A}_{ni} - \frac{2}{3}\partial_i K - \frac{3}{2}\tilde{A}^m{}_i\frac{\partial_m\chi}{\chi} - 8\pi j_i = 0\,,$$

$$\partial_t\chi = \beta^m\partial_m\chi + \frac{2}{3}\chi(\alpha K - \partial_m\beta^m)\,,$$

$$\partial_t\tilde{\gamma}_{ij} = \beta^m\partial_m\tilde{\gamma}_{ij} + 2\tilde{\gamma}_{m(i}\partial_{j)}\beta^m - \frac{2}{3}\tilde{\gamma}_{ij}\partial_m\beta^m - 2\alpha\tilde{A}_{ij}\,,$$

$$\partial_t K = \beta^m\partial_m K - \chi\tilde{\gamma}^{mn}D_mD_n\alpha + \alpha\tilde{A}^{mn}\tilde{A}_{mn} + \frac{1}{3}\alpha K^2 + 4\pi\alpha(S+\rho) - \alpha\Lambda\,,$$

$$\partial_t\tilde{A}_{ij} = \beta^m\partial_m\tilde{A}_{ij} + 2\tilde{A}_{m(i}\partial_{j)}\beta^m - \frac{2}{3}\tilde{A}_{ij}\partial_m\beta^m + \alpha K\tilde{A}_{ij} - 2\alpha\tilde{A}_{im}\tilde{A}^m{}_j + \chi(\alpha\mathcal{R}_{ij} - D_iD_j\alpha - 8\pi\alpha S_{ij})^{\mathrm{TF}}\,,$$

$$\partial_t\tilde{\Gamma}^i = \beta^m\partial_m\tilde{\Gamma}^i + \frac{2}{3}\tilde{\Gamma}^i\partial_m\beta^m - \tilde{\Gamma}^m\partial_m\beta^i + \tilde{\gamma}^{mn}\partial_m\partial_n\beta^i + \frac{1}{3}\tilde{\gamma}^{im}\partial_m\partial_n\beta^n - \tilde{A}^{im}\left(3\alpha\frac{\partial_m\chi}{\chi} + 2\partial_m\alpha\right) + 2\alpha\tilde{\Gamma}^i_{mn}\tilde{A}^{mn} - \frac{4}{3}\alpha\tilde{\gamma}^{im}\partial_m K - 16\pi\frac{\alpha}{\chi}j^i - \sigma\mathcal{G}^i\partial_m\beta^m\,.$$

- Note: there exist slight variations of the exact equations

The BSSN equations

- Auxiliary expressions we have used:

$$\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk} - \frac{1}{2\chi}(\delta^i{}_k\,\partial_j\chi + \delta^i{}_j\,\partial_k\chi - \tilde{\gamma}_{jk}\tilde{\gamma}^{im}\partial_m\chi)$$

$$\mathcal{R}_{ij} = \tilde{R}_{ij} + \mathcal{R}^{\chi}_{ij}\,,$$

$$\mathcal{R}^{\chi}_{ij} = \frac{\tilde{\gamma}_{ij}}{2\chi}\left(\tilde{\gamma}^{mn}\tilde{D}_m\tilde{D}_n\chi - \frac{3}{2\chi}\tilde{\gamma}^{mn}\partial_m\chi\,\partial_n\chi\right) + \frac{1}{2\chi}\left(\tilde{D}_i\tilde{D}_j\chi - \frac{1}{2}\partial_i\chi\,\partial_j\chi\right)\,,$$

$$\tilde{R}_{ij} = -\frac{1}{2}\tilde{\gamma}^{mn}\partial_m\partial_n\tilde{\gamma}_{ij} + \tilde{\gamma}_{m(i}\partial_{j)}\tilde{\Gamma}^m + \tilde{\gamma}^m\tilde{\Gamma}_{(ij)m} + \tilde{\gamma}^{mn}\left[2\tilde{\Gamma}^k_{m(i}\tilde{\Gamma}_{j)kn} + \tilde{\Gamma}^k_{im}\tilde{\Gamma}_{kjn}\right]\,,$$

$$D_iD_j\alpha = \tilde{D}_i\tilde{D}_j\alpha + \frac{1}{\chi}\partial_{(i}\chi\,\partial_{j)}\alpha - \frac{1}{2\chi}\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}\partial_m\partial_n\alpha\,.$$

Beyond BSSN

- BSSN has a zero speed mode in the constraint-subsystem;
  May result in large constraint violations
- BSSN does not have systematic constraint damping
- This can be implemented by considering Generalized Einstein Eqs.
  Bona et al. PRD gr-qc/0302083 "Z4" system
- Conformal version of Z4: Very like BSSN but has constraint damping
  Alic et al. PRD 1106.2254, Hilditch et al. PRD 1212.2901
- Also allows for constraint preserving boundary conditions
  Bona et al. CQG gr-qc/0411110, Ruiz et al. PRD 1010.0523

Progress accelerates: The early 2000s

- BSSN found empirically. Then strong hyperbolicity shown

  Gundlach & Martín-García 2004

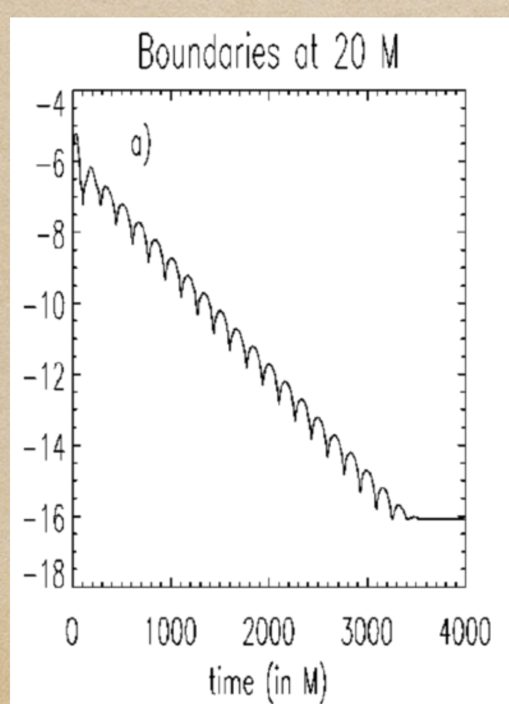
- 1st stable 3+1 evolution of Schwarzschild "simple excision"

