

MATTHEW B WINGATE

STATISTICAL
FIELD THEORY

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Introduction

The study of Statistical Physics teaches us how to connect microscopic physics, i.e. a description of few-particle dynamics and energies, and macroscopic physics, steady-state or near-equilibrium properties of gases, liquids, or magnetic substances, for example. Statistical field theory further develops these ideas, approximating models of microscopic degrees-of-freedom by continuous fields. The ideas of statistical field theory have a natural overlap with classical and quantum field theories you would see in other contexts.

We will focus much of our effort on describing *critical phenomena*, the physics of second-order phase transitions. When a system undergoes a second-order phase transition, the free energy varies continuously, but its second (or higher) derivatives diverges at the critical point. A remarkable *universality* of these divergences was discovered: many disparate systems exhibit the same behaviour near the critical point. The modern theory of renormalization was developed to explain this universality. There we will see how different microscopic theories can be related to each other, and what features can be predicted to be universal.

These developments in the theory of critical phenomena cleared up many of the mysteries of renormalization as it was first developed in quantum field theories. It goes beyond the application of a perturbative expansion and avoids having to “subtract infinities.” There will be several opportunities for us to draw parallels between material in this course and the QFT and AQFT courses.

This course assumes that you have completed an undergraduate course in Statistical Physics. In practice, however, we rely only on a few general ideas and straightforward calculations from such a course. In the rest of this chapter, we very briefly summarize the main ideas which should either be familiar or which should be studied independently alongside these lectures. In the latter case you might consult standard texts on Statistical Physics available in University and College libraries, as well as the notes covering the Part II Course Statistical Physics (and more) by Tong.¹ Some examples questions of relevance to this course will be provided in Examples Sheet 0.

¹ D Tong. Statistical physics: Part II lecture notes. Tong’s DAMTP website, 2012

1.1 Statistical Mechanics

Statistical mechanics allows us to predict macroscopic properties of systems in or near equilibrium by replacing *time averages* over microscopic dynamics with *ensemble averages* over possible microstates states.

Landau²: distribution function ρ . $\rho_{12} = \rho_1\rho_2$, or taking the logarithm of both side, $\log \rho$'s are additive. Therefore $\log \rho$ must depend linearly on constants of the motion. If we imagine a gas in a finite box, the only constant of the motion is energy.

$$\log \rho \propto E. \quad (1.1)$$

For an isolated system E is conserved, say equal to $E = E_0$ and the distribution function is the microcanonical $\rho(E) = \delta(E - E_0)$. For a system in thermal equilibrium with a heatbath

$$\rho(E) = \frac{1}{Z} e^{-\beta E} \quad (1.2)$$

where Z is a normalization factor known as the *partition function* and β is a constant whose physical meaning we will see is given by the reciprocal of *temperature*.

Once we have the probability distribution function and the Hamiltonian we evaluate the expectation value of some macroscopic observable, say O , using

$$\langle O \rangle = \frac{1}{Z} \int O \rho(E) dE. \quad (1.3)$$

In order to avoid ambiguity, we will write $\langle E \rangle \equiv U$.

Depending on the specific calculation, one might change integration variables from E to the natural microscopic degrees of freedom used in the Hamiltonian. For classical gases, the Hamiltonian depends on the positions \vec{x}_i and momenta \vec{p}_i of the constituent particles $E = H(\{\vec{x}\}, \{\vec{p}\})$. (The curly brackets indicate the Hamiltonian is a function of the whole collection of positions and momenta.)

We will also be interested in systems of magnetic spins σ_i , fixed in position so that $E = H(\{\sigma\})$. Finally, one might consider the energy levels to be discrete, e.g. particles in a box, properly taking into account modifications necessary to describe bosons or fermions at low temperatures.

Entropy S is the logarithm of the statistical weight of a system with energy E . within an ensemble.

$$S = -\langle \log \rho \rangle. \quad (1.4)$$

In the canonical ensemble $S = \beta \langle E \rangle + \log Z$.

$$\frac{\partial S}{\partial U} = \beta = \frac{1}{T}. \quad (1.5)$$

This provides us with a definition of temperature T . (We assume energy units for T , i.e. we set Boltzmann's constant $k_B = 1$.)

² L D Landau and E M Lifshitz. *Statistical Physics, Part 1, 3rd Ed.* Pergemon Press, 1980. ISBN 0-08-023038-5

The free energy is define as

$$F = -T \log Z, \quad (1.6)$$

an expression often written $Z = e^{-\beta F}$. This is sometimes referred to as the Helmholtz free energy when we need to distinguish it from other free energies, e.g. the Gibbs free energy.

1.2 Thermodynamics

Historically the study of thermodynamics predates statistical physics; it took some time before the idea of fluids as collections of atoms or molecules gained support. Here we do not take a historical approach. Nevertheless, we will emphasize throughout the course how macroscopic phenomena depend on microscopic details in some ways, but are independent of them in others. Thus here we review some ideas from thermodynamics, but occasionally making reference to the underlying statistical mechanics.

From (1.6) we see the free energy F is explicitly a function of temperature T . For a gas of particles, it would also depend on volume through the probability density in the partition function. For example, a quantum gas in a finite box has quantized energy levels, with energies dependent on the box lengths. Furthermore if the particles interact, the strength of interactions often depends on the distances between particles, which would have a varying mean as the volume changes. In magnetic systems, the degrees-of-freedom interact with magnetic field. Let us denote the strength of an external magnetic field by h , and assume it is given in energy units. Therefore, we generally would write that F is a function of 3 independent variables

$$F = F(T, V, h). \quad (1.7)$$

Further situations could be considered as well, such as varying particle number N , but let us not do so here.

The responses to variations of each of these independent variables are also of interest, as they are proportional to the entropy S , the pressure p , and the magnetization M :

$$\left. \frac{\partial F}{\partial T} \right|_{V,h} = -S, \quad \left. \frac{\partial F}{\partial V} \right|_{T,h} = -p, \quad \left. \frac{\partial F}{\partial h} \right|_{T,V} = -M. \quad (1.8)$$

These 3 dependent variables are conjugate partners to the 3 independent variables.

One can change variables by means of Legendre transforms. This results in new thermodynamic potentials. For example the internal energy (or average energy, from the statistical point of view) $U = U(S, V, h) = F(T, V, h) + TS$, so that $dU = T dS - p dV - M dh$. Similarly, the Gibbs free energy is defined $G = G(T, p, h) = F(T, V, h) + Vp$. The magnetic Gibbs free energy, $\Gamma = \Gamma(T, V, M) = F(T, V, h) + hM$ will be useful later in this course. Note some texts

prefer an internal energy $\tilde{U} = \tilde{U}(S, V, M) = U(S, V, h) + hM$, although they might not use the tilde.

An equation of state is a relationship between thermodynamic variables. Most commonly in Statistical Physics courses, the equation of state gives the pressure of a gas in terms of its temperature and volume: $p = p(V, T)$. The equation-of-state for an a gas of N noninteracting, classical particles, i.e. and ideal gas, is $pv = T$, where $v = V/N$ is the mean volume-per-particle. In this course we will use magnetic systems for most of our examples; in those cases the equation of state will often take the form $M = M(T, h)$.

We shall largely be interested in systems in *thermal equilibrium*. In a thermally isolated system the dynamics of the system obey the second law of thermodynamics. That is, the entropy tends to increase over time: $\Delta S \geq 0$. Consequently, equilibrium is reached when entropy reaches a maximum. This principle leads to consequences for more general situations. When a fixed-volume system is in contact with a heat reservoir of constant temperature, the Helmholtz free energy F which reaches a minimum as the system reaches stable thermal equilibrium. If instead the system's pressure is kept constant, then it is the Gibbs free energy G which is minimized in equilibrium.³

³ F Reif. *Fundamentals of Statistical and Thermal Physics*. McGraw-Hill, 1965

1.3 Phase transitions

A *phase transition* is a discontinuous or otherwise singular change in the thermodynamic properties of a system as thermodynamic variables are changed. The free energy, or one of its derivatives, is a singular function at that point, or set of points. We will be discussing phase transitions throughout this course. Here we simply make contact with what might have appeared in your undergraduate Statistical Physics course.

A phase transition is said to be first order if the first derivative of the free energy has a discontinuity. This occurs when an instability arises. It can be shown that a condition for stability of the system, under volume perturbations, is that the Gibbs free energy satisfy

$$\left. \frac{\partial^2 G}{\partial V^2} \right|_T > 0. \quad (1.9)$$

This implies

$$\left. \frac{\partial p}{\partial V} \right|_T < 0 \quad (1.10)$$

is required for stability. This makes sense intuitively. Think about a gas in thermal contact with a heat reservoir. If we decrease the size of the container, we would expect the pressure to increase: the gas would push back. This is the stable situation. If the opposite is true the gas would collapse. In fact this is not what occurs, instead a phase transition to a liquid takes place.

Let us look at the van der Waal's equation, a modification of the

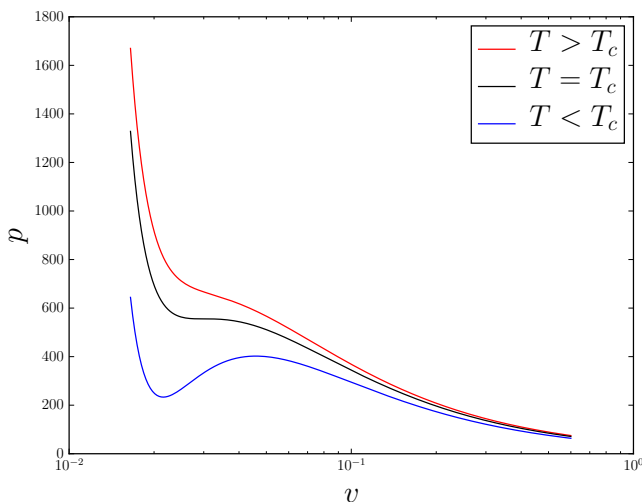


Figure 1.1: Van der Waals isotherms for temperatures above, at, and below the critical temperature. (Arbitrary units, with $(a, b) = (1.5, 0.01)$.)

ideal gas equation of state which reads

$$p = \frac{T}{v - b} - \frac{a}{v^2}. \quad (1.11)$$

The parameters a and b are model parameters tuned for a particular gas in order to account for interactions and finite particle size, respectively. The model is only realistic for $b \ll v$ (equivalent to saying that the mean interparticle separation should be much larger than the linear size of the particles), so it is clear that for large enough T (1.10) holds for all v . As T is lowered, however, the derivative in (1.10) may become nonnegative. In fact, when T reaches the critical value, $T_c = \frac{8a}{27b}$, from above, an instability may develop: $\frac{\partial p}{\partial v} \Big|_T = 0$ when $(v_c, p_c) = \left(3b, \frac{a}{27b^2}\right)$. Figure 1.1 shows 3 isotherms, one of them with $T < T_c$ clearly showing an example an instability signaled by $\frac{\partial p}{\partial v} > 0$.

In fact, when such an instability occurs in the equation-of-state, the system goes through a first-order transition, staying at constant pressure as the volume is increased or decreased. During this transition, there is a mixture of liquid and gas. This can be seen in Figure 1.2, where the data points on the left correspond to the highest densities ρ_g , for a given temperature, at which the system is purely gaseous. On the right, the points indicate the lowest densities at which the system is purely liquid ρ_ℓ . The units in the figure are relative to the critical point (ρ_c, T_c) .

The data show a remarkable universality: the liquid-gas coexistence curves lie on top of each other for a number of gases. It is interesting to consider the difference in densities $\delta\rho = \rho_\ell - \rho_g$, or equivalently the difference in mean volume-per-particle $\delta v = v_g - v_\ell$ (since $v = 1/\rho$). This is an order parameter describing the approach to the critical point. Near $T = T_c$, the curve in Figure 1.2 can be fit to

$$\delta v \sim (T_c - T)^\beta \quad (1.12)$$

where β is a dimensionless *critical exponent*.⁴ From a fit to the data,

⁴ Do not confuse this with the inverse temperature.

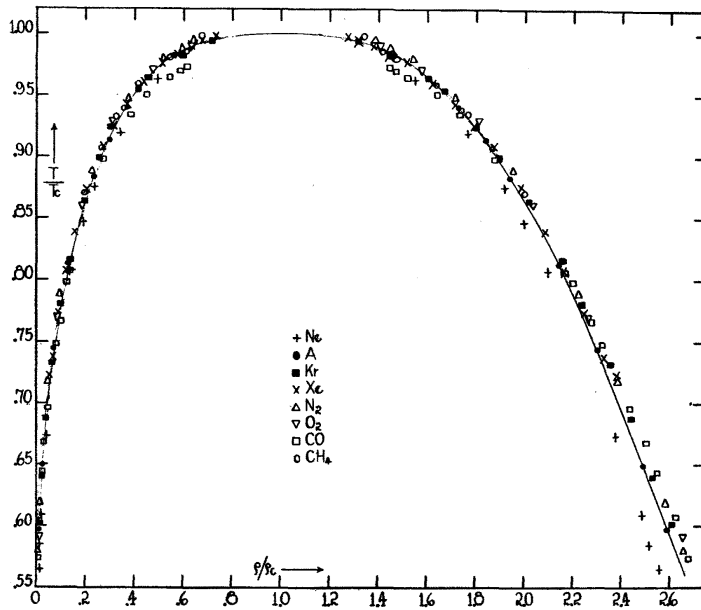


Figure 1.2: Liquid-vapour coexistence curve for 8 systems, with density $\rho = 1/v$ on the horizontal axis and temperature on the vertical axis. Figure reproduced from E A Guggenheim, *J. Chem. Phys.* 13, 253, 1945.

$$\beta \approx \frac{1}{3}.$$

There is a similar situation in magnetic systems, where the order parameter is the magnetization, where

$$M \sim (T_c - T)^\beta. \quad (1.13)$$

We will discuss critical exponents in the context of magnetic systems in more detail in the next chapter.

