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ADVANCED QUANTUM FIELD THEORY

These notes are for use with the Cambridge University Part III lecture course *Advanced Quantum Field Theory*, given in the Department of Applied Mathematics and Theoretical Physics in Lent Term 2020. They draw on the notes of previous lecturers, in particular Hugh Osborn and David Skinner.

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Contents

Introduction

In this course, the second term of Quantum Field Theory, we have three main goals: to introduce and use the path integral formulation of QFT, to understand the need for regularization and renormalization of QFTs, and to begin investigating gauge theories.

Path integral in quantum mechanics

We begin our introduction to path integrals on familiar grounds, quantum mechanics in 1 spatial dimension. We will show that Schrödinger's equation can be reexpressed as an integral over particle trajectories, appropriately weighted.

The Hamiltonian operator $\hat{H}$ depends on position and momentum operators

$$\hat{H} = H(\hat{x}, \hat{p}) \quad \text{with} \quad [\hat{x}, \hat{p}] = i\hbar \,. \tag{1.1}$$

Assume here that we can write the Hamiltonian as the sum of a nonrelativisitic kinetic term and a potential term which depends only on position

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \,. \tag{1.2}$$

The Schrödinger equation governs the time evolution of states

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \,. \tag{1.3}$$

The formal solution can be expressed using the time evolution operator

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\psi(0)\rangle \,. \tag{1.4}$$

In the Schrödinger picture, states depend on time while operators are constant. From the latter statement, it follows that the eigenbasis of operators are also constant. Let's consider the position operator $\hat{x}$ and the fixed basis of position eigenstates $\{|x\rangle\}$. We define the wavefunction to be the complex-valued function

$$\Psi(x, t) = \langle x | \psi(t) \rangle \tag{1.5}$$

where the duplicate use of ψ is standard notation. The action of the Hamiltonian operator on the wavefunction is

$$\langle x | \hat{H} | \psi(t) \rangle = \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \Psi(x, t) \,. \tag{1.6}$$

Our main goal is to replace Schrödinger's differential equation with an integral equation. We start by inserting (1.4) into (1.5), then by inserting a complete set of initial positions x_0:

$$\begin{aligned}
\Psi(x, t) &= \langle x | e^{-i\hat{H}t/\hbar} | \psi(0) \rangle \\
&= \int_{-\infty}^{\infty} dx_0 \, \langle x | e^{-i\hat{H}t/\hbar} | x_0 \rangle \langle x_0 | \psi(0) \rangle \\
&\equiv \int_{-\infty}^{\infty} dx_0 \, K(x, x_0; t) \Psi(x_0, 0)
\end{aligned} \tag{1.7}$$

The last line defines the integration kernel K.

Now we repeat this process multiple times. Let us choose times $t_1 \dots t_n$ in between our initial time $t_0 = 0$ and a final time $t_{n+1} = T$:

$$0 \equiv t_0 < t_1 < \dots < t_n < t_{n+1} \equiv T$$

and factor the time evolution operator into $n+1$ parts

$$e^{-i\hat{H}T/\hbar} = e^{-i\hat{H}(t_{n+1}-t_n)/\hbar} e^{-i\hat{H}(t_n - t_{n-1})/\hbar} \dots e^{-i\hat{H}(t_1 - t_0)/\hbar} . \tag{1.8}$$

We can use this expression in $K(x, x_0; T)$, inserting a complete set of states between each exponential

$$K(x, x_0; T) = \int_{-\infty}^{\infty} \left[\prod_{r=1}^{n} dx_r \langle x_{r+1} | e^{-i\hat{H}(t_{r+1}-t_r)/\hbar} | x_r \rangle \right] \langle x_1 | e^{-i\hat{H}t_1/\hbar} | x_0 \rangle . \tag{1.9}$$

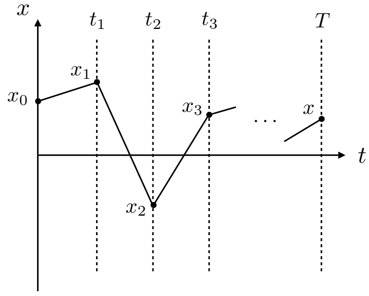


Figure 1.1: An example of a path.

What we have is an integral of amplitudes corresponding to all possible positions for each $t \in \{t_1, \dots, t_n\}$, that is, we integrate over particle "paths" (Fig. 1.1).

To make further progress, let us first consider the free theory, with $V(\hat{x}) = 0$. Denoting the free kernel as K_0 we have between any two points x and x'

$$K_0(x, x'; t) = \langle x | \exp\left(-\frac{i\hat{p}^2 t}{2m\hbar} \right) | x' \rangle \tag{1.10}$$

Inserting a complete set of momentum eigenstates

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} |p\rangle\langle p| = 1$$

and recalling these are plane waves, $\langle x | p \rangle = e^{ipx/\hbar}$, then

$$\begin{aligned} K_0(x, x'; t) &= \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} \langle x | \exp\left(-\frac{i\hat{p}^2 t}{2m\hbar} \right) |p\rangle\langle p | x' \rangle \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} \exp\left(-\frac{ip^2 t}{2m\hbar} \right) e^{ip(x-x')/\hbar} . \end{aligned}$$

Completing the square with the substitution $p' = p - m(x - x')/t$ we have

$$\begin{aligned} K_0(x, x'; t) &= e^{im(x-x')^2/2\hbar t} \int_{-\infty}^{\infty} \frac{dp'}{2\pi\hbar} \exp\left(-\frac{ip'^2 t}{2m\hbar} \right) \\ &= e^{im(x-x')^2/2\hbar t} \sqrt{\frac{m}{2\pi i\hbar t}} . \end{aligned} \tag{1.11}$$

Note that

$$\lim_{t \to 0} K_0(x, x'; t) = \delta(x - x')$$

agreeing with $\langle x | x' \rangle = \delta(x - x')$.

For a nontrivial potential $V(\hat{x}) \neq 0$, we need to use very small time-steps. Recalling the Baker-Campbell-Hausdorff relation

$$e^{\hat{A}} e^{\hat{B}} = \exp\left(\hat{A} + \hat{B} + \tfrac{1}{2}[\hat{A}, \hat{B}] + \dots \right) \neq e^{\hat{A}+\hat{B}} , \tag{1.12}$$

for small $\epsilon \ll 1$

$$e^{\epsilon \hat{A}} e^{\epsilon \hat{B}} = \exp\left(\epsilon \hat{A} + \epsilon \hat{B} + \mathcal{O}(\epsilon^2)\right) . \qquad (1.13)$$

Turning this equation around, we have

$$e^{\epsilon(\hat{A}+\hat{B})} = e^{\epsilon \hat{A}} e^{\epsilon \hat{B}} (1 + \mathcal{O}(\epsilon^2)) . \qquad (1.14)$$

Letting $\epsilon = 1/n$, raising (1.14) to the n-th power, and taking the large n limit, we find

$$e^{\hat{A}+\hat{B}} = \lim_{n\to\infty} \left(e^{\hat{A}/n} e^{\hat{B}/n}\right)^n . \qquad (1.15)$$

We will use this to separate the kinetic and potential terms in the Hamiltonian, in a way often referred to as the Suzuki-Trotter decomposition

Take $t_{r+1} - t_r = \delta t$ for all r with δt very small, and also take n very large, such that $T = n\delta t$. Then

$$\exp\left(-\frac{i\hat{H}\delta t}{\hbar}\right) = \exp\left(-\frac{i\hat{p}^2 \delta t}{2m\hbar}\right) \exp\left(-\frac{iV(\hat{x})\delta t}{\hbar}\right) [1 + \mathcal{O}((\delta t)^2)] . \qquad (1.16)$$

Sandwiching this between position eigenstates, we have

$$\begin{aligned} \langle x_{r+1}| \exp\left(-\frac{i\hat{H}\delta t}{\hbar}\right) |x_r\rangle &= \exp\left(-\frac{iV(x_r)\delta t}{\hbar}\right) \langle x_{r+1}| \exp\left(-\frac{i\hat{p}^2\delta t}{2m\hbar}\right) |x_r\rangle \\ &= \exp\left(-\frac{iV(x_r)\delta t}{\hbar}\right) K_0(x_{r+1}, x_r; \delta t) \\ &= \sqrt{\frac{m}{2\pi i\hbar\delta t}} \exp\left[\frac{i}{2\hbar} m \left(\frac{x_{r+1} - x_r}{\delta t}\right)^2 \delta t - \frac{i}{\hbar} V(x_r)\delta t\right] \end{aligned} \qquad (1.17)$$

having used $V(\hat{x})|x_r\rangle = V(x_r)|x_r\rangle$ and (1.11). Thus, with $T = n\delta t$,

$$K(x, x_0; T) = \int \left[\prod_{r=1}^{n} dx_r\right] \left(\frac{m}{2\pi i\hbar\delta t}\right)^{\frac{n+1}{2}} \exp\left\{\frac{i}{\hbar} \sum_{r=0}^{n} \left[\frac{1}{2} m \left(\frac{x_{r+1} - x_r}{\delta t}\right)^2 - V(x_r)\right] \delta t\right\} . \qquad (1.18)$$

In the limit $n \to \infty$, $\delta t \to 0$, with T fixed, the exponent becomes

$$\frac{1}{\hbar} \int_0^T dt \left[\tfrac{1}{2} m\dot{x}^2 - V(x)\right] \equiv \int_0^T dt\, L(x, \dot{x}) \qquad (1.19)$$

where $L(x, \dot{x})$ is the classical Lagrangian. Defining the classical action at a particular point by $S(x) \equiv \int_0^T dt\, L(x, \dot{x})$ we see that we can write the kernal as a "path integral"

$$\boxed{K(x, x_0; T) = \langle x| \exp\left(-\frac{i\hat{H}t}{\hbar}\right) |x_0\rangle = \int \mathcal{D}x\, e^{\frac{i}{\hbar} S(x)} .} \qquad (1.20)$$

The formal definition of the path integral measure is given by the limit used above

$$\mathcal{D}x = \lim_{\substack{\delta t \to 0 \\ n\delta t = T}} \sqrt{\frac{m}{2\pi i\hbar\delta t}} \prod_{r=1}^{n} \left(\sqrt{\frac{m}{2\pi i\hbar\delta t}}\, dx_r\right) . \qquad (1.21)$$

The normalization factor can be safely ignored, as we will see in the next chapter. Subtleties regarding the definition and existence of path integrals are beyond the scope of this course.

Eq. (1.20) is the main result of the chapter. The amplitude for a particle traveling from point x_0 to point x in time T is equal to integral over all possible positions for each successive moment in time, i.e. the integral over all particle paths, weighted by a phase with argument proportional to the classical action. In the classical limit $\hbar \to 0$, most trajectories will correspond to a highly oscillatory integrand. Only the trajectory (or trajectories) which minimize the classical action have a chance at giving some nonzero integral. Thus we can see how Hamilton's principle of least action for classical dynamics is recovered from the $\hbar \to 0$ limit of path integrals. It is sometimes said that quantum effects such as those seen in the classic double-slit experiment are due to a particle interfering with itself. This is a reference to the phase factor in the path integral (1.20).

In much of this course, we will analytically continue to imaginary time. This is a straightforward mathematical trick as long as the amplitudes of interest are analytic functions or have easy-to-study nonanalyticities. Letting $\tau = it$ yields

$$\langle x|e^{-\hat{H}\tau/\hbar}|x_0\rangle = \int \mathcal{D}x\, e^{-S(x)/\hbar}. \tag{1.22}$$

Written this way, it is much easier to see that, in the $\hbar \to 0$ limit, the integral is dominated by the path which minimizes $S(x)$. Another nice feature is that integrals of the type in (1.22) are more easily shown to be convergent than those of the type in (1.20). Finally, as will become more evident in the next chapter, working in imaginary time shows that many problems in quantum field theory can be expressed as problems in statistical field theory, where $e^{-S/\hbar}$ plays the role of a Boltzmann factor.

Quantum mechanics is essentially quantum field theory in $0+1$ dimensions, with the position operator acting as a field. In 1-dimensional quantum mechanics $\hat{x}(t) : \mathbb{R} \to \mathbb{R}$ is a real field mapping $t \mapsto x$. In 3 dimensions, $\hat{\vec{x}}(t) : \mathbb{R} \to \mathbb{R}^3$.

However, in order to be consistent with Lorentz invariance, space and time must be put on the same footing. QFT does this by demoting the position from an operator to a label. For example, real scalar fields $\phi : \mathbb{R}^{1,3} \to \mathbb{R}$ such that $\phi(t, \vec{x})$ is a real number at each point in 3+1 dimensions.

Summary of main points

1. Quantum mechanical amplitudes can be expressed as path integrals.

Further reading

Many texts present the path integral for quantum mechanics in this way.[1]

[1] L Brown. *Quantum Field Theory*. Cambridge University Press, 1992. ISBN 0-521-40006-4; M Peskin and D Schroeder. *An Introduction to Quantum Field Theory*. Addison-Wesley, 1995. ISBN 0-201-50397-2; and S Weinberg. *Lectures on Quantum Mechanics*. Cambridge University Press, 2013. ISBN 978-0-107-02872-2

Integrals and their diagrammatic expansions

In the last chapter, we introduced the path integral as an alternative way of describing evolution of a quantum mechanical wavefunction in the single variable available, time. In quantum field theory, where the degree(s)-of-freedom are fields defined at every point in spacetime, we will be interested in the behaviour of fields at separated points in spacetime. Generically, we be calculating correlation functions. In the next chapter we will make a better connection between scattering amplitudes and correlation functions.

In the meantime, we will introduce the path integral methods to be used throughout this course. Mainly we will be interested in perturbative expansions of path integrals and the representation of the resulting terms as Feynman diagrams.

We demonstrate the main ideas and methods for a 0-dimensional field $\phi : \{\text{point}\} \to \mathbb{R}$. That is, ϕ is a single, real variable.

Proceeding as we would in imaginary time, we study the integral we would call a partition function in statistical physics

$$Z = \int_{\mathbb{R}} d\phi \, e^{-S(\phi)/\hbar} . \tag{2.1}$$

Let us assume that $S(\phi)$ is a polynomial with even degree and that $S(\phi) \to \infty$ as $\phi \to \pm\infty$. We will be concerned with expectation values of the form

$$\langle f \rangle = \frac{1}{Z} \int d\phi \, f(\phi) \, e^{-S(\phi)/\hbar} . \tag{2.2}$$

f should not grow too rapidly as $|\phi| \to \infty$. Usually f is polynomial in ϕ.

2.1 *Free theory*

In this section, let us work with N fields (variables) ϕ_a with $a = 1, \ldots, N$. This will establish a relation which will provide an interesting comparison when we come to discuss fermions. Consider the action to be

$$S_0(\phi) = \frac{1}{2} \mathcal{M}_{ab} \phi_a \phi_b = \frac{1}{2} \phi^T \mathcal{M} \phi \tag{2.3}$$

where $\mathcal{M}$ is an $N \times N$ symmetric, positive definite[2] matrix.

[2] $\det \mathcal{M} > 0$.

Say we diagonalize $\mathcal{M}$: $\mathcal{M} = P \Lambda P^{\mathsf{T}}$, where Λ is a diagonal matrix of positive definite eigenvalues. For each eigenvalue λ_c

we have a Gaussian integral over the corresponding eigenvector $\chi_c = (P^\mathsf{T})_{ca}\phi_a$. Then the free partition function Z_0 can be integrated

$$
\begin{aligned}
Z_0 &= \int d^N\phi \, \exp\left(-\frac{1}{2\hbar}\,\phi^\mathsf{T}\mathcal{M}\phi\right) \\
&= \int d^N\chi \, \exp\left(-\frac{1}{2\hbar}\,\chi^\mathsf{T}\Lambda\chi\right) \\
&= \prod_c^N \int_{\mathbb{R}} d\chi_c \, \exp\left(-\frac{\lambda_c}{2\hbar}\,\chi_c^2\right) \\
&= \prod_c^N \sqrt{\frac{2\pi\hbar}{\lambda_c}} = \sqrt{\frac{(2\pi\hbar)^N}{\det\mathcal{M}}}\,.
\end{aligned}
\tag{2.4}
$$

(Note that symmetric $\mathcal{M}$ implies orthogonal P, so the Jacobian of the transformation is 1.) We see that the result of a multidimensional Gaussian integral is a square root of a determinant in the denominator. We will find something else when we need to work with fermionic fields. We have a lot more to do with real fields first.

By itself, the partition function is boring; it's just a number. What we want are correlation functions, and we can get them by making a slight modification to the partition function. Introduce an external (N component) source J so that

$$
S_0(\phi) \mapsto S_0(\phi) + J^\mathsf{T}\phi\,. \tag{2.5}
$$

We denote the corresponding integral $Z_0(J) = \int d^N\phi \, \exp\{-\frac{1}{\hbar}[S_0(\phi) + J^\mathsf{T}\phi]\}$. This can be evaluated by completing the square; let $\tilde\phi = \phi + \mathcal{M}^{-1}J$

$$
\begin{aligned}
Z_0(J) &= \int d^N\phi \, \exp\left[-\frac{1}{2\hbar}\phi^\mathsf{T}\mathcal{M}\phi - \frac{1}{\hbar}J^\mathsf{T}\phi\right] \\
&= \exp\left(\frac{1}{2\hbar}J^\mathsf{T}\mathcal{M}^{-1}J\right) \int d^N\tilde\phi \, \exp\left(-\frac{1}{2\hbar}\,\tilde\phi^\mathsf{T}\mathcal{M}\tilde\phi\right) \\
&= Z_0(0)\,\exp\left(\frac{1}{2\hbar}\,J^\mathsf{T}\mathcal{M}^{-1}J\right)\,.
\end{aligned}
\tag{2.6}
$$

This is called a *generating function* because we can obtain correlation functions by differentiating with respect to J. For example, consider

$$
\begin{aligned}
\langle\phi_a\phi_b\rangle &= \frac{1}{Z_0(0)}\int d^N\phi \; \phi_a\phi_b \, \exp\left(-\frac{1}{2\hbar}\,\phi^\mathsf{T}\mathcal{M}\phi - \frac{1}{\hbar}J^\mathsf{T}\phi\right)\bigg|_{J=0} \\
&= \frac{1}{Z_0(0)}\int d^N\phi \, \left(-\hbar\frac{\partial}{\partial J_a}\right)\left(-\hbar\frac{\partial}{\partial J_b}\right)\exp\left(-\frac{1}{2\hbar}\,\phi^\mathsf{T}\mathcal{M}\phi - \frac{1}{\hbar}J^\mathsf{T}\phi\right)\bigg|_{J=0} \\
&= (-\hbar)^2\frac{\partial}{\partial J_a}\frac{\partial}{\partial J_b}\exp\left(\frac{1}{2\hbar}\,J^\mathsf{T}\mathcal{M}^{-1}J\right)\bigg|_{J=0} \\
&= \hbar(\mathcal{M}^{-1})_{ab}\,.
\end{aligned}
\tag{2.7}
$$

That is, the 2-point function is the inverse of the quadratic term in S. In theories with dimension greater than 0, this is the propagator for ϕ (Fig. 2.1).

$\langle\phi_a\phi_b\rangle = a$ ——— $b = \hbar(\mathcal{M}^{-1})_{ab}$

Figure 2.1: Propagator for real ϕ.

We can be more general by inventing a little more notation. Let $\ell(\phi)$ be a linear combination of the N components of ϕ; that is,

$$\ell(\phi) = \sum_{a=1}^{N} \ell_a \phi_a \tag{2.8}$$

where $\ell_a \in \mathbb{R}$ and at least one term is nonzero. We can generalize the same steps used in (2.7) by swapping

$$\ell(\phi) \quad \text{for} \quad \ell\left(-\hbar \frac{\partial}{\partial J}\right) = -\hbar \sum_{a=1}^{N} \ell_a \frac{\partial}{\partial J_a} \,. \tag{2.9}$$

Now consider a correlation function composed of p such terms:

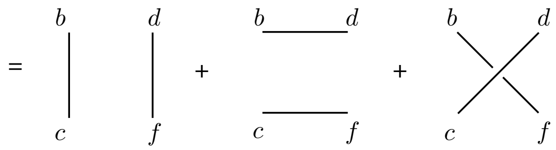
$$\begin{aligned} \langle \ell^{(1)}(\phi) \cdots \ell^{(p)}(\phi) \rangle &= \frac{1}{Z_0(0)} \int d^N\phi \prod_{i=1}^{p} \ell^{(i)}(\phi) \\ &\qquad \times \exp\left(-\frac{1}{2\hbar}\phi^{\mathsf{T}}\mathcal{M}\phi - \frac{1}{\hbar}J^{\mathsf{T}}\phi\right)\Bigg|_{J=0} \\ &= (-\hbar)^p \prod_{i=1}^{p} \ell^{(i)}\left(\frac{\partial}{\partial J}\right) \exp\left(\frac{1}{2\hbar} J^{\mathsf{T}}\mathcal{M}^{-1}J\right)\Bigg|_{J=0} \,. \end{aligned} \tag{2.10}$$

If p is odd, then the integrand is an odd function of at least one ϕ_a and the integral over $\phi_a \in (-\infty, \infty)$ vanishes. For $p = 2k$, we can identify the terms which survive the $J \to 0$ limit as follows. The action of a derivative acting on the exponential is bring down a prefactor of $\mathcal{M}^{-1}J$ multiplying the exponential. The terms which are nonzero as $J \to 0$ are those where a second derivative acts on the prefactor. Therefore, the nonvanishing terms are those where k derivatives act on the exponential, and k remove the J-dependence in the prefactor; that is, those term with k factors of $\mathcal{M}^{-1}$.

Take, for example, the 4-point function.[3] You can check that

[3] Using the notation of the previous paragraph, take $\ell_a^{(1)} = \delta_{ab}$, $\ell_a^{(2)} = \delta_{ac}$, $\ell_a^{(3)} = \delta_{ad}$, $\ell_a^{(4)} = \delta_{af}$.

$$\begin{aligned} \langle \phi_b \phi_c \phi_d \phi_f \rangle = \hbar^2 \Big[&(\mathcal{M}^{-1})_{bc}(\mathcal{M}^{-1})_{df} + (\mathcal{M}^{-1})_{bd}(\mathcal{M}^{-1})_{cf} \\ &+ (\mathcal{M}^{-1})_{bf}(\mathcal{M}^{-1})_{cd} \Big] \end{aligned}$$



(2.11)

Notice that we get 3 terms, one for each way of grouping the 4 fields into pairs. In general, the number of ways of pairing $2k$ elements is $\frac{(2k)!}{2^k k!}$. This can be derived by dividing the number of ways of rearranging all $2k$ points (i.e. $(2k)!$), by the number of ways of rearranging the pairs (i.e. $k!$) and by the number of ways of rearranging the 2 points composing each pair (i.e. 2^k). We will see that combinatorics plays a role in diagrammatic expansions.

Note that if we were working with complex fields, then the matrix $\mathcal{M}$ would be Hermitian rather than symmetric. In that case, the order of the indices in $\mathcal{M}^{-1}$ matters and we would draw the propagator with a directed line (Fig. 2.2).

$$\langle \phi_a \phi_b^* \rangle = a \longrightarrow b = \hbar(\mathcal{M}^{-1})_{ab}$$

Figure 2.2: Propagator for complex ϕ.

2.2 *Interacting theory*

We wish to go beyond the free theory, including higher-terms of ϕ is $S(\phi)$. Exact integration is usually not possible, so we often seek an expansion about the classical result, $\hbar = 0$. However integrals like

$$\int d^N\phi\, f(\phi) e^{-\frac{1}{\hbar}S}$$

do not have a Taylor expansion about $\hbar = 0$. Dyson argued this by contradiction. Assume that the integral did have a Taylor expansion about $\hbar = 0$, then it must have a finite radius of convergence in the complex $\hbar$ plane. However for any Re $\hbar < 0$ the integral clearly diverges.[4] Therefore, the radius of convergence cannot be larger than zero.

[4] Recall we assumed at the start of the Chapter that $S(\phi) \to \infty$ as $\phi \to \pm\infty$.

Consequently the $\hbar$-expansion is at best asymptotic. We say the function $I(\hbar)$ is asyptotic to a power series

$$I(\hbar) \sim \sum_{n=0}^{\infty} c_n \hbar^n \tag{2.12}$$

if and only if, for all N

$$\lim_{\hbar\to 0^+} \frac{1}{\hbar^N} \left| I(\hbar) - \sum_{n=0}^{N} c_n \hbar^n \right| = 0\,. \tag{2.13}$$

That is, for fixed N, the difference between the series and the function vanishes as $\hbar \to 0$ from above. Naturally we are not usually interested in this limit. We wish to successively improve our estimate of the function by increasing N while $\hbar$ is kept fixed. For $\hbar > 0$, the asymptotic series fails to include any transcendental terms such as $\exp(-\frac{1}{\hbar^2})$. These "nonperturbative" terms can be important in some theories, or very small in others.