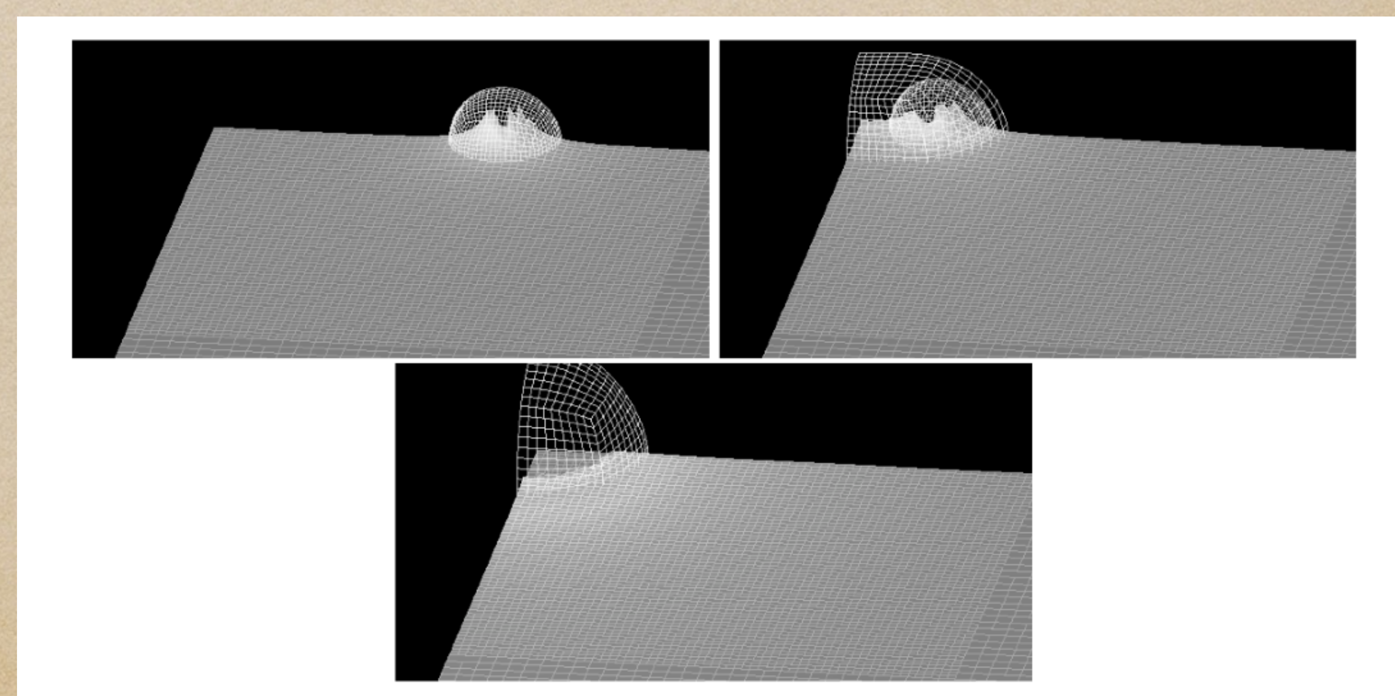
  Alcubierre & Brügmann gr-qc/0008067

- Stable head-on collisions

  US et al gr-qc/0503071, Fiske et al gr-qc/0503100

- BH binary orbit Brügmann et al gr-qc/0312112





Missing pieces I: Constraint damping in GHG

- One can show: GHG constraints related to ADM constraints

$$\mathcal{C}^\alpha = 0, \quad \partial_t \mathcal{C}^\alpha = 0 \ \text{ at } \ t = 0 \quad \Rightarrow \quad \mathcal{H} = 0\,, \quad \mathcal{M}_i = 0$$

- Bianchi identies imply evolution of the $\mathcal{C}^\alpha$:

$$\Box \mathcal{C}_\alpha = -\mathcal{C}^\mu \nabla_{(\mu} \mathcal{C}_{\alpha)} - \mathcal{C}^\mu \left[8\pi \left(T_{\mu\alpha} - \frac{1}{2} T\, g_{\mu\alpha} \right) + \Lambda g_{\mu\alpha} \right] .$$

- In practice: Numerical violations of $\mathcal{C}^\mu = 0 \quad \Rightarrow$ unstable modes!
- Solution: Add constraint damping terms

$$\frac{1}{2} \partial_\mu \partial_\nu g_{\alpha\beta} = -\,\partial_\nu g_{\mu(\alpha} \partial_{\beta)} g^{\mu\nu} - \partial_{(\alpha} H_{\beta)} + H_\mu \Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\nu\alpha} \Gamma^\nu_{\mu\beta} - \Lambda g_{\alpha\beta} - 8\pi \left(T_{\mu\nu} - \frac{1}{2} T\, g_{\alpha\beta} \right) - \kappa \left[2 n_{(\alpha} \mathcal{C}_{\beta)} - \lambda g_{\alpha\beta} n^\mu \mathcal{C}_\mu \right]$$

Gundlach et al CQG (2005)

- E.g. Pretorius PRL gr-qc/0507014 uses $\kappa = 1.25/m\,, \quad \lambda = 1$

Missing pieces II: Gauge in BSSN

- Recall: Einstein's equations say nothing about $\alpha\,, \quad \beta^i$
- Any choice of lapse and shift gives a solution to Einstein's eqs.
- This is the coordinate or gauge freedom of GR
- If the physics do not depend on $\alpha\,, \quad \beta^i$, then why bother?
- Answer: The performance of the numerics DO depend very sensitively on the gauge!

Ingredients for good gauge

- Singularity avoidance
- Avoid slice stretching
- Aim for stationarity in a co-moving frame
- Well-posedness of the system of PDEs
- Generalize "good" gauge, e.g. harmonic
- Lots of good luck!

Bona et al PRL (1995)

Alcubierre et al PRD gr-qc/0206072

Alcubierre CQG gr-qc/0210050

Garfinkle PRD gr-qc/0110013

Moving puncture gauge

- Moving punctures is one of the NR breakthrough methods

  Baker et al PRL gr-qc/0511103; Campanelli et al PRL gr-qc/0511048

- Gauge played a key role

- Variant of $1 + \log$ slicing and $\Gamma-$driver shift

  Alcubierre et al PRD gr-qc/0206072

- Now in use as $\partial_t \alpha = \beta^m \partial_m \alpha - 2\alpha K$

and $$\partial_t \beta^i = \beta^m \partial_m \beta^i + \frac{3}{4} B^i$$

$$\partial_t B^i = \beta^m \partial_m B^i + \partial_t \tilde{\Gamma}^i - \beta^m \partial_m \tilde{\Gamma}^i - \eta B^i$$

or $$\partial_t \beta^i = \beta^m \partial_m \beta^i + \frac{3}{4} \tilde{\Gamma}^i - \eta \beta^i$$

e.g. van Meter et al PRD gr-qc/0605030

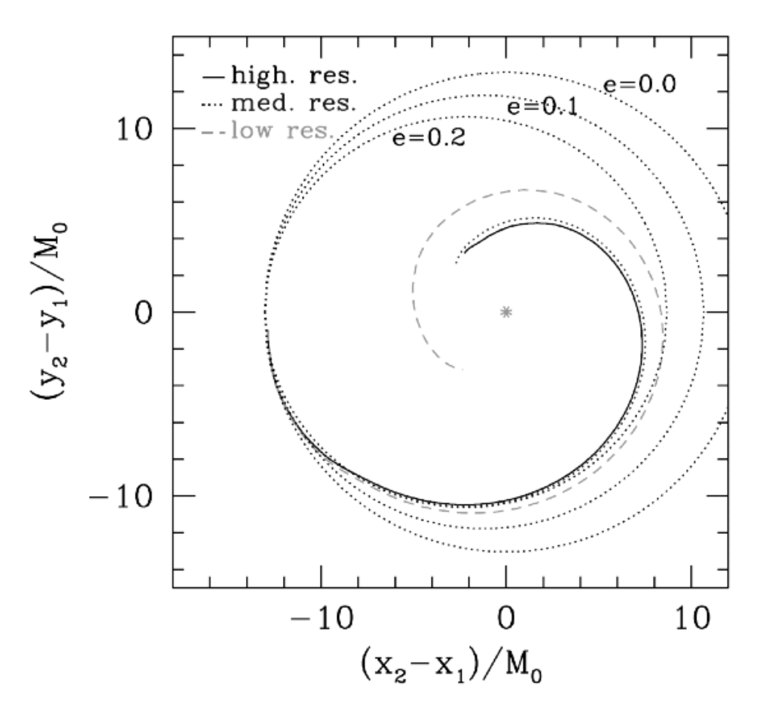
The 2005 breakthroughs

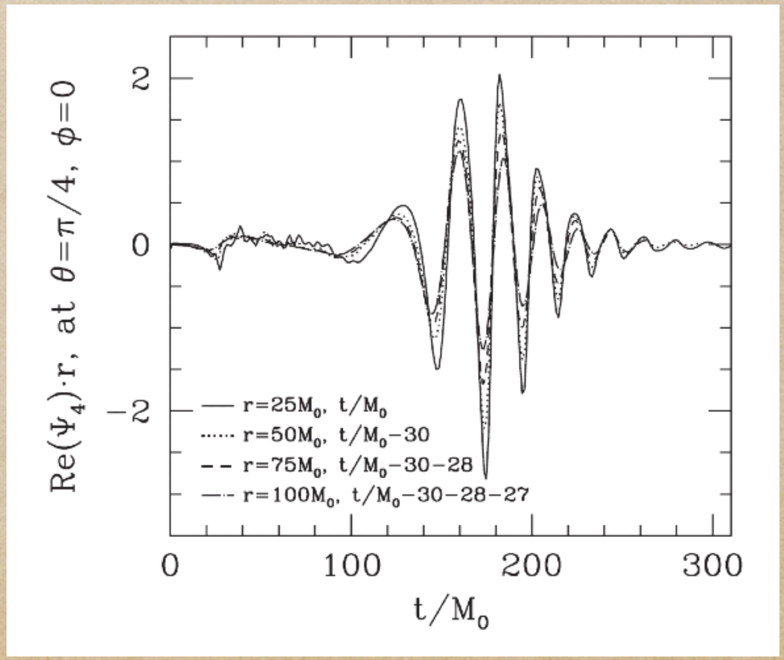
- First simulation of orbiting BBH through merger