Landau theory of phase transitions

Landau developed an elegant model for understanding phase transitions. We will present this using the language of magnetic systems, where the order parameter is the magnetization-per-spin m . In terms of the free energy-per-spin, or free energy density, $\mathcal{F} \equiv F/N$, the magnetization is

$$m = - \left. \frac{\partial \mathcal{F}}{\partial h} \right|_T. \quad (2.1)$$

With the number of spins and the temperature held constant, a stable equilibrium state is one in which \mathcal{F} is a global minimum.⁵

To understand phase transitions as Landau did, we look at how the free energy density behaves as the magnetization is varied away from its equilibrium value (2.1). Let \bar{m} be the variable magnetization, which we imagine we can vary independently of h . We write Landau's free energy density as $\mathcal{A}(T, h, \bar{m})$. Let us assume that \mathcal{A} is a polynomial in \bar{m} , e.g.

$$\mathcal{A}(T, h, \bar{m}) = -h\bar{m} + \frac{1}{2}\mathcal{A}_2\bar{m}^2 + \frac{1}{4}\mathcal{A}_4\bar{m}^4 + \frac{1}{6}\mathcal{A}_6\bar{m}^6 + \dots \quad (2.2)$$

For simplicity, in (2.2) we have assumed that, in the absence of an external field (i.e. $h = 0$) the system does not have a preferred direction and therefore is an even function of m , and also that the term linear in h is sufficient to account for a weak, external magnetic field. There are models where cubic or other odd powers of the order parameter can appear in the free energy.⁶ We define our energy scale so that no constant term appears in (2.2). The magnetization of the equilibrium state m is equal to the value of \bar{m} for which \mathcal{A} is a global minimum: $\mathcal{A}(T, h, m) = \mathcal{F}(T, h)$.

The \mathcal{A}_i coefficients in (2.2) are functions of temperature. In fact, Landau made the assumption that they are all *analytic* in $T - T_c$. Although this is ultimately incorrect, one still develops a qualitatively useful description of phase transitions. In particular, we assume that the critical temperature is defined the point where the coefficient of m^2 vanishes linearly: as $T \rightarrow T_c^-$,

$$\mathcal{A}_2 \sim T - T_c. \quad (2.3)$$

Let us consider two cases described by (2.2) with $h = 0$. Neglecting terms higher order than m^6 , let us assume $\mathcal{A}_6 > 0$ for all temperatures.

⁵ Local minima of \mathcal{F} correspond to *metastable* states; they can exist for finite amounts of time, but given enough time, thermal fluctuations should drive the system to a global minimum of \mathcal{F} .

⁶ As an exercise, you may wish to consider how the discussion below would be altered by the addition of a term $\frac{1}{3}\mathcal{A}_3\bar{m}^3$.

The case $\mathcal{A}_4 > 0$ gives rise to a second order, or continuous, phase transition: as the temperature is lowered (usually) \mathcal{A}_2 decreases from a positive value to zero when $T = T_c$. For $T < T_c$, $\mathcal{A}_2 < 0$ and there are two degenerate minima at $m = \pm m_0 \neq 0$, as shown in Figure 2.1. The degeneracy is broken by any external magnetic field, however small. For example, if $h = \epsilon > 0$ then the $m = m_0 > 0$ state will be stable and the $m = -m_0$ state will only be metastable as it is just a local minimum of \mathcal{A} , not the global minimum.

Neglecting terms higher order than \bar{m}^4 , we can obtain the equation of state from \mathcal{A}

$$\left. \frac{\partial \mathcal{A}}{\partial \bar{m}} \right|_{\bar{m}=m} = 0 \implies m(\mathcal{A}_2 + m^2 \mathcal{A}_4) = h. \quad (2.4)$$

For $h = 0$, the solutions to the equation of state are $m = 0$ and $m = \pm \sqrt{-\frac{\mathcal{A}_2}{\mathcal{A}_4}}$. The first solution is only a solution for $\mathcal{A}_2 > 0$ when the second derivative of \mathcal{A} is positive at $m = 0$. The other solutions are only real when $\mathcal{A}_2 \leq 0$. From (2.3) that, as $T \rightarrow T_c^-$,

$$m \sim (T_c - T)^\beta, \quad \text{with } \beta = \frac{1}{2}. \quad (2.5)$$

This β is one of several *critical exponents*, and its symbol is by convention the same as that used for the inverse temperature; the fact that the critical exponent β is dimensionless and the inverse temperature β has units of inverse energy should be enough context to avoid confusion.

Looking at the equation of state for $T = T_c$, i.e. for $\mathcal{A}_2 = 0$, (2.4) reads $\mathcal{A}_4 m^3 = h$, or as $h \rightarrow 0$,

$$m \sim |h|^{\frac{1}{\delta}}, \quad \text{with } \delta = 3 \quad (2.6)$$

which defines the critical exponent δ . The magnetic susceptibility also has a corresponding critical exponent, γ . For $T > T_c$ we can expand about $m = 0$, neglecting the m^3 term in (2.4) to find

$$\chi = \left. \frac{\partial m}{\partial h} \right|_{h=0} \sim (T - T_c)^\gamma, \quad \text{with } \gamma = 1. \quad (2.7)$$

One finds the same critical exponent for the susceptibility for $T < T_c$, but then one should expand about $m = m_0$.

If $\mathcal{A}_4 < 0$ then the system can have a first order phase transition. This is seen in $\mathcal{A}(T, h, \bar{m})$ when there are minima at $m = 0$ and $m = m_0$ (Figure 2.2). With the normalization of (2.2) such that $\mathcal{A} = 0$ for $m = 0$, we find the ordered phase occurs when there are nonzero solutions to

$$\bar{m}^2 (\mathcal{A}_2 - \frac{1}{2} |\mathcal{A}_4| \bar{m}^2 + \frac{1}{3} \mathcal{A}_6 \bar{m}^4) = 0.$$

In other words, when the discriminant of the quadratic (in \bar{m}^2) term is positive, i.e. $\frac{1}{4} \mathcal{A}_4^2 - \frac{4}{3} \mathcal{A}_2 \mathcal{A}_6 > 0$, there are nonzero solutions and the ordered phase is preferred. The phase transition occurs

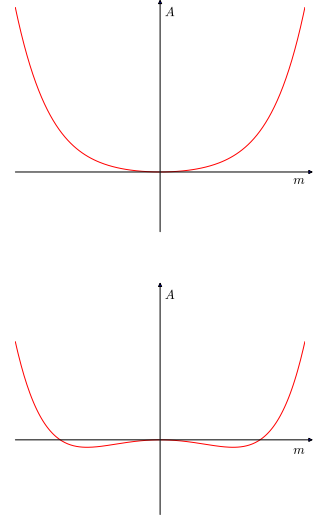


Figure 2.1: Free energy function $\mathcal{A}(\bar{m})$ above and below a continuous phase transition.

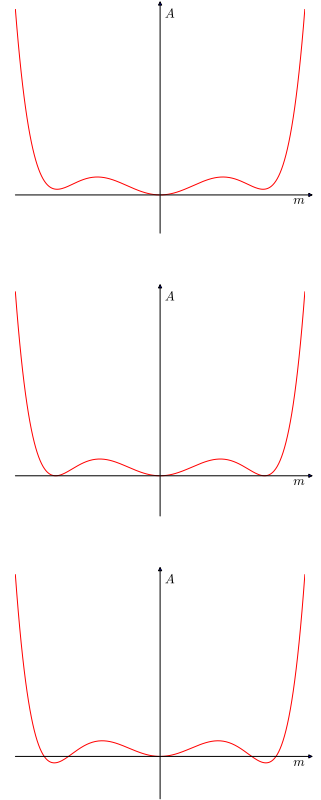


Figure 2.2: Free energy function $\mathcal{A}(\bar{m})$ as coefficients vary across a first order phase transition. In the top figure the minimum corresponds to the disordered phase, in the bottom figure the ordered phase is preferred, and at the phase transition (middle figure) the ordered and disordered phases can coexist.

when $\frac{1}{4}A_4^2 - \frac{4}{3}A_2A_6 = 0$, and the order parameter varies *discontinuously* across the transition. The coexistence of different phases at the phase transition is a feature unique to first order phase transitions.

The coefficients A_2, A_4 , etc. of (2.2) are functions of the independent variables such as temperature T . Let us also suppose there is another variable g . Then a *phase diagram* is the simplest way to summarize the state of a system as a function of T and g . One possible situation is shown in Figure 2.3. There, for $g = 0$, the system has a second order phase transition, corresponding to the case where $A_4 > 0$. As g is increased from 0, the critical temperature T_c is a function of g . Along the line described by $T_c(g)$, the coefficient $A_2 = 0$, while $A_4 > 0$.

In the situation depicted, A_4 is a function of g so that when g becomes large enough and $A_4 = 0$. This point in the phase diagram is known as a *tricritical point* (TCP). Beyond this point, we have the case where $A_4 < 0$ and we have a line of first order transitions, $T_1(g)$. Along this line, $\frac{1}{4}A_4^2 - \frac{4}{3}A_2A_6 = 0$, as discussed above. (Note that the $A_2 = 0$ line will naturally continue through the TCP, but it will lie in the phase where the order has already been broken by the first order transition.

Critical exponents take on different values at a TCP than at an ordinary second order transition (sometimes referred to as a critical point, CP). With $A_4 = 0$ we cannot neglect the \bar{m}^6 term in determining δ , for example. The $T = T_c$ equation of state is $m^5 = h$, therefore $\delta = 5$. Critical exponents for critical points and tricritical points are summarized in Table 2.1.

Landau's phenomenological theory of phase transitions is a very good starting point for thinking about phase transitions. In particular, it shows that critical exponents could be the same for wide varieties of microscopic theories, as long as they result in similar free energies \mathcal{A} . By making a few assumptions about how the coefficients A_i depend on temperature (and any other independent variables) one can make numerical predictions for the critical exponents. It turns out that the numerical predictions are often the incorrect values to describe experimental data (but not in all cases). We will see that Landau's assumptions about analyticity of the free energy coefficients is the culprit, and we will develop a framework for understanding why.

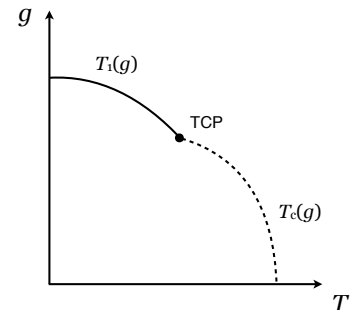


Figure 2.3: Phase diagram showing a tricritical point (TCP). The solid line depicts a first order transition and the dashed line, a second order transition.

Table 2.1: Critical exponents at a critical point (CP) and at a tricritical point (TCP) according to Landau theory.

exponent	definition	CP	TCP
α	$\mathcal{A} \sim t ^{2-\alpha}$	0	$\frac{1}{2}$
β	$m \sim (-t)^\beta$	$\frac{1}{2}$	$\frac{1}{4}$
γ	$\chi \sim t ^{-\gamma}$	1	1
δ	$m \sim h^{1/\delta}$	3	5

Ising model

Some solids, such as iron and nickel, can be magnetized. Atomic magnetic moments align with each other, even in the absence of an external magnetic field. This property is called ferromagnetism. If such a solid is heated above its so-called Curie temperature a phase transition occurs, and the magnetization is lost and they become paramagnetic.

The Ising model is a very simplified model of the ferromagnetic-paramagnetic phase transition, yet it is very instructive. The degrees-of-freedom $\sigma_i = \pm 1$, on $i = 1, \dots, N$ sites, these are interpreted as spins either aligned or anti-aligned in some fixed direction. The spins are located on fixed lattice points; we assume a square-lattice in D dimensions, but one could consider a triangular 2D Ising model, for example. The Hamiltonian is written as

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i \quad (3.1)$$

where $\sum_{\langle ij \rangle}$ means to sum over every pair of nearest-neighbor spins. J is the exchange energy and h represents the external magnetic field (in energy units). For the ferromagnetic model the exchange energy $J > 0$: the energy is lowered by having neighboring spins aligned. Negative J yields the anti-ferromagnetic model, which we will not consider here. The external magnetic field h can have either sign.

We will study the statistical physics of the Ising model. The partition function can be written

$$Z = \sum_{\sigma} \exp(-\beta H) \quad (3.2)$$

where the sum is over all configurations of the N spins and could be written more verbosely as

$$\sum_{\{\sigma_j\}} \text{ or } \sum_{\sigma_j = \pm 1, \forall_j} .$$

3.1 Solution of one-dimensional model

In one dimension the Ising model partition function is

$$Z = \sum_{\sigma_j = \pm 1, \forall_j} \prod_{i=1}^{N-1} \exp \beta \left[J \sigma_i \sigma_{i+1} + \frac{h}{2} (\sigma_i + \sigma_{i+1}) \right] \quad (3.3)$$

Assume periodic boundary conditions, i.e. $\sigma_N = \sigma_0$. Define

$$W_{\sigma_i \sigma_j} = \exp \beta \left[J \sigma_i \sigma_j + \frac{h}{2} (\sigma_i + \sigma_j) \right] \quad (3.4)$$

then

$$Z = \sum_{\sigma_j = \pm 1, \forall j} W_{\sigma_0 \sigma_1} W_{\sigma_1 \sigma_2} \cdots W_{\sigma_{N-1} \sigma_0} \quad (3.5)$$

This expression is of the form of a trace of a product of matrices, $\text{Tr } ABC = \sum_{i,j,k} A_{ij} B_{jk} C_{ki}$. For the Ising model the 4 possible $W_{\sigma_i \sigma_j}$ make up a 2×2 transfer matrix

$$W = \begin{pmatrix} e^{\beta J} e^{\beta h} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} e^{-\beta h} \end{pmatrix} \quad (3.6)$$

The eigenvalues of W are straightforward to find; let us denote them

$$\lambda_{\pm} = e^{\beta J} \cosh \beta h \pm \sqrt{e^{2\beta J} \sinh^2 \beta h + e^{-2\beta J}}. \quad (3.7)$$

Noting that $\lambda_+ > \lambda_-$, in the large N limit we can express the partition function as

$$Z = \text{Tr } W^N = \lambda_+^N + \lambda_-^N = \lambda_+^N \left[1 + \left(\frac{\lambda_-}{\lambda_+} \right)^N \right] \approx \lambda_+^N. \quad (3.8)$$

Free energy per spin

$$\begin{aligned} f &= \frac{F}{N} = -\frac{1}{N\beta} \log Z \\ &= -T \log \lambda_+ \\ &= -T \log \left[e^{\beta J} \cosh \beta h + \sqrt{e^{2\beta J} \sinh^2 \beta h + e^{-2\beta J}} \right] \end{aligned} \quad (3.9)$$

As $T \rightarrow 0$ ($\beta \rightarrow \infty$) we can neglect the smaller of $\exp(\pm \beta h)$ in the sinh and cosh and the $\exp(-2\beta J)$ in the square root to find (written here for the case $h \geq 0$)

$$\lim_{T \rightarrow 0} f = -\lim_{T \rightarrow 0} T \log e^{J/T} e^{h/T} = -J - h. \quad (3.10)$$

We see from this that at $T = 0$ all spins are aligned and that f approaches its minimum value continuously; there is no finite T phase transition.