Once again, we take ϕ to be a real variable and

$$S(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\,. \tag{2.14}$$

We will assume that $\lambda > 0$ so that the integral over ϕ below will be finite as $|\phi| \to \infty$, and that $m^2 > 0$ so that the minimum of S is at $\phi = 0$. Now we expand the exponential in the integrand of the partition function

$$\begin{aligned} Z &= \int d\phi\, \exp\left[-\frac{1}{\hbar}\left(\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right)\right] \\ &= \int d\phi\, e^{\frac{-m^2\phi^2}{2\hbar}} \sum_{V=0}^{\infty} \frac{1}{V!}\left(-\frac{\lambda}{4!\,\hbar}\right)^V \phi^{4V}\,. \end{aligned} \tag{2.15}$$

This infinite series converges and the resulting integral converges. Now we're going to be naughty and swap the order of integration and summation, resulting in an expansion which is asymptotic to Z. We cannot do this with an infinite series, so we first truncate the series. Letting $x = \frac{m^2\phi^2}{2\hbar}$ we write

$$Z \sim \frac{\sqrt{2\hbar}}{m} \sum_{V=0}^{N} \frac{1}{V!}\left(-\frac{\hbar\lambda}{4!\,m^4}\right)^V 2^{2V} \int_0^\infty dx\, e^{-x} x^{2V+\frac{1}{2}-1}\,. \tag{2.16}$$

The integral is the Γ function, $\Gamma(2V+\frac{1}{2}) = \frac{(4V)!\sqrt{\pi}}{4^{2V}(2V)!}$.

$$Z \sim \frac{\sqrt{2\pi\hbar}}{m}\sum_{V=0}^{N}\left(-\frac{\hbar\lambda}{m^4}\right)^V \underbrace{\frac{1}{(4!)^V V!}}_{(1)}\underbrace{\frac{(4V)!}{2^{2V}(2V)!}}_{(2)} \tag{2.17}$$

Stirling's approximation that large V, $V! \approx e^{V\log V}$, so $\frac{1}{(4!)^V V!}\frac{(4V)!}{2^{2V}(2V)!} \approx e^{V\log V} \approx V!$. This factorial growth of coefficients is a sign that the series is not convergent, but asymptotic.

Looking at the two combinatorial factors in (2.17), term (1) comes from the Taylor expansion of the $\frac{\lambda}{4!\hbar}\phi^4$ term in the exponential $e^{-S/\hbar}$. Term (2) is the number of ways of pairing $4V$ elements, i.e. each of the ϕ^4 in the V-th term.

Let us repeat this for the generating function $Z(J)$, employing some methods developed in § 2.1. Denoting $S_0(\phi) = \frac{1}{2}m^2\phi^2$ and $S_1(\phi) = \frac{\lambda}{4!}\phi^4$,

$$\begin{aligned}
Z(J) &= \int d\phi\, \exp\left\{-\frac{1}{\hbar}\left[S_0(\phi)+S_1(\phi)+J\phi\right]\right\} \\
&= \exp\left[-\frac{1}{\hbar}S_1\left(-\hbar\frac{\partial}{\partial J}\right)\right]\int d\phi\, e^{-\frac{1}{\hbar}[S_0(\phi)+J\phi]} \\
&\propto \exp\left[-\frac{\lambda}{4!\,\hbar}\left(\hbar\frac{\partial}{\partial J}\right)^4\right]\exp\left(\frac{1}{2\hbar}J^{\mathsf{T}}\mathcal{M}^{-1}J\right) \\
&\sim \sum_{V=0}^{N}\frac{1}{V!}\left[-\frac{\lambda}{4!\,\hbar}\left(\hbar\frac{\partial}{\partial J}\right)^4\right]^V \sum_{P=0}\frac{1}{P!}\left[\frac{1}{2\hbar}J^{\mathsf{T}}\mathcal{M}^{-1}J\right]^P .
\end{aligned} \tag{2.18}$$

(a) (b)

Figure 2.3: (a) Propagator with external sources at both ends. (b) Vertex.

We can represent the double series (2.18) graphically using Feynman diagrams. Once again, each factor of $\mathcal{M}^{-1} = m^{-2}$ is represented by a line (Fig. 2.3a). Let us use a filled circle at the end of a line to represent a factor of J. For each factor $(\frac{\partial}{\partial J})^4$, which came from the interaction term in the action $S_1(\phi)$, we draw a vertex (Fig. 2.3b).

First let us check that we reproduce the result for $Z = Z(0)$ (2.17), at the same time seeing how the diagrammetic method works. In order to be nonzero when we set $J = 0$, there must be the same number of derivatives coming from the vertices as there are sources at the ends of the propagators, i.e. we must have

$$E \equiv 2P - 4V = 0\,. \tag{2.19}$$

We define E to be the number of external sources left undifferentiated. For $Z(0)$ this must be zero, but when we want n-point functions, we will want $E = n$.

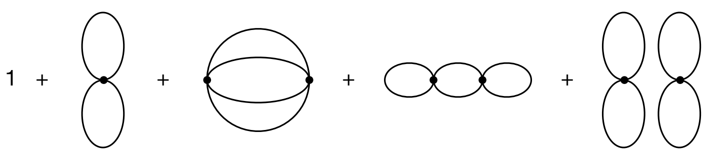


Figure 2.4: First few terms in the expansion of $Z(0)$, normalized by the free, sourceless partition function $Z_0(0)$ given in (2.4).

For the first two nontrivial terms in (2.17) correspond to $(V, P) = (1, 2)$ and $(2, 4)$ (Fig. 2.4). Just as when we multiply algebraic expressions, we combine like terms. Here we have to account the multiple ways in which derivatives from the vertices can act on sources on the propagators. Consider the diagram in Fig. 2.4 with 1 vertex and the number of ways it can be made by multiplying out the terms in (2.18). Let us make a "pre-diagram" where we label the ends of the vertex lines and the propagators (Fig. 2.5). There are $A = 4!$ ways of assigning the sources a, a', b, b' to the derivatives at 1, 2, 3, and 4. Notice this is cancelled by a 4! in the denominator $F = (V!)(4!)^V(P!)\,2^P = 4! \cdot 2 \cdot 2^2$ of (2.17). Therefore the 1-vertex diagram in (Fig. 2.4) comes with a prefactor $\frac{A}{F} = \frac{1}{8}$ (times $-\frac{\hbar\lambda}{m^4}$).

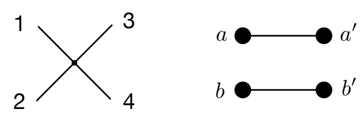


Figure 2.5: Labelled prediagram.

More generally, we can see that the denominator

$$F = (V!)(4!)^V(P!)\,2^P$$

accounts for the permutations of all vertices ($V!$), all the legs on each vertex (4!), all the propagators ($P!$), and both ends on each propagator (2). However, most diagrams have some symmetry which means that some of the permutations in F are identical and have been double-counted in F. In the example above, consider the pairing of vertices $(1a, 2a', 3b, 4b')$. Swapping a with a' at the same time as swapping 1 with 2, we arrive at exactly the same term in (2.18), so it should not be counted twice.

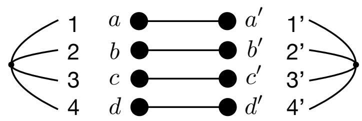


Figure 2.6: Basketball prediagram.

An alternative, and often simpler way to determine $\frac{A}{F}$ is to consider the actions which leave invariant the unlabelled diagram, like the first one in (Fig. 2.4). The diagram is unchanged if we swap the direction we draw the top loop or the bottom loop, or if we swap the top with the bottom loop. That gives us 2^3 ways of drawing the same unlabelled diagram. $\mathcal{S} = 2^3 = 8$.

With these considerations, we can find from the Feynman rules and symmetry factors that the interacting partition function has terms corresponding to those shown in Fig. 2.4 as

$$\begin{aligned}\frac{Z(0)}{Z_0(0)} &= 1 - \frac{\hbar\lambda}{8m^4} + \frac{\hbar^2\lambda^2}{m^8}\left(\frac{1}{48} + \frac{1}{16} + \frac{1}{128}\right) + \ldots \\ &= 1 - \frac{\hbar\lambda}{8m^4} + \frac{35}{384}\frac{\hbar^2\lambda^2}{m^8} + \ldots. \end{aligned} \tag{2.20}$$

Note we have reintroduced the constant of proportionality $Z_0(0) = \frac{\sqrt{2\pi\hbar}}{m}$, the free, $J = 0$ partition function (2.4), not to be confused with the $J = 0$ interacting partition function $Z(0)$ (no subscript).[5]

[5] It is usual to omit the argument when $J = 0$, so that $Z(0) = Z$ and $Z_0(0) = Z_0$.

We next consider how correlation functions can be obtained from diagrammatic expansions of $Z(J)$. The first few contributions to $Z(J)$ with 2 external sources (i.e. $E = 2$) are shown in Figure 2.7. Note that the diagrams with vacuum bubbles can be factored out. The sum of vacuum bubble diagrams is just $Z(0)$. Therefore, they

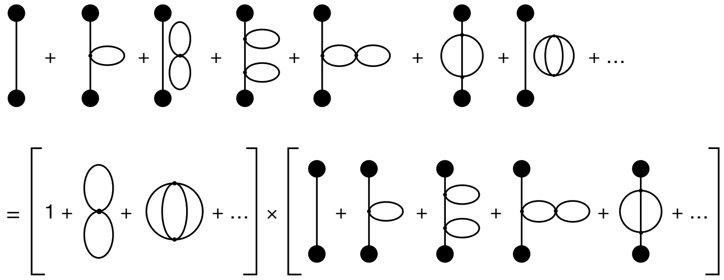
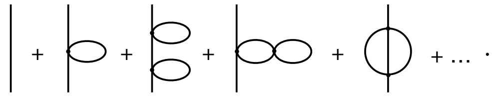


Figure 2.7: First few $E = 2$ terms in the expansion of $Z(J)$. In the second line, we note that the vacuum bubbles can be factored out so that the first line is equal to $Z(0)$ times the sum of diagrams not containing vacuum bubbles.

are divided out in the expectation value

$$\langle \phi^2 \rangle = \frac{(-\hbar)^2}{Z(0)} \left(\frac{\partial}{\partial J} \right)^2 Z(J) \Bigg|_{J=0}$$

$=$  (2.21)

Note the derivative removes the factors of J in the $E = 2$ contributions to $Z(J)$. We represent this by removing the large dots. There is also a factor of 2 associated with the 2 ways of ordering the derivatives; i.e. which derivative acts on which external leg. We can similarly draw diagrams for the first few terms in the expansion of $\langle \phi^4 \rangle$ (Fig. 2.8).

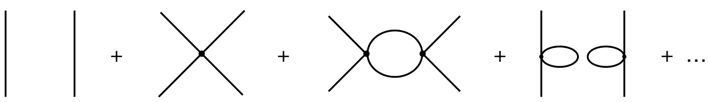


Figure 2.8: First few terms in $\langle \phi^4 \rangle$. Note that there are disconnected diagrams, like the 4th diagram here, but there are contributions from diagrams with vacuum bubbles due to the factor of $[Z(0)]^{-1}$ in the expectation value.

2.3 *Wilsonion effective action*

We will show that the sum of all vacuum diagrams can be generated by considering only *connected* vacuum diagrams. That is,

$$Z = e^{-W/\hbar} \tag{2.22}$$

where Z is the (sourceless) partition function, which we showed could be represented as the asymptotic series of all vacuum diagrams. W is the Wilsonian effective action and will be the sum of all connected vacuum diagrams.

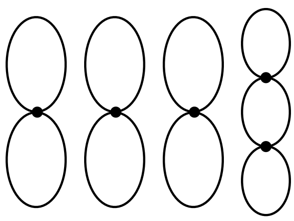


Figure 2.9: A disconnected diagram. Using the numbering of connected contributions appearing in Fig. 2.4, $n_1 = 3$, $n_3 = 1$, and all other $n_I = 0$.

Any diagram D is a product of connected diagrams. Denote the set of connected diagrams by $\{C_I\}$ and assume that each C_I includes its own symmetry factor. Any particular D can be represented as

$$D = \frac{1}{S_D} \prod_I (C_I)^{n_I} \tag{2.23}$$

where $n_I \in \mathbb{N}^0$ depending on how many copies of the I-th connected diagram appear in D. S_D is the additional symmetry factor

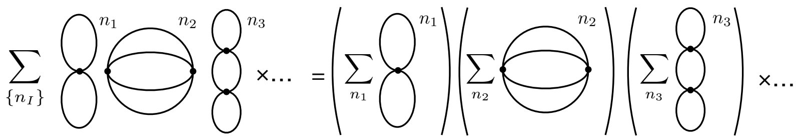


Figure 2.10: Representation of the reordering of the sum over graphs in $Z(0)$ as in (2.25).

associated with rearranging the connected parts, C_I, in such a way as to leave the diagram invariant, i.e.

$$S_D = \prod_I n_I! \,. \tag{2.24}$$

For an example, see Fig. 2.9.

Now the ($J = 0$) partition function, normalized by the free partition function, can be written as the sum over all diagrams. Using (2.23) and (2.24), this sum can then be rearranged

$$\begin{aligned}\frac{Z}{Z_0} &= \sum_{\{n_I\}} \prod_I \frac{1}{n_I!}(C_I)^{n_I} \\ &= \prod_I \sum_{n_I} \frac{1}{n_I!}(C_I)^{n_I} \\ &= \exp\left(\sum_I C_I\right) \equiv e^{-(W-W_0)\hbar}\end{aligned} \tag{2.25}$$

where we implicitly define the Wilsonian effective action

$$W = W_0 - \hbar \sum_I C_I \,. \tag{2.26}$$

Normally we can simply drop the constant W_0 which just accounts for the normalization factor Z_0. In going from the first to the second line in 2.25, we factor the sum over all possible integers n_I for the infinite product of connected graphs, into separate factors for each of the connected graphs (Fig. 2.10).

To see an example of how the effective action is used, let us consider a theory with two real fields[6]

[6] Note that we do not include a factorial in the interation term since the fields are distinguishable.

$$S(\phi, \chi) = \frac{m^2}{2}\phi^2 + \frac{M^2}{2}\chi^2 + \frac{\lambda}{4}\phi^2\chi^2 \tag{2.27}$$

The Feynman rules are shown in Fig. 2.11.

First we work with the full theory, given by action (2.27). The ef-

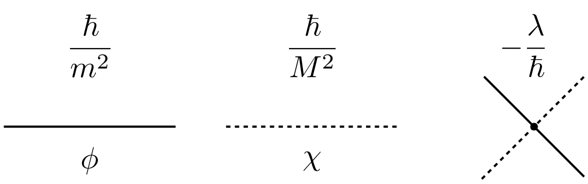
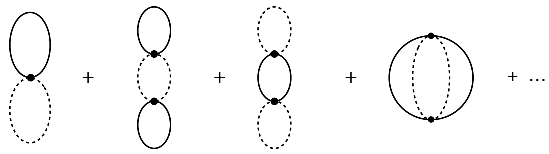


Figure 2.11: Feynman rules for the theory described by (2.27).

fective action W is given by the sum of connected vacuum diagrams

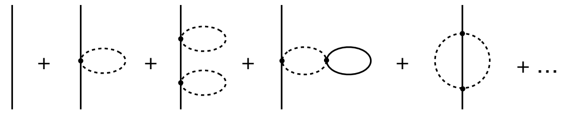
$$-\frac{W}{\hbar} =$$



$$= -\frac{\hbar\lambda}{4m^2M^2} + \frac{\hbar^2\lambda^2}{m^4M^4}\left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8}\right) + \dots \tag{2.28}$$

The 2-point correlation function (or propagator) is given to 2-loops as

$$\langle\phi^2\rangle =$$



$$= \frac{\hbar}{m^2} - \frac{\hbar^2\lambda^2}{2m^4M^4} + \frac{\hbar^3\lambda^3}{m^6M^4}\left(\frac{1}{4} + \frac{1}{4} + \frac{1}{2}\right) + \dots \tag{2.29}$$

Say we want to remove explicit χ-dependence, for example maybe the χ is massive enough not to be produced at experimentally accessible energy scales because $M \gg m$. Integrate out the heavy field. Define W such that

$$e^{-W(\phi)/\hbar} = \int d\chi\, e^{-S(\phi,\chi)/\hbar}. \tag{2.30}$$

The $\phi^2\chi^2$ term is treated as a source term for χ^2 ($J = \phi^2$). Correlation functions only involving ϕ fields can be obtained

$$\langle f(\phi)\rangle = \frac{1}{Z}\int d\phi\, d\chi\, f(\phi)\, e^{-S(\phi,\chi)/\hbar} = \frac{1}{Z}\int d\phi\, f(\phi)\, e^{-W(\phi)/\hbar}. \tag{2.31}$$

It is in this sense that W is an effective action, incorporating the virtual effects of the χ field.

In our example, the integral can be done.

$$\int d\chi\, e^{-S(\phi,\chi)/\hbar} = e^{-\frac{1}{2\hbar}m^2\phi^2}\sqrt{\frac{2\pi\hbar}{M^2 + \frac{\lambda}{2}\phi^2}} \tag{2.32}$$

therefore

$$W(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\hbar}{2}\log\left(1 + \frac{\lambda}{2M^2}\phi^2\right) + \frac{\hbar}{2}\log\frac{M^2}{2\pi\hbar}. \tag{2.33}$$

The constant term cancels in correlation functions, so it is irrelevant in QFT. However, this contributes to the energy density of the universe, to the so-called cosmological constant. Why is the observed value of the cosmological constant so small?

Expand the logarithm

$$\begin{aligned} W(\phi) &= \left(\frac{m^2}{2} + \frac{\hbar\lambda}{4M^2}\right)\phi^2 - \frac{\hbar\lambda^2}{16M^4}\phi^4 + \frac{\hbar\lambda^3}{48M^6}\phi^6 + \dots \\ &\equiv \frac{m_{\text{eff}}^2}{2}\phi^2 + \frac{\lambda_4}{4!}\phi^4 + \frac{\lambda_6}{6!}\phi^6 + \dots + \frac{\lambda_{2k}}{(2k)!}\phi^{2k} + \dots \end{aligned} \tag{2.34}$$

where we define effective mass and couplings

$$m_{\text{eff}}^2 = m^2 + \frac{\hbar\lambda}{2M^2}$$
$$\lambda_{2k} = (-1)^{k+1}\hbar\frac{(2k)!}{2^{k+1}k}\frac{\lambda^k}{M^{2k}}\,. \tag{2.35}$$

In dimensions greater than zero, it is usually not possible to evaluate integrals exactly, so $W(\phi)$ must be calculated perturbatively. The $\frac{\lambda}{4}\phi^2\chi^2$ term can be treated as a source term with 2 legs when doing the χ-integrals (Fig. 2.12).

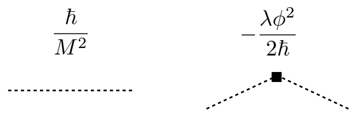
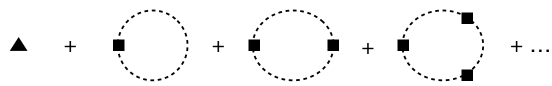


Figure 2.12: Feynman rules for evaluating $W(\phi)$ by integrating out the χ field, treating ϕ^2 as a source term.

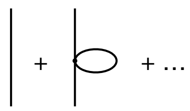
Since W is the $-\hbar$ times the sum of connected diagrams

$W(\phi) =$ 

$$= \frac{m^2}{2}\phi^2 + \frac{1}{2}\frac{\hbar\lambda}{2M^2}\phi^2 - \frac{1}{4}\frac{\hbar\lambda^2}{4M^4}\phi^4 + \frac{1}{3!}\frac{\hbar\lambda}{8M^6}\phi^6 + \ldots \tag{2.36}$$

as before. Using this $W(\phi)$

$$\langle\phi^2\rangle = \frac{1}{Z}\int d\phi\,\phi^2\,e^{-\frac{1}{\hbar}W(\phi)}$$

$=$ 

$$= \frac{\hbar}{m_{\text{eff}}^2} - \frac{\hbar^2\lambda_4}{2m_{\text{eff}}^6} + \ldots \tag{2.37}$$

as before.

2.4 *Quantum effective action*

In this section we introduce the quantum effective action, closely related to the Wilsonian effective action, but with a different interpretation. Consider the mean field configuration in the presence of external source J

$$\Phi = \frac{\partial W}{\partial J} = -\frac{\hbar}{Z(J)}\frac{\partial}{\partial J}\int d\phi\,e^{-\frac{1}{\hbar}(S+J\phi)} = \langle\phi\rangle_J\,. \tag{2.38}$$

We perform Legendre transform, exchanging J as the independent variable for Φ:

$$\Gamma(\Phi) = W(J) - \Phi J\,. \tag{2.39}$$

It may be useful to recall other Legendre transforms used in physics, for example from Lagrangians to Hamiltonians, where time derivatives of coordinates are replaced by conjugate momenta. A more apt analogy comes from the statistical physics of magnetic systems. Consider the partition function for a system of spins s_i in the presense of an external magnetic field h_i

$$Z(h) = \sum_{\{s_i\}} \exp\left[-\beta\left(\sum_{i,j,\ldots}\mathcal{H}(s_i,s_j,\ldots) + \sum_i h_i s_i\right)\right] \tag{2.40}$$

where $\mathcal{H}$ is the Hamiltonian[7] and β is the inverse temperature. The Helmholtz free energy $F(h)$ plays the role of the Wilsonian effective action:

$$F(h) = -\frac{1}{\beta} \log Z(h) \,. \tag{2.41}$$

The magnetization M plays the role of the average field in the presence of the field h

$$M(h) = -\frac{dF}{dh} = \sum_i \langle s_i \rangle \,. \tag{2.42}$$

However, we can swap which variable is independent through a Legendre transform to the Gibbs free energy

$$G(M) = F(h) + Mh \,. \tag{2.43}$$

Note that

$$\frac{dG}{dM} = \frac{dF}{dh}\frac{dh}{dM} + h + M\frac{dh}{dM} = h \tag{2.44}$$

where the last equality follows from (2.42). Read from right-to-left, this formally gives the external field as a function of the magnetization $h(M)$; however, it is most useful to think about what (2.44) implies about $G(M)$. For example, the minimum of $G(M)$ coincides with the equilibrium magnetization of the system when $h = 0$.

We can similarly look at the gradient of the quantum effective action

$$\begin{aligned}
\frac{\partial \Gamma}{\partial \Phi} &= \frac{\partial W}{\partial \Phi} - J - \Phi \frac{\partial J}{\partial \Phi} \\
&= \underbrace{\frac{\partial W}{\partial J}}_{=\Phi} \frac{\partial J}{\partial \Phi} - J - \Phi \frac{\partial J}{\partial \Phi} = -J \,.
\end{aligned} \tag{2.45}$$

For example

$$\left.\frac{\partial \Gamma}{\partial \Phi}\right|_{J=0} = 0 \tag{2.46}$$

that is, in the absence of a source, $\Phi = \langle \phi \rangle$ corresponds to an extremum of the function $\Gamma(\Phi)$.

We must apply usual caveat regarding Legendre transforms for physical systems: it is assumed that the functions being transformed are convex. In cases where this is not so, it is possible that one could apply a kind of Maxwell construction, as one sees in the statistical physics of first-order transitions. In field theory, we rarely have enough information to know beforehand whether such an approach is safe. As they say, then the proof of the pudding is in the eating.[8]

We can develop a perturbative expansion of $\Gamma(\Phi)$ building upon our earlier perturbative expansions. Let us consider a Wilsonian effect action constructed, not using the classical action $S(\phi)$ which we always compare to the physical Planck's constant $\hbar$, but instead using the quantum effective action $\Gamma(\Phi)$ compared to some ficticious

[7] e.g. the Ising Hamiltonian $\mathcal{H} \propto -s_i s_j$ where the sum would be over all nearest neighbors $\langle ij \rangle$

[8] Translation: If the results of the calculation are correct, then the assumptions in the calculation were (probably) correct. Otherwise, not.

Planck-like constant g:

$$e^{-W_\Gamma(J)/g} = \int d\Phi \, \exp\left[-\frac{1}{g}(\Gamma(\Phi) + J\Phi)\right] . \tag{2.47}$$

W_Γ is the sum of connected diagrams with Φ propagators and vertices derived from $\Gamma(\Phi)$. A diagram with ℓ loops will enter at order g^ℓ, so we can write the series as

$$W_\Gamma(J) = \sum_{\ell=0} g^\ell W_\Gamma^{(\ell)}(J) . \tag{2.48}$$

Tree diagrams compose $W_\Gamma^{(0)}(J)$. In the $g \to 0$ limit only tree diagrams contribute. Also as $g \to 0$, the integral over Φ will be dominated by the minimum of the exponent in (2.47), i.e. the Φ such that

$$\frac{\partial \Gamma}{\partial \Phi} = -J . \tag{2.49}$$

Therefore $W_\Gamma(J) = W_\Gamma^{(0)}(J) = \Gamma(\Phi) + J\Phi = W(J)$, the last equality from (2.39). The sum of connected diagrams $W(J)$ can be obtained from the sum of tree diagrams using the action $\Gamma(\Phi) + J\Phi$.

In order for this to be true, $\Gamma(\Phi)$ must be the sum of a class of diagrams called *one-particle irreducible* or "1PI" for short. These are diagrams which contain no "bridges;" an internal line (that is, an edge) of a connected graph is a *bridge* if cutting it would make the graph disconnected (e.g. see Fig. 2.13).