  Pretorius PRL gr-qc/0507014

  - GHG
  - Initial data: Scalar field
  - BH excision
  - Radiated energy $\sim 3\,\%\,M$
  - Eccentricity $e \sim 0 \ldots 0.2$

- Presented at Banff conference



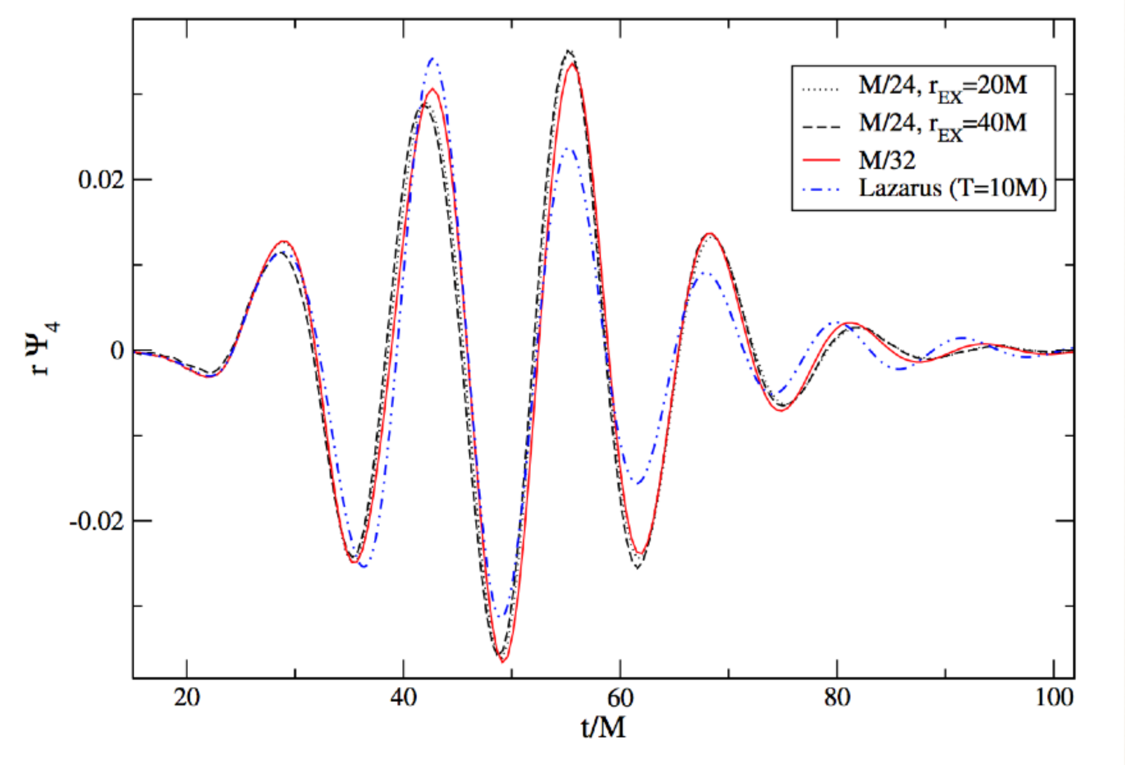


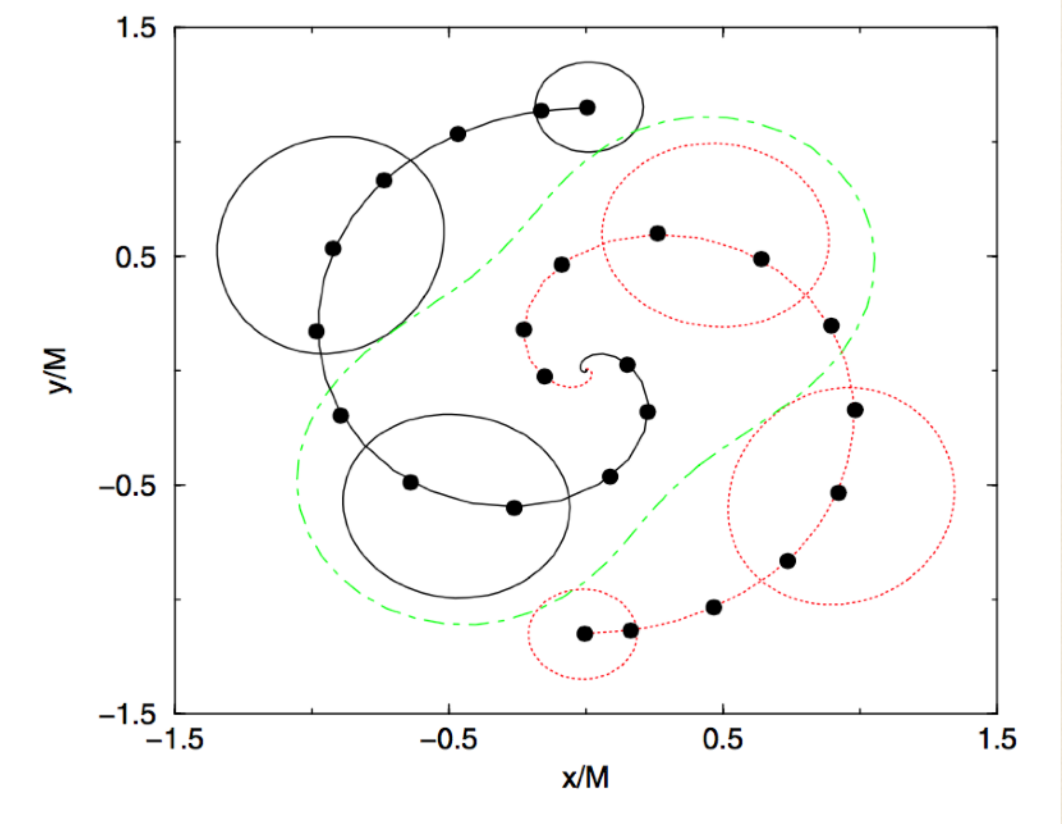
The 2005 breakthroughs

- Moving puncture breakthrough by Brownsville and Goddard groups

  Campanelli et al PRL gr-qc/0511048; Baker et al PRL gr-qc/0511103

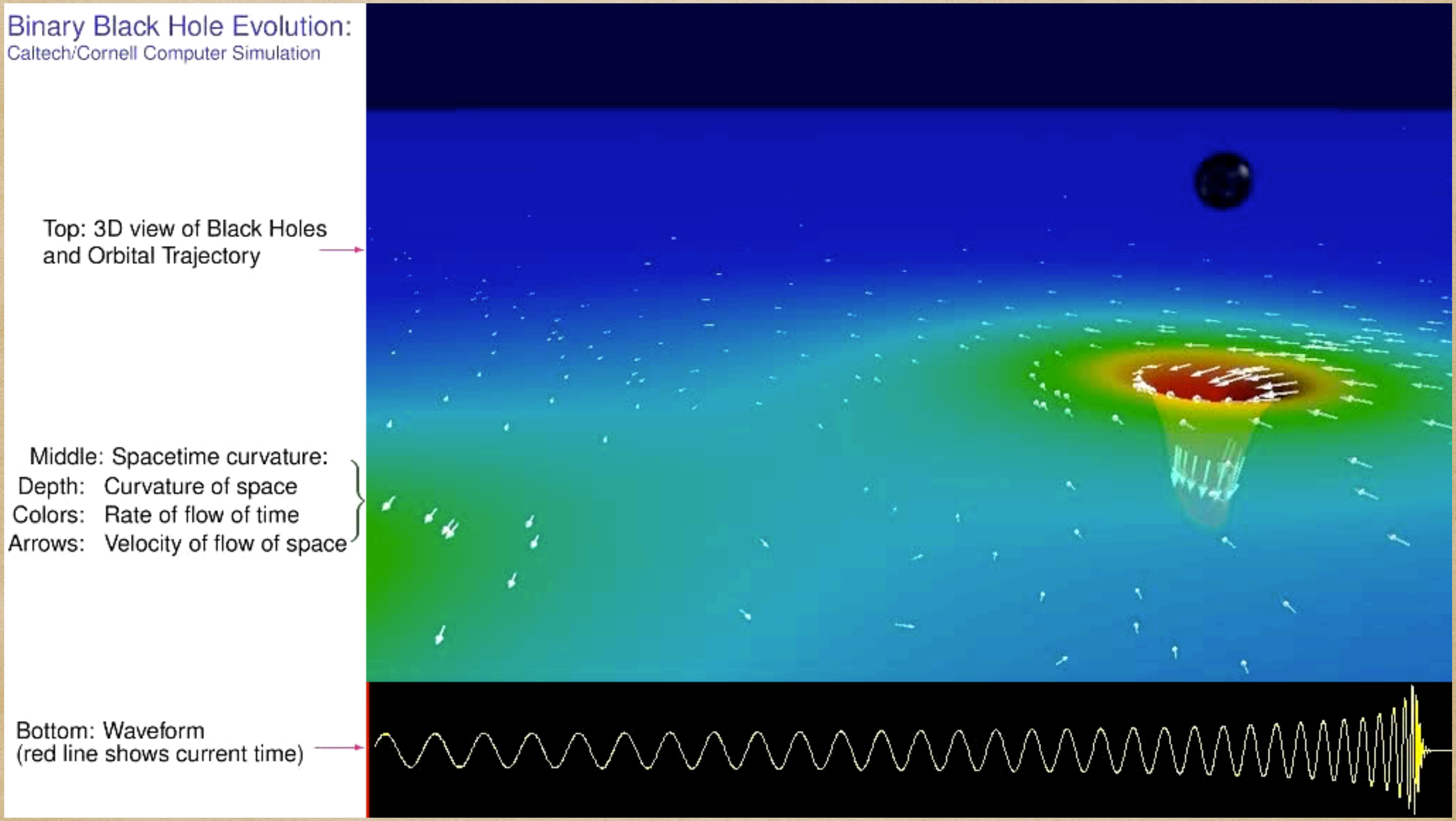
  - BSSN
  - Bowen-York initial data
  - Moving puncture gauge
  - Radiated energy $\sim 3\,\%\,M$