The magnetization (per spin)

$$\begin{aligned} m &= \frac{1}{N} \left\langle \sum_i \sigma_i \right\rangle \\ &= \frac{1}{NZ} \sum_{\sigma} \sum_i \sigma_i e^{\beta J \sum_{(ij)} \sigma_i \sigma_j + \beta h \sum_i \sigma_i} \\ &= \frac{1}{NZ} \sum_{\sigma} \frac{1}{\beta} \frac{\partial}{\partial h} e^{\beta J \sum_{(ij)} \sigma_i \sigma_j + \beta h \sum_i \sigma_i} \\ &= \frac{1}{N\beta} \frac{\partial}{\partial h} \log Z \\ &= T \frac{\partial}{\partial h} \log \lambda_+ \\ &= \frac{e^{\beta J} \sinh \beta h}{\sqrt{e^{2\beta J} \sinh^2 \beta h + e^{-2\beta J}}}. \end{aligned} \quad (3.11)$$

The expression simplifies in the high T limit (keeping J fixed, but leaving h arbitrary): $m \rightarrow \tanh \beta h$. As T is lowered, $m(h)$ approaches a step function. If one first takes the $T \rightarrow 0$ limit before taking $h \rightarrow 0$ we find

$$\lim_{h \rightarrow 0^\pm} \lim_{T \rightarrow 0} m = \frac{\sinh \beta h}{\sqrt{\sinh^2 \beta h}} = \pm 1. \quad (3.12)$$

Thus we find that, for $h = 0$, $m = 0$ unless we first take the zero temperature limit. The $D = 1$ Ising model is paramagnetic at all finite temperatures, but has a phase transition at $T = 0$ to a ferromagnetic ground state.

Let us find the same result using the transfer matrix. We can straightforwardly diagonalize W . Since W is symmetric, the diagonalization is done via a rotation matrix P

$$W = P \Lambda P^{-1} \quad (3.13)$$

where

$$\Lambda = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \quad \text{and} \quad W \vec{e}_\pm = \lambda_\pm \vec{e}_\pm \quad (3.14)$$

The eigenvectors, and hence the rotation matrix, can be found explicitly

$$P = (\vec{e}_+, \vec{e}_-) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \quad (3.15)$$

however it is quicker to observe that $\cot 2\phi = e^{2\beta J} \sinh \beta h$.⁷

Define

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (3.16)$$

then

$$\begin{aligned} m &= \langle \sigma_i \rangle \\ &= \frac{1}{Z} \sum_{\sigma} W_{\sigma_0 \sigma_1} \cdots W_{\sigma_{i-1} \sigma_i} W_{\sigma_i \sigma_{i+1}} \cdots W_{\sigma_{N-1} \sigma_0} \\ &= \frac{\text{Tr} S W^N}{\text{Tr} W^N}. \end{aligned} \quad (3.17)$$

After diagonalizing

$$\begin{aligned} m &= \frac{\text{Tr} P^{-1} S P \Lambda^N}{\text{Tr} \Lambda^N} \\ &= \frac{\cos 2\phi (\lambda_+^N - \lambda_-^N)}{\lambda_+^N + \lambda_-^N}. \end{aligned} \quad (3.18)$$

In the large N limit

$$m = \cos 2\phi = \frac{e^{\beta J} \sinh \beta h}{\sqrt{e^{2\beta J} \sinh^2 \beta h + e^{-2\beta J}}} \quad (3.19)$$

as we found before.

⁷ Exercise: Prove that the angle ϕ which rotates from the standard basis to the eigenvector basis of a symmetric matrix $W = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$, satisfies $\cot 2\phi = \frac{A-C}{2B}$.

The transfer matrix method generalizes nicely to allow us to calculate more complicated expectation values. Let us look at the *connected* Green's function or correlation function

$$G(r) = \langle \sigma_0 \sigma_r \rangle_c = \langle \sigma_0 \sigma_r \rangle - \langle \sigma_0 \rangle \langle \sigma_r \rangle. \quad (3.20)$$

The word “connected” has its origins in Feynman diagrams, which will be encountered later (possibly) and in QFT.

$$\begin{aligned} \langle \sigma_0 \sigma_r \rangle &= \frac{1}{Z} \sum_{\sigma} \sigma_0 W_{\sigma_0 \sigma_1} \cdots W_{\sigma_{r-1} \sigma_r} \sigma_r W_{\sigma_r \sigma_{r+1}} \cdots W_{\sigma_{N-1} \sigma_0} \\ &= \frac{\text{Tr} S W^r S W^{N-r}}{\text{Tr} W^N} \\ &= \frac{\text{Tr} P^{-1} S P \Lambda^r P^{-1} S P \Lambda^{N-r}}{\lambda_+} + \dots \\ &= \cos^2 2\phi + \sin^2 2\phi \left(\frac{\lambda_-}{\lambda_+} \right)^r. \end{aligned} \quad (3.21)$$

Thus

$$G(r) = \sin^2 2\phi \left(\frac{\lambda_-}{\lambda_+} \right)^r.$$

Define

$$\zeta = \frac{1}{\log(\lambda_+/\lambda_-)}. \quad (3.22)$$

For $h = 0$,

$$\frac{\lambda_+}{\lambda_-} = \frac{1}{\tanh \beta J} \Rightarrow \zeta \sim e^{2\beta J} = e^{2J/T} \quad (3.23)$$

which diverges as $T \rightarrow 0$, showing that there indeed is a 2nd order phase transition at $T = 0$. The Green's function behaves like

$$G(r) = \sin^2 2\phi e^{-r/\zeta} \quad (3.24)$$

so for $r \gg \zeta$ the spin degrees-of-freedom are uncorrelated. Note that as ζ diverges, we see the onset of long-range order.

As an exercise, use (3.11) and translational invariance to show that the magnetic susceptibility per spin can be written

$$\chi = \left. \frac{\partial m}{\partial h} \right|_T = \beta \sum_r G(r). \quad (3.25)$$

3.2 Mean field theory

Consider the Ising model in D dimensions, with Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i. \quad (3.26)$$

Once again, the first sum is over all nearest-neighbour pairs. Let us denote by q the number of nearest neighbors any given site has; e.g. for sites on a square/cubic/hypercubic lattice, each site has $q = 2D$ nearest neighbours.

Let m be the mean magnetization, and denote the difference from the mean of the spin at site i by

$$\delta\sigma_i = \sigma_i - m. \quad (3.27)$$

Then the interaction term in H can be rewritten

$$\begin{aligned}\sigma_i \sigma_j &= (m + \delta\sigma_i)(m + \delta\sigma_j) \\ &= m^2 + m(\delta\sigma_i + \delta\sigma_j) + \delta\sigma_i \delta\sigma_j \\ &= -m^2 + m(\sigma_i + \sigma_j) + \delta\sigma_i \delta\sigma_j.\end{aligned}\quad (3.28)$$

The mean field approximation is made by neglecting the last term, resulting in the mean-field Ising Hamiltonian⁸

$$H = \frac{1}{2}qJNm^2 - (qJm + h) \sum_i \sigma_i. \quad (3.29)$$

Hence the partition function can be solved in the MFT approximation

$$\begin{aligned}Z &= e^{-\beta qJNm^2/2} \sum_{\sigma} e^{\beta(qJm+h) \sum_i \sigma_i} \\ &= e^{-\beta qJNm^2/2} [2 \cosh \beta(qJm + h)]^N.\end{aligned}\quad (3.30)$$

We may similarly obtain an expression for the magnetization

$$\begin{aligned}m = \langle \sigma_j \rangle &= \frac{\sum_{\sigma_j=\pm 1} \sigma_j e^{\beta(qJm+h)\sigma_j}}{\sum_{\sigma_j=\pm 1} e^{\beta(qJm+h)\sigma_j}} \\ &= \tanh \beta(qJm + h).\end{aligned}\quad (3.31)$$

In order to investigate solutions to this transcendental equation, let us rearrange it as

$$\tanh^{-1} m = \frac{qJ}{T} m + \frac{h}{T} \quad (3.32)$$

noting that the right-hand side (RHS) is the equation of a line with gradient qJ/T and y -intercept h/T .

First considering the $h = 0$ case, we see $m = 0$ is always a real solution. Since

$$\frac{d}{dm} \tanh^{-1} m = \frac{1}{1-m^2} \geq 1 \quad \text{for } -1 < m < 1 \quad (3.33)$$

we immediately see that if $qJ/T < 1$, there will be no other intersections of the RHS of (3.32) with the LHS (e.g. see Fig. 3.1). For temperatures such that $qJ/T > 1$ there are 2 nonzero solutions; let us denote these as $m = \pm m_0$. We will later discuss the stability of these solutions, and it turns out that the physical solutions are

$$m = \begin{cases} \pm m_0 & \text{for } T < T_c \\ 0 & \text{for } T > T_c \end{cases} \quad (3.34)$$

where $T_c = qJ/k$.

In the $h \neq 0$ case, there is always a solution with $m \neq 0$ for all temperatures (Fig. 3.2). Therefore, there is no phase transition to a paramagnetic state. As T decreases from large to small, the physical solution moves smoothly from small, but finite m to large m , with the same sign as h . At some point solutions with m having the opposite sign to h appear, but we will see later that these are not realized physically.

⁸ Up to the σ -independent term, this could have been derived just by replacing $\sum_{\langle ij \rangle} \sigma_i \sigma_j \rightarrow qm \sum_i \sigma_i$ in (3.26). This term is needed to get the correct free energy as a function of m . However, this term is not needed for mean-field expectation values or for computations involving derivatives with respect to σ_j .

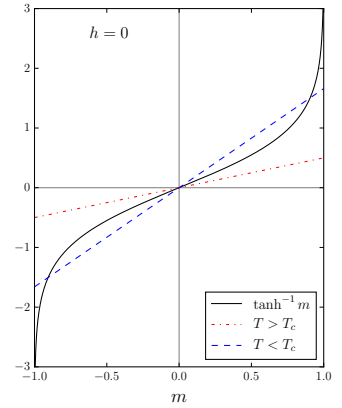


Figure 3.1: Graphical solution of mean field magnetization, $h = 0$ case.

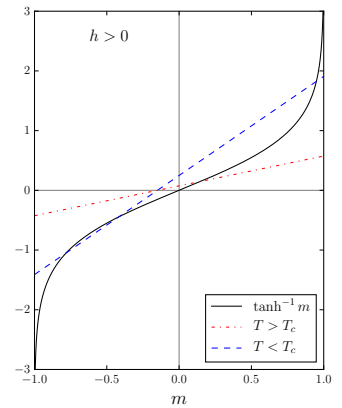


Figure 3.2: Graphical solution of mean field magnetization, $h \neq 0$ case.

Let us investigate critical behaviour which occurs in the $h \rightarrow 0$ limit. Then (3.32) is given by

$$\tanh^{-1} m = \frac{T_c}{T} m.$$

Expanding about small m and using $T/T_c = 1 + t$ we find

$$\begin{aligned} m + \frac{1}{3}m^3 &= \frac{m}{1+t} \\ m(3t + m^2) &= 0 \end{aligned}$$

where in obtaining the first line we neglect terms of $O(m^5)$ and in the second we neglect another small term of $O(tm^3)$.

$$m = \begin{cases} \pm\sqrt{-3t} & \text{for } T < T_c \\ 0 & \text{for } T > T_c \end{cases}. \quad (3.35)$$

Approaching T_c from the $T < T_c$, the magnetization behaves as

$$m \sim \pm(T - T_c)^{\frac{1}{2}}. \quad (3.36)$$

The singularity in $\frac{dm}{dT}$ as $T \rightarrow T_c^-$ indicates a phase transition.

Let us examine the magnetic susceptibility at $h = 0$. Define

$$\chi = \left. \frac{\partial m}{\partial h} \right|_{h=0}. \quad (3.37)$$

Now

$$\begin{aligned} \tanh^{-1} m &= \frac{T_c}{T} m + \frac{h}{T} \\ (1+t) \tanh^{-1} m &= m + \frac{h}{T_c}. \end{aligned} \quad (3.38)$$

For $T > T_c$ the magnetization is very small, so we can approximate $\tanh^{-1} m \approx m$ in (3.38) and find

$$m = \frac{h}{T - T_c} \implies \chi \sim (T - T_c)^{-1}. \quad (3.39)$$

For $T < T_c$, the magnetization is very close to the zero-field magnetization $m_0 = \pm\sqrt{-3t}$, so let $m = m_0 + \epsilon$. Eq. (3.38) is then

$$(1+t) \left[m_0 + \epsilon + \frac{1}{3}(m_0 + \epsilon)^3 + \dots \right] = m_0 + \epsilon + \frac{h}{T_c}$$

which implies

$$\epsilon = -\frac{h}{2(T - T_c)} \implies \chi = \frac{1}{2(T_c - T)} \sim (T_c - T)^{-1}. \quad (3.40)$$

Set $T = T_c$ and ask how m vanishes as $h \rightarrow 0$. Using the fact that m vanishes at the critical temperature, eq. (3.38) implies

$$m = \left(\frac{3h}{T_c} \right)^{\frac{1}{3}} \sim h^{\frac{1}{3}}. \quad (3.41)$$

The comparison between mean field theory and exact (or numerical) results is not very encouraging. Define the critical exponents β , γ , δ as follows

$$\begin{aligned} m &\sim (T_c - T)^\beta \text{ for } T < T_c \\ \chi &\sim |T - T_c|^{-\gamma} \\ m &\sim h^{\frac{1}{\delta}}. \end{aligned} \tag{3.42}$$

As shown in Table 3.2, the mean field results are not very close for $D = 2$, but we observe they are a bit better for $D = 3$. By the end of the course we will see in more detail why mean field theory works better in larger numbers of dimensions. In the meantime, we note that we will wish to go beyond mean field theory in order to understand critical exponents in 2, and even 3, dimensions. We cannot neglect the effects of fluctuations in the way done here.

Table 3.2: Comparison of critical exponents and temperature.