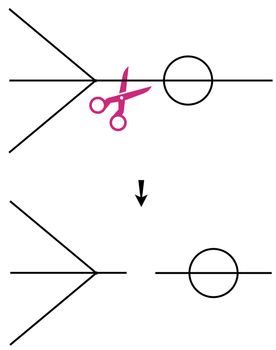


Figure 2.13: Example of a diagram with a bridge. The first diagram is not one-particle irreducible because it becomes disconnected if an internal line is cut.

The quantum effective action $\Gamma(\Phi)$ is the result of summing all the 1PI graphs with propagators and vertices derived from the action $S(\phi)$, yielding many effective vertices for $\Gamma(\Phi)$. Then correlation functions can be formed by considering only tree graphs constructed using the vertices in $\Gamma(\Phi)$ (e.g. see Fig. 2.14).

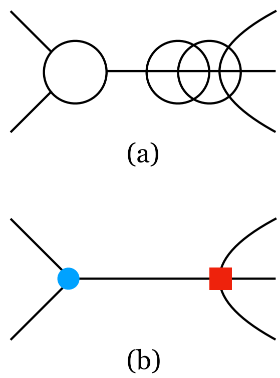


Figure 2.14: (a) A complicated diagram in a theory where $S(\phi)$ contains ϕ^3 and ϕ^4 terms. (b) A tree diagram, with the vertices coming from Φ^3 and Φ^4 in $\Gamma(\Phi)$. Diagram (a) is one of many contained in diagram (b), since the vertices of $\Gamma(\Phi)$ are the sum of 1PI diagrams with the corresponding number of external legs.

For example, in a theory with N fields ϕ_a, $a = 1, \ldots, N$, we can find the connected two-point function

$$\begin{aligned}\langle \phi_a \phi_b \rangle_J^{\text{conn}} &= \langle \phi_a \phi_b \rangle_J - \langle \phi_a \rangle_J \langle \phi_b \rangle_J \\ &= -\hbar \frac{\partial^2 W}{\partial J_a \partial J_b}\end{aligned}$$

using relations above, transforming from J to Φ as the independent variables

$$\begin{aligned}&= -\hbar \frac{\partial \Phi_b}{\partial J_a} = -\hbar \left(\frac{\partial J_a}{\partial \Phi_b}\right)^{-1} \\ &= \hbar \left(\frac{\partial^2 \Gamma}{\partial \Phi_b \partial \Phi_a}\right)^{-1} .\end{aligned} \tag{2.50}$$

In words, the full ϕ propagator, including loops, is equal to $\hbar$ times the inverse of the quadratic term in $\Gamma(\Phi)$. Similar relations can be derived for n-point functions, with $n > 2$.

2.5 Fermions

$$\theta_a\theta_b = -\theta_b\theta_a \tag{2.51}$$

For any scalar $\phi_b \in \mathbb{C}$

$$\theta_a\phi_b = \phi_b\theta_a \tag{2.52}$$

i.e. Grassmann numbers communte with complex numbers. Note that $\theta_a^2 = 0$, which implies any function of n Grassmann numbers can be written as a finite sum

$$F(\theta) = f + \rho_a\theta_a + \frac{1}{2!}g_{ab}\theta_a\theta_b + \ldots + \frac{1}{n!}h_{a_1a_2\cdots a_n}\theta_{a_1}\theta_{a_2}\cdots\theta_{a_n} \tag{2.53}$$

where the coefficients $g, \ldots, h$ are totally antisymmetric in their indices, e.g. $g_{ab} = -g_{ba}$.

Differentiation obeys a product rule with an additional minus sign

$$\frac{\partial}{\partial\theta_a}[\theta_b F(\theta)] = -\theta_b\frac{\partial F}{\partial\theta_a} + \delta_{ab}F(\theta) \tag{2.54}$$

with δ_{ab} the familiar Kronecker delta.

Integration. Require translational invariance, for constant Grassmann η

$$\int d\theta(\theta + \eta) = \int d\theta\,\theta \tag{2.55}$$

This implies

$$\int d\theta = 0 \quad \text{and} \quad \int d\theta\,\theta = 1 \tag{2.56}$$

where the latter equality includes a choice of normalization. Note the similarity between differentiation and integration. These rules are attributed to Berezin. One helpful identity is

$$\int d\theta\,\frac{\partial}{\partial\theta}F(\theta) = 0\,. \tag{2.57}$$

Useful for integrating by parts.

With n Grassmann variables θ_a, the only nonvanishing integrals involve exactly 1 power of each integration variable

$$\int d^n\theta\,\theta_1\theta_2\cdots\theta_n \equiv \int d\theta_n d\theta_{n-1}\cdots d\theta_1\,\theta_1\theta_2\cdots\theta_n = 1 \tag{2.58}$$

In general

$$\int d^n\theta\,\theta_{a_1}\theta_{a_2}\cdots\theta_{a_n} = \epsilon^{a_1a_2\cdots a_n} \tag{2.59}$$

where ϵ is completely antisymmetric, i.e. equal to $+1$ if the indices are an even permutation of $1, 2, \ldots, n$, to -1 if they are an odd permutation, or to 0 if any indices are repeated. Say we have a change of variables: $\theta'_a = A_{ab}\theta_b$. Then

$$\begin{aligned}\int d^n\theta\,\theta'_{a_1}\cdots\theta'_{a_n} &= A_{a_1b_1}A_{a_2b_2}\cdots A_{a_nb_n}\int d^n\theta\,\theta_{b_1}\cdots\theta_{b_n}\\ &= A_{a_1b_1}A_{a_2b_2}\cdots A_{a_nb_n}\,\epsilon^{b_1\cdots b_n}\\ &= \det A\,\epsilon^{a_1\cdots a_n}\\ &= \det A\int d^n\theta'\,\theta'_{a_1}\cdots\theta'_{a_n}\end{aligned} \tag{2.60}$$

Therefore a change of variables is accompanied by a determinant in the numerator.

$$d^n\theta = \det A \, d^n\theta' \,. \tag{2.61}$$

In order to move to field theories with fermions, we consider the possible forms an action can take. With 2 Grassmann variables, we have an action

$$S(\theta) = \frac{1}{2} A\theta_1\theta_2 \tag{2.62}$$

with $A \in \mathbb{R}$. The partition function is

$$\begin{aligned} Z_0 &= \int d^2\theta \, e^{-\frac{1}{\hbar}S(\theta)} \\ &= \int d^2\theta \left(1 - \frac{A}{2\hbar}\theta_1\theta_2\right) = -\frac{A}{2h} \,. \end{aligned} \tag{2.63}$$

With $n = 2m$ fermion fields

$$S(\theta) = \frac{1}{2} A_{ab}\theta_a\theta_b \,, \tag{2.64}$$

A is an antisymmetric matrix, and

$$\begin{aligned} Z_0 &= \int d^{2m}\theta \, e^{-\frac{1}{\hbar}S} \\ &= \int d^{2m}\theta \sum_{j=0}^{m} \frac{(-1)^j}{(2\hbar)^j j!} (A_{ab}\theta_a\theta_b)^j \\ &= \frac{(-1)^m}{(2\hbar)^m m!} \int d^{2m}\theta \, A_{a_1a_2}A_{a_3a_4} \cdots A_{a_{2m-1},a_{2m}} \, \theta_{a_1} \cdots \theta_{2m} \\ &= \frac{(-1)^m}{(2\hbar)^m m!} \epsilon^{a_1a_2\cdots a_{2m}} A_{a_1a_2} \cdots A_{a_{2m-1},a_{2m}} \\ &= \frac{(-1)^m}{\hbar^m} \mathrm{Pf}A = \pm\sqrt{\frac{\det A}{\hbar^n}} \,. \end{aligned} \tag{2.65}$$

Compare this to (2.4) for bosonic (i.e. ordinary) integration, where the determinant appears in the denominator. Above we have made use of the Pfaffian of a $2m \times 2m$ antisymmetric matrix A, defined to be

$$\mathrm{Pf}A = \frac{1}{2^m m!} \epsilon^{a_1a_2\cdots a_{2m}} A_{a_1a_2} \cdots A_{a_{2m-1},a_{2m}} \tag{2.66}$$

For example, $\mathrm{Pf}\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} = a$.

LSZ reduction formula

[We adopt natural units $\hbar = c = 1$ in this chapter.]

Here we briefly sketch how the Lehmann-Symanzik-Zimmermann reduction formula appears. This gives us a connection between scattering amplitudes and correlations functions, in particular vacuum expectation values.

We will work through the case of $2 \to 2$ scattering. In order to get a feel for what is going on, we imagine that our theory is very weakly interacting, so that we can use our knowledge of the free theory as a good approximation. At the end, we will comment on how interactions introduce deviations from the free theory. The conclusions we reach here can be shown more rigourously, but in a way which does not allow us to use our intuition from free theory. You are invited to investigate these discussions at the end of the course.

Let us consider a real scalar field, in Minkowski spacetime using the mostly minus metric,

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3\, 2E} \left[a(\vec{k})\, e^{-ik\cdot x} + a^\dagger(\vec{k})\, e^{ik\cdot x} \right] \tag{3.1}$$

where $E^2 = |\vec{k}|^2 + m^2$ and $k \cdot x = Et - \vec{k} \cdot \vec{x}$. The creation and annihilation operators in (3.1) are relativistically normalized.

Invert to find expressions for $a(\vec{k})$

$$\begin{aligned} \int d^3x\, e^{ik\cdot x} \phi(x) &= \frac{1}{2E}\, a(\vec{k}) + \frac{1}{2E}\, e^{2iEt}\, a^\dagger(-\vec{k}) \\ \int d^3x\, e^{ik\cdot x} \partial_0 \phi(x) &= -\frac{i}{2}\, a(\vec{k}) + \frac{i}{2}\, e^{2iEt}\, a^\dagger(-\vec{k}) \end{aligned} \tag{3.2}$$

which can be solved to give

$$\begin{aligned} a(\vec{k}) &= \int d^3x\, e^{ik\cdot x} \left[i\, \partial_0 \phi(x) + E\phi(x) \right] \\ a^\dagger(\vec{k}) &= \int d^3x\, e^{-ik\cdot x} \left[-i\, \partial_0 \phi(x) + E\phi(x) \right] \end{aligned} \tag{3.3}$$

For a free theory, a one-particle state is created

$$|k\rangle = a^\dagger(\vec{k}) |\Omega\rangle \tag{3.4}$$

where the vacuum $|\Omega\rangle$ satisfies $a(\vec{k})|\Omega\rangle$ for all $\vec{k}$, and $\langle \Omega | \Omega \rangle = 1$.

Introduce a Gaussian wave packet, i.e. define a creation operator

$$a_1^\dagger \equiv \int d^3k\, f_1(\vec{k}) a^\dagger(\vec{k}) \;\; \text{with} \;\; f_1(\vec{k}) \propto \exp\left[-\frac{(\vec{k} - \vec{k}_1)}{4\sigma^2} \right] \tag{3.5}$$

with some mean momentum $\vec{k}_1$ and width σ. Similarly for a second particle, define

$$a_2^\dagger \equiv \int d^3k\, f_2(\vec{k}) a^\dagger(\vec{k})\,, \tag{3.6}$$

with $\vec{k}_2 \neq \vec{k}_1$. This is a temporary step which keeps things well-behaved later. We can imagine evolving the Gaussians in the far distant past and future, to times when the overlap between Gaussians in coordinate space vanishes. We assume that this works even when interactions are included. One complication in the presence of interactions is that the operators $a_1^\dagger$ and $a_2^\dagger$ are time-dependent. However, in the distant past or future, we assume they coincide with their free theory expressions.

Define initial and final states (in/out states) to be

$$\begin{aligned} |i\rangle &= \lim_{t\to-\infty} a_1^\dagger(t) a_2^\dagger(t)|\Omega\rangle \\ |f\rangle &= \lim_{t\to\infty} a_{1'}^\dagger(t) a_{2'}^\dagger(t)|\Omega\rangle \end{aligned} \tag{3.7}$$

with $\langle i|i\rangle = 1 = \langle f|f\rangle$, $\vec{k}_1 \neq \vec{k}_2$ and $\vec{k}'_1 \neq \vec{k}'_2$.

Let us look at the following difference:[9]

$$\begin{aligned} a_1^\dagger(\infty) - a_1^\dagger(-\infty) &= \int_{-\infty}^{\infty} dt\, \partial_0 a_1^\dagger(t) \\ &= \int d^3k\, f_1(\vec{k}) \int d^4x\, \partial_0 \left[e^{-ik\cdot x}(-i\partial_0\phi + E\phi) \right] \\ &= -i \int d^3k\, f_1(\vec{k}) \int d^4x\, e^{-ik\cdot x}(\partial_0^2 + E^2)\phi \\ &= -i \int d^3k\, f_1(\vec{k}) \int d^4x\, e^{-ik\cdot x}(\partial_0^2 - \nabla^2 + m^2)\phi \\ &= -i \int d^3k\, f_1(\vec{k}) \int d^4x\, e^{-ik\cdot x}(\partial^2 + m^2)\phi(x)\,. \end{aligned} \tag{3.8}$$

[9] Line 1: Fundamental theorem of calculus. Line 2: Equation (3.3). Line 3: Differentiation. Line 4: Dispersion relation, $|k^2|$ as Laplacian acting (leftward) on the exponential, then integration by parts twice with $f_1(\vec{k})$ ensuring surface terms vanish.

(In a free theory, the Klein-Gordon equation $(\partial^2 + m^2)\phi = 0$ implies $a_1^\dagger(\infty) = a_1^\dagger(-\infty)$.) This relation (3.8) will be used below to integrate creation (annihilation) operators from the distant past (future) to the distant future (past).

Now let's consider the $2 \to 2$ scattering amplitude of interest

$$\langle f|i\rangle = \langle\Omega|\mathcal{T} a_{1'}(\infty)\, a_{2'}(\infty)\, a_1^\dagger(-\infty) a_2^\dagger(-\infty)|\Omega\rangle \tag{3.9}$$

Note we can just insert the time-ordering, as the operators are already time-ordered. We then use equations like (3.8) to substitute

$$\begin{aligned} a_j^\dagger(-\infty) &= a_j^\dagger(\infty) + i \int d^3k\, f_j(\vec{k}) \int d^4x\, e^{-ik\cdot x}(\partial^2 + m^2)\phi(x) \\ a_{j'}(\infty) &= a_{j'}(-\infty) + i \int d^3k\, f_j(\vec{k}) \int d^4x\, e^{ik\cdot x}(\partial^2 + m^2)\phi(x) \end{aligned} \tag{3.10}$$

The time-ordering moves the $a_j(\infty)$ to the left, annihilating $\langle\Omega|$ and the $a_{j'}(-\infty)$ to the right, annihilating $|\Omega\rangle$. The only nonzero term is the one with the products of the integrals:

$$\boxed{\begin{aligned} \langle f|i\rangle =& (i)^4 \int d^4x_1\, d^4x_2\, d^4x'_1\, d^4x'_2\, e^{-ik_1\cdot x_1}\, e^{-ik_2\cdot x_2}\, e^{ik'_1\cdot x'_1}\, e^{ik'_2\cdot x'_2} \\ &\times (\partial_1^2 + m^2)(\partial_2^2 + m^2)(\partial_{1'}^2 + m^2)(\partial_{2'}^2 + m^2) \\ &\times \langle\Omega|\mathcal{T}\phi(x_1)\phi(x_2)\phi(x'_1)\phi(x'_2)|\Omega\rangle \end{aligned}} \tag{3.11}$$

having taken the narrow-width limit, $\sigma \to 0$, of the Gaussian wave-packets, such that $f(\vec{k}_j) \to \delta^{(3)}(\vec{k} - \vec{k}_j)$.[10]

Eq. (3.11) is the LSZ reduction formula. It says that all the interesting information describing scattering is contained correlation functions like $\langle \Omega | \mathcal{T} \phi(x_1) \cdots \phi(x_n) \phi(x'_1) \cdots \phi(x'_{n'}) | \Omega \rangle$, written here for $n \to n'$ scattering. Everything else in the formula (3.11) is independent of the details of the interactions.

We should examine our assumptions that interactions do not change the $t \to \pm\infty$ in and out states. In fact, we only need the following weaker assumptions:

[10] Note the signs in the exponentials in (3.11). We have negative signs for the initial-state, or "incoming," momenta: $e^{-ik\cdot x}$, and positive signs for the final-state, or "outgoing," momenta: $e^{ik'\cdot x'}$. These signs are a consequence of our sign conventions for Fourier transforms (9.1) and for the metric. In this chapter we have used the mostly minus Minkowski metric. We will find the opposite sign for Euclidean metric (as we would for the mostly plus Minkowski metric).

1. Assume a unique ground state, and that the first excited state is a single-particle state.

2. We want $\phi|\Omega\rangle$ to be a single-particle state, i.e. that

$$\langle \Omega | \phi | \Omega \rangle = 0 \,. \tag{3.12}$$

If $\langle \Omega | \phi | \Omega \rangle = v \neq 0$, then let $\tilde{\phi} = \phi - v$ and work with this field.

3. We want ϕ normalized so that it creates a plane wave with unit amplitude

$$\langle k | \phi(x) | \Omega \rangle = e^{ik\cdot x} \tag{3.13}$$

as in the free case. Interactions may require us to rescale the field, e.g. $\phi \mapsto Z_\phi^{1/2} \phi$.

With these, and careful thought about multiparticle states, the LSZ formula still applies. These observations also hint at the fact that the field and couplings we write down in the classical Lagrangian will need to be "renormalized" when including effects due to interactions. For example, we will need to introduce renormalization factors Z_ϕ, Z_m, and Z_λ in ϕ^4 theory

$$\mathcal{L} = \frac{Z_\phi}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{Z_m}{2} m^2 \phi^2 - \frac{Z_\lambda}{4!} \lambda \, \phi^4 \,. \tag{3.14}$$

Summary of main points

1. The interesting contributions to scattering amplitudes are given by correlation functions (3.11).

2. We want to work with a field ϕ whose vacuum expectation value is 0. If that is not the case, then we would define a shifted field $\phi - v$.

3. We want our field to be normalized so that $\phi(0)|\Omega\rangle$ is a correctly normalized, single-particle state. In the presence of interactions this might mean rescaling the field by a factor $Z_\phi^{1/2}$.

Further reading

This treatment follows Srednicki, where a few more of the subtleties are discussed.[11] Other texts give a proper nonperturbative derivation, but such a discussion would be more appropriate at the end of this course.[12]

[11] M Srednicki. *Quantum Field Theory*. Cambridge University Press, 2007. ISBN 978-0-521-86449-7

[12] S Weinberg. *The Quantum Theory of Fields I*. Cambridge University Press, 1995. ISBN 0-521-55001-7; and M Peskin and D Schroeder. *An Introduction to Quantum Field Theory*. Addison-Wesley, 1995. ISBN 0-201-50397-2

Scalar field theory

We now have an appreciable toolbox of functional integral methods and are ready to apply them to a quantum field theory. Let's begin with scalar field theory in 4-dimensions.

4.1 Wick rotation

In Minkowski space, with mostly minus metric the Lagrange density is

$$\mathcal{L}[\phi] = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - V[\phi] \tag{4.1}$$

where the interactions arise from

$$V[\phi] = \frac{1}{2}m^2\phi^2 + \sum_{n>2} \frac{1}{n!} V^{(n)} \phi^n \,. \tag{4.2}$$

Writing the Lagrangian as $L = \int d^3x \, \mathcal{L}$, the partition function is

$$Z = \int \mathcal{D}\phi \, \exp\left(i \int dx^0 \, L \right) \,. \tag{4.3}$$

The free propagator is obtained by Fourier transforming $\tilde{\phi}(k) = \int d^4x \, e^{ik\cdot x} \phi(x)$ and looking at the quadratic term. As you would have discussed last term, a prescription is needed to avoid poles along the real k axis, so you wrote the Minkowski propagator as

$$\frac{i}{k^2 - m^2 + i\epsilon} = \frac{i}{(k^0)^2 - |\vec{k}|^2 - m^2 + i\epsilon} \,. \tag{4.4}$$

We can avoid the need for an $i\epsilon$ prescription and make the integrals more convergent by working in imaginary time $ix^0 = x_4$. This is effectively a rotation of integration contours in such a way that avoids crossing any nonanalyticities. We arrange the signs so that in Euclidean space, our Lagrangian and partition function are

$$\mathcal{L}[\phi] = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi + V[\phi] \tag{4.5}$$

and

$$Z = \int \mathcal{D}\phi \, \exp\left(- \int dx_4 \, L \right) \,. \tag{4.6}$$

The rotation in momentum space can be inferred from requiring $k \cdot x$ to be invariant:

$$k_0 x^0 - \vec{k}\cdot\vec{x} = -ik_0 x_4 - \vec{k}\cdot\vec{x} = -k_4 x_4 - \vec{k}\cdot\vec{x} \tag{4.7}$$

where $k_4 = ik_0$. Therefore when we see an $ik \cdot x$ in mostly-minus Minkowski spacetime, we should write it as $-ik \cdot x$ in Euclidean spacetime. The free propagator in momentum space is

$$\tilde{\Delta}_0(k) = \frac{1}{k^2 + m^2} = \frac{1}{(k_4)^2 + |\vec{k}|^2 + m^2} . \tag{4.8}$$

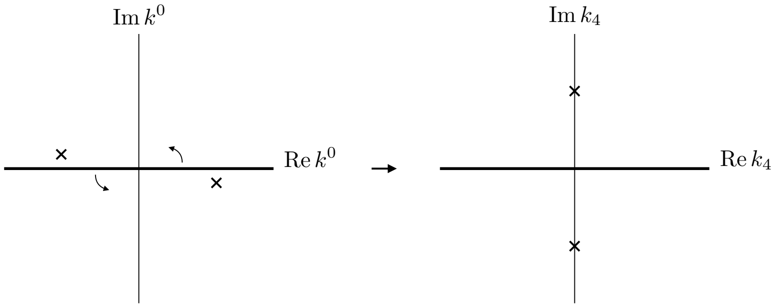


Figure 4.1: Wick rotation from Minkowski to Euclidean spacetime.

4.2 *Feynman rules*

The derivation of the free propagator follows § 2.1, with the complication that now the fields depend on spacetime position

$$S_0[\phi, J] = \int d^4x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + J(x)\phi(x) \right] . \tag{4.9}$$

We solve for the free propagator by transforming to momentum space. Following the sign convention for Fourier transforms, Eq. (9.1) extended into Euclidean spacetime, we write $\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} \tilde{\phi}(k)$ and substitute into (4.9), taking care to use unique integration variables for each substitution, to find the position integral yields a Dirac δ-function in the two momenta. After one of the momentum integrations, we find

$$\begin{aligned} S_0 &= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left[\tilde{\phi}(-k)(k^2 + m^2)\tilde{\phi}(k) + \tilde{J}(-k)\tilde{\phi}(k) + \tilde{J}(k)\tilde{\phi}(-k) \right] \\ &= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left[\tilde{\chi}(-k)(k^2 + m^2)\tilde{\chi}(k) - \frac{\tilde{J}(-k)\tilde{J}(k)}{k^2 + m^2} \right] . \end{aligned} \tag{4.10}$$

We arrived at the first line by symmetrizing the Fourier transfer of $\frac{1}{2}[J(x)\phi(x) + J(x)\phi(x)]$, choosing one momentum assignment for the first term and the opposite for the second. The second line is obtained as before by completing the square using $\tilde{\chi} = \tilde{\phi} + \tilde{J}/(k^2 + m^2)$. The path integral over $\tilde{\chi}$ is Gaussian. We find (assuming a normalization $Z_0[0] = 1$)

$$Z_0[\tilde{J}] = \exp \left[\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\tilde{J}(-k)\tilde{J}(k)}{k^2 + m^2} \right] . \tag{4.11}$$

We obtain the Feynman propagator by taking the functional derivative of $Z_0[\tilde{J}]$ twice[13]

[13] Note that while $\frac{\delta}{\delta J(x)} J(y) = \delta^{(4)}(x - y)$, in momentum space $\frac{\delta}{\delta \tilde{J}(q)} \tilde{J}(k) = (2\pi)^4 \delta^{(4)}(q - k)$.

$$\tilde{\Delta}_0(q) = \left. \frac{\delta^2 Z_0[\tilde{J}]}{\delta \tilde{J}(-q)\, \delta \tilde{J}(q)} \right|_{\tilde{J}=0} = \frac{1}{q^2 + m^2} \,. \tag{4.12}$$

We can Fourier transform to obtain the propagator in position space

$$\Delta_0(x - x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-x')}}{k^2 + m^2} \,. \tag{4.13}$$

Thus we can write the generating functional as a position space integral

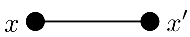
$$Z_0[J] = \exp\left[\frac{1}{2}\int d^4x\, d^4x'\, J(x)\, \Delta_0(x - x')\, J(x')\right] . \tag{4.14}$$

When we include interactions, we have $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ with $\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}m^2\phi^2$. Generalizing (2.18) we have the double expansion

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi \,\exp\left[-\int d^4x(\mathcal{L}_0 + \mathcal{L}_1 + J\phi)\right] \\ &= \exp\left\{-\int d^4y\, \mathcal{L}_1\left[-\frac{\delta}{\delta J(y)}\right]\right\} \\ &\quad \times \exp\left[\frac{1}{2}\int d^4x\, d^4x\;\; J(x)\Delta_0(x-x')J(x')\right] \\ &\sim \sum_{V=0}^{N} \frac{1}{V!}\left(-\int d^4y\, \mathcal{L}_1\left[-\frac{\delta}{\delta J(y)}\right]\right)^V \\ &\quad \times \sum_{P=0} \frac{1}{P!}\left[\frac{1}{2}\int d^4x\, d^4x'\, J(x)\Delta_0(x-x')J(x')\right]^P . \end{aligned} \tag{4.15}$$

Our representation of terms in this expansion as Feynman graphs is as before, with the addition of position-dependent labels.