The goldrush years

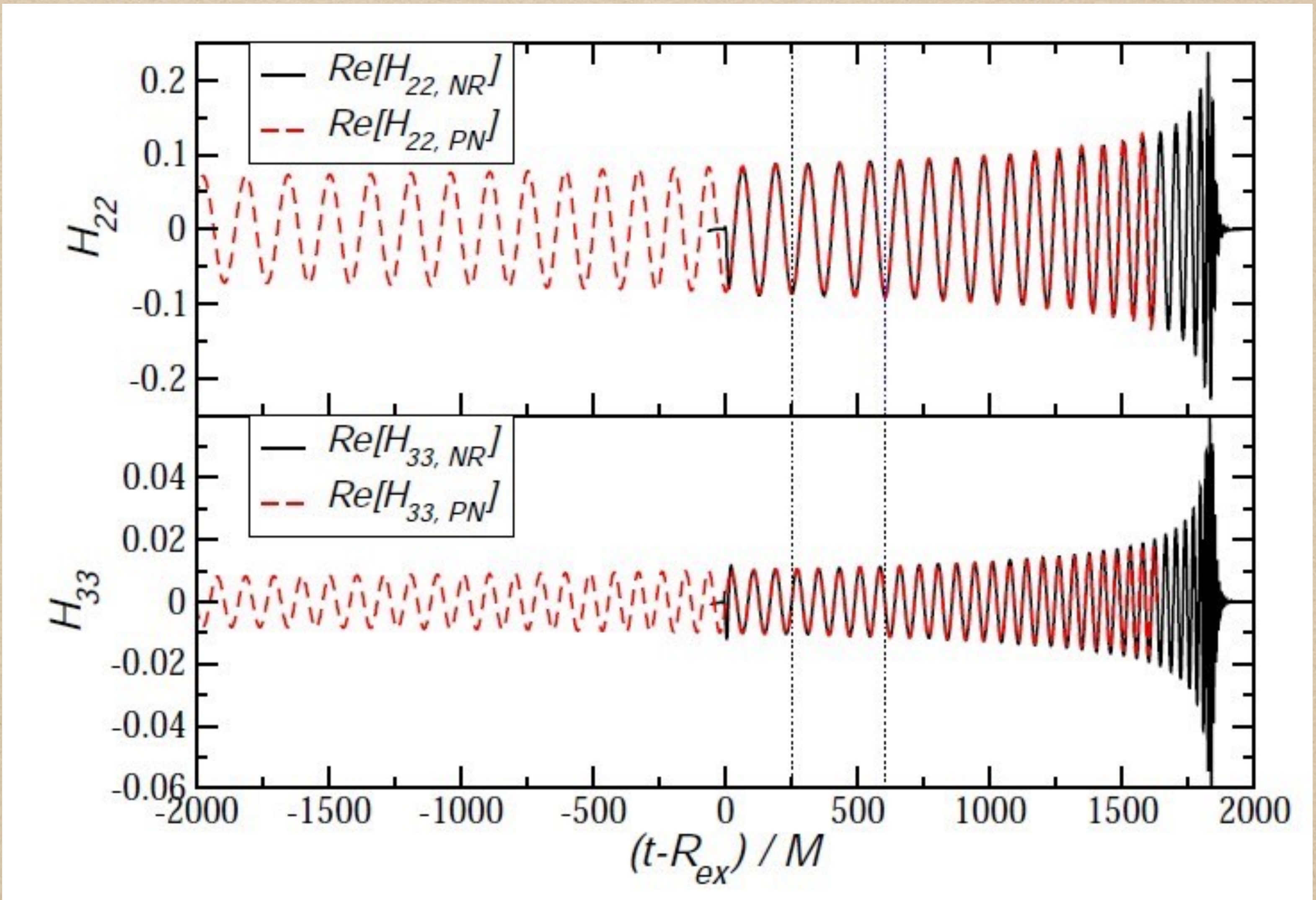
Anatomy of a BHB inspired



Thanks to Caltech-Cornell groups

Hybrid waveforms and catalogs

- Stitch together PN and NR waveforms



US et al CQG 2011

- Mass produce waveforms; Hinder et al CQG 1307.5307;
  Mroué et al PRL 1004.4697

Gravitational recoil

- Anisotropic GW emission $\Rightarrow$ recoil of remnant BH

Bonnor & Rotenburg Proc.R.Soc.Lond.A. (1961);

Peres PR (1962); Bekenstein ApJ (1973)

- Escape velocities:

Globular clusters	~ 30 km/s
dSph	$20 \ldots 100$ km/s
dE	$100 \ldots 300$ km/s
Giant galaxies	$\sim 1\,000$ km/s