	MFT	$D = 2$ Ising	$D = 3$ Ising
β	$\frac{1}{2}$	$\frac{1}{8}$	0.3...
γ	1	$\frac{7}{4}$	1.2...
δ	3	15	5.0...
T_c	$4J$	$2.27J$	

Scaling

Landau's assumption that the free energy is a polynomial in the order parameter, with coefficients analytic in $T - T_c$ leads to the correct power law behaviour, albeit with generally incorrect critical exponents (below some upper critical dimension). In this chapter, we first derive the mean field exponents from Landau's free energy using simple scaling arguments. We then relax Landau's analyticity assumption and show the effect on the scaling analysis and the critical exponents.

For this discussion we focus only on second order transitions, such that $\mathcal{A}_4 > 0$ in (2.2) and we may neglect higher order terms. The steps in this chapter may be repeated, keeping the $\mathcal{A}_6 \bar{m}^6$ term, to study the critical exponents near a tricritical point.

4.1 Mean field

Consider a scaling transformation, where we scale our basic length unit, the lattice spacing a , by a factor b

$$a \rightarrow a' = ba \quad (4.1)$$

while keeping physical, macroscopic quantities such as the free energy or the system size fixed. The requirement that a physical length L remain fixed means that the number of degrees will change,⁹ e.g. in 1 dimension

$$L = Na = N'a'.$$

In D dimensions we look at the number of spins in a fixed volume

$$L^D = Na^D = N'a'^D. \quad (4.2)$$

While the free energy remains constant under this scaling, the free energy per spin transforms as

$$\mathcal{F} \rightarrow \mathcal{F}' = b^D \mathcal{F}. \quad (4.3)$$

and equivalently, $\mathcal{A} \rightarrow \mathcal{A}' = b^D \mathcal{A}$. We say that the *scaling dimension* of \mathcal{F} (and \mathcal{A}) is $-D$. The sign is just a convention.

Let us introduce some simplifying notation. We use square brackets to denote scaling dimension: $[Q] = q$ implies the quantity Q has scaling dimension q , i.e.

$$[Q] = q \implies Q \rightarrow Q' = b^{-q} Q. \quad (4.4)$$

⁹ Later we will see explicit examples where we thin out the degrees-of-freedom by neglecting every other spin. If the correlation length is much larger than the lattice spacing, then neighboring degrees-of-freedom are highly correlated and we can use a single spin as a representative of its neighbors.

Suppose the magnetization (our order parameter) scales under the transformation (4.1) with dimension d_m : $[\bar{m}] = d_m$. Together, $[\mathcal{A}] = -D$ and $[\bar{m}] = d_m$ implies that a term $\mathcal{A}_n \bar{m}^n$ appearing in (2.2) has a coefficient with scaling dimension

$$[\mathcal{A}_n] = -(D + nd_m). \quad (4.5)$$

Similarly $[h] = -D - d_m$. In mathematical terms, the underlying assumption behind this scaling hypothesis is that $\mathcal{A}(\bar{m})$ is a homogenous function.

In order to find illuminating *scaling forms* for the free energy \mathcal{A} we will carry out a standard dimensional analysis, using scaling dimensions. First we find a combination of h , \mathcal{A}_2 , and \mathcal{A}_4 which is invariant under (4.1). Let us raise \mathcal{A}_2 and \mathcal{A}_4 to powers p_2 and p_4 , respectively. Then

$$\left[h \mathcal{A}_2^{p_2} \mathcal{A}_4^{p_4} \right] = -D(1 + p_2 + p_4) - d_m(1 + 2p_2 + 4p_4). \quad (4.6)$$

Requiring the scaling dimension of $h \mathcal{A}_2^{p_2} \mathcal{A}_4^{p_4}$ to be equal to 0 for general D and d_m implies $p_2 = -\frac{3}{2}$ and $p_4 = \frac{1}{2}$.

In Landau theory, at a second order transition $\mathcal{A}_2 \sim t = \frac{(T-T_c)}{T_c}$ (2.3) while \mathcal{A}_4 remains positive and, we assume, slowly varying in T . Therefore, in Landau's approximation, the scale invariant combination of interest is

$$\frac{h}{|t|^{3/2}}. \quad (4.7)$$

We introduce the absolute value signs since we will treat the $T \rightarrow T_c^-$ and $T \rightarrow T_c^+$ cases separately, defining different sets of functions.

We will be interested in approaching the critical point $(t, h) = (0, 0)$ along two different lines: 1. setting $h = 0$ first and studying $t \rightarrow 0$; 2. setting $t = 0$ and then $h \rightarrow 0$.

1. In studying singularities as $t \rightarrow 0$ with the intention of taking $h \rightarrow 0$ first, there will be some unknown function which depends on the scale invariant ratio (4.7). To see how the free energy behaves near the critical point we need a quantity with scaling dimension D . Using only \mathcal{A}_2 and \mathcal{A}_4 (since we need something useful when $h = 0$) we find

$$\left[\frac{|\mathcal{A}_2|^2}{\mathcal{A}_4} \right] = -D.$$

Near $t = 0$, $\mathcal{A}_2 \sim |t|$ while \mathcal{A}_4 is assumed to be slowly varying, therefore the scaling hypothesis implies the equilibrium free energy vanishes quadratically as $t \rightarrow 0$, holding the ratio $h/|t|^{3/2}$ fixed; i.e.

$$\mathcal{F} = c_1 |t|^2 f_{\gtrless} \left(\frac{h}{|t|^{3/2}} \right) \quad (4.8)$$

with $f_{<}(0) = f_{>}(0)$ and c_1 a constant.

2. Studying singularities as $h \rightarrow 0$, perhaps with $t = 0$, then it is sensible to use the reciprocal of (4.7) as the independent scale-invariant ratio. At $t = 0$ only the following has scaling dimension $-D$

$$\left[\frac{h^{4/3}}{\mathcal{A}_4^{1/3}} \right] = -D.$$

Therefore, the scaling form useful to study the $h \rightarrow 0$ critical behaviour is

$$\mathcal{F} = c_2 h^{\frac{4}{3}} g_{\geq} \left(\frac{|t|^{3/2}}{h} \right) \quad (4.9)$$

again with $g_{<}(0) = g_{>}(0)$ and c_2 a constant.

Note that the exponents of t and h in (4.8) and (4.9) are completely determined from the scaling hypothesis and should therefore be universal. On the other hand the constants c_1 and c_2 , and the functions f_{\geq} and g_{\geq} are nonuniversal.

We can obtain mean field exponents from the scaling forms (4.8) and (4.9).

$$m = - \left. \frac{\partial \mathcal{F}}{\partial h} \right|_{h \rightarrow 0^+} \sim |t|^2 \frac{1}{|t|^{3/2}} f'_{\geq}(0^+)$$

implies $m \sim |t|^\beta$ with $\beta = \frac{1}{2}$ for the ordered phase, when $t < 0$ (i.e. $f'_{<}(0^+) > 0$). Of course $m = 0$ for the disordered phase, when $t > 0$ and therefore $f'_{>}(0^+) = 0$. The susceptibility

$$\chi = \frac{\partial m}{\partial h} \sim |t|^{-\gamma}$$

with $\gamma = 1$. Finally, the specific heat

$$C = -T \left. \frac{\partial^2 \mathcal{F}}{\partial T^2} \right|_{h=0} \sim t^\alpha$$

with $\alpha = 0$.

4.2 Departure from mean field

In fact the mean field scaling form of § 4.1 does not fit experimental or numerical results in most cases. However, one does find agreement with data if we relax the values of the exponents we inferred from dimensional analysis, so that

$$\mathcal{F} = |t|^{2-\alpha} f_{\geq} \left(\frac{h}{|t|^\Delta} \right) \quad (4.10)$$

(we also absorbed any dimensionful constant into the function f_{\geq}). The critical exponents now differ from their mean field values.

$$\begin{aligned} m &= - \left. \frac{\partial \mathcal{F}}{\partial h} \right|_{h \rightarrow 0^+} \sim |t|^{2-\alpha-\Delta} \implies \beta = 2 - \alpha - \Delta \\ \chi|_{h=0} &= \left. \frac{\partial m}{\partial h} \right|_{h \rightarrow 0^+} \sim |t|^{2-\alpha-2\Delta} \implies \gamma = -2 + \alpha + 2\Delta. \end{aligned}$$

We can rearrange the expressions above to obtain

$$\alpha + 2\beta + \gamma = 2 \quad (4.11)$$

$$\beta + \gamma = \Delta. \quad (4.12)$$

The first of these expressions is known as Rushbrooke's identity since it satisfies the inequality derived using only thermodynamic arguments.¹⁰

To find δ , we again need to focus on how \mathcal{F} behaves as $h \rightarrow 0$ after the $t \rightarrow 0$ limit has already been taken. Looking at the magnetization in the ordered phase, $t < 0$, we see

$$\begin{aligned} m &= |t|^\beta f'_<\left(\frac{h}{|t|^\Delta}\right) \\ &= h^{\frac{\beta}{\Delta}} \left(\frac{|t|^\Delta}{h}\right)^{\frac{\beta}{\Delta}} f'_<\left(\frac{h}{|t|^\Delta}\right) \\ &\equiv h^{\frac{\beta}{\Delta}} \phi\left(\frac{|t|^\Delta}{h}\right) \end{aligned} \quad (4.13)$$

where the last line implicitly defines the new function ϕ . For $t = 0$ and assuming $\phi(0)$ is finite, we see $m \sim h^{\beta/\Delta}$, and we conclude that

$$\delta = \frac{\Delta}{\beta} = 1 + \frac{\gamma}{\beta}. \quad (4.14)$$

This is known as Widom's identity.

The exponents α and Δ will be determined using renormalization group theory. More precisely, we will determine the scaling dimensions y_t and y_h , defined such that

$$[t] = -y_t \quad \text{and} \quad [h] = -y_h. \quad (4.15)$$

With these definitions

$$[\mathcal{F}] = -D \quad \implies \quad 2 - \alpha = \frac{D}{y_t} \quad (4.16)$$

$$\left[\frac{h}{|t|^\Delta}\right] = 0 \quad \implies \quad \Delta = \frac{y_h}{y_t}. \quad (4.17)$$

Further reading

Arguments similar to those presented in § 4.1 above are given by Goldenfeld¹¹ in his Chapter 7, including a brief explanation (at the same level as here) of how to understand the failure of naive dimensional analysis when fluctuations become important. Section 4.1 of Kardar¹² gives a nice, alternative derivation of the scaling form beyond mean field theory based on the assumption that the free energy is a homogeneous function. This was the original approach taken by Widom. The scaling laws are also derived in §§ 8.1-2 of Goldenfeld.

¹⁰ C.f. examples sheet. The observation that this and other inequalities derived from thermodynamics appeared to be satisfied with equality inspired much of the theoretical progress in critical phenomena.

¹¹ N Goldenfeld. *Lectures on Phase Transitions and the Renormalization Group*. Addison-Wesley, 1992. ISBN 0-201-55409-7

¹² M Kardar. *Statistical Physics of Fields*. Cambridge University Press, 2007. ISBN 978-0-521-87341-3

Real space renormalization

In this chapter we give a derivation of the scaling behaviour conjectured in § 4.2. Given a microscopic model, such as the Ising model, we carry out a *renormalization group transformation* in real space. We will study those which can be thought of as “blocking transformations” The idea is to remove some degrees-of-freedom while maintaining the critical behaviour of the original model.

This procedure can be understood intuitively. If the correlation length is very large compared to the spacing between degrees-of-freedom, then we should be able to represent a neighborhood of correlated degrees-of-freedom by a single variable. The quantitative details carry some interesting features which may seem surprising, but correctly explain universality of critical exponents.

Below we illustrate the main ideas with a simple, solvable example, the Ising model in one dimension. Afterward, we discuss the general theory, keeping in mind that most models cannot be solved exactly. Nevertheless, there are reasonable approximations which can be made and provide a method for quantitative understanding of critical phenomena in many systems. Furthermore, these ideas underpin the renormalization of quantum field theories.

5.1 $D = 1$ Ising model

Consider 2 one-dimensional lattices, both with periodic boundary conditions, one lattice with $N = 2^k$ sites and the other with $N' = 2^{k-1}$ sites. The couplings are respectively J, h and J', h' .

We will relate the two lattice using a *blocking function* $T(\sigma', \sigma)$, where σ and σ' are the Ising spins, equal to ± 1 , on the respective lattices. The blocking function determines the mapping between the collection $\{\sigma'_\ell\}$, $\ell \in [1, N']$ of spins on the “blocked” lattice and the collection of original spins $\{\sigma_j\}$, $j \in [1, N]$.