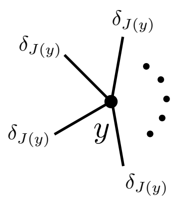
- For each term in the sum over P we have a propagator connecting sources at two points x and x'



$$= \Delta_0(x - x') \tag{4.16}$$

- For each term in the sum over V we have a vertex at an integration point y

$$-\mathcal{L}_1\left[-\frac{\delta}{\delta J(y)}\right] = \tag{4.17}$$

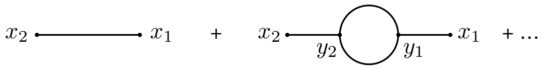


  where we used the shorthand $\delta_{J(y)} \equiv -\frac{\delta}{\delta J(y)}$. Note also that here we have assumed the simple case where $\mathcal{L}_1$ is a monomial in ϕ. If this is not the case, one just has to take care to keep track of the combinations of different types of vertices; usually this is straightforward but can become tedious.

- One integrates over all internal points of the graph, i.e. the y variables in (4.15).

- There are symmetry factors as discussed in Chapter 2.

For example, the 2-point function may be written as a diagrammatic exansion

$\langle \phi(x_2)\phi(x_1)\rangle =$ 

(4.18)

The first term is just $\Delta_0(x_2 - x_1)$ as in (4.16). The second term is

$$\lambda^2 \int d^4y_1\, d^4y_2\, \Delta_0(x_2 - y_2)\Delta_0(y_1 - x_1)[\Delta_0(y_2 - y_1)]^2 . \tag{4.19}$$

It is rare that the perturbative expansion is carried out in position space. It is much more likely that we are interested in initial and final momenta rather than initial and final positions. Therefore we should carry out a Fourier transform.

$$\langle \tilde{\phi}(p_2)\tilde{\phi}(p_1)\rangle = \int d^4x_1\, d^4x_2\, e^{-i(p_1\cdot x_1 + p_2\cdot x_2)} \langle \phi(x_2)\phi(x_1)\rangle \tag{4.20}$$

This convention for the signs of the p_1 and p_2 in the Fourier transforms corresponds to choosing the momenta to be directed outward (Fig. 4.2). Note that this is consistent with the signs appearing in the phases of (3.11) once we account for the Wick rotation (4.7) from mostly-minus Minkowski to Euclidean spacetime.

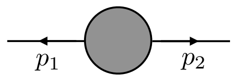


Figure 4.2: Momentum directions for the 2-point function using the Fourier transform conventions of (4.20).

Thinking about momentum conservation, it is natural to set $p_1 = -p_2 = -p$ and draw the momentum flowing through the diagram from left to right. Nevertheless it is useful to remember a general sign convention, and then make adjustments afterward based on the particular physics of interest.

The contribution to $\langle \tilde{\phi}(p_2)\tilde{\phi}(p_1)\rangle$ from (4.19) is then

$$\begin{aligned}\tilde{D}^{(1)} \equiv \lambda^2 &\int d^4x_1\, d^4x_2\, e^{-i(p_1\cdot x_1 + p_2\cdot x_2)} \int d^4y_1\, d^4y_2 \\ &\times \int \left[\prod_{j=1}^{4} \frac{d^4k_j}{(2\pi)^4}\right] e^{ik_1\cdot(y_1-x_1)} e^{ik_2\cdot(x_2-y_2)} e^{i(k_3+k_4)\cdot(y_2-y_1)} \\ &\times \tilde{\Delta}_0(k_1)\tilde{\Delta}_0(k_2)\tilde{\Delta}_0(k_3)\tilde{\Delta}_0(k_4) .\end{aligned} \tag{4.21}$$

The x_1 and x_2 integrations give, respectively, factors of $(2\pi)^4\delta^{(4)}(p_1 + k_1)$ and $(2\pi)^4\delta^{(4)}(p_2 - k_2)$. After doing the corresponding integrals over k_1 and k_2, we can see that the y_1 and y_2 integrals similarly yield $(2\pi)^4\delta^{(4)}(p_1 + k_3 + k_4)$ and $(2\pi)^4\delta^{(4)}(p_2 - k_3 - k_4)$. Finally, we can integrate over k_3, say, to find, dropping the subscript on the final integration variable

$$\tilde{D}^{(1)} = \lambda^2 \int \frac{d^4k}{(2\pi)^4} (2\pi)^4\delta^{(4)}(p_1 + p_2)\, \tilde{\Delta}_0(-p_1)\tilde{\Delta}_0(p_2)\tilde{\Delta}_0(p_2 - k)\tilde{\Delta}_0(k) . \tag{4.22}$$

Realizing that overall momentum will be conserved for physical processes, it is common to set $p_2 = -p_1 \equiv p$ (Fig. 4.3).

It is straightforward to see that the steps of the example generalize to other diagrams in momentum space. To summarize, the momentum-space Feynman rules for real scalar theory are

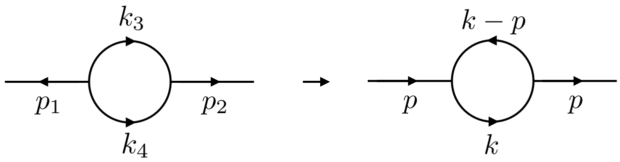


Figure 4.3: Momentum in diagram $\tilde{D}^{(1)}$, before and after integrating over δ-functions.

- Lines are propagators with factors of $(k^2 + m^2)^{-1}$.
- n-point vertices correspond to factors of $-V^{(n)}$ in (4.2).
- Momentum is conserved at each vertex.
- For each loop, there is an integral over a single loop momentum.
- Overall momentum is conserved, so there is an overall factor $(2\pi)^4\delta^{(4)}(\sum_j p_j)$ containing all external momenta.
- One must account for the graph's symmetry factor.

4.3 *Vertex functions*

The effective actions introduced in § 2.3 and § 2.4, become functionals. The Wilsonian effective action $W[J]$, now a functional of an external source field $J(x)$, is still the sum of connected Feynman diagrams (times $-\hbar^{-1}$). The quantum effective action $\Gamma[\Phi]$ depends on the mean field $\Phi(x)$ in the presence of $J(x)$, and is the sum of single-particle irreducible (1PI) graphs. The relation between them is the Legendre transform

$$\Gamma[\Phi] = W[J] - \int d^4x\, J(x)\Phi(x)\,, \tag{4.23}$$

which implies

$$\frac{\delta W[J]}{\delta J(x)} = \Phi(x) \quad \text{and} \quad \frac{\delta \Gamma[\Phi]}{\delta \Phi(x)} = -J(x)\,. \tag{4.24}$$

Note that these relations can be read two ways: one where $J(x)$ is an independent external source, in which case the mean field $\Phi(x)$ depends on $J(x)$; the other where we think of $\Phi(x)$ as independent with the corresponding $J(x)$ changing accordingly as $\Phi(x)$ is varied.

Let us introduce some notation. For the connected n-point functions we write

$$G^{(n)}(x_1, \ldots, x_n) \;=\; (-1)^{n+1} \prod_{i=1}^{n} \frac{\delta}{\delta J(x_i)} W[J] \tag{4.25}$$

and we define the vertex functions

$$\Gamma^{(n)}(x_1, \ldots, x_n) \;=\; (-1)^{n} \prod_{i=1}^{n} \frac{\delta}{\delta \Phi(x_i)} \Gamma[\Phi]\,. \tag{4.26}$$

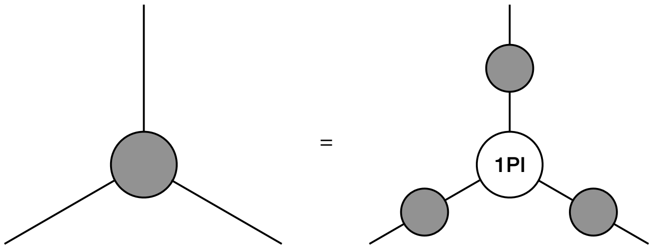


Figure 4.4: Graphical representation of (4.29).

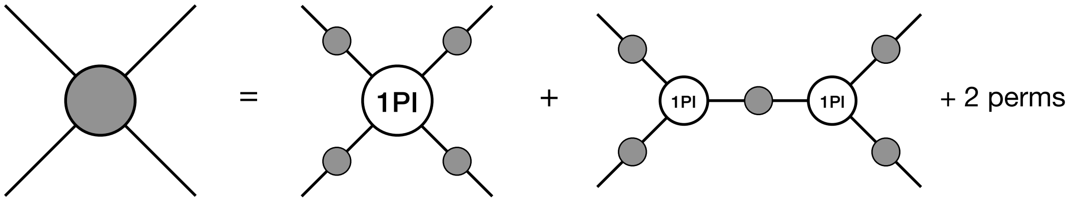


Figure 4.5: Graphical representation relating $G^{(4)}$ to integrals over $\Gamma^{(4)}$ and $\Gamma^{(3)}$. The 2 terms not shown are the other two ways of connecting the two $\Gamma^{(3)}$ vertices.

Consider $n = 2$.

$$G^{(2)}(x,y) = -\frac{\delta^2 W}{\delta J(x)\delta J(y)} = -\frac{\delta\Phi(y)}{\delta J(x)}$$
$$\Gamma^{(2)}(y,z) = \frac{\delta^2\Gamma}{\delta\Phi(y)\delta\Phi(z)} = -\frac{\delta J(z)}{\delta\Phi(y)}\,. \tag{4.27}$$

These are inverses of each other:

$$\int d^4y\, G(x,y)\Gamma(y,z) = \delta^{(4)}(x-z)\,. \tag{4.28}$$

As you showed in an example sheet problem,

$$G^{(3)}(x_1,x_2,x_3) = \int d^4z_1\, d^4z_2\, d^4z_3\, G^{(2)}(x_1,z_1)G^{(2)}(x_2,z_2) \times G^{(2)}(x_3,z_3)\,\Gamma^{(3)}(z_1,z_2,z_3)\,. \tag{4.29}$$

This tells us that the connected 3-point function can be constructed from convolving the 3-point vertex function with three 2-point functions. In the diagrammatic picture, the connected 3-point function is equal to the 1PI graphs with 3 legs with each leg equal to $G^{(2)}$ and the internal points integrated over. We can think of the 3-point vertex as an "amputated" n-point function.

Note that for $n > 3$, $G^{(n)}$ is composed of terms coming from products of $\Gamma^{(m)}$ with $3 \leq m \leq n$. In Figure 4.5 for example, the connected 4-point function $G^{(4)}$ can be written schematically as a sum of four terms: one with the amputated 4-point vertex $\Gamma^{(4)}$ and three terms coming from the three ways of inserting two 3-point vertices $\Gamma^{(3)}$ (recall that the external legs carry distinguishable momenta).

The expression (4.29) can be inverted

$$\Gamma^{(3)}(y_1,y_2,y_3) = \int d^4x_1\, d^4x_2\, d^4x_3\, \Gamma^{(2)}(x_1,y_1)\Gamma^{(2)}(x_2,y_2) \times \Gamma^{(2)}(x_3,y_3)\, G^{(3)}(x_1,x_2,x_3)\,. \tag{4.30}$$

We will often work in momentum space. The Fourier transform of the two-point function is

$$\begin{aligned}\langle\tilde{\phi}(p_1)\tilde{\phi}(p_2)\rangle &= \int d^4x_1\, d^4x_2\, e^{-ip_1\cdot x_1}\, e^{-ip_2\cdot x_2}\langle\phi(x_1)\phi(x_2)\rangle \\ &= \int d^4x_1\, d^4x_2\, e^{-ip_1\cdot x_1}\, e^{-ip_2\cdot x_2}\langle\phi(x_1-x_2)\phi(0)\rangle \\ &= \int d^4y\, d^4x_2\, e^{-ip_1\cdot y}\, e^{-i(p_1+p_2)\cdot x_2}\langle\phi(y)\phi(0)\rangle \\ &= \int d^4y\, \delta^{(4)}(p_1+p_2)\, e^{-ip_1\cdot y}\langle\phi(y)\phi(0)\rangle \qquad (4.31)\end{aligned}$$

where we have assumed translational invariance in going from the first line to the second. Note this implies that momentum is conserved.[14] The connected two-point function can also be written as a Fouier transform

[14] Recall that the momentum operator is said to be the generator of translations.

$$\tilde{G}^{(2)}(p) = \int d^4x\, e^{-ip\cdot x} G(x,\phi(0))\,. \qquad (4.32)$$

It is conventional to make use of momentum conservation to write $\tilde{G}^{(2)}$ as a function of a single momentum.

We can similarly consider Fourier transforms $\tilde{\Gamma}^{(n)}$ of the vertex functions $\Gamma^{(n)}$.

4.4 *Renormalization*

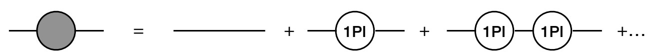
Let us consider the following classical action for a real scalar field in 4 dimensions is

$$S[\phi] = \int d^4x\, \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right]\,. \qquad (4.33)$$

In this section we will begin to compute terms in the quantum effective action using perturbation theory.

Let us consider the perturbative expansion of the propagator. As discussed above, the full propagator can be written as the sum of all tree diagrams composed with vertices derived from the quantum effective action. Graphically, these are trees connected by blobs containing all relevant 1PI diagrams. In the case of the propagator, this is a geometric series

$\tilde{G}^{(2)}(p) =$



$$\begin{aligned} &= \frac{1}{p^2+m^2} + \frac{1}{p^2+m^2}\Pi(p^2)\frac{1}{p^2+m^2} \\ &\quad + \frac{1}{p^2+m^2}\Pi(p^2)\frac{1}{p^2+m^2}\Pi(p^2)\frac{1}{p^2+m^2} + \ldots \\ &= \frac{1}{p^2+m^2-\Pi(p^2)}\,. \qquad (4.34)\end{aligned}$$

We have implicitly defined the *vacuum polarization* $\Pi(p^2)$ as the amputated two-point 1PI amplitude, and used Lorentz invariance to conclude that it should depend on p only through p^2.

Using (4.28) the 2-point vertex function can be inferred from $\Pi(p^2)$ to be

$$\tilde{\Gamma}^{(2)}(p) = \left[\tilde{G}^{(2)}(p)\right]^{-1} = p^2 + m^2 - \Pi(p^2)\,. \tag{4.35}$$

Expanding in a perturbative series, the 1-loop contribution (Fig. 4.6) is

$$\begin{aligned}\Pi_1(p^2) &= -\frac{\lambda}{2}\int^{\Lambda}\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2} \\ &= -\frac{\lambda S_4}{2(2\pi)^4}\int_0^{\Lambda}\frac{k^3\,dk}{k^2+m^2} \\ &= -\frac{\lambda m^2}{32\pi^2}\int_0^{\Lambda^2/m^2}\frac{u\,du}{1+u} \\ &= -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2\log\left(1+\frac{\Lambda^2}{m^2}\right)\right]\,. \end{aligned}\tag{4.36}$$

Notation in the first line is shorthand for $|k| < \Lambda$. Surface area $S_d = 2\pi/\Gamma(\frac{d}{2})$, where Γ here is the Γ-function. We note that this is divergent as $\Lambda \to \infty$.

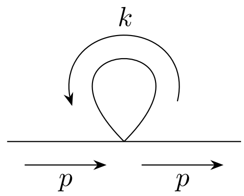


Figure 4.6: One-loop vacuum polarization diagram in ϕ^4 theory.

$$\tilde{\Gamma}_1^{(4)}(p_1,p_2,p_3,p_4) = \frac{\lambda^2}{2}\int^{\Lambda}\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\sum_P\frac{1}{(P+k)^2+m^2} \tag{4.37}$$

where the sum P runs over p_1+p_2, p_1+p_3, and p_1+p_4. At this moment, we wish to investigate the ultraviolet divergence in (**??**). As Λ and therefore the loop momentum gets arbitrarily large, the external momenta are negilgible. Thus it suffices to consider

$$\begin{aligned}\tilde{\Gamma}_1^{(4)}(0,0,0,0) &= \frac{3\lambda^2}{2}\int^{\Lambda}\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+m^2)^2} \\ &= \frac{3\lambda^2}{32\pi^2}\left[\log\left(1+\frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{\Lambda^2+m^2}\right]\,. \end{aligned}\tag{4.38}$$

On general grounds, we expect the full propagator to have the form

$$\begin{aligned}\tilde{G}^{(2)}(p) &= \sum_n \frac{|\langle\Omega|\tilde{\phi}(0)|n\rangle|^2}{p^2+m_n^2} \\ &= \frac{|\langle\Omega|\tilde{\phi}(0)|1\rangle|^2}{p^2+m_{\text{phys}}^2} + \ldots \end{aligned}\tag{4.39}$$

where the top line includes a sum over a set of complete states (which would include a continuum of scattering states), and the second line assumes that the first excited state is a single particle with a mass measured to be m_{phys}. For a field properly normalized as discussed in the context of the LSZ formula, we would also expect $\langle\Omega|\phi(0)|1\rangle = 1$ (see the discussion around (3.13)). However, generally speaking, corrections from loop diagrams will mean that $\langle\Omega|\phi(0)|1\rangle \neq 1$ and the mass in our Lagrangian $m \neq m_{\text{phys}}$. In fact, we find divergences.

Clearly we have to do something about the divergences found in the one-loop calculations above. A practical approach is to prescribe a way of separating the divergences from finite pieces. In doing so, we will sacrifice some predictability of the theory, however not as much as you might think.

In order to emphasize that the parameters and fields used above give rise to divergent contributions, let us add a '0' subscript. The 'original' Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}(\partial\phi_0)^2 + \frac{1}{2}m_0^2\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4 \tag{4.40}$$

is written here in terms of the original field ϕ_0 and the original mass m_0 and coupling λ_0.

Let us first define a rescaled field ϕ such that $\phi_0 = Z_\phi^{1/2}\phi$. The constant Z_ϕ can be determined by requiring the rescaled field to satisfy $\langle\Omega|\tilde{\phi}(0)|1\rangle = 1$. In terms of this field

$$\mathcal{L}_0 = \frac{Z_\phi}{2}(\partial\phi)^2 + \frac{Z_\phi}{2}m_0^2\phi^2 + \frac{Z_\phi^2\lambda_0}{4!}\phi^4 \,. \tag{4.41}$$

Next we assert that we can divide the original Lagrangian into two sets of terms: a renormalized Lagrangian and a set of 'counterterms'

$$\mathcal{L}_0 = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\delta Z_\phi}{2}(\partial\phi)^2 + \frac{1}{2}\delta m^2\phi^2 + \frac{\delta\lambda}{4!}\phi^4 \,. \tag{4.42}$$

We might write this as $\mathcal{L}_0 = \mathcal{L}_{\text{ren}} + \mathcal{L}_{\text{ct}}$. Equating coefficients in (4.41) and (4.42) we have

$$\delta Z_\phi = Z_\phi - 1\,, \quad \delta m^2 = Z_\phi m_0^2 - m^2\,, \quad \text{and} \quad \delta\lambda = Z_\phi^2\lambda_0 - \lambda\,. \tag{4.43}$$

We can now use (4.42) to do perturbative calculations of n-point vertices $\tilde{\Gamma}^{(n)}$. The Feynman rules for the terms in the renormalized Lagrangian are the same as in the original Lagrangian. In addition to these, we treat the counterterms as additional vertices (Fig. 4.7).

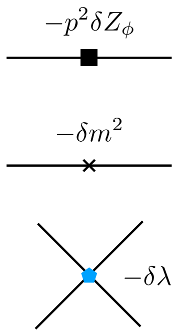


Figure 4.7: Vertices arising from counterterms.

The counterterms are all proportional to $\hbar$ (were we to restore it), assuming that the δZ_ϕ, δm^2, and $\delta\lambda$ are nonzero at one-loop order.[15] Therefore at the same time as we consider one-loop graphs involving the renormalized Lagrangian, we have to include tree-diagrams with counterterm vertices.

[15] Actually, δZ is only nonzero at 2-loop order in ϕ^4 theory, but let's ignore that for the moment.

Let's return to the 2-point vertex and $\Pi_1(p^2)$. We already have the contribution from using the renormalized Lagrangian – just interpret the m and λ in (4.36) as the renormalized mass and coupling. To that we add the contribution from the counterterm tree-graph

$$\Pi_{1,\text{ct}} = -\delta m^2 \,. \tag{4.44}$$

We can have a finite result for $\Pi_1 + \Pi_{1,\text{ct}}$ if we choose δm^2 to cancel the divergence in (4.36), for example let

$$\delta m^2 = -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2\log\left(1 + \frac{\Lambda^2}{m^2}\right)\right] \,. \tag{4.45}$$

With this choice $\Pi_1 + \Pi_{1,\text{ct}} = 0$.

This is not a unique choice. We are free to add any finite term to δm^2. Whatever prescription we choose for the finite terms here and in the other counterterms, constitutes a 'renormalization scheme.' The scheme used in (4.45) is called 'on-shell' scheme, because with it

$$\tilde{G}^{(2)}(p) = \frac{1}{p^2 + m^2 - \Pi_1 - \Pi_{1,\mathrm{ct}}} = \frac{1}{p^2 + m^2} \tag{4.46}$$

and the renormalized mass can be identified with the mass of the single particle created by ϕ: $m = m_{\mathrm{phys}}$. The full statement of an on-shell renormalization scheme are that the pole of $\tilde{G}^{(2)}(p)$ should be at $p^2 = -m^2_{\mathrm{phys}}$ and that the residue of the pole should be 1. In terms of $\Pi_{\mathrm{ren}}(p^2) = \Pi(p^2) + \Pi_{\mathrm{ct}}(p^2)$ these amount to

$$\Pi_{\mathrm{ren}}(-m^2_{\mathrm{phys}}) = m^2 - m^2_{\mathrm{phys}} \text{ (usually } = 0) \tag{4.47}$$

$$\left.\frac{\partial \Pi_{\mathrm{ren}}}{\partial p^2}\right|_{p^2 = -m^2_{\mathrm{phys}}} = 0\,. \tag{4.48}$$

In the case of ϕ^4 we have

$$\Pi_{\mathrm{ren}} = -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \log\left(1 + \frac{\Lambda^2}{m^2}\right)\right] - p^2 \delta Z_\phi - \delta m^2\,. \tag{4.49}$$

Because the only 1-loop diagram has a loop integral independent of the external momentum, we did not get a contribution – in particular a divergence – proportional to p^2 in Π_1. Therefore, $\delta Z = 0$ here and (4.48) is satisfied. With our choice of δm^2 (4.45), we satisfy (4.47).

Next we must choose $\delta\lambda$ so that the divergence in $\tilde{\Gamma}^{(4)}_1(0,0,0,0)$ (4.38) is cancelled. The contribution from the counterterm vertex at tree-level is just $\tilde{\Gamma}^{(4)}_{1,\mathrm{ct}} = -\delta\lambda$. Choosing

$$\delta\lambda = \frac{3\lambda^2}{32\pi^2}\left(\log\frac{\lambda^2}{m^2} - 1\right) \tag{4.50}$$

gives

$$\begin{aligned} \lambda_{\mathrm{eff}} \equiv \tilde{\Gamma}^{(4)}(0,0,0,0) &= \lambda + \tilde{\Gamma}^{(4)}_1(0,0,0,0) + \tilde{\Gamma}^{(4)}_{1,\mathrm{ct}} \\ &= \lambda - \frac{3\lambda^2}{32\pi^2}\left[\log\left(1 + \frac{m^2}{\Lambda^2}\right) + \frac{m^2}{m^2 + \Lambda^2}\right] \end{aligned} \tag{4.51}$$

which we can think of as an effective coupling, in fact is the coefficient of the quartic term in the quantum effective action. Note that $\lambda_{\mathrm{eff}} = \lambda$ in the $\Lambda \to \infty$ limit. This is due our choice of finite term in (4.50).

4.5 *Dimensional regularization*

A momentum cutoff is not compatible with gauge invariance in nonabelian gauge theories. In this section we introduce a more sophisticated method of regulating divergent loop integrals: varying the dimension. Let us work through the example of ϕ^4 theory in 4 dimensions. Actually, we will work in $d = 4 - \epsilon$ dimensions, where we will treat ϵ as a small parameter.

First, we perform a little dimensional analysis in d dimensions, where the couplings acquire non-standard dimensionality. In natural units $\hbar = c = 1$, we know we must have a dimensionless action, here

$$S = \int d^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \right] . \tag{4.52}$$

Let us use $[A]$ to give the mass dimension of quantity A. Given $[S] = 0$ and $[\partial] = -[x] = [m] = 1$, the mass term in (4.52), $[m^2\phi^2] = d$, implies that $[\phi] = \frac{d}{2} - 1$. Next looking at the interaction term, we can infer that $[\lambda] = 4 - d = \epsilon$.

It is more convenient to work with a dimensionless couplings. We introduce an arbitrary mass scale μ; this is not a cutoff to be taken to infinity, but it is a scale associated with the regulator. Then we can write

$$\lambda = \mu^\epsilon g(\mu) \tag{4.53}$$

where $g(\mu)$ is a dimensionless coupling whose value depends on μ.