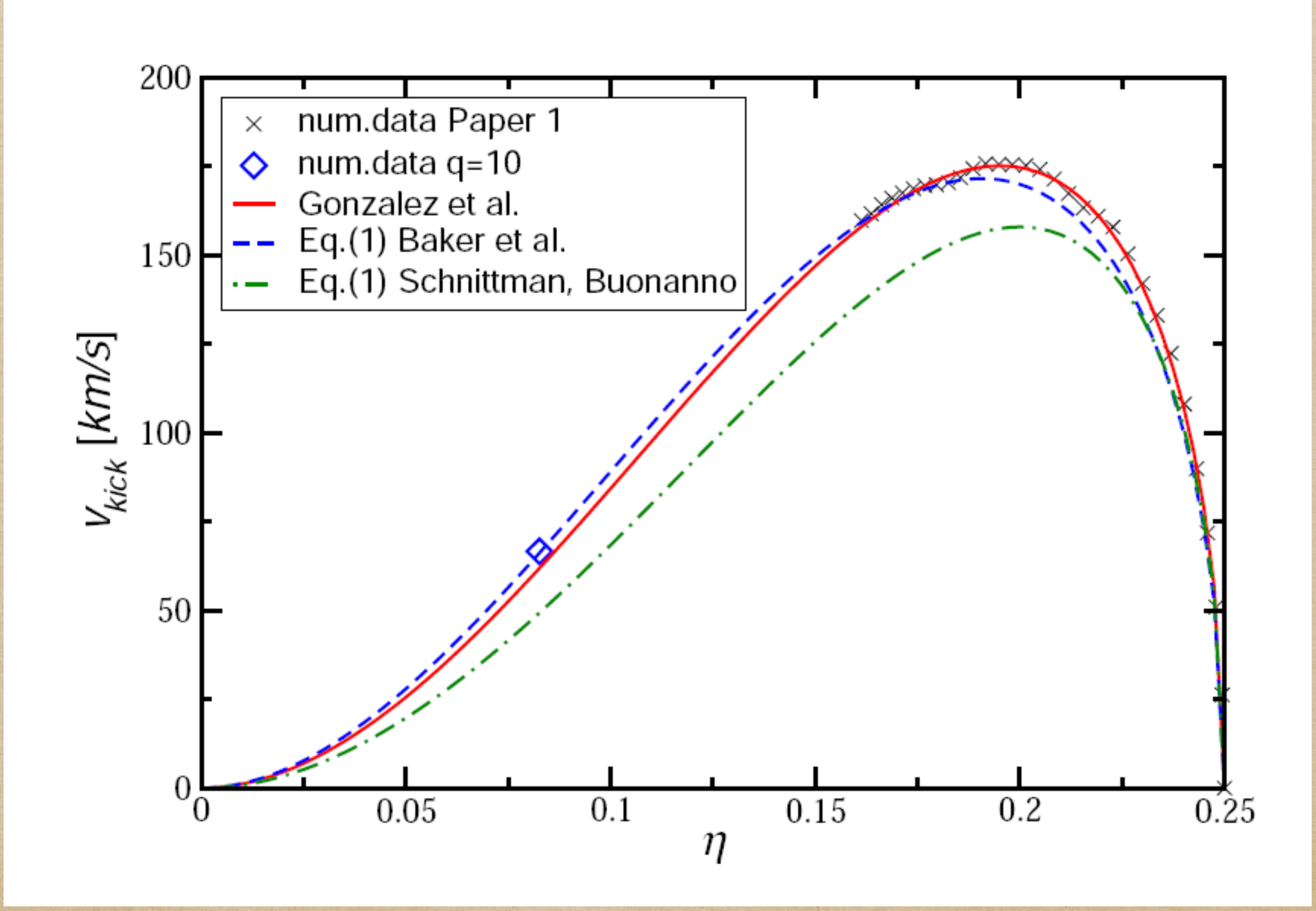
- Ejection/displacement of BHs affects
    - Growth history of SMBHs
    - BH populations, IMBHs
    - galaxy structure
    - observational "footprints"

Komossa Adv.Astron. 1202.1977

Kicks from non-spinning BH binaries

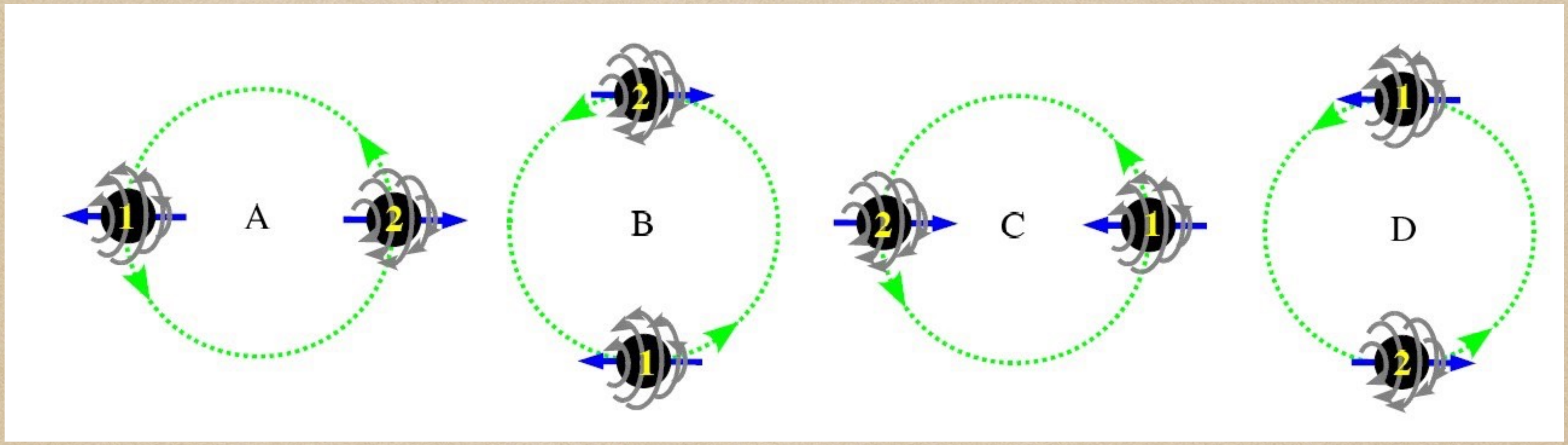
- Maximal kick: $\sim 180\ \mathrm{km/s}$ pretty harmless!

González et al PRL gr-qc/0610154



Spinning BHs: Superkicks

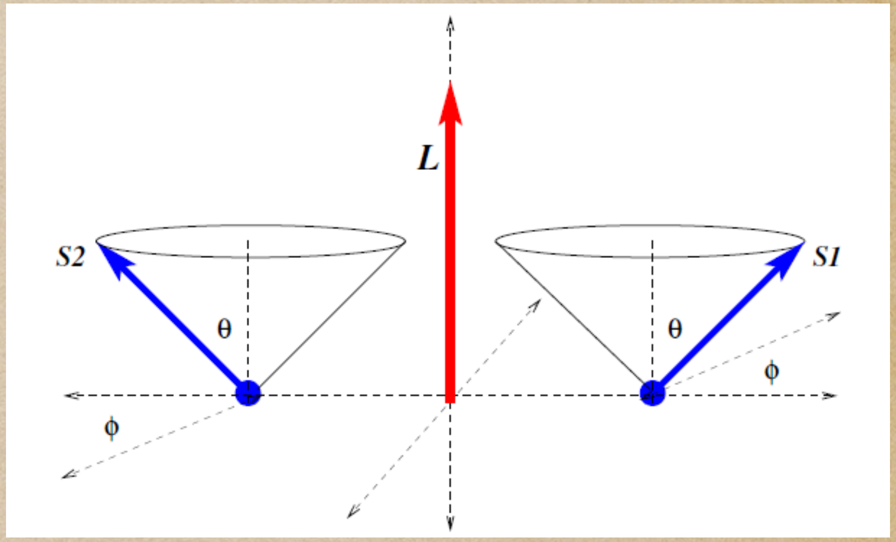
- Superkick configurations; Kidder gr-qc/9506022; Pretorius 0710.1338