One time of blocking procedure is called a *majority rule* transformation: we require that a group of nearby spins on the original lattice be represented by a blocked spin which is aligned with the majority of the original group of nearby spins (perhaps “flip a coin” in case of a tie). This type of procedure makes more sense for a model with more than 2 neighbors. Here let us choose $T(\sigma', \sigma)$ to be a *decimation* procedure: we require that $\sigma'_\ell = \sigma_{2\ell}$. This procedure eliminates, or “thins out” $\sigma_{2\ell+1}$, the spins on odd sites of the original lattice. The decimation blocking function described above can

be written mathematically as

$$T(\sigma', \sigma) = \prod_{\ell=1}^{N'} \delta_{\sigma'_\ell \sigma_{2\ell}} \quad (5.1)$$

with $\delta_{\alpha\beta}$ the Kronecker δ . Note that

$$\sum_{\sigma'} T(\sigma', \sigma) = 1.$$

Let us assume that the Hamiltonians for both the original and blocked theories take the same form

$$H(\tilde{J}, \tilde{h}, \tilde{N}, \tilde{\sigma}) = -\tilde{J} \sum_n \tilde{\sigma}_n \tilde{\sigma}_{n+1} - \tilde{h} \sum_n \tilde{\sigma}_n \quad (5.2)$$

where the tilde stands for either unprimed or primed variables. Then the *renormalization group transformation* implicitly determines J' and h' by requiring that the partition functions for the two theories are equal. This can be done looking at a particular configuration of blocked spins and demanding that

$$e^{-\beta H(J', h', N', \sigma')} e^{-\beta N' K'} = \sum_{\sigma} T(\sigma', \sigma) e^{-\beta H(J, h, N, \sigma) - \beta N K}. \quad (5.3)$$

We can check that the partition functions are indeed equal

$$\begin{aligned} Z(J', h', N', \sigma') &= \sum_{\sigma'} e^{-\beta H(J', h', N', \sigma') - \beta N' K'} \\ &= \sum_{\sigma'} \sum_{\sigma} T(\sigma', \sigma) e^{-\beta H(J, h, N, \sigma) - \beta N K} \\ &= \sum_{\sigma} e^{-\beta H(J, h, N, \sigma) - \beta N K} = Z(J, h, N, \sigma). \end{aligned} \quad (5.4)$$

Recall the transfer matrix elements for the Ising model were given to be (3.4)

$$W_{\sigma_i \sigma_j} = \exp \beta \left[J \sigma_i \sigma_j + \frac{h}{2} (\sigma_i + \sigma_j) \right] e^{-\beta K}.$$

Let us introduce the shorthand $z = e^{\beta J}$, $\mu = e^{\beta h}$, and $k = e^{-\beta K}$. Then the transfer matrix (3.6) can be written

$$W = k \begin{pmatrix} z\mu & z^{-1} \\ z^{-1} & z\mu^{-1} \end{pmatrix}$$

and similarly for W' , with appropriate definitions for z' , μ' , and k' . Thus $Z(J, h, N, \sigma) = \text{Tr } W^N$ and $Z(J', h', N', \sigma') = \text{Tr } W'^{N'}$. The term in (5.4) containing σ'_ℓ and $\sigma'_{\ell+1}$ reads

$$\begin{aligned} W'_{\sigma'_\ell \sigma'_{\ell+1}} &= \sum_{\sigma_{2\ell+1}} \delta_{\sigma'_\ell \sigma_{2\ell}} \delta_{\sigma'_{\ell+1} \sigma_{2\ell+2}} W_{\sigma_{2\ell} \sigma_{2\ell+1}} W_{\sigma_{2\ell+1} \sigma_{2\ell+2}} \\ W' &= W^2. \end{aligned} \quad (5.5)$$

That is, the blocked transfer matrix is equal to the original transfer matrix squared. Equating matrix elements, that implies the follow-

ing relations between couplings

$$\begin{aligned} k'z'\mu' &= k^2 \left(z^2\mu^2 + \frac{1}{z^2} \right) \\ \frac{k'}{z'} &= k^2 \left(\mu + \frac{1}{\mu} \right) \\ \frac{k'z'}{\mu'} &= k^2 \left(\frac{z^2}{\mu^2} + \frac{1}{z^2} \right). \end{aligned}$$

Define $x = z^{-4} = e^{-4\beta J}$, $y = \mu^{-2} = e^{-2\beta h}$, and $w = k^{-4} = e^{4\beta K}$, and similarly for x' , y' , and w' , then $W' = W^2$ implies

$$\begin{aligned} x' &= \frac{x(1+y)^2}{(x+y)(1+xy)} \\ y' &= \frac{y(x+y)}{1+xy} \\ w' &= \frac{w^2 x^2 y^2}{(1+y)^2(x+y)(1+xy)}. \end{aligned} \quad (5.6)$$

The set of equations (5.6) are the *renormalization group equations* for the transformation described above.

Let us study the renormalization group equations by considering the effect of repeated iterations of the transformations. Given some initial (x, y, w) , we obtain (x', y', w') which we can then use as new inputs to the renormalization group equations to obtain new values (x'', y'', w'') , and so on. In this way we map out *trajectories* in coupling constant space. Some example trajectories are shown in Fig. 5.1.

We are especially interested in fixed points of the renormalization group equations. Are there any values of (x, y, w) such that $(x', y', w') = (x, y, w)$? In the case of (5.6) we can easily check that there are two classes of fixed points.

1. $x = 1 \forall y$. Here $\beta J = J/T = 0$ which implies the system is at infinite temperature. The spins are decoupled, or random.
2. $(x, y) = (0, 1)$ implying $T = 0$ and $h = 0$.

The correlation length ζ , like other macroscopic properties, should remain fixed when measured in physical units. Therefore, when measured with respect to the successively growing lattice spacings of each blocking step, the correlation length will generally decrease. That is

$$a \mapsto b^p a \quad \implies \quad \zeta \mapsto \zeta_p = b^{-p} \zeta. \quad (5.7)$$

At a fixed point of the RG transformation, the coupling constants are unchanged, so the correlation length is fixed, not just in physical units but also lattice units: $\zeta_{p+1} = \zeta_p$. Therefore, either $\zeta = 0$ or $\zeta = \infty$. In the first case, degrees-of-freedom are independent of their neighbours; they are randomized, as they would be in an infinite temperature system, or one with all interactions turned off. We say a fixed point with $\zeta = 0$ is trivial. Nontrivial fixed points

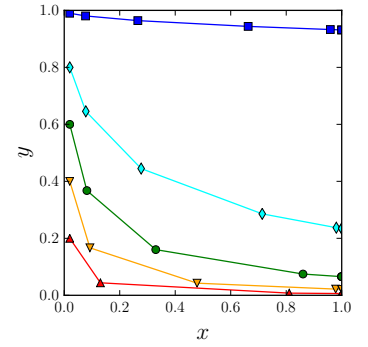


Figure 5.1: Renormalization group trajectories in the x - y plane for several different initial values of (x, y) , indicated by different colours and symbols. Iterations of the renormalization group equations (5.6) follow points along a line from left to right.

are those where $\xi = \infty$; these correspond to second order phase transitions.

Linearize about the fixed points. In case (i), we let $x = 1 - \epsilon$ for all y , with $0 < \epsilon \ll 1$. Then after one RG step $\epsilon' = \epsilon^2 y / (1 + y)^2$. There is no linear term and convergence to the infinite temperature limit is especially fast.

In case (ii), let $x = \epsilon$ and $y = 1 - \eta$ then to linear order

$$\begin{pmatrix} \epsilon' \\ \eta' \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \epsilon \\ \eta \end{pmatrix}$$

The RG transformation is already diagonal and we can read off the eigenvalues 4 and 2. It is usual to express the eigenvalues as powers of b , from which we infer critical exponents

$$\epsilon' = b^{y_t} \epsilon \quad \text{and} \quad \eta' = b^{y_h} \eta$$

with $y_t = 2$ and $y_h = 1$ (since $b = 2$).

Let us investigate the role of $K = \frac{1}{4} T \log w$. The free energy per spin can be inferred from (5.4) to be

$$\mathcal{F}(J, h, K) = f(J, h) + K$$

where

$$f(J, h) = -\frac{1}{N\beta} \log Z(J, h, K = 0, N).$$

After blocking one step

$$\exp(-\beta N \mathcal{F}(J, h, K)) = \exp(-\beta N' \mathcal{F}(J', h', K'))$$

implying

$$\mathcal{F}(J, h, K) = \frac{N'}{N} \mathcal{F}(J', h', K') = b^{-1} \mathcal{F}(J', h', K').$$

After p steps, and introducing the shorthand $u_p = (J_p, h_p)$ such that $u_0 = (J, h)$, $u_1 = (J', h')$, etc.,

$$\mathcal{F}(u_0, K_0) = b^{-p} \mathcal{F}(u_p, K_p)$$

or

$$f(u_0) = b^{-p} f(u_p) + b^{-p} K_p - K_0.$$

We can interpret the K_p term as the contribution to the free energy from those degrees-of-freedom which have been thinned out by the RG transformation. From $K = \frac{1}{4} T \log w$ and defining the function $g(x, y)$ so that the last line of (5.6) reads

$$w' = (we^{g(x,y)})^b$$

then $K_p = bK_{p-1} + bg(u_{p-1})$ or

$$b^{-p} K_p = \sum_{q=0}^{p-1} b^{-q} g(u_q) + K_0.$$

Therefore

$$f(u_0) = b^{-p} f(u_p) + \sum_{q=0}^{p-1} b^{-q} g(u_q). \quad (5.8)$$

Near critical point $(x, y) = (0, 1)$, that is for $(x, y) = (\epsilon, 1 - \eta)$

$$\begin{aligned} w' &= \frac{1}{4} w^2 \epsilon [(1 - 2\epsilon) + \dots] \\ K' &= 2K + kT \left(\frac{1}{4} \log \epsilon - \frac{1}{2} \log 2 - \frac{1}{2} \epsilon + \dots \right) \\ g(u) &= kT \left(\frac{1}{8} \log \epsilon - \frac{1}{4} \epsilon - \frac{1}{4} \log 2 + \dots \right) \end{aligned} \quad (5.9)$$

The blocking transformation of the $D = 1$ Ising model introduced most of the ideas of RG analysis in real space. We would like to build upon this concrete example to develop a more general theory.

5.2 General theory

Here we generalize the previous section. We work in D dimensions, considering a system initially defined on a lattice with spacing a . We will label the degrees-of-freedom (spins or fields) σ_r where the r subscript somehow labels its location.

We introduce a generic notation for “operators,” by which we mean terms which can appear in the Hamiltonian. Let these be denoted $\mathcal{O}_i(\{\sigma\})$, where the i simply labels different operators. For example, we could define operators to represent nearest-neighbour interactions, next-to-nearest-neighbour interactions, or even some kind of 3-body operator: $\mathcal{O}_1 = \sum_{\langle jk \rangle} \sigma_j \sigma_k$, $\mathcal{O}_2 = \sum_{\langle\langle JK \rangle\rangle} \sigma_J \sigma_K$, and $\mathcal{O}_3 = \sum_{\ell, m, n} \sigma_\ell \sigma_m \sigma_n$ (where we leave the details of the 3-body interaction unspecified here).

We write a generic Hamiltonian as the sum of operators times associated couplings u_i :

$$H(\vec{u}, \sigma) = \sum_i u_i \mathcal{O}_i(\{\sigma\}). \quad (5.10)$$

Note we use the vector notation to represent the collection of couplings. This will be natural as we come to think of RG transformations as trajectories in the space of coupling-constants. The partition function is

$$Z(\vec{u}, K, N) = \sum_{\sigma} \exp[-\beta H(\vec{u}, \sigma) - \beta NK]. \quad (5.11)$$

As in the previous section, Eq. (5.3), the RG transformation with blocking kernel $T(\sigma', \sigma)$ is defined by requiring

$$e^{-\beta H(\vec{u}', \sigma') - \beta N' K'} = \sum_{\sigma} T(\sigma', \sigma) e^{-\beta H(\vec{u}, \sigma) - \beta NK}. \quad (5.12)$$

This defines a mapping of original couplings \vec{u} to new values, \vec{u}' , in the blocked Hamiltonian. If we want to consider multiple iterations of the RG transformation, we will replace dashes by a subscript, as

in \vec{u}_p , to represent the coupling constant vector after p iterations of the RG transformation. We write the generic RG transformation, going from the p th to the $(p+1)$ st iteration as

$$\vec{u}_p \mapsto \vec{u}_{p+1} = R(\vec{u}_p). \quad (5.13)$$

Under a blocking transformation, the lattice spacing

$$a \mapsto a' = ba, \quad \text{with } b^D = \frac{N}{N'} \quad (5.14)$$

(e.g. $b = 2$ in the $D = 1$ Ising example). Requiring

$$Z(\vec{u}_p, K_p) = Z(\vec{u}_{p-1}, K_{p-1})$$

for all $p > 0$ implies the free energy density obeys

$$F(\vec{u}_0, K_0) = b^{-pD} F(\vec{u}_p, K_p). \quad (5.15)$$

Writing $F(\vec{u}, K) = f(\vec{u}) + K$ we have

$$\begin{aligned} f(\vec{u}_0) &= b^{-pD} f(\vec{u}_p) + b^{-pD} K_p - K_0 \\ &= b^{-pD} f(\vec{u}_p) + \sum_{q=0}^{p-1} b^{-qD} g(\vec{u}_q) \end{aligned} \quad (5.16)$$

where

$$g(\vec{u}_q) = b^{-D} K_{q+1} - K_q. \quad (5.17)$$

We will return to (5.16) soon.

Distances, measured in lattice units, scale with b as

$$\vec{r} \mapsto \vec{r}' = b^{-1} \vec{r}$$

for example the correlation length. Unlike in the case of the Ising model, where degrees-of-freedom are either just ± 1 , in other theories it will be necessary, or at least desirable, to scale the variables σ . These are just summation (or integration) variables inside of partition functions or expectation values. Therefore, we can relabel them after blocking if we wish. Furthermore, we can make a change of variables to rescale the degrees-of-freedom after blocking. This is most applicable to real-valued variables, as opposed to integer-valued ones. Thus we can decide to write the spin variables after 1 RG step in terms of the original spins as

$$\sigma_{\vec{r}'} = \Xi(b) \sigma_{\vec{r}} \quad (5.18)$$

where $\vec{r}(\vec{r}')$ labels the position of the spin in the unblocked (blocked) lattice. This is equivalent to defining the scaling dimension for σ . If we define d_σ through $\Xi = b^{d_\sigma}$, then $[\sigma] = -d_\sigma$.

For example, consider 2-point correlation function after a $b = 2$ blocking:

$$\langle \sigma_{\vec{r}} \sigma_0 \rangle^{(u)} = \langle \sigma'_{r/2} \sigma'_0 \rangle^{(u')} = \Xi^2(b) \langle \sigma_{r/2} \sigma_0 \rangle^{(u')} = \Xi^2(b) \langle \sigma_{r'} \sigma_0 \rangle^{(u')}$$

where the superscripts denote which coupling constants to use in the expectation value. The vectors \vec{r} and \vec{r}' represent the same physical displacement, but are measured in different lattice units, i.e. $\vec{r}' a' = \vec{r} a$.

$$G(\vec{r}, \vec{u}) = \Xi^2(b) G(b^{-1} \vec{r}, \vec{u}'). \quad (5.19)$$

5.3 Linearization near a fixed point

Fixed point

$$\vec{u}_* = R(\vec{u}_*).$$

At a fixed point, the physics is unchanged by an RG transformation – the couplings are unchanged. In particular the correlation length ζ is unchanged. Under an RG transformation $\zeta' = b^{-1}\zeta$, so if $\zeta' = \zeta$, as it must at a fixed point, then $\zeta = 0$ or ∞ .

Linearize by considering the RG transformation near \vec{u}_* . Let

$$\vec{u} = \vec{u}_* + \delta\vec{u} \quad \text{and} \quad \vec{u}' = \vec{u}_* + \delta\vec{u}'$$

with $\delta\vec{u}$ and $\delta\vec{u}'$ small.