In a moment we will return to considering the integrals arising from 1-loop diagrams of § 4.4. However first, let us collect a few mathematical observations which will be useful to us.

1. The definition of the (surface) area S_d of a unit sphere in d dimensions: Consider a Gaussian integral in $d \in \mathbb{Z}^+$ dimensions

$$\begin{aligned} (\sqrt{\pi})^d &= \int_{\mathbb{R}^d} \prod_{i=1}^{d} dx_i \, e^{-x_i^2} \\ &= S_d \int_0^\infty dr \, r^{d-1} \, e^{-r^2} \\ &= \frac{S_d}{2}\Gamma\left(\frac{d}{2}\right) . \end{aligned} \tag{4.54}$$

We use this expression to analytically continue the definition of S_d to $d \in \mathbb{C}$

$$S_d = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} . \tag{4.55}$$

2. We recall some useful properties of the Γ function (also analytically continued). For $\alpha > 0$ we have

$$\begin{aligned} \Gamma(\alpha) &= \int_0^\infty dx \, x^{\alpha-1} e^{-x} \\ &= \frac{1}{\alpha}\int_0^\infty dx \left(\frac{d}{dx}\right) e^x \\ &= \frac{1}{\alpha}\left[x^\alpha e^{-x}\right]_0^\infty + \frac{1}{\alpha}\int_0^\infty dx \, x^\alpha e^{-x} \\ &= 0 + \frac{1}{\alpha}\Gamma(\alpha+1) . \end{aligned} \tag{4.56}$$

We can use this recursion relation to analytically continue to $\mathrm{Re}\,\alpha < 0$, with poles when $\mathrm{Re}\,\alpha \in \mathbb{Z}^- \cup \{0\}$. Other useful facts which follow from the above are that $\Gamma(1) = 1$, hence $\Gamma(n) = (n-1)!$ for $n \in \mathbb{Z}^+$, and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

3. The Taylor expansion about small α of the logarithm of the Γ function can be shown to be

$$\log \Gamma(1+\alpha) = -\gamma\alpha - \sum_{k=2}^{\infty} (-1)^k \frac{1}{k} \zeta(k) \alpha^k \tag{4.57}$$

where $\gamma \approx 0.577216$ is the Euler-Mascheroni constant and $\zeta(k) = \sum_{n=1}^{\infty} n^{-k}$ is the Riemann ζ function. We will usually use just the first term of this along with (4.56) to write $\alpha\Gamma(\alpha) \approx e^{-\gamma\alpha} \approx 1 - \gamma\alpha$ or

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + \ldots . \tag{4.58}$$

4. We will encounter integrals which are called Euler Beta functions[16]

[16] with a capital β.

$$B(s,t) = \int_0^1 du\, u^{s-1} (1-u)^{t-1} \tag{4.59}$$

which can be shown to be related to the Γ function via[17]

[17] Work with the integral representation of the product of two Γ functions and make an inspired change of variables [see e.g. Wikipedia].

$$B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} . \tag{4.60}$$

Now we are ready to evaluate our 1-loop integrals using dimensional regularization.

$$\begin{aligned} \Pi_1 &= -\frac{1}{2} g(\mu) \mu^{\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2} \\ &= -\frac{1}{2} g(\mu) \mu^{\epsilon} \frac{S_d}{2(2\pi)^d} \int_0^{\infty} \frac{k^{d-1} dk}{k^2 + m^2} . \end{aligned} \tag{4.61}$$

Focusing on the integral

$$\begin{aligned} \mu^{\epsilon} \int_0^{\infty} \frac{k^{d-1} dk}{k^2 + m^2} &= \frac{\mu^{\epsilon}}{2} \int_0^{\infty} \frac{(k^2)^{\frac{d}{2}-1} dk^2}{k^2 + m^2} \\ &= \frac{m^2}{2} \left(\frac{\mu}{m}\right)^{\epsilon} \int_0^1 du\, u^{-\frac{d}{2}} (1-u)^{\frac{d}{2}-1} \\ &= \frac{m^2}{2} \left(\frac{\mu}{m}\right)^{\epsilon} \frac{\Gamma(1-\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(1)} \end{aligned} \tag{4.62}$$

where we substituted $u = m^2/(k^2+m^2)$ and recognized the integral as $B(1-\frac{d}{2}, \frac{d}{2})$.

The final step involves looking at the small ϵ limit. We have

$$\begin{aligned} \Gamma(1-\tfrac{d}{2}) = \Gamma(\tfrac{\epsilon}{2} - 1) &= -\frac{1}{1-\frac{\epsilon}{2}} \Gamma(\tfrac{\epsilon}{2}) \\ &= -\frac{1}{1-\frac{\epsilon}{2}} \left(\frac{2}{\epsilon} - \gamma + O(\epsilon) \right) \\ &= -\frac{2}{\epsilon} + \gamma - 1 + O(\epsilon) \end{aligned} \tag{4.63}$$

as well as

$$\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon/2} = 1 + \frac{\epsilon}{2} \log \frac{4\pi\mu^2}{m^2} + O(\epsilon^2) . \tag{4.64}$$

Putting it all together, we find

$$\Pi_1 = -\frac{gm^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + 1 + \log\frac{4\pi\mu^2}{m^2}\right) + O(\epsilon)\,. \tag{4.65}$$

Counterterm. We have a choice regarding what to do with the finite terms. Minimal subtraction (MS): just subtract the divergence.

$$\delta m^2 = -\frac{gm^2}{16\pi^2\epsilon} \tag{4.66}$$

Modified minimal subtraction ($\overline{\text{MS}}$): also subtract the associated constants

$$\delta m^2 = -\frac{gm^2}{32\pi^2\epsilon}\left(\frac{2}{\epsilon} - \gamma + \log 4\pi\right)\,. \tag{4.67}$$

On-shell schemes are also possible in dimensional regularization.

We also wish to look at the 4-point vertex function

$$\begin{aligned}\tilde{\Gamma}^{(4)}(0,0,0,0) &= \frac{3g^2\mu^{2\epsilon}}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+m^2)^2} \\ &= \frac{3g^2\mu^{\epsilon}}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log\frac{4\pi\mu^2}{m^2}\right) + O(\epsilon)\end{aligned} \tag{4.68}$$

For the $\overline{\text{MS}}$ counterterm, let

$$\delta g = \frac{3g^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log 4\pi\right)\,. \tag{4.69}$$

Let us now explore effects due to the introduction of the renormalization scale μ. There are two different ways we can examine this. The first follows from the discussion of the previous section. We can look at how we have split the original Lagrangian $\mathcal{L}_0$ (4.40) into

$$\mathcal{L}_{\text{ren}} + \mathcal{L}_{\text{ct}} = \frac{1+\delta Z_\phi}{2}(\partial\phi)^2 + \frac{m^2+\delta m^2}{2}\phi^2 + \frac{(g+\delta g)\mu^\epsilon}{4!}\phi^4\,. \tag{4.70}$$

We can equate coefficients as before: $m_0^2 = m^2 + \delta m^2$ and $\lambda_0 = (g+\delta g)\mu^\epsilon$. The original parameters do not depend on μ, so m^2 and g^2 must compensate for the μ-dependence of the counterterm coefficients.

Let's work in the MS scheme to save writing. The calculation for $\overline{\text{MS}}$ is essentially the same. Defining $\beta(g) \equiv \frac{dg}{d\log\mu} = \mu\frac{dg}{d\mu}$,

$$\begin{aligned}0 &= \mu\frac{d}{d\mu}[(g+\delta g)\mu^\epsilon] \\ &= \epsilon g\left(1 + \frac{3g}{16\pi^2\epsilon}\right) + \beta(g)\left(1 + \frac{3g}{8\pi^2\epsilon}\right)\end{aligned} \tag{4.71}$$

We solve for $\beta(g)$ under the assumption that we can treat the terms proportional to g/ϵ as small, yielding

$$\beta(g) = \frac{3g^2}{16\pi^2} - g\epsilon + O\left(\frac{g^2}{\epsilon}, 2\text{ loops}\right)\,. \tag{4.72}$$

The divergences we have ignored here will be cancelled if we carry out renormalization to 2-loop order.

Note that $\beta(g) > 0$ here. Integrating the differential equation between μ and μ',

$$\frac{1}{g(\mu')} = \frac{1}{g(\mu)} - \frac{3}{16\pi^2}\log\frac{\mu'}{\mu} \tag{4.73}$$

or

$$g(\mu') = \frac{g(\mu)}{1 - \frac{3g(\mu)}{16\pi^2}\log\frac{\mu'}{\mu}} = g(\mu) + \frac{3g^2(\mu)}{16\pi^2}\log\frac{\mu'}{\mu}\,. \tag{4.74}$$

In the last step, we assumed $g(\mu)$ is small.

Renormalization group

Quantum field theory is not fully defined by its Lagrangian. It must be regulated somehow. A regularization scheme introduces an associated, unphysical scale. Renormalization conditions must be imposed in order to uniquely set parameters in the theory. Then we can make physical predictions.

Physical predictions should be independent of the specific, arbitrary choices made. That is, they should be scheme and scale independent. The renormalization group[18] studies how theories with different renormalization scales can give the same physical predictions. In the language of statistical field theories, the regularization takes place in the ultraviolet, or very short-distance scales, while we are interested in physics at lower energies compared to some artificial cutoff. We think of the interesting physics as the infrared, or long-distance, side of our regulated theory. Requiring scheme and scale independence of QFT predictions is comparable to studying universal macroscopic physics emerging from statistical systems with different microscopic details.

[18] Not a group.

In this chapter we will write down a general action for a real scalar field in d dimensions. Take Λ_0 as our initial hard momentum cutoff. Then we can write the action as a sum over an infinity of terms

$$S_{\Lambda_0}[\phi] = \int d^d x \left[\frac{1}{2}(\partial\phi)^2 + \sum_i \frac{g_{i0}}{\Lambda_0^{d_i - d}} O_i(x) \right] . \qquad (5.1)$$

The $O_i(x)$ represent local terms of ϕ or its derivatives raised to some power, $O_i = (\partial\phi)^{r_i}\phi^{s_i}$, and $d_i > 0$ is the mass dimension of O_i. The factors of Λ_0 are there so that the g_{i0} are dimensionless. Note that in this way of writing the action, the mass term is included in the sum over generic operators.

5.1 Effective actions

The partition function is

$$\mathcal{Z}_{\Lambda_0}(g_{i0}) = \int^{\Lambda_0} \mathcal{D}\phi \, e^{-S_{\Lambda_0}[\phi]} . \qquad (5.2)$$

The integral is over fields such that $|p| \leq \Lambda_0$. That is, the $\phi(x)$ in $\mathcal{Z}$ are

$$\begin{aligned} \phi(x) &= \int_{|p|\leq\Lambda_0} \frac{d^d p}{(2\pi)^d}\, e^{ip\cdot x}\, \tilde{\phi}(p) \\ &= \int_{|p|\leq\Lambda} \frac{d^d p}{(2\pi)^d}\, e^{ip\cdot x}\, \tilde{\phi}(p) + \int_{\Lambda<|p|\leq\Lambda_0} \frac{d^d p}{(2\pi)^d}\, e^{ip\cdot x}\, \tilde{\phi}(p) \\ &= \phi^-(x) \;+\; \phi^+(x)\,. \end{aligned} \tag{5.3}$$

After the first line, we have introduced an intermediate scale $\Lambda < \Lambda_0$, and separated ϕ into low momentum (IR) and high momentum (UV) modes. Note that these sets of modes are disjoint, so $\mathcal{D}\phi = \mathcal{D}\phi^-\mathcal{D}\phi^+$.

Integrating out the high energy modes results in an effective Wilsonian action

$$S^{\text{eff}}_{\Lambda}[\phi] = -\log \int_{\Lambda}^{\Lambda_0} \mathcal{D}\phi^+\, e^{-S_{\Lambda_0}[\phi^- + \phi^+]}\,. \tag{5.4}$$

This equation tells us how one action maps to another as UV modes are integrated out. We will soon iterate this procedure.

One thing to note, especially if carrying out explicit calculation, is that the kinetic and mass terms do not couple high and low momentum modes. That is, we may write

$$S_{\Lambda_0}[\phi^- + \phi^+] = S^0[\phi^-] + S^0[\phi^+] + S^{\text{int}}_{\Lambda_0}[\phi^-, \phi^+] \tag{5.5}$$

with free action

$$S^0[\phi] = \int d^d x\, \frac{1}{2}\left[(\partial\phi)^2 + m^2\phi^2\right]\,. \tag{5.6}$$

Since ϕ^- and ϕ^+ have disjoint support in momentum space, there is no quadratic term $\phi^-\phi^+$ in S_{Λ_0}. In momentum space, the corresponding term in the action would be $\tilde{\phi}^-(k)\,\tilde{\phi}^+(k')\,\delta(k+k')$, which would vanish upon integrating over the δ-function, due to this disjoint support. Contrariwise, cubic terms may not vanish: $\tilde{\phi}^-(k)\,\tilde{\phi}^-(k')\,\tilde{\phi}^+(k'')\,\delta(k+k'+k'')$ would allow $\Lambda < |k''| < \Lambda_0$ with both $|k|$, $|k'| < \Lambda$. The effective interaction at cutoff scale Λ can be written as

$$S^{\text{int}}_{\Lambda}[\phi] = -\log\left\{\int \mathcal{D}\phi^+\, e^{-S^0[\phi^+] - S^{\text{int}}_{\Lambda_0}[\phi^-, \phi^+]}\right\}\,. \tag{5.7}$$

In the next sections we explore what this integration over modes means for coupling constants and vertex functions.

5.2 *Running couplings*

Physics should be independent of whether we use a cutoff of Λ_0 or Λ. Therefore the corresponding partition functions should be equal

$$\begin{aligned} \int^{\Lambda} \mathcal{D}\phi\, e^{-S^{\text{eff}}_{\Lambda}[\phi]} &= \int^{\Lambda_0} \mathcal{D}\phi\, e^{-S_{\Lambda_0}[\phi]} \\ \mathcal{Z}_{\Lambda}(g_i(\Lambda)) &= \mathcal{Z}_{\Lambda_0}(g_{i0}; \Lambda_0)\,. \end{aligned} \tag{5.8}$$

Note the right-hand side is independent of Λ; therefore, the left-hand side must also be independent of Λ. The renormalized couplings $g_i(\mu)$ must "run" in order to cancel for any explicit Λ dependence

$$\Lambda \frac{d\mathcal{Z}_\Lambda}{d\Lambda} = \left(\Lambda \frac{\partial}{\partial \Lambda}\bigg|_{g_i} + \frac{dg_i}{d\Lambda}\frac{\partial}{\partial g_i}\bigg|_{\Lambda} \right) \mathcal{Z}_\Lambda(g) = 0\,. \tag{5.9}$$

The action S_{Λ_0} (5.1) is completely general, so S^{eff}_Λ should have the same form.

$$S^{\text{eff}}_\Lambda[\phi] = \int d^d x \left[\frac{Z_\Lambda}{2}(\partial\phi)^2 + \sum_i \frac{Z_\Lambda^{n_i/2}}{\Lambda_0^{d_i - d}} g_i(\Lambda) O_i(x) \right] \tag{5.10}$$

where we introduce a field renormalization factor $Z_\Lambda^{1/2}$ to account for the fact that the integration over UV modes will generically change the normalization, and n_i is the number of fields in O_i. Let

$$\phi^r = Z_\Lambda^{1/2}\phi \tag{5.11}$$

be the renormalized field. The remaining variations in terms is described by the Λ-dependence of the couplings, for each of which we have a β-function

$$\beta_i^{\text{qu}} = \Lambda \frac{dg_i}{d\Lambda}\,. \tag{5.12}$$

Note that $\beta_i^{\text{qu}} = \beta_i^{\text{qu}}(\{g_j\})$ can depend on the whole collection of couplings, and that the classical, leading-order result $\beta_i^{\text{cl}} = (d_i - d)g_i$, giving the full $\beta_i = \beta_i^{\text{cl}} + \beta_i^{\text{qu}}$.

5.3 *Vertex functions*

The anomalous dimension of the field ϕ is defined as

$$\gamma_\phi = -\frac{\Lambda}{2}\frac{d}{d\Lambda}\log Z_\Lambda\,. \tag{5.13}$$

Let us consider further integrating out more UV modes, from Λ down to $s\Lambda$, with $0 < s < 1$. Vertex functions (4.26) should be independent of the cutoff, so we should have

$$Z_{s\Lambda}^{-n/2}\Gamma_{s\Lambda}^{(n)}(x_1,\ldots,x_n;g_i(s\Lambda)) = Z_\Lambda^{-n/2}\Gamma_\Lambda^{(n)}(x_1,\ldots,x_n;g_i(\Lambda)) \tag{5.14}$$

We want to consider the limit where s is infinitesimally close to 1. Let $s\Lambda = \Lambda'$ with Λ fixed, then differentiate with respect to s. We have

$$s\frac{d}{ds}Z_{s\Lambda}^{-n/2} = -\frac{n}{2}Z_{s\Lambda}^{-n/2} s\frac{d}{ds}\log Z_{s\Lambda} = n\gamma_\phi \tag{5.15}$$

using $s\frac{d}{ds} = \Lambda'\frac{d}{d\Lambda'}$ and relabeling $\Lambda' \to \Lambda$. We find

$$\Lambda\frac{d}{d\Lambda}\Gamma_\Lambda^{(n)}(x_1,\ldots,x_n;g_i(\Lambda)) = \left(\Lambda\frac{\partial}{\partial\Lambda} + \beta_i\frac{\partial}{\partial g_i} + n\gamma_\phi\right)\Gamma_\Lambda^{(n)}(x_1,\ldots,x_n;g_i(\Lambda))\,. \tag{5.16}$$

We have done the first step of a renormalization group action: 1) Integrate out momentum modes in $(s\Lambda, \Lambda)$. There is one more step remaining: 2) rescale coordinates by defining $x^\mu \to s x^\mu$. The momentum scale is then rescaled $\Lambda \to \Lambda s$. This allows us to compare the computations using coupling constants before and after the RG action. Under this rescaling, we should ensure the kinetic term should be invariant, so that

$$\phi^r(sx) = s^{1-\frac{d}{2}}\phi^r(x) \;\text{ or }\; \phi^r(x) \to s^{\frac{d}{2}-1}\phi^r(sx) \tag{5.17}$$

The rest of the action is invariant.

Let us look at the n-point vertex function $\Gamma^{(n)}_{s\Lambda}(x_1,\ldots,x_n;g_i(s\Lambda))$:

$$\begin{aligned}\Gamma^{(n)}_{\Lambda}(x_1,\ldots,x_n;g_i(\Lambda)) &= \left(\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{n/2}\Gamma^{(n)}_{s\Lambda}(x_1,\ldots,x_n;g_i(s\Lambda)) \\ &= \left(s^{2-d}\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{n/2}\Gamma^{(n)}_{\Lambda}(sx_1,\ldots,sx_n;g_i(s\Lambda))\end{aligned} \tag{5.18}$$

Note that in this rescaling step, the numerical values for $Z_{s\Lambda}$ and $g_i(s\Lambda)$ are not changed. Now reconsider the initial coordinates. Instead of looking at x_i, look at x_i/s. Then

$$\Gamma^{(n)}_{\Lambda}\left(\frac{x_1}{s},\ldots,\frac{x_n}{s};g_i(\Lambda)\right) = \left(s^{2-d}\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{n/2}\Gamma^{(n)}_{\Lambda}(x_1,\ldots,x_n;g_i(s\Lambda)) \tag{5.19}$$

As $s \to 0$ we are integrating out more UV modes. From this equation, we see on the left-hand side, that we are looking at vertex functions where $|x_i - x_j|/s$ is getting large (probing the IR). This is equated to vertex functions with fixed positions, but with the coupling running to lower scales $s\Lambda$.

We can also examine the prefactor. For infinitesimal $\delta s = 1 - s$

$$\left(s^{2-d}\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{1/2} = 1 + \left(\frac{d-2}{2} + \gamma_\phi\right)\delta s\,. \tag{5.20}$$

That is, the fields behave as if their mass dimension were

$$\frac{d-2}{2} + \gamma_\phi \equiv \Delta_\phi\,. \tag{5.21}$$

The field's "scaling dimension" is equal to its classical, or "engineering" mass dimension plus its anomalous dimension.

5.4 *Renormalization group flow*

Coupling constant space.

Renormalization group trajectories are lines in coupling constant space traced out as the renormalization scale is varied, e.g. as momentum modes are integrated out. The lines are the solutions to the set of differential equations given by the beta functions $\beta_i(g_j)$. By construction theories lying along the same RG trajectories describe the same long-distance (or IR) physics.

Fixed points. $\beta_i|_{\{g_j^*\}} = 0$ for all i.

$$\beta_i(\{g_j\}) = (d_i - d)g_i + \Lambda \frac{dg_i}{d\Lambda}(\{g_j\}) \tag{5.22}$$

Scale invariance at a fixed point. $\gamma_\phi(\{g_i^*\}) = \gamma_\phi^*$. Callan-Symanzik implies

$$\Lambda \frac{\partial}{\partial \Lambda} \Gamma_\Lambda^{(2)}(x, y; \{g_i^*\}) = -2\gamma_\phi^* \Gamma^{(2)}(x, y; \{g_i^*\}) \,. \tag{5.23}$$

Translational and Lorentz invariance implies $\Gamma_\Lambda^{(2)}(x,y) = \Gamma_\Lambda^{(2)}(|x - y|)$. Like $\langle \phi(x)\phi(y)\rangle$, $\Gamma_\Lambda^{(2)}$ has mass (engineering) dimensions of $d-2$. Therefore, in order to satisfy (5.23)

$$\Gamma_\Lambda^{(2)}(x, y; \{g_i^*\}) = \frac{\Lambda^{d-2}}{\Lambda^{2\Delta_\phi}} \frac{c(\{g_i^*\})}{|x-y|^{2\Delta_\phi}} \tag{5.24}$$

where

$$\Delta_\phi = \frac{1}{2}(d-2) + \gamma_\phi^* \tag{5.25}$$

is the field's scaling dimension and c is a dimensionless function of the fixed-point couplings. This power-law behaviour for $\Gamma_\Lambda^{(2)}$ is characteristic of scale-invariant theories. In contrast, theories with a mass scale have a finite correlation length $\xi \propto M^{-1}$ and the two-point functions decay exponentially $\Gamma^{(2)} \sim \exp(-M|x-y|)/|x-y|^{2\Delta_\phi}$.

We can analyze the RG flow near a fixed point by linearizing the RG equations. Let $\delta g_j = g_j - g_j^*$.

$$\Lambda \frac{dg_i}{d\Lambda}\bigg|_{g_j^* + \delta g_j} = B_{ij}\delta g_j + O((\delta g)^2) \,. \tag{5.26}$$

Let the eigenvectors and eigenvalues of the matrix B_{ij} be denoted σ_i and $\Delta_i - d$, where we refer to Δ_i as the scaling dimension of σ_i. In general, the σ_i represent a linear combination of some operators O_j in the action.

Under the RG transformation, the linearized flow near $\{g_j^*\}$ is

$$\begin{aligned} \Lambda \frac{d\sigma_i}{d\Lambda} &= (\Delta_i - d)\,\sigma_i \\ \implies \sigma_i(\Lambda) &= \left(\frac{\Lambda}{\Lambda_0}\right)^{\Delta_i - d} \sigma_i(\Lambda_0) \end{aligned} \tag{5.27}$$

with some initial $\Lambda_0 > \Lambda$.

5.5 *Continuum limit and renormalizability*

Further reading

Schwartz[19] covers the RG in Chapter 23; §23.6 in particular has a nice summary of Wilsonian RG. Banks[20] has a nice discussion in Chapter 9, going well beyond what we have been able to cover here.

[19] M D Schwartz. *Quantum Field Theory and the Standard Model*. Cambridge University Press, 2014. ISBN 978-1-107-03473-0

[20] T Banks. *Modern Quantum Field Theory*. Cambridge University Press, 2008. ISBN 978-0-521-85082-7

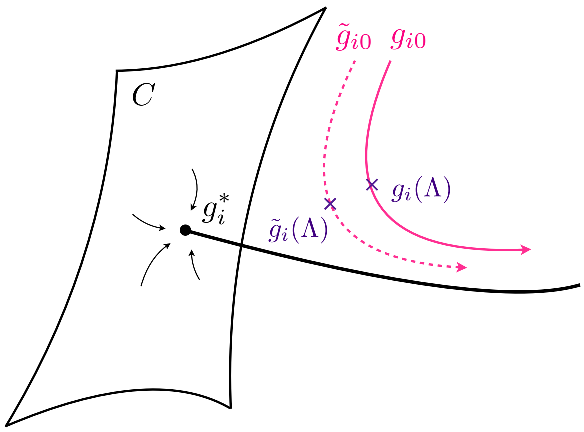


Figure 5.1: RG flow near a fixed point. As we tune the couplings $g_{i0} \to \tilde{g}_{i0}$ in the initial action closer to the critical surface C, the greater the ratio $\frac{\tilde{\Lambda}_0}{\Lambda} > \frac{\Lambda_0}{\Lambda}$ can be with the effective couplings $g_i(\Lambda)$ still in the vicinity of the fixed point.