- Kicks up to $v_{\rm max} \approx 4\,000$ km/s

González et al PRL gr-qc/0702052

Campanelli et al PRL gr-qc/0702133



- Yet larger kicks for partially aligned spins

$v_{\rm max} \approx 5\,000$ km/s

Lousto et al PRL 1108.2009

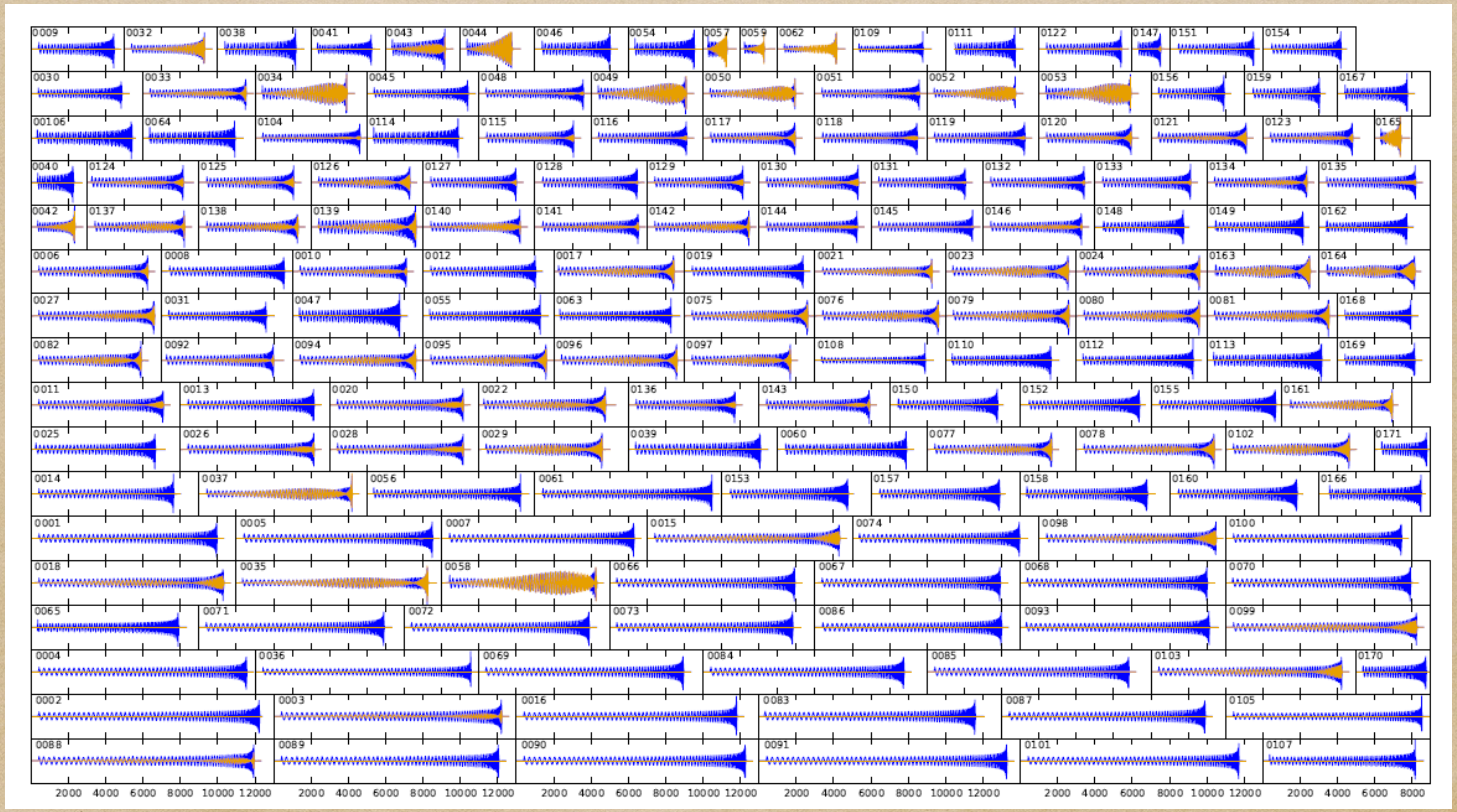
Towards new horizons

Tools of mass production

- Explore seven-dim. parameter space. E.g. SpEC catalogue:

  171 waveforms: $m_1/m_2 \leq 8$ up to 34 orbits

Mroué et al PRL 1304.6077



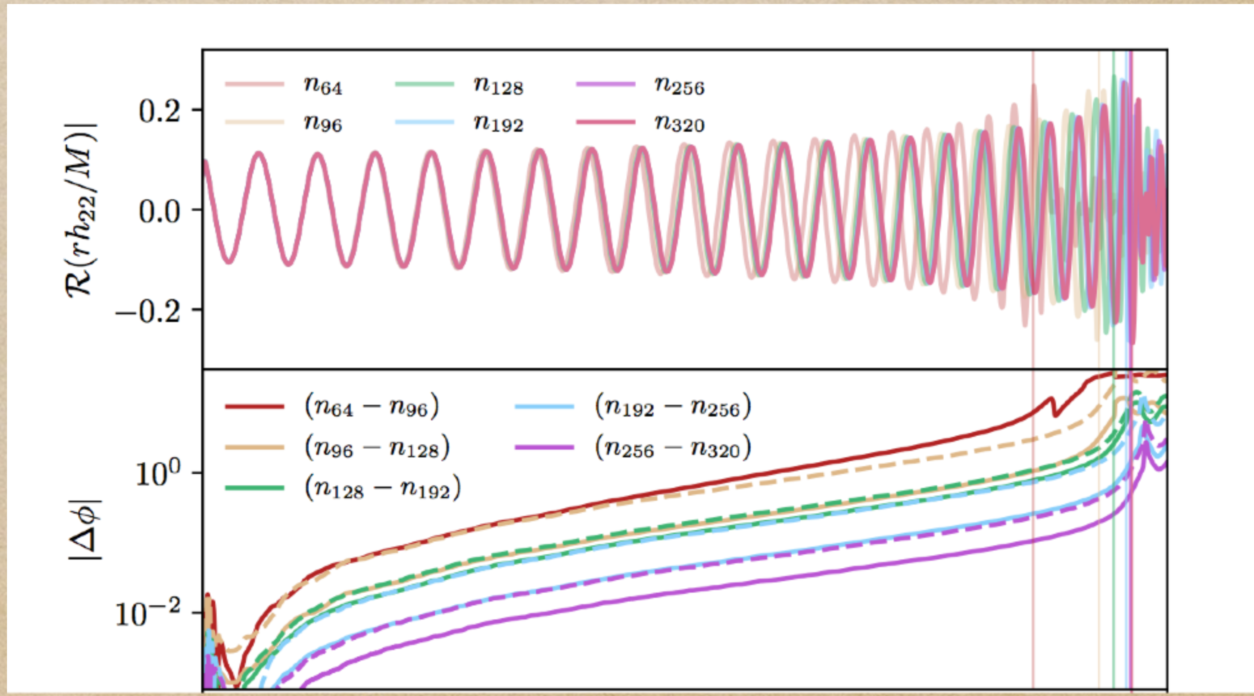
Neutron star binaries

- Other challenges: No spacetime singularities, but shocks!
  - HRSC treatment needs flux conservative Eqs. $\partial_t \boldsymbol{u} + \partial_i \boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{s}$
  - Solutions not unique $\rightarrow$ entropy conditions!
- The first NS binary inspirals preceded the BH breakthrough!

