Advanced Quantum Field Theory — Mathematical Tripos, Part III
Dr M B Wingate — Lent Term 2020

Examples Sheet 4

1. Show that the QED correlation function $\langle j^\mu(x) j^\nu(y)\rangle$, when written in momentum space, is proportional to

$$\Pi^{\mu\nu}(k) + \Pi^{\mu\rho}(k) D_{\rho\sigma}(k) \Pi^{\sigma\nu}(k) + \cdots$$

where $j^\mu = \bar{\psi}\gamma^\mu\psi$, $\Pi^{\mu\nu}$ is the photon self-energy (the sum of 1PI diagrams with 2 amputated, external photons), and $D_{\rho\sigma}$ is the tree-level photon propagator. Hence show that $\Pi^{\mu\nu}$ is transverse, i.e. that $k_\mu \Pi^{\mu\nu} = 0$.

2. Complete the one-loop renormalization of nonabelian gauge theory with n_f flavours of fermions using dimensional regularization in $d = 4 - \epsilon$ dimensions. That is, show that the following counterterms must be added (assuming Feynman gauge, $\xi = 1$)

   (a) Gauge boson propagator counterterm

$$\delta_3 = \frac{g^2}{(4\pi)^2} \frac{2}{\epsilon} \left(\frac{5}{3} C_A - \frac{4}{3} n_f T_F \right) ;$$

   (b) Fermion propagator counterterm

$$\delta_2 = -\frac{g^2}{(4\pi)^2} \frac{2}{\epsilon} C_F ;$$

   (c) Vertex counterterm

$$\delta_1 = -\frac{g^2}{(4\pi)^2} \frac{2}{\epsilon} \left(C_A + C_F \right) .$$

   Note $\mathrm{Tr}\, T^a T^b = T_F \delta^{ab}$, $T^a_{ij} T^a_{jk} = C_F \delta_{ik}$, and $f^{acd} f^{bcd} = C_A \delta^{ab}$.

   [The full calculation is tedious, but you should do enough to be sure you are able to correctly evaluate contributions due to individual diagrams.]

3. Consider Yang-Mills theory with a complex scalar field $\phi(x)$ which transforms under representation r of the gauge group. Show that the Feynman rules involving ϕ are a straightforward modification of those from scalar QED (Sheet 3). Compute the β-function for the coupling g. [You should reuse results from Sheet 3 and from earlier this sheet.]

4. Consider a gauge-fixed action for a free Abelian gauge field A^μ of the form

$$S = \int d^d x \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + B\, \partial_\mu A^\mu - \frac{\xi}{2} B^2 - \bar{c}\, \partial^2 c \right]$$

where B is an auxiliary boson field and $(c, \bar{c})$ are anticommuting ghost and antighost fields.

(a) Verify that this action is invariant under the BRST transformation: $\delta A^\mu = \theta \partial^\mu c$, $\delta \bar{c} = \theta B$, and $\delta c = \delta B = 0$. Also verify that the transformation is nilpotent.

(b) Show that the action can be written in the form

$$S = \int d^d x \left[\frac{1}{2} \Phi^T \Delta \Phi - \bar{c}\, \partial^2 c \right]$$

where

$$\Phi = \begin{pmatrix} A^\nu \\ B \end{pmatrix} , \qquad \Delta = \begin{pmatrix} -\partial^2 \delta_{\mu\nu} + \partial_\mu \partial_\nu & -\partial_\nu \\ \partial_\mu & -\xi \end{pmatrix}$$

and Φ^T is the transpose of Φ.

(c) Obtain equations or the normalized correlation functions $\langle \Phi(x) \Phi^T(0) \rangle$ and $\langle c(x) \bar{c}(0) \rangle$ and show that

$$\int d^d x\, e^{-ip\cdot x} \langle \Phi(x) \Phi^T(0) \rangle = \frac{1}{p^2} \begin{pmatrix} \delta_{\mu\nu} - (1-\xi) \frac{p_\mu p_\nu}{p^2} & -ip_\mu \\ ip_\nu & 0 \end{pmatrix}$$

$$\int d^d x\, e^{-ip\cdot x} \langle c(x) \bar{c}(0) \rangle = \frac{1}{p^2} \, .$$

(d) Assuming that $\langle \delta Y \rangle = 0$ for any Y, consider $\langle \delta(\Phi(x) \bar{c}(0)) \rangle$ and show that we must have $\langle B(x) B(0) \rangle = 0$. Obtain a relation between $\langle c(x) \bar{c}(0) \rangle$ and $\langle A^\mu(x) B(0) \rangle$ which should be verified.

Please e-mail me at `M.Wingate@damtp.cam.ac.uk` with any comments, especially any errors.