Quantum electrodynamics

QED is the original field theory, describing the electromagnetic interactions of relativistic electrons as they exchange photons.

The Euclidean spacetime, classical action is

$$S[\psi,\bar{\psi},A] = \int d^4x \left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(\not{D}+m)\psi\right] \tag{6.1}$$

where $\not{D} = \gamma^\mu(\partial_\mu + ieA_\mu)$, ψ and $\bar{\psi}$ are 4-spin-component Grassmann fields, and the field strength tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It is straightforward to check that the action is invariant under U(1) gauge transformations

$$\begin{aligned} \psi(x) &\to e^{ie\alpha(x)}\psi(x) \\ \bar{\psi}(x) &\to \bar{\psi}(x)e^{-ie\alpha(x)} \\ A_\mu(x) &\to A_\mu(x) - \partial_\mu\alpha(x)\,. \end{aligned} \tag{6.2}$$

The partition function is

$$\mathcal{Z} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A\, e^{-S[\psi,\bar{\psi},A]}\,. \tag{6.3}$$

Note that ψ and $\bar{\psi}$ are independent Grassmann variables.

In Euclidean spacetime, the Dirac matrices satisfy the anticommutation relation

$$\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu} \tag{6.4}$$

and we take them to be Hermitian: $\gamma_\mu^\dagger = \gamma_\mu$. A typical representation is

$$\begin{aligned} \gamma_j &= \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix} \\ \gamma_4 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \gamma_5 &= \gamma_1\gamma_2\gamma_3\gamma_4 \end{aligned} \tag{6.5}$$

with σ_j being the Pauli matrices.

6.1 Feynman rules

It is useful to work in momentum space

$$\begin{aligned} \psi(x) &= \int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot x}\tilde{\psi}(p) \\ \bar{\psi}(x) &= \int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot x}\tilde{\bar{\psi}}(p)\,. \end{aligned} \tag{6.6}$$

Then the free electron action in momentum space is

$$S_f[\tilde{\psi},\tilde{\bar{\psi}}] = \int \frac{d^4p}{(2\pi)^4}\, \tilde{\bar{\psi}}(-p)(i\not{p}+m)\tilde{\psi}(p)\,. \tag{6.7}$$

The generating functional with Grassmann sources η and $\bar{\eta}$ is

$$\begin{aligned}\mathcal{Z}_0[\tilde{\eta},\tilde{\bar{\eta}}] &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, \exp\left\{-\int \frac{d^4p}{(2\pi)^4}\left[\tilde{\bar{\psi}}(i\not{p}+m)\tilde{\psi} - \tilde{\bar{\eta}}\tilde{\psi} + \tilde{\bar{\psi}}\eta\right]\right\} \\ &= \mathcal{Z}_0[0,0]\, \exp\left\{-\int \frac{d^4p}{(2\pi)^4}\, \tilde{\bar{\eta}}(i\not{p}+m)^{-1}\tilde{\eta}\right\}\end{aligned} \tag{6.8}$$

following similar steps as in the scalar theory.

The free electron propagator is obtained from

$$\begin{aligned}\tilde{S}_F(p) &= \left.\frac{\delta}{\delta\tilde{\bar{\eta}}}\frac{\delta}{\delta\tilde{\eta}}\mathcal{Z}[\tilde{\eta},\tilde{\bar{\eta}}]\right|_{\tilde{\eta},\tilde{\bar{\eta}}=0} \\ &= \frac{1}{i\not{p}+m}\,.\end{aligned} \tag{6.9}$$

Making the spin indices explicit, we would write

$$\tilde{S}_F^{\alpha\beta}(p) = \frac{-i\not{p}^{\,\alpha\beta} + m\,\delta^{\alpha\beta}}{p^2+m^2}\,. \tag{6.10}$$

This propagator is drawn as a solid line with an arrow.

For the photon generating functional, we couple an external source following Maxwell's equations

$$\partial_\nu F^{\mu\nu} = J^\mu \tag{6.11}$$

so that we have

$$\mathcal{Z}_0[J] = \int \mathcal{D}A\, \exp\left\{-\int d^4x\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J^\mu A_\mu\right]\right\}\,. \tag{6.12}$$

To check gauge invariance of the second term in the action (6.12), let $A'_\mu(x) = A_\mu(x) - \partial_\mu\alpha(x)$ and perform the following integral by parts

$$\begin{aligned}\int d^4x\, J^\mu(A'_\mu - A_\mu) &= \int d^4x\, J^\mu\,\partial_\mu\alpha \\ &= \int d^4x\,\partial_\mu(J^\mu\alpha) - \int d^4x\,(\partial_\mu J^\mu)\alpha = 0\end{aligned} \tag{6.13}$$

where the first integral vanishes for suitable boundary conditions at infinity. The second term vanishes because J^μ is a conserved current: $\partial_\mu\partial_\nu F^{\mu\nu} = 0$ due to F being antisymmetric in its Lorentz indices.

Fourier transforming the gauge field and source, the action is

$$\begin{aligned}S_g[\tilde{A},\tilde{J}] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\Big[&\tilde{A}_\mu(-k)\left(k^2\delta^{\mu\nu} - k^\mu k^\nu\right)\tilde{A}_\nu(k) \\ &+ \tilde{J}^\mu(-k)\tilde{A}_\mu(k) + \tilde{J}^\mu(k)\tilde{A}_\mu(-k)\Big]\,.\end{aligned} \tag{6.14}$$

We can see propagator $\tilde{D}^{\mu\nu}$ must be the inverse of the quadratic term, i.e. it must solve

$$(k^2\delta^{\mu\nu} - k^\mu k^\nu)\tilde{D}_{\nu\rho}(k) = \delta^\mu{}_\rho\,. \tag{6.15}$$

One solution is

$$\tilde{D}^{\mu\nu}(k) = \frac{1}{k^2}\left(\delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\,. \tag{6.16}$$

There is a subtle point, though, in that there are zero modes which have vanishing action and hence give a divergent contribution to path integrals. Consider gauge fields which are gauge transforms of the field $A_\mu(x) = 0$; that is, $A_\mu(x) = \partial_\mu\alpha(x)$. In momentum space these are fields which can be written as $\tilde{A}_\mu(k) = k_\mu\tilde{\alpha}(k)$. Let us define

$$P^{\mu\nu}(k) = \delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \tag{6.17}$$

and note that it is a 4×4 projection matrix: $P^\mu{}_\nu(k)P^{\nu\rho}(k) = P^{\mu\rho}(k)$. As such, its eigenvalues are either 0 or 1.[21] The action vanishes for gauge fields parallel to the zero eigenvector k^μ:

[21] $P^2 v = Pv = \lambda v$ implies $\lambda^2 = \lambda$.

$$P^{\mu\nu}(k)k_\nu = 0\,. \tag{6.18}$$

Note also that fields satisfying $\tilde{A}_\mu(k) = k_\mu\tilde{\alpha}(k)$ also give no contribution to the source terms in the action (6.14) since $k_\mu\tilde{J}^\mu = 0$ from current conservation ($\partial_\mu J^\mu = 0$).

We can see that the other three eigenvalues of $P^{\mu\nu}$ must be equal to 1, since the sum of all the eigenvalues is equal to the trace

$$\delta_{\mu\nu}P^{\mu\nu}(k) = 3\,. \tag{6.19}$$

Therefore, we make sense of the path integral over A_μ by requiring that we do not integrate over the fields gauge-equivalent to $A_\mu(x) = 0$. This means we only integrate over fields which are orthogonal to k^μ in the sense that

$$k^\mu A_\mu(k) = 0 \quad \text{or} \quad \partial^\mu A_\mu(x) = 0\,. \tag{6.20}$$

This is called Lorenz or Landau gauge.

In the subspace specified by (6.20) $P^{\mu\nu}(k)$ is just the identity, so the inverse of $k^2P^{\mu\nu}$ is easy to write down and just gives the Landau-gauge propagator (6.16). The generating functional is

$$\mathcal{Z}_0[\tilde{J}] = \exp\left\{\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\,\tilde{J}_\mu(-k)\,\frac{P^{\mu\nu}}{k^2}\,\tilde{J}_\nu(k)\right\}\,. \tag{6.21}$$

Noting once again that $k^\mu J_\mu = 0$, we see we can drop the 2nd term in (6.17) in (6.21) and write

$$\mathcal{Z}_0[\tilde{J}] = \exp\left\{\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{\tilde{J}_\mu(-k)\tilde{J}^\mu(k)}{k^2}\right\} \tag{6.22}$$

which is equivalent to working in Feynman gauge. In fact one can write the photon propagator in a more general gauge as

$$\tilde{D}^{\mu\nu}(k) = \frac{1}{k^2}\left(\delta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right) \tag{6.23}$$

where $\xi = 0$ corresponds to Landau gauge and $\xi = 1$ to Feynman gauge. This R_ξ gauge, or covariant gauge, is more useful in the context of nonabelian gauge theories.

The interaction term in the Lagrangian density is

$$ieA_\mu(x)\bar{\psi}^\alpha(x)(\gamma^\mu)^{\alpha\beta}\psi^\beta(x) \tag{6.24}$$

where we have written the spin indices explicitly. Following the procedure developed in § 2.2, we can write the interacting generating functional for QED as

$$\begin{aligned} Z[\eta,\bar{\eta},J] \propto \exp&\left[ie\int d^4x \left(\frac{\delta}{\delta J(x)}\right)\left(\frac{\delta}{\delta\eta^\alpha(x)}\right)(\gamma^\mu)^{\alpha\beta}\left(\frac{\delta}{\delta\bar{\eta}^\beta(x)}\right)\right] \\ &\times Z_0[\eta,\bar{\eta}]Z_0[J]\,. \end{aligned} \tag{6.25}$$

It is a tedious exercise to show that, for every fermion loop, we pick up a factor of (-1) due to the necessity of anticommuting the Grassmann sources appropriately.

6.2 *Vacuum polarization*

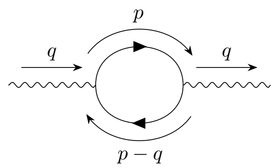


Figure 6.1: Vacuum polarization one-loop diagram.

Let us begin the discussion of QED renormalization by considering quantum corrections to the photon propagator; this is known as the polarization of the vacuum due to quantum fluctuations. As with the scalar two-point function, the photon two-point function can be written as a geometric series.

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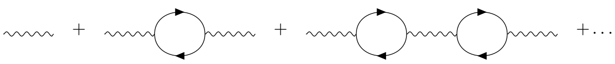


(6.26)

The key quantity to calculate at one-loop order is the amputated diagram with a fermion loop (Fig. 6.1).

We will use dimensional regularization, so define a dimensionless coupling g through

$$e^2 = g^2(\mu)\,\mu^\epsilon \tag{6.27}$$

where $\epsilon = 4 - d$ in d dimensions. The 1-loop vacuum polarization is

$$\begin{aligned} \Pi_1^{\mu\nu}(q) &= -(-ig)^2\mu^\epsilon\int\frac{d^dp}{(2\pi)^d}\,\mathrm{tr}\left(\frac{1}{i\not{p}+m}\right)\gamma^\mu\left(\frac{1}{i(\not{p}-\not{q})+m}\right)\gamma^\nu \\ &= g^2\mu^\epsilon\int\frac{d^dp}{(2\pi)^d}\frac{\mathrm{tr}\left\{(-i\not{p}+m)\gamma^\mu[-i(\not{p}-\not{q})+m]\gamma^\nu\right\}}{(p^2+m^2)[(p-q)^2+m^2]}\,. \end{aligned} \tag{6.28}$$

Use Feynman's trick

$$\frac{1}{AB} = \int_0^1 dx\int_0^1 dy\,\frac{\delta(x+y-1)}{[Ay+Bx]^2} \tag{6.29}$$

to write the integral of the denominator as

$$\int_0^1 \frac{dx}{\{(p^2+m^2)(1-x)+[(p-q)^2+m^2]x\}^2} = \int_0^1 \frac{dx}{[(p-qx)^2+m^2+q^2x(1-x)]^2} \tag{6.30}$$

Shift the momentum integration variable to $\ell = p - qx$ to obtain

$$\Pi_1^{\mu\nu}(q) = g^2\mu^\epsilon \int \frac{d^d\ell}{(2\pi)^d}\int_0^1 dx \frac{\mathrm{tr}\,\{[-i(\not\ell+\not q x)+m]\gamma^\mu[-i[\not\ell-\not q(1-x)]+m]\gamma^\nu\}}{(\ell^2+\Delta)^2} \tag{6.31}$$

where we introduce the abbreviation $\Delta = m^2 + q^2x(1-x)$. We need the following spin traces

$$\mathrm{tr}\gamma^\mu\gamma^\nu = 4\delta^{\mu\nu} \tag{6.32}$$

$$\mathrm{tr}\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma = 4(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\nu}\delta^{\rho\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho})\,. \tag{6.33}$$

Therefore the trace in the numerator of (6.31) becomes

$$\mathrm{tr}\{\cdot\} = 4\left\{-(\ell+qx)^\mu[\ell-q(1-x)]^\nu + (\ell+qx)\cdot[\ell-q(1-x)]\delta^{\mu\nu} - (\ell+qx)^\nu[\ell-q(1-x)]^\mu + m^2\delta^{\mu\nu}\right\}\,. \tag{6.34}$$

As $d \to 4$, integrals over odd powers of p vanish, so we neglect these terms. For the same reason only the diagonal parts of $\ell^\mu\ell^\nu$ have nonzero interals. The nonvanishing terms can be obtained by the following replacements

$$\ell^\mu\ell^\nu \to \frac{1}{d}\delta^{\mu\nu}\ell^2$$

$$\ell^\mu\ell^\rho\ell^\nu\ell^\sigma \to \frac{(\ell^2)^2}{d(d+2)}\left(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\nu}\delta^{\rho\sigma} + \delta^{\mu\sigma}\delta^{\rho\nu}\right)\,. \tag{6.35}$$

Now the integrand is rotationally invariant, so the angular part of the integral can be done easily. This amounts to replacing the measure

$$\int \frac{d^d\ell}{(2\pi)^d} \to S_d \frac{\ell^{d-1}d\ell}{(2\pi)^d} = \frac{(\ell^2)^{\frac{d}{2}-1}d\ell^2}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\,. \tag{6.36}$$

Putting these ingredients together gives

$$\Pi_1^{\mu\nu}(q) = \frac{4g^2\mu^\epsilon}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\int_0^1 dx \int_0^\infty d\ell^2 \frac{(\ell^2)^{\frac{d}{2}-1}}{(\ell^2+\Delta)^2}\left[\ell^2\left(1-\frac{2}{d}\right)\delta^{\mu\nu} + \left(2q^\mu q^\nu - q^2\delta^{\mu\nu}\right)x(1-x) + m^2\delta^{\mu\nu}\right]\,. \tag{6.37}$$

With the substitution $u = \frac{\Delta}{\ell^2+\Delta}$, we can identify that these integrals are proportional to Euler's Beta functions, (4.59) $\Delta^{\frac{d}{2}-1}B(1-\frac{d}{2}, 1+\frac{d}{2})$ and $\Delta^{\frac{d}{2}-2}B(2-\frac{d}{2}, \frac{d}{2})$, and use the result (4.60). We find

$$\begin{aligned}\Pi_1^{\mu\nu}(q) &= \frac{4g^2\mu^\epsilon}{(4\pi)^{\frac{d}{2}}}\Gamma(\tfrac{\epsilon}{2})\int_0^1 dx \frac{1}{\Delta^{\epsilon/2}}\left\{\delta^{\mu\nu}\left[m^2 - x(1-x)q^2\right] - \delta^{\mu\nu}\left[m^2 + x(1-x)q^2\right] + 2x(1-x)q^\mu q^\nu\right\} \\ &= \frac{8g^2\mu^\epsilon}{(4\pi)^{\frac{d}{2}}}\Gamma(\tfrac{\epsilon}{2})\int_0^1 dx \frac{1}{\Delta^{\epsilon/2}}(-q^2\delta^{\mu\nu} + q^\mu q^\nu)x(1-x) \\ &\equiv (q^2\delta^{\mu\nu} - q^\mu q^\nu)\pi_1(q^2)\end{aligned} \tag{6.38}$$

where we implicitly define the Lorentz-invariant function

$$\pi_1(q^2) = -\frac{8g^2\Gamma(\frac{\epsilon}{2})}{(4\pi)^{\frac{d}{2}}}\int_0^1 dx\, x(1-x)\left(\frac{\mu^2}{\Delta}\right)^{\frac{\epsilon}{2}}\,. \tag{6.39}$$

Note that $\Pi_1^{\mu\nu}(q)$ has the same Lorentz structure as the free photon propagator, i.e. $q_\mu \Pi_1^{\mu\nu}(q) = 0$. In the $d \to 4$ ($\epsilon \to 0$) limit

$$\pi_1(q^2) = -\frac{g^2}{2\pi^2} \int_0^1 x(1-x) \left(\frac{2}{\epsilon} - \gamma + \log \frac{4\pi\mu^2}{\Delta} \right) + O(\epsilon) \,. \tag{6.40}$$

In order to deal with the divergence we write the orginal action as the sum of a renormalized action plus a counterterm action.

$$S_0 = S + S_{\text{ct}} \tag{6.41}$$

with

$$\begin{aligned} e_0 &= Z_e \\ m_0 &= Z_m m \\ \psi_0 &= \sqrt{Z_2}\psi \\ A_0 &= \sqrt{Z_3} A. \end{aligned} \tag{6.42}$$

We write the right-hand side of (6.41) as

$$S + S_{\text{ct}} = \int d^4x \left[\frac{1}{4} Z_3 F^2 + Z_2 \bar{\psi} \partial\!\!\!/ \psi + Z_m Z_2 m \bar{\psi}\psi + ie Z_1 \bar{\psi} A\!\!\!/ \psi \right] \tag{6.43}$$

where $Z_1 = Z_e Z_2 \sqrt{Z_3}$. Let $Z_k = 1 + \delta Z_k$ for $k = e, m, 1, 2, 3$ and δZ_k small. Then $\delta Z_e = \delta Z_1 - \delta Z_2 - \frac{1}{2}\delta Z_3$. We will later show that gauge invariance implies $Z_1 = Z_2$ so that $\bar{\psi}(\partial\!\!\!/ + ieA\!\!\!/)\psi$ is renormalized by a single, overall factor. Given that this is the case $\delta Z_e = -\frac{1}{2}\delta Z_3$.

Counterterm diagram

$$\delta Z_3 = -\frac{g^2(\mu)}{12\pi^2} \left(\frac{2}{\epsilon} - \gamma + \log 4\pi \right) \tag{6.44}$$

in the $\overline{\text{MS}}$ scheme.

$$\pi_1^{\text{ren}}(q^2) = \frac{g^2(\mu)}{2\pi^2} \int dx\, x(1-x) \log \left(\frac{m^2 + x(1-x)q^2}{\mu^2} \right) \tag{6.45}$$

Note about branch cut.

6.3 *QED β function*

$$g_0 = Z_3^{-\frac{1}{2}} g \mu^{\frac{\epsilon}{2}} = \left(1 - \frac{1}{2}\delta Z_3 \right) g\mu^{\frac{\epsilon}{2}} \tag{6.46}$$

$$0 = \mu \frac{dg_0}{d\mu} = \left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) \left[1 + \frac{g^2}{24\pi^2} \left(\frac{2}{\epsilon} - \gamma + \log 4\pi \right) \right] g\mu^{\frac{\epsilon}{2}} \tag{6.47}$$

which implies

$$\frac{\epsilon g}{2} \left(1 + \frac{g^2}{12\pi^2\epsilon} \right) + \beta(g) \left(1 + \frac{g^2}{4\pi^2\epsilon} \right) = 0 \tag{6.48}$$

or

$$\beta(g) = -\left(\frac{\epsilon g}{2} + \frac{g^3}{24\pi^2}\right)\left(1 + \frac{g^2}{4\pi^2\epsilon}\right)^{-1}$$
$$= -\frac{\epsilon g}{2} + \frac{g^3}{12\pi^2} \; + \text{ 2-loop} \qquad (6.49)$$

The QED β function is positive, so the coupling is marginally irrelevant. Integrate from μ to μ'

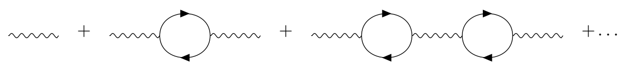
$$\frac{1}{g^2(\mu')} = \frac{1}{g^2(\mu)} + \frac{1}{6\pi^2}\log\frac{\mu}{\mu'}\,. \qquad (6.50)$$

Let Λ_{QED} be the scale at which the coupling diverges:

$$g^2(\mu) = \frac{6\pi^2}{\log\frac{\Lambda_{\text{QED}}}{\mu}}\,. \qquad (6.51)$$

Given experimental measurements $m_e = 0.511$ MeV and $\alpha = \frac{1}{4\pi}g^2(m_e) = \frac{1}{137}$, implies that $\Lambda_{\text{QED}} \approx 10^{286}$ GeV. COMMENTS.

$$G^{(2)}_{\mu\nu}(q) = \int d^d x\, e^{iq\cdot x}\langle A_\mu(x)A_\nu(0)\rangle$$

$=$ 

$$= D_{\mu\nu} + D_{\mu\nu}\Pi^{\mu\nu}D_{\mu\nu} + D_{\mu\nu}\Pi^{\mu\nu}D_{\mu\nu}\Pi^{\mu\nu}D_{\mu\nu} + \ldots$$
$$= D_{\mu\nu}(q)(1 + \pi(q^2) + \pi^2(q^2) + \ldots)$$
$$= \frac{D_{\mu\nu}}{1-\pi(q^2)}\,. \qquad (6.52)$$

At one loop, $\Pi^{\mu\nu} = \Pi_1^{\mu\nu}$ and $\pi = \pi_1$ we found earlier in the chapter.

$G^{(2)}_{\mu\nu}(q)$ can be found from differentiating the quantum effective action. Using the one-loop results above, this differentiation is on the following term

$$\Gamma[\psi,\bar\psi,A] \supset \int \frac{d^d p}{(2\pi)^d}[1-\pi(p^2)](p^2\delta^{\mu\nu} - p^\mu p^\nu)\frac{1}{2}\tilde{A}_\mu(-p)\tilde{A}_\nu(p)\,. \qquad (6.53)$$

Rescale $A_\mu \to \frac{1}{e}A_\mu$ and consider position space to find

$$\Gamma[\psi,\bar\psi,A] \supset \int d^d x\left[\frac{1-\pi(0)}{4e^2}F^2 + \partial^2F^2 \text{ term}\right] \qquad (6.54)$$

The coefficients in $\Gamma[\psi,\bar\psi,A]$ should be μ-independent. The physical coupling should just be read off from the coefficient of $\frac{1}{4}F^2$, say

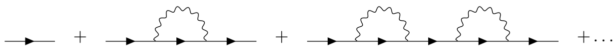
$$\frac{1}{e^2_{\text{phys}}} = \frac{1-\pi(0)}{e^2} = \frac{1}{g^2\mu^\epsilon}\left[1 - \frac{g^2}{2\pi}\int_0^1 dx\, x(1-x)\log\frac{\Delta}{\mu^2}\right]. \qquad (6.55)$$

Taking the logarithmic derivative $\frac{d}{d\log\mu}$ of both side will give the same β-function we obtained earlier.

6.4 *Full one-loop renormalization*

$$G(\not{p}) = \int d^4x\, e^{ip\cdot x} \langle \psi(x)\bar{\psi}(0)\rangle$$

$$=$$



$$= \frac{1}{i\not{p} + m - \Sigma(\not{p})} \tag{6.56}$$

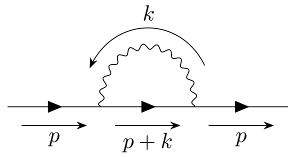


Figure 6.2: One-loop fermion self-energy diagram Σ_1.

$$\Sigma_1(\not{p}) = -\frac{g^2}{16\pi^2}\int_0^1 dx\,[(2-\epsilon)x(i\not{p}) + (4-\epsilon)m] \times \left[\frac{2}{\epsilon} - \gamma + \log\frac{4\pi\mu^2}{\Delta} + O(\epsilon)\right] \tag{6.57}$$

where here $\Delta = x(1-x)p^2 + (1-x)m^2$.[22] Note that $\Sigma_1(\not{p})$ is not proportional to $i\not{p} + m$. The two independent counterterms δZ_2 and δZ_m are introduced to remove the two divergences. Counterterm Feynman rule: 2-point fermion vertex with coefficient $-[\delta Z_2 i\not{p} + (\delta Z_2 + \delta Z_m)m]$. Renormalization conditions, either $\overline{\text{MS}}$ (or MS), or on-shell. In the latter case we require

[22] There is also an infrared divergence at $k = 0$. This can be regulated by introducing a small mass for the photon, then taking the limit of it vanishing.

$$\Sigma(\not{p})\big|_{\not{p}=im_{\text{phys}}} = 0 \quad \text{and} \quad \left.\frac{d\Sigma}{d\not{p}}\right|_{\not{p}=im_{\text{phys}}} = 0\,. \tag{6.58}$$

The statement $\not{p} = im_{\text{phys}}$ is a shorthand, which can be understood formally. What we really mean is that the renormalization constants should be set so that $G(\not{p})$(6.56) should have a pole on the imaginary axis corresponding to the physical electron mass, and its residue should be 1. To be careful we would rationalize the denominator in (6.56) so that it carried no spin indices.