Shibata+ PRD 2003, Marronetti+ PRL 2004, Miller+ PRD 2004

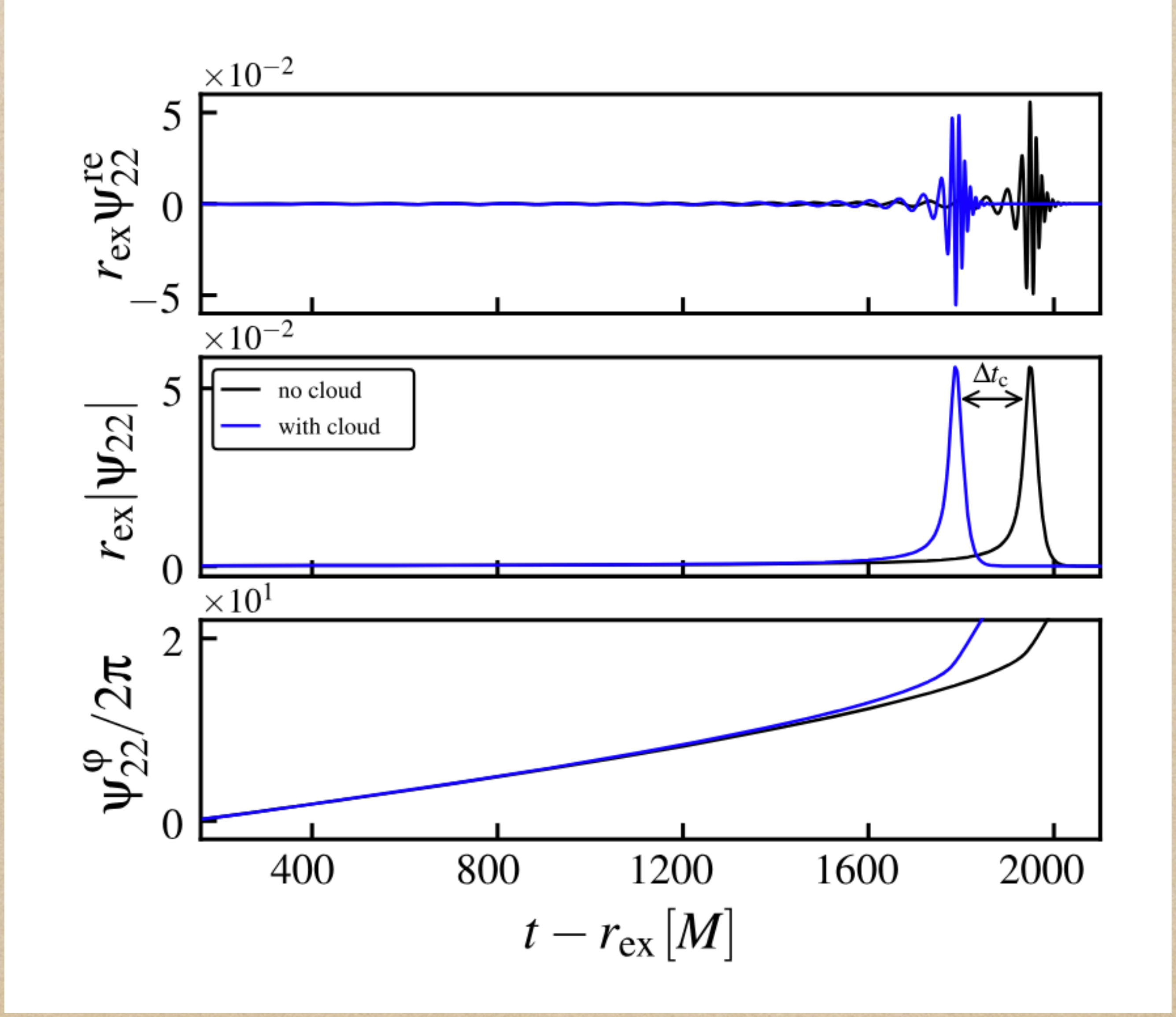
- Template constructions: e.g. Dietrich et al 1905.06011

`NRTidalv2` approximant



BBH inspiral in dark-matter environments

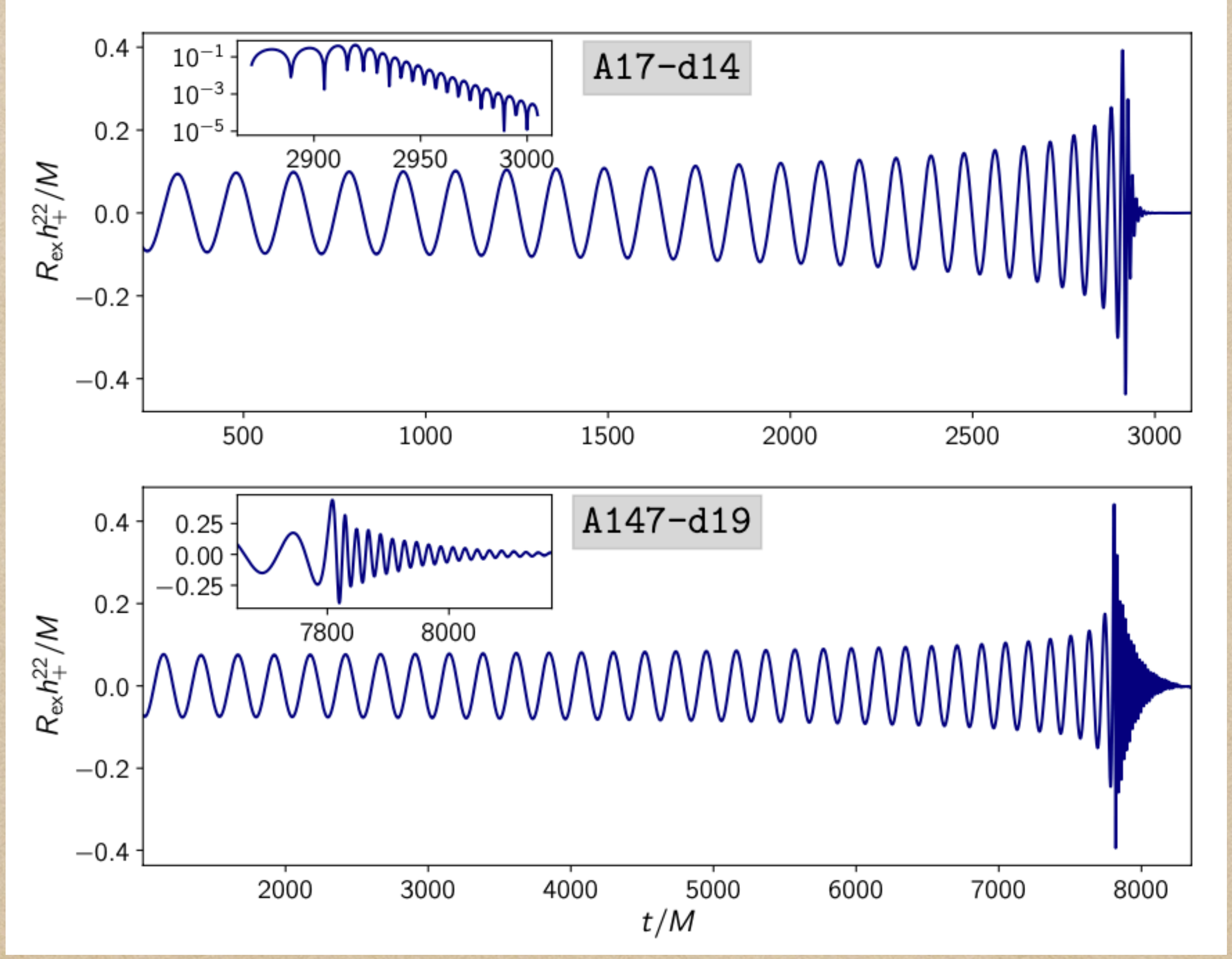
- Equal-mass BBHs + complex, massive scalar field
- Dephasing maximal if Compton wavelength ≈ orbital separation



Aurrekoetxea et al PRL 2024

Boson-star binaries

- Phase error $\approx 0.1 \dots 0.2$
- Amplitude error $\lesssim 3\,\%$
- Eccentricity $\approx 0.002 \dots 0.005$