For ease of writing components we put the index on R as follows

$$u'_i = [R(\vec{u})]_i \equiv R_i(\vec{u}).$$

$$u_{*i} + \delta u'_i = R_i(\vec{u}_* + \delta\vec{u}) = R_i(\vec{u}_*) + \left. \frac{\partial R_i}{\partial u_j} \right|_{\vec{u}_*} \delta u_j + \dots \quad (5.20)$$

Since $R(\vec{u}_*) = \vec{u}_*$,

$$\delta u'_i = \left. \frac{\partial R_i}{\partial u_j} \right|_{\vec{u}_*} \delta u_j \equiv M_{ij}(\vec{u}_*) \delta u_j. \quad (5.21)$$

$M_{ij} = \frac{\partial R_i}{\partial u_j}$ is the matrix describing the linearized RG transformation at the fixed point \vec{u}_* . Usually M is diagonalizable and here let us assume we find real eigenvalues λ_α along with corresponding eigenvectors \vec{e}_α . In the eigenvector bases we can find the components $\psi_\alpha^{(l)}$

$$\delta\vec{u} = \sum_\alpha \psi_\alpha \vec{e}_\alpha \quad \text{and} \quad \delta\vec{u}' = \sum_\alpha \psi'_\alpha \vec{e}_\alpha.$$

Then (5.21) becomes

$$\psi'_\alpha = \lambda_\alpha \psi_\alpha. \quad (5.22)$$

The ψ_α are called scaling fields or variables and parametrize the distance to the fixed point along the \vec{e}_α directions. In the $D = 1$ Ising model these were $\psi_\alpha = x, y$.

From the eigenvalues we can infer critical exponents.

$$y_\alpha = \frac{\log \lambda_\alpha}{\log b} \implies \psi'_\alpha = b^{y_\alpha} \psi_\alpha. \quad (5.23)$$

Using the notation of scaling dimensions, $[\psi_\alpha] = -y_\alpha$.

$$\begin{aligned} H(\vec{u}) &= H(\vec{u}_*) + \sum_i \delta u_i \mathcal{O}_i(\{\sigma\}) \\ &= H(\vec{u}_*) + \sum_\alpha \psi_\alpha \mathcal{O}_\alpha(\{\sigma\}) \end{aligned} \quad (5.24)$$

where

$$\mathcal{O}_\alpha(\{\sigma\}) = \sum_i (\vec{e}_\alpha)_i \mathcal{O}_i(\{\sigma\})$$

are the scaling operators, linear combinations of the original operators in H .

The behaviour of the various operators \mathcal{O}_α depend on the signs of each of the y_α .

1. $y_\alpha > 0$, $|\psi_\alpha|$ increases with each RG iteration, flow away from the fixed point in the \vec{e}_α direction. In this case \mathcal{O}_α is said to be a *relevant* operator.
2. $y_\alpha < 0$, $|\psi_\alpha|$ decreases, the flow is toward the fixed point in the \vec{e}_α . \mathcal{O}_α is said to be an *irrelevant* operator.
3. $y_\alpha = 0$. This is a marginal case, and \mathcal{O}_α is said to be a *marginal* operator. Higher order, beyond linear, will determine whether the operator is marginally relevant or irrelevant.

Figure 5.2 illustrates the situation where there are 2 irrelevant operators and 1 relevant operator. There is a 2-dimensional *critical surface* made up by the set of points which, lie on RG trajectories which flow into the fixed point \vec{u}_* . All of the points on the critical surface, including the fixed point, describe theories tuned to criticality, i.e. to $\xi = \infty$. Coming out of the surface from the fixed point, in the direction of the relevant eigenvector, is a trajectory called the *renormalized trajectory*.

Figure 5.3 shows a 2-dimensional cross-section cutting through \vec{u}_* . When we study a specific Hamiltonian and vary one (or more) of its couplings so that we cross a second-order transition, the couplings move across the critical surface,

Some comments:

1. Every theory with couplings tuned to be on the critical surface has $\xi = \infty$, i.e. is at a continuous phase transition.
2. Every such phase transition is equivalent to the one described by $H(\vec{u}_*)$; we see the same critical behaviour. This gives us a greater understanding of the universality of systems sharing the same critical exponents.
3. To tune any $H(\vec{u})$ to a phase transition, we need to vary the couplings of the theory (e.g. by changing the temperature or external magnetic field) to decrease the relevant variables $|\psi_\alpha|$. Sometimes the number of relevant variables is given a label, say κ , and can be thought of as the codimension of the critical surface (or manifold, speaking more generally). Usually κ is just a few.

We now wish to look at the free energy in the neighborhood of a fixed point u_* . As in the Ising model and other similar magnetic systems, let us denote the relevant scaling variables as t and h . Let ψ represent an irrelevant scaling variable. Let $\vec{u} = (u_1, u_2, u_3)$ be parameters in the Hamiltonian H .¹³ The phase transition occurs at

$$h = 0, \quad t = \frac{T - T_c(\psi)}{T_c(\psi)} = 0.$$

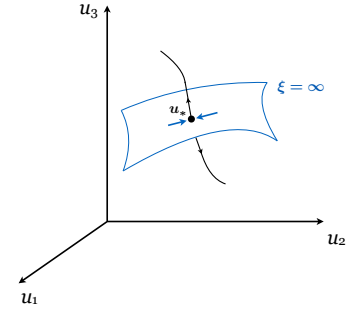


Figure 5.2: Critical surface (blue) containing a fixed point and renormalized trajectory (black) leaving the critical surface at the fixed point, in the direction of the relevant eigenvector.

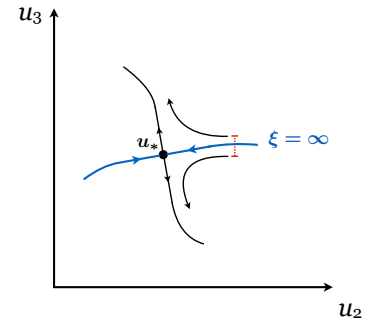


Figure 5.3: Cross-section of Fig. 5.2. Cross-section of the critical surface (blue) and renormalized trajectory (thick black). The thin black lines show RG flow with initial conditions slightly off of the critical surface. The red line depicts how we might vary the coupling of a particular Hamiltonian across a phase transition.

¹³ It is usual to absorb the inverse temperature β into the definitions of the couplings.

Recall the RG transformation R is a mapping in coupling constant space

$$\vec{u}_p \mapsto \vec{u}_{p+1} = R(\vec{u}_p). \quad (5.25)$$

We can relate the p -th point in coupling constant space to some initial point \vec{u}_0

$$\vec{u}_p = R(\vec{u}_{p-1}) = R(R(\vec{u}_{p-2})) = \dots = R^p(\vec{u}_0). \quad (5.26)$$

As we did for the specific case of the Ising model, we divide the free energy into

$$\mathcal{F}(\vec{u}_p, K_p) = f(\vec{u}_p) + K_p. \quad (5.27)$$

As before we find

$$f(\vec{u}_0) = b^{-pD} f(\vec{u}_p) + \sum_{q=0}^{p-1} b^{-qD} g(\vec{u}_q) \quad (5.28)$$

where we write $\mathcal{F}(\vec{u}, K) = f(\vec{u}) + K$ and define g through

$$K_p = b^D K_{p-1} + b^D g(\vec{u}_{p-1}). \quad (5.29)$$

Repeated iterations of the RG transformation should reach the neighbourhood of the fixed point in a finite number of iterations. Irrelevant variables flow to their fixed point values quickly. Denote by \bar{p} the number of iterations to get near enough to \vec{u}_* so that a linear approximation is a good one. Then

$$\mathcal{F}(\vec{u}_0, K_0) = b^{-\bar{p}D} \mathcal{F}(\vec{u}_{\bar{p}}, K_{\bar{p}}). \quad (5.30)$$

We can write this as

$$\mathcal{F}(\vec{u}_0, K_0) = B f(a_t t, a_h h, a_\psi \psi) \quad (5.31)$$

for the following reasons

1. Near the fixed point \vec{u}_* let us expand \vec{u} linearly in the scaling variables

$$\vec{u}_{\bar{p}} = \vec{u}_* + (a_t t, a_h h, a_\psi \psi). \quad (5.32)$$

For simplicity we also assume here a diagonal transformation between the explicit couplings of the Hamiltonian. This is generally not the case; a 3×3 matrix is necessary to relate (u_1, u_2, u_3) to (t, h, ψ) , but we do not wish to complicate the notation any further. Either we can define our coordinates so that \vec{u}_* is at the origin, or we could absorb \vec{u}_* dependence into the definition of the function f .

2. Choose initial K_0 so that $K_{\bar{p}}$ vanishes.
3. $b^{-\bar{p}D} = B$ is a fixed constant after we fix \bar{p} .

For simplicity we set the uninteresting constants a_t , a_h , a_ψ , and B equal to 1 below; this can be interpreted as absorbing them into a new definition for f .

The discussion above outlined \bar{p} RG iterations which gave $\vec{u}_{\bar{p}}$ in the neighbourhood of the fixed point, where the RG transformation could be linearized. Then we tidied up the notation. Now let us perform another p' iterations of the RG transformation.

$$\mathcal{F}(\vec{u}_0, K_0) = b^{-p'D} f(b^{p'y_t} t, b^{p'y_h} h, b^{p'y_\psi} \psi) + \sum_{q=0}^{p'-1} b^{-qD} g(b^{qy_t} t). \quad (5.33)$$

Note g only depends on t , not on h (or ψ). We assume this is so for a well-designed RG transformation. External fields like h (and presumably whatever ψ represents) typically couple only to long wavelength modes, so do not contribute to the inhomogeneous term arising from the thinned-out degrees-of-freedom.

As the RG trajectory approaches a fixed point, \vec{u}_p is a slowly-moving function of p . The trajectory is almost continuous. Therefore we can treat $b^{p'}$ (and b^q) as a continuous variable, denoted \hat{b} . This gives us the freedom to choose an arbitrary value for $b^{p'}$, not necessarily an integer power of b . Then we write the singular contribution to $\mathcal{F}(\vec{u}_0, K_0)$ as

$$\mathcal{F}_s = \hat{b}^{-D} f(\hat{b}^{y_t} t, \hat{b}^{y_h} h, \hat{b}^{y_\psi} \psi). \quad (5.34)$$

Now we choose a specific value for \hat{b} , one such that we flow to a reference value for $|b^{y_t} t|$. The specific value is not important, but for convenience let us set \hat{b} such that

$$\hat{b}^{y_t} t = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases}. \quad (5.35)$$

This is the point denoted at the end of the trajectory in Fig. 5.4. We need not worry whether this reference point is within the linear regime: for RG trajectories which approach the fixed point, vastly many more iterations are made in the linear regime than outside – if not, then we need to choose a better initial condition!

Given that ψ was assumed to represent an irrelevant variable, $y_\psi < 0$ so $\hat{b}^{y_\psi} \psi \approx 0$. We then have for (5.34)

$$\mathcal{F}_s = |t|^{D/y_t} f\left(\pm 1, \frac{h}{|t|^{y_h/y_t}}, 0\right) \quad (5.36)$$

arriving at the scaling form of the previous chapter (4.10)

$$\mathcal{F} = |t|^{2-\alpha} f_{\geq}\left(\frac{h}{|t|^\Delta}\right).$$

From this we can identify

$$f_{\geq}(x) = f(\pm 1, x, 0) \equiv f_{\pm}(x).$$

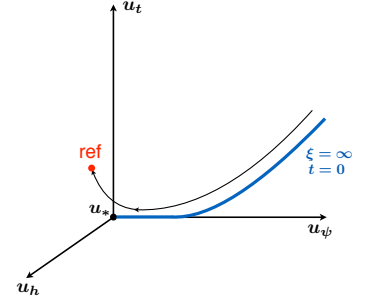


Figure 5.4: Given an initial set of couplings near a critical surface (thick blue line), the thin black line depicts a blocking transformations toward the fixed point, and then to a reference point as determined by (5.35), for example.

By differentiating \mathcal{F} , we can read off critical exponents

$$\begin{aligned}\alpha &= 2 - \frac{D}{y_t} \quad \text{and} \quad \Delta = \frac{y_h}{y_t} \\ \beta &= 2 - \alpha - \Delta = \frac{D - y_h}{y_t} \\ \gamma &= 2\Delta - 2 + \alpha = \frac{2y_h - D}{y_t} \\ \delta &= \frac{\Delta}{\beta} = \frac{y_h}{D - y_t}.\end{aligned}\tag{5.37}$$

The inhomogenous term can spoil things. As the RG trajectory becomes nearly continuous, let $s = b^{qy_t}|t|$. Then

$$\begin{aligned}\sum_{q=0}^{p-1} b^{-qD} g(b^{y_t}t) &\rightarrow |t|^{D/y_t} \int_{|t|}^1 ds \left| \frac{ds}{dq} \right|^{-1} s^{-D/y_t} g(\pm s) \\ &= \frac{|t|^{D/y_t}}{y_t \log b} \int_{|t|}^1 ds s^{-(D-y_t)/y_t} g(\pm s).\end{aligned}\tag{5.38}$$

The prefactor $|t|^{D/y_t}$ is the same as in the singular term, but we cannot make general statements about how the integral behaves as $|t| \rightarrow 0$. Sometimes it converges, sometimes it does not. Thus we see a sign that the Renormalization Group analysis is not a *proof* of universality; however, it is a powerful and successful framework for understanding universal phenomena where they appear.

Making further contact with § 4.2, where we assumed $[\mathcal{A}] = -D$ and $[m] = d_m$. Here we obtain similar scaling dimensions from the RG flow near a fixed point. The transformation is defined such that

$$\vec{r} \mapsto \vec{r}' = b^{-1}r \implies [\vec{r}] = 1.\tag{5.39}$$

By definition of the y exponents:

$$t' = b^{y_t}t \implies [t] = -y_t\tag{5.40}$$

$$h' = b^{y_h}h \implies [h] = -y_h.\tag{5.41}$$

From (5.36) and $[t] = -y_t$ we see that we agree that the free energy has dimension $[\mathcal{F}] = -D$.

Further more, it follows from $m \sim |t|^\beta$ that

$$d_m = [m] = \beta[t] = -D + y_h.\tag{5.42}$$

The magnetic susceptibility $\chi = \frac{\partial m}{\partial h}$ then has dimension

$$[\chi] = [m] - [h] = -D + 2y_h.\tag{5.43}$$

Its critical exponent, from $\chi \sim |t|^{-\gamma}$ is then as given in (5.37).