Further reading

Srednicki[23], e.g. §§54, 57, 58. Schwartz[24] Chapter 14.

[23] M Srednicki. *Quantum Field Theory*. Cambridge University Press, 2007. ISBN 978-0-521-86449-7

[24] M D Schwartz. *Quantum Field Theory and the Standard Model*. Cambridge University Press, 2014. ISBN 978-1-107-03473-0

Symmetries and path integrals

Path integral integral quantization makes use of the classical action. We recall from Noether's theorem that symmetries of the theory can be explored by studying variations of the classical action. In this chapter we investigate the quantum effects of theories which possess symmetries at the classical level.

7.1 Schwinger-Dyson equations for scalars

Consider a free, massless scalar theory described by the action

$$S = \frac{1}{2}\int d^4y\, \partial_\mu\phi\partial^\mu\phi = -\frac{1}{2}\int d^4y\, \phi\partial^2\phi\,; \tag{7.1}$$

we will find it more useful to use the second form of the action, obtained by integrating by parts.

Now consider the vacuum expectation value of the field $\langle\phi(x)\rangle$. Inside the path integral, let us shift the field $\phi \to \phi + \varepsilon$, which leaves the measure invariant. We will assume that $\varepsilon(x)$ is a small variation, so that we can neglect terms $O(\varepsilon^2)$.

$$\langle\phi(x)\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}\phi\, [\phi(x) + \varepsilon(x)]\, e^{\frac{1}{2}\int d^4y\, (\phi+\varepsilon)\partial^2(\phi+\varepsilon)} \tag{7.2}$$

Expand about small ε so that the exponential above can be written as

$$e^{\frac{1}{2}\int d^4y\, \phi\partial^2\phi}\left[1 + \frac{1}{2}\int d^4z\left(\phi\partial^2\varepsilon + \varepsilon\partial^2\phi\right)\right] = e^{\frac{1}{2}\int d^4y\, \phi\partial^2\phi}\left(1 + \int d^4z\, \varepsilon\partial^2\phi\right) \tag{7.3}$$

where we have integrated by parts twice. Now

$$\langle\phi(x)\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}\phi\, e^{\frac{1}{2}\int d^4y\, \phi\partial^2\phi}\left[\phi(x) + \varepsilon(x) + \phi(x)\int d^4z\, \varepsilon(z)\partial_z^2\phi(z)\right]. \tag{7.4}$$

Note we have put a subscript z on the latter partial derivative to make it clear we differentiate with respect to z and not x. (The derivative with respect to y in the exponential will not appear after another step.) Note that the first term on the right hand side is simply $\langle\phi(x)\rangle$, so the remaining terms must sum to 0. Writing $\varepsilon(x) = \int d^4z\, \varepsilon(z)\delta^{(4)}(z-x)$, we can factor the $\epsilon(z)$ out

$$\int d^4z\, \varepsilon(z)\int \mathcal{D}\phi\, e^{\frac{1}{2}\int d^4y\, \phi\partial^2\phi}\left[\phi(x)\partial_z^2\phi(z) + \delta^{(4)}(z-x)\right] = 0\,. \tag{7.5}$$

Since this is true for all $\varepsilon(z)$, and since the path integral is independent of z, except at the field insertion, it must be that

$$\partial_z^2 \langle \phi(z)\phi(x) \rangle = -\delta^{(4)}(z-x)\,. \tag{7.6}$$

This is the Schwinger-Dyson equation for a free, massless scalar field. In fact this is nothing but the Klein Gordon equation for the Green's function which yields the Feynman propagator. The steps above can be repeated for general n-point functions, for example

$$\partial_z^2 \langle \phi(z)\phi(x)\phi(y) \rangle = -\delta^{(4)}(z-x)\langle \phi(y) \rangle - \delta^{(4)}(z-y)\langle \phi(x) \rangle\,. \tag{7.7}$$

When we add an interaction term

$$S = \int d^4y \left(-\frac{1}{2}\phi\partial^2\phi + \mathcal{L}_{\text{int}}[\phi] \right) \tag{7.8}$$

we can simply modify the steps above, Taylor expanding

$$\mathcal{L}_{\text{int}}[\phi + \varepsilon] = \mathcal{L}_{\text{int}}[\phi] + \varepsilon\mathcal{L}'_{\text{int}}[\phi] \tag{7.9}$$

where $\mathcal{L}'_{\text{int}}[\phi(z)] = \frac{\delta}{\delta\phi(z)}\mathcal{L}_{\text{int}}$. The resulting Schwinger-Dyson equation is

$$\partial_z^2 \langle \phi(z)\phi(x) \rangle = \langle \mathcal{L}'_{\text{int}}[\phi(z)]\phi(x) \rangle - \delta^{(4)}(z-x)\,. \tag{7.10}$$

Again, this is the classical equation of motion $(\partial^2 - \mathcal{L}'_{\text{int}})\phi = 0$ up to a contact term. The interaction term appears similarly for n-point functions, for example

$$\begin{aligned}\partial_z^2 \langle \phi(z)\phi(x)\phi(y) \rangle = \langle \mathcal{L}'_{\text{int}}[\phi(z)]\phi(x)\phi(y) \rangle - \delta^{(4)}(z-x)\langle \phi(y) \rangle \\ - \delta^{(4)}(z-y)\langle \phi(x) \rangle\,.\end{aligned} \tag{7.11}$$

This can be put into a nice form if we recall a few things from Noether's theorem. Under $\phi \to \phi + \varepsilon$, the Lagrange density transforms as $\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}$, with

$$\delta\mathcal{L}(y) = \frac{\delta\mathcal{L}(y)}{\delta\phi(y)}\varepsilon(y) + \frac{\delta\mathcal{L}(y)}{\delta(\partial_\mu\phi)}\partial_\mu\varepsilon(y)\,. \tag{7.12}$$

Let us write the variation of the Lagrangian with respect to $\varepsilon(z)$ as

$$\frac{\partial\mathcal{L}(y)}{\delta\varepsilon(z)} = \frac{\delta\mathcal{L}(y)}{\delta\phi(y)}\delta^{(4)}(z-y) + \frac{\delta\mathcal{L}(y)}{\delta(\partial_\mu\phi(y))}\partial_\mu\delta^{(4)}(z-y)\,. \tag{7.13}$$

The variation of the action $S = \int d^4y\mathcal{L}$ is then

$$\begin{aligned}\frac{\delta S}{\delta\varepsilon(z)} &= \int d^4y\, \frac{\delta\mathcal{L}(y)}{\delta\varepsilon(z)} \\ &= \int d^4y \left[\frac{\delta\mathcal{L}}{\delta\phi(y)}\delta^{(4)}(z-y) - \partial_\mu \left(\frac{\delta\mathcal{L}(y)}{\delta(\partial_\mu\phi(y))} \right) \delta^{(4)}(z-y) \right] \\ &= \frac{\delta\mathcal{L}(z)}{\delta\phi(z)} - \partial_\mu \left(\frac{\delta\mathcal{L}(z)}{\delta(\partial_\mu\phi(z))} \right)\,.\end{aligned} \tag{7.14}$$

In going from the firs to the second line, we integrated by parts in order to move the derivative operator from the δ-function to its prefactor. Now use this to replace the first term in (7.12), with appropriate changes to the coordinate

$$\begin{aligned}\delta\mathcal{L}(y) &= \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi(y))}\varepsilon(y)\right]+\frac{\delta S}{\delta\varepsilon(y)}\varepsilon(y)\\ &\equiv \partial_\mu j^\mu+\frac{\delta S}{\delta\varepsilon(y)}\varepsilon(y)\end{aligned} \tag{7.15}$$

where we have identified the Noether current as

$$j^\mu(y)=\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi(y))}\varepsilon(y)\,. \tag{7.16}$$

Now back to our action (7.8). Here (7.14) reads

$$\frac{\delta S}{\delta\varepsilon(z)}=\mathcal{L}'_{\text{int}}[\phi]-\partial^2\phi\,. \tag{7.17}$$

Therefore the Schwinger-Dyson equation (7.10) becomes

$$\left\langle\frac{\delta S}{\delta\varepsilon(z)}\phi(x)\right\rangle=\delta^{(4)}(z-x)\,. \tag{7.18}$$

All this discussion is most useful for transformations which leave the classical Lagrangian invariant: that is, those for which $\delta\mathcal{L}=0$. In this case (7.15) becomes

$$\partial_\mu j^\mu=-\frac{\delta S}{\delta\varepsilon(z)}\varepsilon(z) \tag{7.19}$$

and (7.18) becomes

$$\frac{\partial}{\partial z^\mu}\langle j^\mu(z)\phi(x)\rangle=-\delta^{(4)}(z-x)\langle\varepsilon(x)\rangle\,. \tag{7.20}$$

This is the form of the Ward-Takahasi we will use in the next section.

Before leaving this section, it is worth remarking that the Schwinger-Dyson equations provide a link between path integral and canonical quantization. Consider repeating the steps we used earlier, this time for the generating functional $\mathcal{Z}[J]$.

$$\begin{aligned}\mathcal{Z}[J] &= \int\mathcal{D}\phi\,\exp-\int d^4y\left[-\frac{1}{2}(\phi+\varepsilon)\partial^2(\phi+\varepsilon)+\mathcal{L}_{\text{int}}[\phi+\varepsilon]+J(\phi+\varepsilon)\right]\\ &= \int\mathcal{D}\phi\,e^{-\int d^4y(\mathcal{L}+J\phi)}\left[1-\int d^4z\,\varepsilon(z)\left(-\partial_z^2\phi+\mathcal{L}'_{\text{int}}+J\right)+\ldots\right]\\ &= \mathcal{Z}[J]\quad\text{for any }\varepsilon(z).\end{aligned} \tag{7.21}$$

That implies the $O(\varepsilon)$ terms must sum to 0, i.e.

$$\begin{aligned}-\partial_z^2\int\mathcal{D}\phi\,\phi\,e^{-\int d^4y(\mathcal{L}+J\phi)} &= \int\mathcal{D}\phi\,(\mathcal{L}'_{\text{int}}+J)\,e^{-\int d^4y(\mathcal{L}+J\phi)}\\ \partial_z^2\frac{\delta\mathcal{Z}}{\delta J(z)} &= \left\{\mathcal{L}'_{\text{int}}\left[-\frac{\delta}{\delta J(z)}\right]+J(z)\right\}\mathcal{Z}[J]\,.\end{aligned} \tag{7.22}$$

This differential equation can be used to define the generating functional, and hence the whole theory. $\mathcal{Z}[J]$ is the unique solution (for appropriate boundary conditions) to the differential equation.

A similar procedure can be done in canonical quantization. A generating function for time-ordered operator vacuum matrix elements can be defined. Then it can be shown that it is a solution to the same Schwinger-Dyson equation (7.22). Since both generating function(al)s solve the same differential equation, they must describe the same physics. Canonical and path integral quantization are equivalent formulations.

7.2 *Schwinger-Dyson equations for fermions*

$$\mathcal{L} = \bar{\psi}\partial\!\!\!/\psi + \text{non-derivative terms in } \psi \tag{7.23}$$

Under

$$\psi(x) \to e^{i\alpha(x)}\psi(x) \quad \text{and} \quad \bar{\psi}(x) \to \bar{\psi}(x)e^{-i\alpha(x)} \tag{7.24}$$

the path integral measure is invariant, but the kinetic term in the Lagrangian is not:

$$\bar{\psi}\partial\!\!\!/\psi \to \bar{\psi}\partial\!\!\!/\psi + i\bar{\psi}\gamma^\mu\psi\,\partial_\mu\alpha\,. \tag{7.25}$$

Consider $\langle\psi(x_1)\bar{\psi}(x_2)\rangle$. We expand in small α as in §7.1, to find that the $O(\alpha)$ term must vanish, i.e.

$$\begin{aligned} 0 = &\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-S}\left[i\int d^4x\bar{\psi}(x)\gamma^\mu\psi(x)\partial_\mu\alpha(x)\right]\psi(x_1)\bar{\psi}(x_2) \\ &+\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-S}\left[i\alpha(x_1) - i\alpha(x_2)\right]\psi(x_1)\bar{\psi}(x_2) \end{aligned} \tag{7.26}$$

or, integrating the first integral by parts, and replacing $\alpha(x_k) = \int d^4x\,\alpha(x)\delta^{(4)}(x - x_k)$ in the second,

$$\begin{aligned} &\int d^4x\,\alpha(x)\,\partial_\mu\left[\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-S}\,\bar{\psi}(x)\gamma^\mu\psi(x)\right]\psi(x_1)\bar{\psi}(x_2) \\ &\quad = -\int d^4x\,\alpha(x)\left[\delta^{(4)}(x-x_1) - \delta^{(4)}(x-x_2)\right]\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-S}\psi(x_1)\bar{\psi}(x_2) \end{aligned} \tag{7.27}$$

Since this holds for all $\alpha(x)$

$$\partial_\mu\langle j^\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle = -\left[\delta^{(4)}(x-x_1) - \delta^{(4)}(x-x_2)\right]\langle\psi(x_1)\bar{\psi}(x_2)\rangle \tag{7.28}$$

where $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$ is the Noether current corresponding to the transformation (7.24).

$$\begin{aligned} \mathcal{M}^\mu(p, q_1, q_2) &\equiv \int d^4x\, d^4x_1\, d^4x_2\, e^{ip\cdot x}e^{iq_1\cdot x_1}e^{-iq_2\cdot x_2}\langle j^\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle \\ \mathcal{M}(q_1, q_2) &\equiv \int d^4x_1\, d^4x_2\, e^{iq_1\cdot x_1}e^{-iq_2\cdot x_2}\langle\psi(x_1)\bar{\psi}(x_2)\rangle \end{aligned} \tag{7.29}$$

Note

$$\begin{aligned} \mathcal{M}(q_1 + p, q_2) \equiv &\int d^4x\, d^4x_1\, d^4x_2\, e^{ip\cdot x}e^{iq_1\cdot x_1}e^{-iq_2\cdot x_2} \\ &\times \delta^{(4)}(x - x_1)\langle\psi(x_1)\bar{\psi}(x_2)\rangle \end{aligned} \tag{7.30}$$

and similarly for $\mathcal{M}(q_1, q_2 - p)$.

7.3 *Ward-Takahashi identity and renormalization*

The renormalized QED Lagrangian is

$$\mathcal{L} = \frac{1}{4} Z_3 F^2 + Z_2 \bar{\psi} \partial\!\!\!/ \psi + Z_2 Z_m m \bar{\psi}\psi + Z_1 e \bar{\psi} A\!\!\!/ \psi \,. \tag{7.31}$$

Further reading

Srednicki[25] discusses symmetries and Ward-Takahashi identities for scalar theories in §22. Schwartz[26] Sections 14.7, 14.8, and 19.5.

[25] M Srednicki. *Quantum Field Theory*. Cambridge University Press, 2007. ISBN 978-0-521-86449-7

[26] M D Schwartz. *Quantum Field Theory and the Standard Model*. Cambridge University Press, 2014. ISBN 978-1-107-03473-0

Nonabelian gauge theory

Gauge theories naturally describe long-range interactions between scalar or spinor matter; the gauge symmetry implies a massless gauge boson as the carrier of the force. However, as we found for QED, abelian gauge theories in 4-dimensions generically have positive β-functions: the interactions are weak at long distances and become strong only at very short distance.

This is the opposite behaviour to what is seen in hadronic physics, where hadronic constituents are seen to become weakly interacting at *high* energies (short distances), while these interactions are strong at low energies (long distances). The key to finding theories with this type of behaviour (in 4 dimensions) is to have the gauge boson interact with itself. We will see that this require the gauge symmetry to be that of a nonabelian group.

8.1 Lie groups

A Lie group is a group with an infinite number of elements which is also a differentiable manifold. Group elements continuously connected to the identity can be written as

$$U = \exp(i\theta^a T^a)\mathbb{1} \tag{8.1}$$

where the T^a are the group generators and the θ^a are numbers parametrizing the group element (a sum over generators is implied by the repeated index).

The generators T^a of a Lie group form a Lie algebra under an operation generically called a Lie bracket

$$[T^a, T^b] = if^{abc}T^c \tag{8.2}$$

where f^{abc} are the structure constants. When we represent the generators by matrices, the Lie bracket is just the usual commutator, $[A, B] = AB - BA$. The Lie brackets satisfy the Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \tag{8.3}$$

which implies

$$f^{abd}f^{dce} + f^{bcd}f^{dae} + f^{cad}f^{dbe} = 0\,. \tag{8.4}$$

We use a normalization for the generators such that

$$f^{acd}f^{bcd} = N\delta^{ab}\,. \tag{8.5}$$

We will primarily be interested in unitary groups, those whose group elements satisfy $U^\dagger U = \mathbb{1}$. Special unitary groups are those whose elements are unitary and have determinant equal to 1. In this case all elements are continuously connected to the identity and can be written as in (8.1). The group $G = \mathrm{SU}(N)$ has $N^2 - 1$ generators, so it has dimension $d(G) = N^2 - 1$. Other finite-dimensional Lie groups are orthogonal, symplectic, or the five exceptional Lie groups.

The fundamental representation is the smallest nontrivial representation of a Lie algebra. For $\mathfrak{su}(N)$ these are the $N \times N$ traceless, Hermitian matrices. Under an infinitesimal $\mathrm{SU}(N)$ transformation, an N-component field in the fundamental representation transforms as

$$\phi_i \to \phi_i + i\alpha^a (T^a_{\text{fund}})_{ij}\,\phi_j \tag{8.6}$$

with real α^a. Note we use $i, j, \ldots$ as representation indices (i.e. color, isospin or flavour) and $a, b, \ldots$ to index generators. The complex-conjugate field transforms under the anti-fundamental representation for which $T^a_{\text{afund}} = -(T^a_{\text{fund}})^*$ and hence

$$\phi_i^* \to \phi_i^* + i\alpha^a (T^a_{\text{afund}})_{ij}\,\phi_j^* = \phi_i^* - i\alpha^a \phi_j^* (T^a_{\text{fund}})_{ji}. \tag{8.7}$$

The last step uses the fact that the generators are Hermitian. From now on, we drop the "fund" subscript on the generators when we work with the fundamental representation

$$T^a = T^a_{\text{fund}}\,. \tag{8.8}$$

The adjoint representation is the one which acts on the vector space spanned by the generators themselves. The matrix elements of the generators are given by the structure constants:

$$(T^a_{\text{adj}})_{ij} = -if^{aij}\,. \tag{8.9}$$

We will see that the gauge fields transform in the adjoint representation.

The Index of a representation $T(R)$ defined to via an inner product of generators

$$tr(T^a_R T^b_R) = (T^a_R)_{ij}(T^b_R)_{ji} = T(R)\delta^{ab}\,. \tag{8.10}$$

For the fundamental representation

$$T^a_{ij}T^b_{ji} = \frac{1}{2}\delta^{ab} \implies T(\text{fund}) = T_F = \frac{1}{2}\,, \tag{8.11}$$

while for the adjoint representation

$$f^{acd}f^{bcd} = N\delta^{ab} \implies T(\text{adj}) = T_A = N\,. \tag{8.12}$$

Quadratic Casimir for representation R, $C_2(R)$, satisfies

$$T^a_R T^a_R = C_2(R)\mathbb{1}\,. \tag{8.13}$$

In (8.10) if we set $a = b$ and sum over a we find

$$T(R)\, d(G) = C_2(R)\, d(R) \tag{8.14}$$

where $d(R)$ is the dimension of representation R and $d(G)$ is the dimension of the group. Therefore, the quadratic Casimirs for the fundamental and adjoint representations are respectively

$$C_2(\text{fund}) = C_F = \frac{N^2 - 1}{2N} \quad \text{and} \quad C_2(\text{adj}) = C_A = N\,. \tag{8.15}$$

8.2 *Gauge invariance and Wilson lines*

Under a $U(1)$ transformation

$$\psi(x) \to e^{i\alpha(x)}\psi(x) \quad \text{and} \quad \bar{\psi}(x) \to \bar{\psi}(x)e^{-i\alpha(x)} \tag{8.16}$$

$\bar{\psi}\partial\!\!\!/\psi$ is not invariant. Consider a derivative acting in the direction of a unit vector n^μ

$$n^\mu \partial_\mu \psi = \lim_{a\to 0} \frac{1}{a}\left[\psi(x + an) - \psi(x)\right]\,. \tag{8.17}$$

Noting

$$\psi(x + an) - \psi(x) \to e^{i\alpha(x+an)}\psi(x + an) - e^{i\alpha(x)}\psi(x) \tag{8.18}$$

suggests how we might construct a gauge covariant derivative, one which would transform like $\psi(x)$, i.e.

$$D_\mu\psi(x) \to e^{i\alpha(x)}D_\mu\psi(x)\,. \tag{8.19}$$

Define a Wilson line, $W(y, x)$, going from point x to y to transform as

$$W(y, x) \to e^{i\alpha(y)}W(y, x)e^{-i\alpha(x)}\,. \tag{8.20}$$

The Wilson line a type of parallel transporter on the gauge manifold. Choose a normalization such that $W(x, x) = 1$. Then W can be written as an element of $U(1)$, a phase:

$$W(y, x) = e^{i\phi(y,x)} \tag{8.21}$$

with real ϕ. We also choose a convention that $W(x, y) = (W(y, x))^*$. We can use the Wilson line to define a covariant derivative

$$n^\mu D_\mu = \lim_{a\to 0} \frac{1}{a}\left[\psi(x + an) - W(x + an, x)\psi(x)\right]\,. \tag{8.22}$$

It is clear that $\bar{\psi}\, D\!\!\!\!/\,\psi$ is gauge invariant.

Let us make contact with our previous formulation of QED. For small a, define A_μ as the field at the midpoint along a small Wilson line

$$W(x + an, x) = \exp\left[iean^\mu A_\mu(x + \tfrac{a}{2}n)\right]\,. \tag{8.23}$$

Taking the $a \to 0$ limit we have

$$D_\mu\psi(x) = \left[\partial_\mu - ieA_\mu(x)\right]\psi(x)\,. \tag{8.24}$$

The gauge transformation property of $D_\mu \psi(x)$ implies

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)\,,. \tag{8.25}$$

Note $D_\nu(D_\mu\psi)$ also transforms just as ψ, therefore

$$[D_\mu, D_\nu]\psi(x) \to e^{i\alpha(x)}[D_\mu, D_\nu]\psi\,. \tag{8.26}$$

Noting that this commutator is not a differential operator,

$$[D_\mu, D_\nu] = -ie(\partial_\mu A_\nu - \partial_\nu A_\mu) = -ieF_{\mu\nu}\,, \tag{8.27}$$

tells us that the phase in (8.26) must be due only to the transformation of ψ and that $F_{\mu\nu}$ is gauge invariant.

PARAGRAPH ABOUT PLAQUETTE

$$P_{12}(x) = W(y_1, y_4)W(y_4, y_3)W(y_3, y_2)W(y_2, y_1)\,. \tag{8.28}$$

Expand about small a

$$P_{12}(x) = 1 - iea^2 F_{12}(x) + O(a^2)\,. \tag{8.29}$$

In the case of nonabelian gauge transformations

$$\psi(x) \to V(x)\psi(x) \quad \text{and} \quad \bar\psi(x) \to \bar\psi(x)V^\dagger(x) \tag{8.30}$$

with $V(x) \in G$, the Wilson line transforms as

$$W(y, x) \to V(y)W(y, x)V^\dagger(x)\,. \tag{8.31}$$

Again we choose a normalization such that $W(x, x) = \mathbb{1}$, the identity of G. For small a

$$W(x + an, x) = 1 + igan^\mu A^a_\mu T^a + O(a^2)\,. \tag{8.32}$$

To find the gauge transformation of the field A_μ, i.e. the mapping from A^a_μ to field A'^a_μ, we require $D_\mu\psi \to VD_\mu\psi$ so that

$$\begin{aligned}
(\partial_\mu - igA'_\mu)V\psi &= V(\partial_\mu - igA_\mu)\psi \\
\partial_\mu V - igA'_\mu V &= -igVA_\mu \\
\implies A'_\mu &= VA_\mu V^{-1} - \frac{i}{g}(\partial_\mu V)V^{-1}\,.
\end{aligned} \tag{8.33}$$

We have used the abbreviated notation $A_\mu = A^a_\mu T^a$.