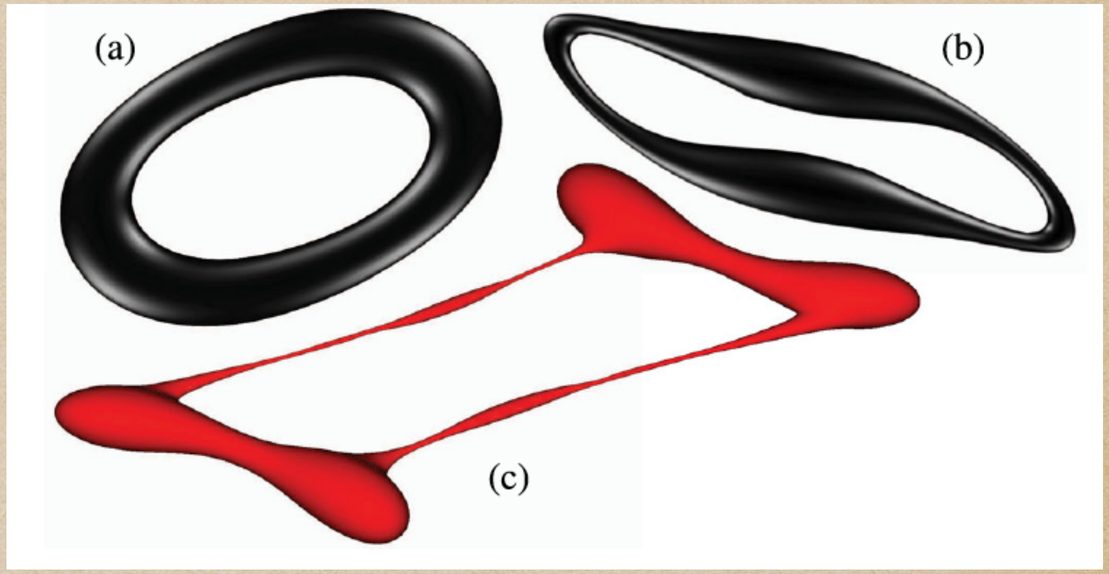
Evstafyeva et al PRL 2024

Cosmic censorship in D=5

Figueras, Kunesch & Tunyasuvunakool PRL 1512.04532

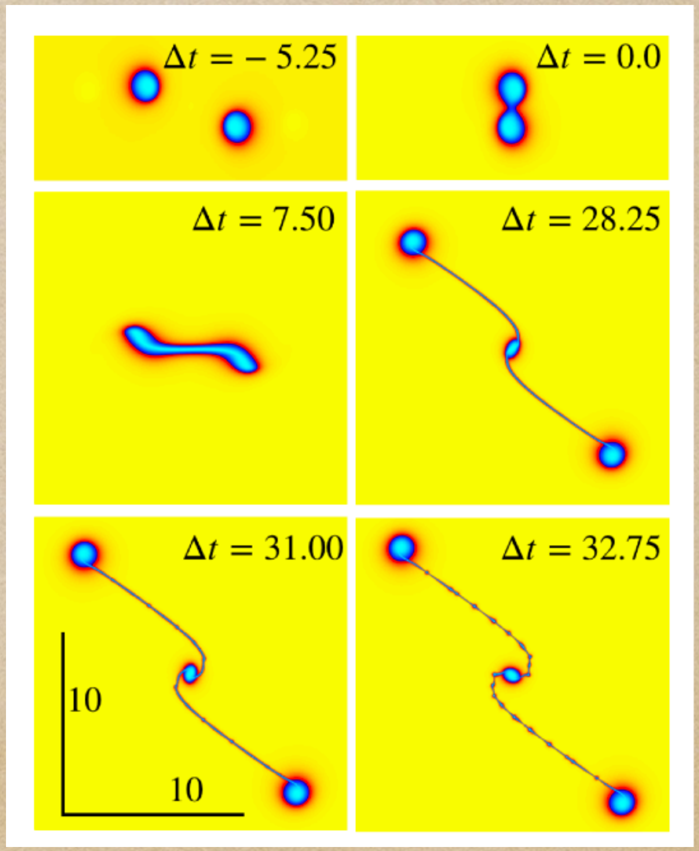
- 5D simulations (mod.cartoon)
- Conformal Z4 system
- Black ring: assympt.flat!
- Gregory-Laflamme instability

  $\Rightarrow$ Violation of CC!



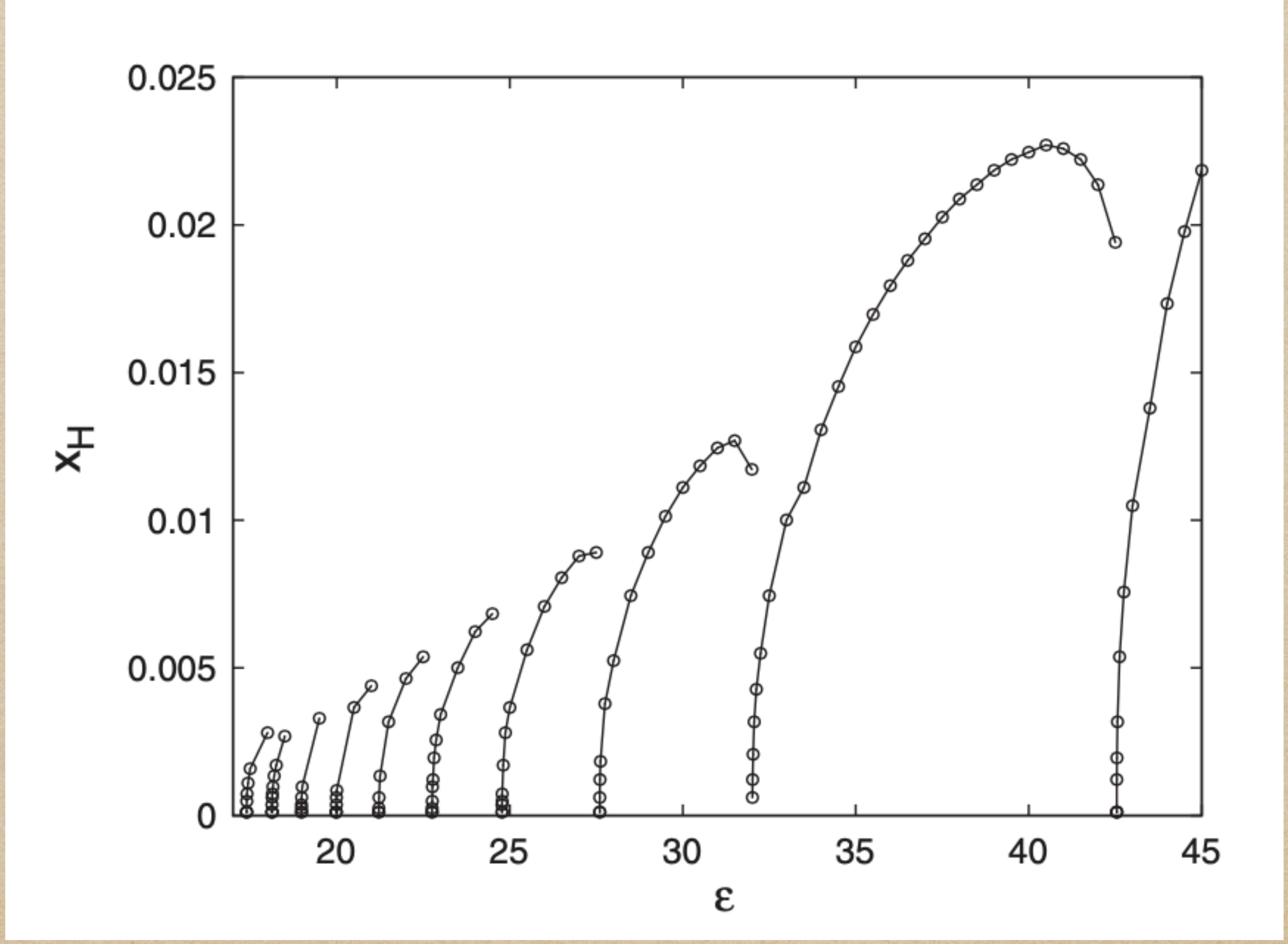
Andrade, Figueras & US JHEP 2022

- 7D BH collisions
- No finetuning!



Critical collapse in AdS

- Einstein-massless-scalar field equations with $\Lambda < 0$
- Rightmost branch = Choptuik PRL 1993
- AdS unstable due to energy shift from low to high frequencies



Bizon & Rostworowski PRL 2011

The future…

- Waveform catalogs for binaries containing BHs, NSs
  - Precessing BHBs, high-mass ratios
  - NSNS, BHNS systems

- Waveform predictions for compact objects in modified gravity
- Model GW signatures of dark matter candidates
- Exotic compact objects
- NR in cosmology
- Applications in AdS/CFT
- Critical collapse in >1 dimensions
- Higher dimensional GR; is D=4 special?