The correlation length exponent, from $\xi \sim |t|^{-\nu}$, can be inferred from the condition that we block to a reference point (5.35); \hat{b} is chosen so that $|\hat{b}t| = 1$. Denote the correlation length in the unblocked theory by ξ . After a finite number \bar{p} steps, the theory is in the linear regime and $\xi_{\bar{p}} = b^{-\bar{p}}\xi$. Then we block many times (almost continuously), ending up at a reference theory with (much

shorter) correlation length ξ_0 . The original correlation length is related to the reference correlation length by

$$\xi = b^{\bar{p}} \hat{b} \xi_0. \quad (5.44)$$

Therefore

$$\xi \sim |t|^{-1/y_t} \implies \nu = \frac{1}{y_t}. \quad (5.45)$$

Finally the critical exponent η , parametrizing the power-law behaviour of the correlation function $G(r)$. Later we will further justify having the following form for $G(r)$

$$G(r) \sim \frac{g(r/\xi)}{r^{D-2+\eta}}. \quad (5.46)$$

Usually, for $r > \xi$, $g(r/\xi) \propto \exp(-r/\xi)$, i.e. degrees-of-freedom become decorrelated exponentially quickly in the separation between them, as long as they are separated by at least 1 correlation length. Near a critical point, though, $\xi \rightarrow \infty$ and the correlations fall off much more slowly, characterized by the η critical exponent.

Referring back to the discussion leading to (5.19), we can see that the initial \bar{p} steps of our RG procedure yield

$$G(\vec{r}, \vec{u}) = \prod_{q=1}^{\bar{p}} \Xi^2(b) G(b^{-\bar{p}} \vec{r}, \vec{u}_*). \quad (5.47)$$

The last factor is universal, say $\hat{G}(b^{-\bar{p}})$, independent of the initial position in coupling constant space, as long as the flow reaches the neighborhood of the fixed point. We can define the new correlation function in this linear region as

$$\bar{G}(\vec{r}) = \prod_{q=1}^{\bar{p}} \Xi^2(b) \hat{G}(b^{-\bar{p}})$$

and after a further p blockings to the reference model, with $\hat{b} = b^p$, we have

$$\bar{G}(\vec{r}) = \Xi^{2p} \bar{G}(b^{-p} r) = \hat{b}^{2d_\sigma} \bar{G}(\hat{b}^{-1} r) \quad (5.48)$$

In the last step we wrote Ξ in terms of the scaling dimension of σ (see text after (5.18)). From (5.48) we conclude that $\bar{G}(r) \sim 1/r^{-2d_\sigma}$, and that

$$\eta = 2 - D - 2d_\sigma. \quad (5.49)$$

Note the 6 critical exponents satisfy 4 scaling laws

$$\begin{aligned} \alpha + 2\beta + \gamma &= 2 && \text{Rushbrooke} \\ \beta + \gamma &= \beta\delta && \text{Widom} \\ 2 - \alpha &= \nu D && \text{Josephson} \\ \gamma &= \nu(2 - \eta) && \text{Fisher.} \end{aligned} \quad (5.50)$$

Therefore, one need only measure or calculate any 2 critical exponents to be able to determine all 6. In this analysis we saw that

determining y_t and y_h was sufficient to determine the critical exponents above.

A final remark on dimensions. The 2-point Green's function $G(r)$ should carry dimensions of $[\text{length}]^{2-D}$ – see (5.46) with $\eta = 0$; we refer to these as the *engineering dimension* of $G(r)$. For $\eta \neq 0$, one refers to the difference between the scaling dimension and the engineering dimension as the *anomalous dimension*. The short distance scale, the lattice spacing, implicitly fixes the mismatch in units, so restoring a explicitly

$$G(r) \sim \frac{a^\eta g(r/\xi)}{r^{d-2+\eta}}.$$

Universality is the statement that different short distance details can lead to the same long distance physics. That is not to say the short distance physics is totally irrelevant, just that its effects contribute in a very specific way, observable deviation from naive scaling in the form of anomalous dimensions, and hence critical exponents different from mean field predictions.

Landau-Ginzburg theory

In this chapter, we study a model which appears to be quite different from the Ising model. However, we will see that it gives, in a certain approximation, the same critical behaviour as we found for the Ising model in the mean field approximation. Furthermore, we will see this theory, due to Landau and Ginzburg, is a framework for including interactions between fluctuations, and gives a general field-theoretic approach. The text by Le Bellac gives an introduction starting from mean field theory and building from there.¹⁴

¹⁴ M Le Bellac. *Quantum and Statistical Field Theory*. Oxford University Press, 1991. ISBN 0-19-853964-9

6.1 Hamiltonian

Consider a square lattice with spacing a in D dimensions, with N sites, at each of which there is a scalar variable

$$-\infty < \phi_i \equiv \phi(\vec{x}_i) < \infty. \quad (6.1)$$

Let us assume periodic boundary conditions, so that $\phi(\vec{x}_i + L_\nu \hat{\nu}) = \phi(\vec{x}_i)$, with $\hat{\nu}$ a unit vector along one of the lattice axes and L_ν the box-length in that direction.

Let us define a shorthand notation for the forward finite difference operator

$$\Delta_\nu \phi(\vec{x}_i) = \frac{1}{a} [\phi(\vec{x}_i + a\hat{\nu}) - \phi(\vec{x}_i)]. \quad (6.2)$$

The Hamiltonian (times β , although we absorb this factor into the definition of the coupling constants) is

$$H(\{\phi_i\}) = H(\phi) = a^D \sum_{i=1}^N \left[\frac{1}{2} (\Delta\phi(\vec{x}_i))^2 + \frac{1}{2} r_0 \phi_i^2 + \frac{1}{4!} u_0 \phi_i^4 \right] \quad (6.3)$$

and the partition function is

$$Z = \int \left(\prod_{i=1}^N d\phi_i \right) e^{-H(\phi) + \sum_i h_i \phi_i}. \quad (6.4)$$

Note $h_i = h(\vec{x}_i)$ allows for nonuniform external magnetic field.

A few comments are in order before a more detailed investigation. First, if we think about a constant field $\phi_i = \phi_0$, the difference term in H vanishes and the exponent in the integrand of Z takes the form we saw in for the free energy (2.2). Second, for non-constant ϕ_i , the difference term $(\Delta_\nu \phi(\vec{x}_i))^2$ introduces a coupling between nearest neighbors as in the Ising model, i.e. a term proportional to $-\phi(\vec{x}_i + a\hat{\nu})\phi(\vec{x}_i)$.

Having defined the theory on the lattice this way, and motivated the terms in the Hamiltonian, we will make analysis easier by approximating the lattice theory by one in the continuum. Thus we move from a theory of N scalar variables to a scalar field theory.

We have argued a few times in these lectures that, if the correlation length is much larger than the spacing between individual degrees-of-freedom, then we can replace the original variables by representative variables. This time we will approximate the highly correlated variables by a slowly varying continuous field. Thus we consider there to be a real-valued field $\phi(\vec{x})$ for any point in our spatial volume, $\vec{x} \in V$. We also replace the difference term $\Delta_\nu \phi(\vec{x}_i)$ in (6.3) by $\partial_\nu \phi(\vec{x})$, where ∂_ν is shorthand for $\frac{\partial}{\partial x_\nu}$.

Thus we arrive at the Landau-Ginzburg Hamiltonian

$$H = \int d^D x \left[\frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} r_0 \phi^2 + \frac{1}{4!} u_0 \phi^4 \right]. \quad (6.5)$$

Similar to the discussion in Landau's theory of phase transitions, when we study second order phase transitions, we will end up assuming that the quadratic coupling $r_0(T)$ changes sign as the temperature is varied through the critical temperature while $u_0(T) > 0$ varies slowly. The Hamiltonian (6.5) can be amended, e.g. with a ϕ^6 term in order to study a tricritical point, but this is beyond what we will do in these lectures, although you may have an opportunity to explore this. For the moment we have not introduced a variable coefficient in front of the derivative term $(\nabla \phi)^2$. We will do so in the next Chapter, where we will ultimately conclude that we can choose the coefficient to be fixed as in (6.5).

6.2 Functional differentiation and integration

Let us work in 1 dimension here; the extension to D dimensions is straightforward. Also let $I[\phi]$ be a *functional* of $\phi(x)$. For example, I could represent one of the terms in (6.5)

$$\int dx \left(\frac{d\phi}{dx} \right)^2 \quad \text{or} \quad \int dx \phi^{2n}.$$

We can define the functional derivative $\frac{\delta I}{\delta \phi(x)}$ of some functional $I[\phi]$ implicitly by using methods from variational principles. Let ϵ be a small parameter and $\epsilon \eta(x)$ be a small variation. Then we can expand

$$I[\phi + \epsilon \eta] = I[\phi] + \epsilon \int dx \frac{\delta I}{\delta \phi(x)} \eta(x) + O(\epsilon^2).$$

One can use this as a starting point to more carefully prove the identities we use. The calculus of variations can be built from this framework, in which the usual algebraic rules for derivatives are obeyed by the functional derivative.

We may also think of the functional derivative as the continuum limit of derivatives of I with respect to the function (or field) ϕ at

point x :

$$\frac{\delta I}{\delta \phi(x)} = \lim_{a \rightarrow 0} \frac{1}{a} \frac{\partial J}{\partial \phi_i}. \quad (6.6)$$

Where J is a discrete version of I . We provide a few examples; for a given I , we write down a discrete sum J which approaches I in the continuum limit. From the derivative of J we obtain the functional derivative of I .

1. Take

$$I = \int dy f(y) \phi^n(y) \quad \text{and} \quad J = a \sum_j f_j \phi_j^n. \quad (6.7)$$

Note that the lattice spacing a in the expression for J is necessary to carry the dimensions of length, ensuring $J \rightarrow I$ in the continuum limit (assuming it exists). Then

$$\frac{\partial J}{\partial \phi_i} = a n f_i \phi_i^{n-1} \implies \frac{\delta I}{\delta \phi(x)} = n f(x) \phi^{n-1}(x). \quad (6.8)$$

2. For

$$I = \int dy \left(\frac{d\phi}{dy} \right)^2 \quad \text{and} \quad J = a \sum_j \frac{1}{a^2} (\phi_{j+1} - \phi_j)^2 \quad (6.9)$$

we have

$$\frac{\partial J}{\partial \phi_i} = \frac{2}{a} (2\phi_i - \phi_{i+1} - \phi_{i-1}) \implies \frac{\delta I}{\delta \phi(x)} = -2 \frac{d^2 \phi}{dx^2}. \quad (6.10)$$

3.

$$I = \int dy V(\phi(y)) \implies \frac{\delta I}{\delta \phi(x)} = V'(\phi(x)). \quad (6.11)$$

4.

$$\frac{\partial \phi_i}{\partial \phi_j} = \delta_{ij} \implies \frac{\delta \phi(x)}{\delta \phi(y)} = \delta(x - y). \quad (6.12)$$

5.

$$\frac{\partial J}{\partial \psi_i} = \sum_j \frac{\partial J}{\partial \phi_j} \frac{\partial \phi_j}{\partial \psi_i} \implies \frac{\delta I}{\delta \psi(x)} = \int dy \frac{\delta I}{\delta \phi(y)} \frac{\delta \phi(y)}{\delta \psi(x)}. \quad (6.13)$$

Functional integration is the extension of the integral over a discrete number of variables ϕ_i with $i \in [1, N]$, to an integration over all possible field configurations. We write

$$\mathcal{D}\phi = \lim_{a \rightarrow 0} \mathcal{N}(a) \prod_{i=1}^N d\phi_i. \quad (6.14)$$

Here a field-independent normalization $\mathcal{N}(a)$ can be introduced to give something finite in the continuum limit. There are many subtleties which can be important in functional integration; however, these will not play a role in the physics we study in this course.

6.3 Landau approximation

Partition function

$$\begin{aligned} Z &= \int \mathcal{D}\phi \exp \left\{ - \int d^D x \left[\frac{1}{2} (\nabla\phi)^2 + \frac{r_0}{2} \phi^2 + \frac{u_0}{4!} \phi^4 - h\phi \right] \right\} \\ &= \int \mathcal{D}\phi \exp \left\{ - \int d^D x [\mathcal{H}[\phi(\vec{x})] - h\phi(\vec{x})] \right\} \end{aligned} \quad (6.15)$$

Landau's approximation is a saddle point approximation of the integral. We find the specific field configuration $\phi_0(\vec{y})$ satisfying

$$\left. \frac{\delta}{\delta\phi(\vec{y})} \int d^D x [\mathcal{H}[\phi(\vec{x})] - h(\vec{x})\phi(\vec{x})] \right|_{\phi_0(\vec{y})} = 0 \quad (6.16)$$

that is,

$$-\nabla^2 \phi_0(\vec{y}) + r_0 \phi_0(\vec{y}) + \frac{u_0}{3!} \phi_0^3(\vec{y}) - h(\vec{y}) = 0. \quad (6.17)$$

Now we approximate Z by $\exp \left\{ - \int d^D x [\mathcal{H}(\phi_0) - h\phi_0] \right\}$. The Helmholtz free energy

$$\mathcal{F}[h] = -\log Z = \int d^D x [\mathcal{H}(\phi_0) - h\phi_0]. \quad (6.18)$$

Note that the magnetization

$$m(\vec{y}) = -\frac{\delta \mathcal{F}}{\delta h(\vec{y})} = \phi_0(\vec{y}). \quad (6.19)$$

The Legendre transform to the magnetic Gibbs free energy gives

$$\begin{aligned} \Gamma[m] &= \mathcal{F}[h] - \int d^D y h(\vec{y}) m(\vec{y}) \\ &= \int d^D y \left[\frac{1}{2} (\nabla m)^2 + \frac{r_0}{2} m^2 + \frac{u_0}{4!} m^4 \right]. \end{aligned} \quad (6.20)$$

Differentiating with respect to the magnetization at a point \vec{x} ,

$$\frac{\delta \Gamma}{\delta m(\vec{x})} = -\nabla_x m(\vec{x}) + r_0 m(\vec{x}) + \frac{u_0}{3!} [m(\vec{x})]^3 = h(\vec{x}) \quad (6.21)$$

where ∇_x is the gradient operator at \vec{x} .

