**Examples Sheet 2**

1. Consider the theory given by the action

$$S[\phi] = \int d^d x \left[ \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{g}{3!} \phi^3 + \frac{\lambda}{4!} \phi^4 \right]$$

where $\phi$ is a real scalar field.

(a) Determine all one-loop 1PI graphs, complete with their appropriate symmetry factors, which contribute to

$$\langle \phi(x)\phi(y) \rangle, \ \langle \phi(x)\phi(y)\phi(z) \rangle, \text{ and } \langle \phi(x)\phi(y)\phi(z)\phi(w) \rangle,$$

expressing your answers in terms of integrals over $d$-dimensional loop momenta. [You are not required to evaluate the integrals.]

2. Consider a scalar field $\phi$ with potential $V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{6} \mu^{\epsilon/2} g(\mu) \phi^3$ in dimension $d = 6 - \epsilon$. Here $\mu$ is an arbitrary mass scale introduced so that the coupling $g(\mu)$ is dimensionless.

(a) Draw the one-loop 1PI graph which contributes to the propagator at order $g^2$.

(b) Using dimensional regularization, show that the divergent part of the corresponding integral for the 6-dimensional theory is

$$-\frac{1}{\epsilon} \frac{g^2}{(4\pi)^3} \left( m^2 + \frac{p^2}{6} \right)$$

where $p$ is the external momentum. Also compute the divergence corresponding to the 1PI one-loop graph that gives a $g^3$ correction to the 3-point function, and find the one-loop divergence for the 1-point function.
*[Hint: use the Feynman parametrization $\frac{1}{AB} = \int_0^1 dx \int_0^1 dy \frac{\delta(x+y-1)}{(xA+yB)^2}$, then change of variables to get denominators of integrands into the form $(\ell^2 + \Delta)^2$, where $\ell$ is the loop momentum. You can generalize, introducing more Feynman parameters for integrals which need it.]*

(c) Show that in 6 dimensions all these divergences may be cancelled by introducing the counterterm action

$$S_{\text{ct}}[\phi] = \int d^d x \, \mathcal{L}_{\text{ct}} = -\frac{1}{\epsilon} \frac{1}{6(4\pi)^3} \int d^d x \left[ \frac{g^2}{2} \partial_\mu \phi \, \partial^\mu \phi + \mu^{-\epsilon} (V''(\phi))^3 \right].$$

Check that $\mathcal{L}_{\text{ct}}$ has dimension $d$.

(d) Determine the $\beta$-function for the coupling $g$ and show $\beta(g) < 0$ at small $g$.

3. Show that under the redefinition $g_i \mapsto g'_i(g_j)$ of the couplings of a theory at scale $\Lambda$, the $\beta$-functions transform as

$$\beta_i \mapsto \beta'_i = \frac{\partial g'_i}{\partial g_j}\, \beta_j\,.$$

Show that in a theory with a single coupling $g$, the first two terms in the $\beta$-function $\beta(g) = ag^3 + cg^5 + \mathcal{O}(g^7)$ are invariant under any coupling constant redefinition of the form $g \mapsto g' = g + \mathcal{O}(g^3)$.

Show that, in a neighbourhood of $g = 0$, it is possible to choose this redefinition so as to remove all terms *except* these first two. [*Hint: consider setting* $g'(g) = g + g^3 f(g)$ *where* $f(0) = 1$.] What is the significance of this calculation?

4. Consider a four-dimensional theory whose only couplings are a mass parameter $m^2$ and a marginally relevant coupling $g$.

   (a) Write down generic expressions for the $\beta$-functions in such a theory to lowest non-trivial order. (You should be able to identify the *values* of the classical contributions to the $\beta$-functions and the *sign* of the leading-order quantum correction to $\beta(g)$.

   (b) Sketch the RG flows for this theory.

   (c) Suppose that $g(\Lambda') = 0.1$ when the cut-off $\Lambda'$ is fixed at $10^5$ GeV. If $m^2(\Lambda')$ is measured to be 100 GeV, what value of $m^2(\Lambda)$ would be needed at the higher scale $\Lambda = 10^{19}$ GeV?

   (d) Suppose you changed your value of $m^2(\Lambda)$ by one part in $10^{20}$. What would be the change in $m^2(\Lambda')$?

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