For infinitesimal transformations $V(x) = 1 + i\alpha^a(x)T^a + \ldots$

$$\begin{aligned}
\psi(x) &\to (1 + i\alpha^a(x)T^a)\psi(x) \\
A^a_\mu(x) &\to A^a_\mu(x) + \frac{1}{g}\partial_\mu\alpha^a(x) + f^{abc}A^b_\mu(x)\alpha^c(x) \\
&= A^a_\mu(x) + \frac{1}{g}\left[\partial_\mu\delta^{ac} - igA^b_\mu(-if^{bac})\right]\alpha^c(x) \\
&= A^a_\mu(x) + \frac{1}{g}\left[\partial_\mu\delta^{ac} - igA^b_\mu(T^b_{\text{adj}})^{ac}\right]\alpha^c(x) \\
&= A^a_\mu(x) + \frac{1}{g}D^{ac}_\mu\alpha^c(x)
\end{aligned} \tag{8.34}$$

where $D_\mu^{ac} \equiv \partial_\mu \delta^{ac} - igA_\mu^b (T_{\text{adj}}^b)^{ac}$ is the covariant derivative in the adjoint representation, and we have used (8.9). The field strength tensor, defined from

$$[D_\mu, D_\nu] = -igF_{\mu\nu}^a T^a \tag{8.35}$$

is then

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c \,. \tag{8.36}$$

Under infinitesimal transformations, F is not gauge invariant:

$$F_{\mu\nu}^a \to F_{\mu\nu}^a - f^{abc}\alpha^b F_{\mu\nu}^c \,; \tag{8.37}$$

however $F_{\mu\nu}^a F^{a,\mu\nu} = \sum_a (F^a)^2$ is. (Usually the explicit summation symbol is dropped, since we are never interested in the square of only a single term.) Thus we arrive at the Lagrangian for a non-abelian gauge field coupled to fermions transforming in the fundamental representation

$$\begin{aligned} \mathcal{L} &= \frac{1}{4}(F^a)^2 + \bar{\psi}_i \left(\partial\!\!\!/ \delta_{ij} - igA\!\!\!/^a T_{ij}^a + m\delta_{ij} \right) \psi_j \\ &= \frac{1}{4}(F^a)^2 + \bar{\psi}(D\!\!\!\!/ + m)\psi \,. \end{aligned} \tag{8.38}$$

In fact there is another gauge invariant term which can appear in the Lagrangian:

$$\mathcal{L}_\theta = \theta \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F^{a,\rho\sigma} = 2\theta \partial_\mu (\epsilon^{\mu\nu\rho\sigma} A_\nu^a F^{a,\rho\sigma}) \,. \tag{8.39}$$

As a total derivative, it appears this would contribute only a boundary term to the path integral, and be unimportant. Indeed it can be shown that total-derivative interactions cannot contribute at any order in perturbation theory.[27] However, $\mathcal{L}_\theta$ can give a nonperturbative contribution due to nontrivial topological configurations of the A_μ field. This term breaks CP, and one outstanding puzzle in the Standard Model is why the coefficient θ in the QCD Lagrangian is so small, i.e. consistent with zero to high accuracy. This is known as the strong CP problem.

[27] See Schwartz §7.4.2, for example.

8.3 Fadeev-Popov gauge fixing

Say we have an ordinary integral over two variables, where the integrand only depends on one

$$\mathcal{Z} \propto \int da\, db\, e^{-S(a)} \,. \tag{8.40}$$

The integral over b is redundant and gives a divergent contribution to $\mathcal{Z}$ if it is taken over $(-\infty, \infty)$. In § 6.1, we simply dropped the integral over redundant degrees of freedom, finding a way to write the equivalent of

$$\mathcal{Z} = \int da\, e^{-S(a)} \,. \tag{8.41}$$

Things are more difficult for nonabelian gauge theory, where we won't be able to be so slick. Instead we can introduce a δ-function

$$\mathcal{Z} = \int da\, db\, \delta(b)\, e^{-S(a)}\,. \tag{8.42}$$

This is unchanged if we shift the argument of the δ-function by an arbitrary function of a

$$\mathcal{Z} = \int da\, db\, \delta(b - f(a))\, e^{-S(a)}\,. \tag{8.43}$$

In fact, we do not need to specify $f(a)$. Instead, we could suppose that $b = f(a)$ is the solution, for fixed a to some equation $G(a, b) = 0$. Then using the composition rule for the δ-function

$$\delta(G(a,b)) = \left|\frac{\partial G}{\partial b}\right|^{-1} \delta(b - f(a))\,, \tag{8.44}$$

we have

$$\mathcal{Z} = \int da\, db\, \frac{\partial G}{\partial b}\, \delta(G(a,b))\, e^{-S(a)} \tag{8.45}$$

where we have assumed that $\frac{\partial G}{\partial b} > 0$. This argument can be generalized to n-component variables a and b to read

$$\mathcal{Z} = \int d^n a\, d^n b\, \det\left(\frac{\partial G}{\partial b}\right) \left[\prod_{i=1}^{n} \delta(G_i)\right] e^{-S(a)}\,. \tag{8.46}$$

In the context of gauge theory, the b variables in (8.46) represent the gauge-equivalent degrees-of-freedom, the redundancies in the path integral over the field $A_\mu(x)$ due to gauge transformations. The the gauge-fixed partition function is then

$$\mathcal{Z} = \int \mathcal{D}A\, \det\left(\frac{\delta G}{\delta \alpha}\right) \left[\prod_{x,a} \delta^{(4)}(G^a)\right] e^{-S_{\text{YM}}[A]}\,. \tag{8.47}$$

The function $G(a, b)$ is the gauge-fixing condition we apply to remove those degeneracies. For nonabelian fields, we need one for each generator. Let

$$G^a(x) = \partial^\mu A^a_\mu(x) - \omega^a(x) \tag{8.48}$$

where $\omega^a(x)$ is function of x, independent of the gauge field. Given the gauge transformation of the A_μ field (8.34), G transforms as

$$G^a(x) \to G^a(x) + \frac{1}{g} \partial^\mu D^{ab}_\mu \alpha^b(x)\,. \tag{8.49}$$

The functional derivative with respect to $\alpha^b(y)$ is

$$\frac{\delta G^a(x)}{\delta \alpha^b(y)} = \frac{1}{g} \partial^\mu D^{ab}_\mu \delta^{(4)}(x - y)\,, \tag{8.50}$$

with the understanding that the derivatives on the right-hand side are with respect to x.

In order to evaluate the determinant appearing in (8.46), we introduce Grassmann variables $c^a(x)$ and $\bar{c}^a(x)$ to write the *Fadeev-Popov* determinant as

$$\det \frac{\delta G^a(x)}{\delta \alpha^b(y)} \propto \int \mathcal{D}c\, \mathcal{D}\bar{c}\, e^{-S_{\text{gh}}} \tag{8.51}$$

with $S_{\text{gh}} = \int d^4x\, \mathcal{L}_{\text{gh}}$ and

$$\begin{aligned} \mathcal{L}_{\text{gh}} &= -\bar{c}^a\, \partial^\mu D_\mu^{ab} c^b \\ &= \partial^\mu \bar{c}^a\, D_\mu^{ab} c^b \\ &= \partial^\mu \bar{c}^a\, \partial_\mu c^a - ig\partial^\mu \bar{c}^a A_\mu^c (T^c_{\text{adj}})^{ab} c^b \\ &= \partial^\mu \bar{c}^a\, \partial_\mu c^a - g f^{abc} A_\mu^c \partial^\mu \bar{c}^a\, c^b \end{aligned} \tag{8.52}$$

where we have implicitly integrated by parts in going from lines 1 to 2 and 3 to 4, and have dropped total derivatives. Because the c fields are Grassmann valued but have the Lagrangian of a scalar field, they are called *ghost* fields. They would violate the spin-statistics theorem, but they are unphysical. We will discuss the interpretation further later.

Note that for Abelian gauge theory, $f^{abc} = 0$. Therefore, the ghosts do not couple to the photon field in QED.[28] The Fadeev-Popov determinant just contributes an overall multiplicative factor to the QED generating functional and can be ignored. In nonabelian gauge theories, though, we have to include the ghost fields in our Feynman rules.

[28] Or to the electron, for that matter.

The final step of this chapter is to exchange the δ-function in (8.47) for a term which can appear in the action. For this purpose, it was convenient to introduce the $\omega^a(x)$ in (8.48). These are arbitrary functions; any choice is as good as any other. Now we decide instead of taking a single choice, we will average over the choice of $\omega^a(x)$ with a Gaussian weight. That is take the $\mathcal{Z}$ from (8.47), multiply and integrate:

$$\int \mathcal{D}\omega\, \exp\left(-\int d^4x\, \frac{\omega^2}{2\xi}\right) \mathcal{Z}\,. \tag{8.53}$$

Now the integral over ω can be done first, with the δ-function forcing $\omega^a(x) = \partial^\mu A_\mu^a(x)$. Now up to a multiplicative constant the resulting partition function (which we also call $\mathcal{Z}$ because it gives the same correlation functions) is

$$\mathcal{Z} \propto \int \mathcal{D}A\, \mathcal{D}\bar{c}\, \mathcal{D}c\, \exp\left(-S_{\text{YM}} - S_{\text{gh}} - S_{\text{gf}}\right) \tag{8.54}$$

where

$$\begin{aligned} S_{\text{YM}} &= \int d^4x\, \frac{1}{4}(F^a)^2 \\ S_{\text{gh}} &= \int d^4x\, \left(\partial^\mu \bar{c}^a\, \partial_\mu c^a - g f^{abc} A_\mu^c \partial^\mu \bar{c}^a\, c^b\right) \\ S_{\text{gf}} &= \int d^4x\, \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 \end{aligned} \tag{8.55}$$

8.4 One-loop renormalization

A compact way of writing the Lagrangian for a nonabelian gauge field, let's refer to it as the *gluon* field, coupled to fermions, after gauge-fixing, is

$$\mathcal{L} = \frac{1}{4}(F^a)^2 + \bar{\psi}(\not{D} + m)\psi + \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 - \bar{c}^a \partial^\mu D^{ab}_\mu c^b \,. \tag{8.56}$$

To obtain the Feynman rules, it is useful to unpack this a little bit. The quadratic pieces give the propagators and can be written

$$\mathcal{L}_{\text{quad}} = -\frac{1}{2} A_\mu \left[\partial^2 \delta^{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial^\mu \partial^\nu\right] A_\nu + \bar{\psi}\,(\not{\partial} + m)\psi - \bar{c}\partial^2 c \,. \tag{8.57}$$

Using a and b for indices in the adjoint representation, and i and j for indices in the fundamental representation, and suppressing spin indices (fermion propagator only) we find the following for the propagators:[29]

[29] Recall the Fourier transform maps derivatives to factors of $i \times$ momentum.

$$\begin{aligned}
\text{gluon prop} &= \frac{1}{k^2}\left[\delta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right]\delta^{ab} \\
\text{fermion prop} &= \frac{1}{i\not{p} + m}\,\delta^{ij} \\
\text{ghost prop} &= \frac{1}{q^2}\,\delta^{ab} \,.
\end{aligned} \tag{8.58}$$

Interactions come from the the other terms in $\mathcal{L}$. Expanding D_μ and $F^a_{\mu\nu}$ as in (8.24) and (8.36), respectively, as well as using the steps in (8.52), we find

$$\begin{aligned}
\mathcal{L}_{\text{int}} = {} & g f^{abc}(\partial_\mu A^a_\nu) A^b_\mu A^c_\nu + \frac{g^2}{4}(f^{abc} A^b_\mu A^c_\nu)(f^{ade} A^{d,\mu} A^{e,\nu}) \\
& - i\bar{\psi}_i g \not{A}^a (T^a_{ij})\psi_j - g f^{abc} A^c_\mu (\partial_\mu \bar{c}^a) c^b \,.
\end{aligned} \tag{8.59}$$

Our momentum-space Feynman rules (with all momenta flowing into the vertex) are therefore[30,31]

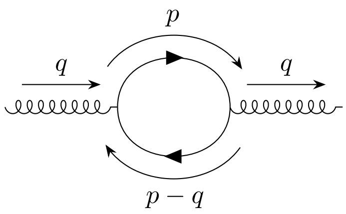
[30] Recall these vertices get an extra minus sign due to the fact that the generating functional contains a factor $\exp(-S)$.

[31] Let us introduce the abbreviated notation $\delta^{\mu[\rho}\delta^{\sigma]\nu} \equiv \delta^{\mu\rho}\delta^{\sigma\nu} - \delta^{\mu\sigma}\delta^{\rho\nu}$.

$$\begin{aligned}
\text{3-gluon vertex} &= -igf^{abc}\left[\delta^{\mu\nu}(k-p)^\rho + \delta^{\nu\rho}(p-q)^\mu + \delta^{\rho\mu}(q-k)^\nu\right] \\
\text{4-gluon vertex} &= -g^2\Big(f^{abe}f^{cde}\delta^{\mu[\rho}\delta^{\sigma]\nu} + f^{ace}f^{bde}\delta^{\mu[\nu}\delta^{\sigma]\rho} \\
&\qquad + f^{ade}f^{bce}\delta^{\mu[\nu}\delta^{\rho]\sigma}\Big) \\
\text{fermion vertex} &= ig\gamma^\mu (T^a)_{ij} \\
\text{ghost vertex} &= igf^{abc}p^\mu \,.
\end{aligned} \tag{8.60}$$

The contribution to the vacuum polarization coming from fermion loops is just as in QED.

$$\mathcal{M}_F^{ab\mu\nu} =$$



$$= -\mathrm{Tr}(T^a T^b)(ig)^2 \mu^\epsilon \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p-q)^2 + m^2} \frac{1}{p^2 + m^2}$$
$$\times \, \mathrm{tr}\left[(-i\not{p} + m)\gamma^\mu[-i(\not{p} - \not{q}) + m]\gamma^\nu\right] \qquad (8.61)$$

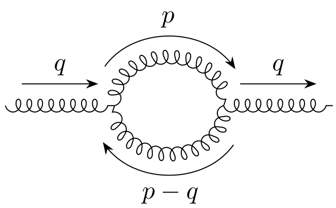
The integral is just what we saw for the QED vacuum polarization (6.38); the only new feature is the need to trace over the generators in the fundamental representation (8.11). We find

$$\mathcal{M}_F^{ab\mu\nu} = -T_F \delta^{ab}(q^2\delta^{\mu\nu} - q^\mu q^\nu)\frac{g^2}{2\pi^2}\int_0^1 dx\, x(1-x)\left(\frac{2}{\epsilon} - \gamma + \log\frac{4\pi\mu^2}{\Delta}\right) \qquad (8.62)$$

where $\Delta = m^2 + q^2 x(1-x)$ and $T_F = \frac{1}{2}$.

The gluon self-interactions complicate things. Using Feynman gauge, $\xi = 1$,

$$\mathcal{M}_3^{ab\mu\nu} =$$



$$= \frac{g^2\mu^\epsilon}{2}\int \frac{d^d p}{(2\pi)^d}\frac{1}{p^2(p-q)^2} f^{acd} f^{bcd} \mathcal{N}^{\mu\nu}\,. \qquad (8.63)$$

Note that we have a symmetry factor of $\frac{1}{2}$ since the gluon is its own antiparticle, so the gluon lines are not oriented in the same way fermion and ghost propagators are. We will also use (8.12) to write $f^{acd} f^{bcd} = N\delta^{ab}$. The numerator in (8.63) is

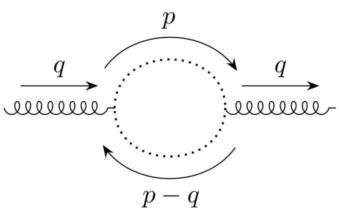
$$\mathcal{N}^{\mu\nu} = [\delta^{\mu\sigma}(q+p)^\rho + \delta^{\rho\sigma}(q-2p)^\mu + \delta^{\rho\mu}(p-2q)^\sigma]$$
$$\times \, [\delta^{\nu\rho}(q+p)^\sigma + \delta^{\rho\sigma}(-q-2p)^\nu + \delta^{\sigma\nu}(p-2q)^\rho] \qquad (8.64)$$

Introduce a Feynman parameter to write

$$\frac{1}{p^2(p-q)^2} = \int_0^1 \frac{dx}{[(1-x)p^2 + x(p-q)]^2} = \int_0^1 \frac{dx}{(\ell^2 + \Delta)^2} \qquad (8.65)$$

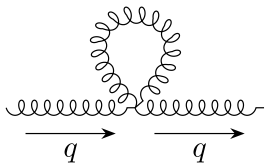
having shifted $\ell = p + xq$ and defined $\Delta = x(1-x)q^2$.

$$\mathcal{M}^{ab\mu\nu}_{\text{gh}} =$$



$$= \frac{g^2 C_A \delta^{ab}}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx \, \frac{1}{\Delta^{2-\frac{d}{2}}} \left[\Gamma(1-\tfrac{d}{2})\tilde{f}(x) + \Gamma(2-\tfrac{d}{2})\tilde{g}(x)\right] \delta^{\mu\nu} q^2 \tag{8.66}$$

$$\mathcal{M}^{ab\mu\nu}_{\text{gh}} =$$



$$= 0 \tag{8.67}$$

Renormalized Lagrangian

$$\begin{aligned}
\mathcal{L} = &\frac{1}{4} Z_3 (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 \\
&+ Z_{3g} g f^{abc} (\partial_\mu A^a_\nu) A^b_\mu A^c_\nu + \frac{1}{4} Z_{4g} g^2 f^{abe} f^{cde} A^a_\mu A^b_\nu A^{\mu,c} A^{\nu^d} \\
&+ Z_{2'} \bar{c} \partial^2 c - Z_{1'} g f^{abc} (\partial^\mu \bar{c}^a) A^b_\mu c^c \\
&+ Z_2 \bar{\psi} \partial\!\!\!/ \psi + Z_m m \bar{\psi}\psi + Z_1 g \bar{\psi} A\!\!\!/^a T^a \psi
\end{aligned} \tag{8.68}$$

$$\begin{aligned}
\beta(g) &= -\frac{\epsilon g}{2} - g\mu \frac{d}{d\mu}\left(\frac{Z_1}{Z_2 Z_3^{1/2}}\right) \\
&= g\left[-\frac{\epsilon}{2} - \mu \frac{d}{d\mu}\left(\delta_1 - \delta_2 - \frac{\delta_3}{2}\right)\right]
\end{aligned} \tag{8.69}$$

The fermion self-energy diagram (Fig. 8.1) gives a correction to $Z_2 = 1 + \delta_2$ of

$$\delta_2 = -\frac{g^2}{8\pi^2}\frac{1}{\epsilon} T_F \,. \tag{8.70}$$

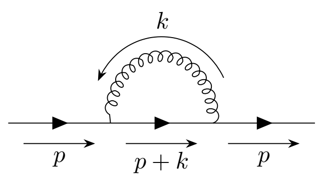


Figure 8.1: Fermion self-energy.

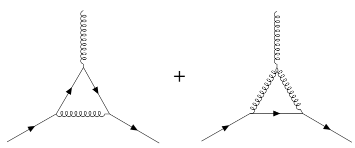


Figure 8.2: One-loop fermion-gauge vertex corrections.

The one-loop corrections to the fermion-gauge boson vertex (Fig. 8.2) give

$$\delta_1 = -\frac{g^2}{8\pi^2}\frac{1}{\epsilon}(T_F + C_A) \,. \tag{8.71}$$

We can use the leading order β-function: $\mu \frac{d}{d\mu} g = -\frac{\epsilon}{2} g$ to write (8.69) as

$$\begin{aligned}
\beta(g) &= -\frac{\epsilon}{2} + \frac{\epsilon}{2} g^2 \frac{\partial}{\partial g}\left(\delta_1 - \delta_2 - \frac{\delta_3}{2}\right) \\
&= -\frac{\epsilon}{2} - \frac{g^3}{16\pi^2}\left(\frac{11}{3}C_A - \frac{4}{3} n_f T_F\right)
\end{aligned} \tag{8.72}$$

where we have allowed for there to be n_f flavours of fermions.

8.5 BRST symmetry

Becchi, Rouet, Stora, and, working independently, Tyutin.

QED Lagrangian

$$\mathcal{L} = \frac{1}{4}F^2 + \bar{\psi}(\not{D} + m)\psi + \frac{1}{2\xi}(\partial^\mu A_\mu)^2 - \bar{c}\partial^2 c \tag{8.73}$$

Under

$$\begin{aligned} \psi(x) &\to \psi(x) + \delta\psi(x) && \text{with} \quad \delta\psi(x) = i\alpha(x)\psi(x) \\ A_\mu(x) &\to A_\mu(x) + \delta A_\mu(x) && \text{with } \delta A_\mu(x) = \frac{1}{e}\partial_\mu\alpha(x) \end{aligned} \tag{8.74}$$

the gauge-fixing term transforms as

$$(\partial^\mu A_\mu)^2 \to (\partial^\mu A_\mu + \frac{1}{e}\partial^2\alpha)^2 \tag{8.75}$$

This is only invariant for a subset of gauge transformations, those for which $\partial^2\alpha = 0$.

Notice that $\bar{c}\partial^2 c$ in the Lagrangian implies the equations of motion for these fields are $\partial^2 c = 0 = \partial^2\bar{c}$. This serves as inspiration to consider a transformation $\alpha(x) = \theta c(x)$ with θ any global Grassmann number. Then

$$(\partial^\mu A_\mu)^2 \to (\partial^\mu A_\mu)^2 + \frac{2}{\epsilon}(\partial^\mu A_\mu)(\theta\partial^2 c) + \frac{1}{e^2}(\theta\partial^\mu c)(\theta\partial_\mu c)\,. \tag{8.76}$$

The last term vanishes since $\theta^2 = 0$. If we transform

$$\bar{c}(x) \to \bar{c}(x) + \delta\bar{c}(x) \qquad \text{with} \quad \delta\bar{c}(x) = -\frac{\theta}{e\xi}\partial^\mu A_\mu(x) \tag{8.77}$$

then $\mathcal{L}$ will be invariant. Note this is a global, not local, symmetry because it is parameterized by a single global variable θ.

QCD Lagrangian

$$\mathcal{L} = \frac{1}{4}(F^a)^2 + \bar{\psi}(\not{D} + m)\psi + \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 - \bar{c}\partial^\mu D_\mu c \tag{8.78}$$

where $D_\mu c^a = \partial_\mu c^a + g f^{abc} A^b_\mu c^c$. Let $\alpha^a(x) = \theta c^a(x)$ then

$$\begin{aligned} \delta\psi_i &= i\theta c^a T^a_{ij}\psi_j \\ \delta A^a_\mu &= \frac{\theta}{g} D^{ab}_\mu c^b \\ \delta\bar{c}^a &= -\frac{\theta}{g\xi}\partial^\mu A^a_\mu\,. \end{aligned} \tag{8.79}$$

Due to the A_μ in $D_\mu c$,

$$D_\mu c^a \to D_\mu c^a - \theta f^{abc}(D_\mu c^b)c^c\,. \tag{8.80}$$

In order to make this invariant, we now must transform $c(x)$ with

$$\delta c^a = -\frac{\theta}{2} f^{abc} c^b c^c\,. \tag{8.81}$$

It can be convenient to introduce an auxilliary scalar field $B^a(x)$, called the Nakanishi-Lautrup field.

$$\mathcal{L} = \frac{1}{4}(F^a)^2 + \bar{\psi}(\not{D} + m)\psi - \frac{\xi}{2}(B^a_\mu)^2 + B^a \partial^\mu A^a_\mu - \bar{c}\partial^\mu D_\mu c \qquad (8.82)$$

such that path integration over the B^a field gives (8.78). Now the transformation rules (8.79) read

$$\begin{aligned} \delta \bar{c}^a &= \theta B^a \\ \delta B^a &= 0\,. \end{aligned} \qquad (8.83)$$

Further reading

The Wilson line is used to derive a gauge-invariant Lagrangian in Schwartz and in Peskin & Schroeder. Srednicki[32] has a concise treatment of Fadeev-Popov gauge fixing in §71, which we followed, with more involved treatments given by Schwartz[33] Sections 14.5 and 25.4 and Peskin & Schroeder[34] in §9.4 and §16.2.

[32] M Srednicki. *Quantum Field Theory*. Cambridge University Press, 2007. ISBN 978-0-521-86449-7

[33] M D Schwartz. *Quantum Field Theory and the Standard Model*. Cambridge University Press, 2014. ISBN 978-1-107-03473-0

[34] M Peskin and D Schroeder. *An Introduction to Quantum Field Theory*. Addison-Wesley, 1995. ISBN 0-201-50397-2

Conventions

$c = 1$ mostly everywhere. We tend to keep $\hbar$ explicit through much of the notes.

When in Minkowski spacetime, we use the "mostly minus" metric $(+ - - -)$.

Fourier transforms in D Euclidean dimensions: functions are composed as weighted sums (integrals) over a continuum of Fourier $\vec{k}$-modes. Using notation familiar in 3-dimensions, we adopt the convention common in theoretical physics

$$f(\vec{x}) = \int \frac{d^D k}{(2\pi)^D} \tilde{f}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} . \tag{9.1}$$

Free scalar fields in Minkowski spacetime, using mostly minus metric

$$\phi(x) = \int \frac{d^3 p}{(2\pi\hbar)^3\, 2E} \left[a(\vec{p})\, e^{-ip\cdot x/\hbar} + a^\dagger(\vec{p})\, e^{ip\cdot x/\hbar} \right] . \tag{9.2}$$

Note the creation and annihilation operators are relativisitically normalized (hence the additional $(2E)^{-\frac{1}{2}}$) and that the time-dependent field is in the Heisenberg picture (hence the temporal phases $e^{\mp iEt/\hbar}$). When we work in Euclidean metric, the scalar product $ik \cdot x \mapsto -ik \cdot x$ – see the discussion around (4.7). Note also that if we had used the mostly plus Minkowski metric, the signs of the scalar products in (9.2) would be reversed; this ease of Wick rotating between $(- + + +)$ and $(+ + + +)$ is one reason put forward for preferring the mostly plus convention.[35]

[35] However many of us grew up with the convention that $p \cdot p = m^2$, not $-m^2$.

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