The connected Green's function can be obtained by differentiating twice

$$\begin{aligned} G(\vec{x}, \vec{y}) &= \langle \phi(\vec{y}) \phi(\vec{x}) \rangle_c \\ &= \frac{\delta^2}{\delta h(\vec{y}) \delta h(\vec{x})} \log Z \\ &= -\frac{\delta}{\delta h(\vec{y})} \frac{\delta}{\delta h(\vec{x})} \mathcal{F} \\ &= \frac{\delta}{\delta h(\vec{y})} m(\vec{x}) \end{aligned} \quad (6.22)$$

Differentiating (6.21) with respect to $h(\vec{y})$ gives

$$\left[-\nabla_x^2 + r_0 + \frac{u_0}{2} m^2 \right] G(\vec{x}, \vec{y}) = \delta(\vec{x} - \vec{y}). \quad (6.23)$$

We can solve this by Fourier transform. If we let h be uniform in space, then translational invariance implies that $G(x, y) = G(x - y)$. Let

$$G(\vec{x}) = \int \frac{d^D q}{(2\pi)^D} \tilde{G}(\vec{q}) e^{-i\vec{q} \cdot \vec{x}}. \quad (6.24)$$

Insert this into (6.23) to find

$$\tilde{G}(\vec{q}) = \frac{1}{q^2 + r_0 + \frac{u_0}{2} m^2}. \quad (6.25)$$

If, near $T \approx T_c$, we can write $r_0 = c(T - T_c)$ with constant c , (6.21) implies $c(T - T_c)m + \frac{u_0}{3!} m^3 = 0$ and therefore

$$\tilde{G}(\vec{q}) = \begin{cases} \frac{1}{q^2} \left(1 + \frac{2c(T_c - T)}{q^2}\right)^{-1} & \text{for } T < T_c \\ \frac{1}{q^2} \left(1 + \frac{c(T_c - T)}{q^2}\right)^{-1} & \text{for } T > T_c \end{cases}. \quad (6.26)$$

Note the $1/q^2$ divergence for $q^2 \rightarrow 0$ as $T \rightarrow T_c$. This is a consequence of long range, or small wavevector, correlations, as we expect at a second order phase transition.

More generally one observes a $1/q^{2-\eta}$ divergence, where η is the correlation function critical exponent. That is, we can write

$$\tilde{G}(\vec{q}) = \frac{1}{q^{2-\eta}} \tilde{f}(q\tilde{\zeta}) \quad (6.27)$$

for small q . $\tilde{f}(q\tilde{\zeta})$ is a dimensionless function of the dimensionless product of q with the correlation length $\tilde{\zeta}$, and \tilde{f} should be finite as $q\tilde{\zeta} \rightarrow \infty$. In the case of Landau-Ginzburg theory, we can infer that $\eta = 0$ and $\nu = \frac{1}{2}$ from rearranging (6.26) into the form of (6.27).¹⁵

Generalizing an examples sheet problem, one can show that the Fourier transform back to position space yields for low q (large r)

$$G(r) = \frac{f(r/\tilde{\zeta})}{\tilde{\zeta}^{D+\eta-2}} = \frac{g(r/\tilde{\zeta})}{r^{D+\eta-2}}. \quad (6.28)$$

Typically one finds exponentially decaying $G(r)$ for $r \gg \tilde{\zeta}$; i.e., $g \sim e^{-r/\tilde{\zeta}}$.

¹⁵ Recall ν is the critical exponent for the correlation length: $\tilde{\zeta} \sim (T - T_c)^{-\nu}$.

Renormalization in Fourier space

7.1 Gaussian model

We begin applying renormalization group (RG) methods to the Landau-Ginzburg theory with no interactions: free scalar field theory, also called in this context the Gaussian model. (We take ϕ to be real.) It will be useful to introduce an external source $h(\vec{x})$ which couples to $\phi(\vec{x})$:

$$H_0[\phi, h] = \int d^D x \left[\frac{1}{2} \alpha^{-1} (\nabla \phi)^2 + \frac{1}{2} r_0 \phi^2 - h(\vec{x}) \phi(\vec{x}) \right] \quad (7.1)$$

usually taking the $h \rightarrow 0$ limit to obtain results. Note that, we have introduced a parameter in front of the kinetic term $(\nabla \phi)^2$.

Fourier transform

$$\tilde{\phi}(\vec{p}) = \int d^D x e^{-i\vec{p}\cdot\vec{x}} \phi(\vec{x}) \quad (7.2)$$

and similarly for h . Note that $\phi(\vec{x})$ being real implies

$$\tilde{\phi}^*(\vec{p}) = \tilde{\phi}(-\vec{p}). \quad (7.3)$$

The transformed Hamiltonian is

$$H_0[\tilde{\phi}, \tilde{h}] = \frac{1}{2} \int^\Lambda \frac{d^D p}{(2\pi)^D} \left[(\alpha^{-1} p^2 + r_0) |\tilde{\phi}(\vec{p})|^2 - (\tilde{h}(-\vec{p}) \tilde{\phi}(\vec{p}) + \tilde{h}(\vec{p}) \tilde{\phi}(-\vec{p})) \right]. \quad (7.4)$$

Note we impose a momentum cutoff Λ . On a spatial lattice, each momentum component $p_i < \pi/a$. For simplicity, here we have done the Fourier transform in the continuum and imposed a spherical cutoff in momentum space.

The partition function is given by

$$Z_0 = \int \mathcal{D}\tilde{\phi} e^{-H_0[\tilde{\phi}]}. \quad (7.5)$$

This is somewhat loose notation for what is essentially a functional integral. Imagine working with discrete momentum space, due to putting the system in a finite volume. Then the partition function is the integral over all values of $\tilde{\phi}$ at each discrete value of p . Then take the infinite volume limit. (I hope to write an appendix on functional integrals.) Note the exponential is diagonal in \vec{p} ; there is no coupling between modes with different momenta.

Denote

$$\tilde{\Delta}(\vec{p}) = \alpha^{-1} p^2 + r_0 \quad (7.6)$$

Complete the square

$$\begin{aligned} \tilde{\phi} \tilde{\Delta} \tilde{\phi}^* - (\tilde{h}^* \tilde{\phi} + \tilde{h} \tilde{\phi}^*) &= [\tilde{\phi} - \tilde{h} \tilde{\Delta}^{-1}] \tilde{\Delta} [\tilde{\phi}^* - \tilde{h}^* \tilde{\Delta}^{-1}] - \tilde{h} \tilde{\Delta}^{-1} \tilde{h}^* \\ &\equiv 2\mathcal{H}_G[\tilde{\phi}] - \tilde{h} \tilde{\Delta}^{-1} \tilde{h}^*. \end{aligned} \quad (7.7)$$

with the last line defining \mathcal{H}_G .

Thus

$$Z_0[\tilde{h}] = Z_G \exp \left[\frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \tilde{h} \tilde{\Delta}^{-1} \tilde{h}^* \right] \quad (7.8)$$

where

$$Z_G = \int \mathcal{D}\phi \exp \left[\int \frac{d^D p}{(2\pi)^D} \mathcal{H}_G[\tilde{\phi}] \right]. \quad (7.9)$$

The source term h in H provides an easy way to compute expectation values

$$\begin{aligned} \langle \phi(y) \rangle &= \frac{1}{Z_0} \int \mathcal{D}\phi \phi(y) \exp \left\{ - \int d^D x \left[\frac{1}{2\alpha} (\nabla\phi)^2 + \frac{1}{2} r_0 \phi^2 \right] \right\} \\ &= \frac{1}{Z_0} \frac{\delta Z_0[h]}{\delta h(y)} \Big|_{h=0} = \frac{\delta}{\delta h(y)} \log Z_0[h] \Big|_{h=0} \end{aligned} \quad (7.10)$$

Even more useful is the connected 2-point function

$$\begin{aligned} \langle \phi(x) \phi(y) \rangle_c &= \langle \phi(x) \phi(y) \rangle - \langle \phi(x) \rangle \langle \phi(y) \rangle \\ &= \left[\frac{1}{Z_0} \frac{\delta^2 Z_0[h]}{\delta h(x) \delta h(y)} - \left(\frac{1}{Z_0} \frac{\delta Z_0[h]}{\delta h(x)} \right) \left(\frac{1}{Z_0} \frac{\delta Z_0[h]}{\delta h(y)} \right) \right] \Big|_{h=0} \\ &= \frac{\delta^2}{\delta h(x) \delta h(y)} \log Z_0[h] \Big|_{h=0}. \end{aligned} \quad (7.11)$$

Fourier transform the connected 2-point function

$$\begin{aligned} \langle \tilde{\phi}(\vec{q}) \tilde{\phi}(\vec{p}) \rangle_c &= \int d^D x d^D y e^{-i\vec{q}\cdot\vec{x}} e^{-i\vec{p}\cdot\vec{y}} \langle \phi(x) \phi(y) \rangle_c \\ &= \int d^D x d^D x' e^{-i(\vec{q}+\vec{p})\cdot\vec{x}} e^{-i\vec{p}\cdot\vec{x}'} G_0(\vec{x}') \\ &= (2\pi)^D \delta^{(D)}(\vec{q} + \vec{p}) \tilde{G}_0(\vec{p}) \end{aligned} \quad (7.12)$$

From (7.11) and (7.8) we also have

$$\begin{aligned} \langle \tilde{\phi}(\vec{q}) \tilde{\phi}(\vec{p}) \rangle_c &= (2\pi)^D \frac{\delta^2}{\delta \tilde{h}(-\vec{q}) \delta \tilde{h}(-\vec{p})} \log Z_0[\tilde{h}] \Big|_{\tilde{h}=0} \\ &= (2\pi)^D \delta^{(D)}(\vec{q} + \vec{p}) \tilde{\Delta}^{-1}(\vec{p}) \end{aligned} \quad (7.13)$$

This implies

$$\tilde{G}_0(\vec{p}) = \tilde{\Delta}^{-1}(\vec{p}) = \frac{\alpha}{p^2 + \alpha r_0}. \quad (7.14)$$

Thus we can transform back to position space

$$G_0(\vec{x}) = \int \frac{d^D p}{(2\pi)^D} e^{i\vec{p}\cdot\vec{x}} \frac{\alpha}{p^2 + \alpha r_0}. \quad (7.15)$$

Let us write $\xi^{-2} = \alpha r_0$ and consider limits where $r = |\vec{x}|$ is much larger and much smaller than ξ we find the asymptotic behaviour

$$G_0(r) \sim \begin{cases} \frac{1}{r^{D-2}} & r \ll \xi \\ \frac{\xi e^{-r/\xi}}{(r\xi)^{(D-1)/2}} & r \gg \xi. \end{cases} \quad (7.16)$$

To explore scaling behaviour, we will implement a 2-step RG transformation: (1) Thinning/removing small wavelength degrees-of-freedom and (2) Rescaling units. First, let us work with a constant external source $h(\vec{x}) = h$, so that it only couples to the zero-momentum mode. The source term in the Hamiltonian then becomes

$$\int d^D x h(\vec{x}) \phi(\vec{x}) = h \int d^D x \phi(\vec{x}) = h \tilde{\phi}(0). \quad (7.17)$$

(1) Thinning We divide the field ϕ into long and short wavelength parts, respectively

$$\tilde{\phi}(\vec{p}) = \tilde{\phi}_<(\vec{p}) + \tilde{\phi}_>(\vec{p}) \quad (7.18)$$

with

$$\tilde{\phi}_<(\vec{p}) = \begin{cases} \tilde{\phi}(\vec{p}) & 0 \leq p \leq \Lambda/b \\ 0 & \Lambda/b < p \leq \Lambda \end{cases}$$

$$\tilde{\phi}_>(\vec{p}) = \begin{cases} 0 & 0 \leq p \leq \Lambda/b \\ \tilde{\phi}(\vec{p}) & \Lambda/b < p \leq \Lambda \end{cases}.$$

Because H does not couple different momentum modes (it is diagonal in \vec{p}) we can separate the two functional integrals

$$Z_0 = \int \mathcal{D}\tilde{\phi}_< e^{-H_0[\tilde{\phi}_<]} \int \mathcal{D}\tilde{\phi}_> e^{-H_0[\tilde{\phi}_>]}. \quad (7.19)$$

The integrals over $\tilde{\phi}_>$ are Gaussian, and just give an overall multiplicative factor

$$Z_0 = e^{-F_>} \int \mathcal{D}\tilde{\phi}_< e^{-H_0[\tilde{\phi}_<]} \quad (7.20)$$

with

$$H_0[\tilde{\phi}_<] = \int^{\Lambda/b} \frac{d^D p}{(2\pi)^D} \left[\frac{1}{2} (\alpha^{-1} p^2 + r_0) |\phi_<(\vec{p})|^2 + h \tilde{\phi}_<(0) \right].$$

(2) Rescaling

Let us set $h = 0$ temporarily. We now rescale so that the momentum cutoff is again Λ :

$$\vec{p} = \vec{p}'/b$$

$$\vec{x} = b \vec{x}'$$

$$\tilde{\phi}_<(\vec{p}) = b^{-\tilde{d}_\phi} \tilde{\phi}'(\vec{p}'). \quad (7.21)$$

We write the thinned, rescaled Hamiltonian, in 2 ways

$$H'_0 = \int \frac{d^D p'}{(2\pi)^D} b^{-D-2\tilde{d}_\phi-2} \left[\frac{1}{2} \alpha^{-1} p'^2 + \frac{1}{2} b^2 r_0 \right] |\phi'(\vec{p}')|^2 \quad (7.22)$$

$$= \int \frac{d^D p'}{(2\pi)^D} b^{-D-2\tilde{d}_\phi} \left[\frac{1}{2b^2} \alpha^{-1} p'^2 + \frac{1}{2} r_0 \right] |\phi'(\vec{p}')|^2. \quad (7.23)$$

In position space we would rescale the field

$$\phi_{<}(\vec{x}) = b^{-d_\phi} \phi'(\vec{x}'). \quad (7.24)$$

The fact that $\phi_{<}$ and $\tilde{\phi}_{<}$ are Fourier transforms allows one to show that $\tilde{d}_\phi = d_\phi - D$. The position space equivalents of (7.22) and (7.23) are

$$H'_0 = \int d^D x' b^{D-2d_\phi-2} \left[\frac{1}{2} \alpha^{-1} (\nabla' \phi')^2 + \frac{1}{2} b^2 r_0 \phi'^2 \right] \quad (7.25)$$

$$= \int d^D x' b^{D-2d_\phi} \left[\frac{1}{2b^2} \alpha^{-1} (\nabla' \phi')^2 + \frac{1}{2} r_0 \phi'^2 \right]. \quad (7.26)$$

These 2 ways of writing H'_0 make clear the 2 fixed points of the renormalization group transformation we made, that is, the points in coupling constant space where $H'_0 = H_0$. Near these 2 fixed points, the field ϕ scales differently.

Looking at (7.26) we see that if the scaling dimension of the field satisfies $D - 2d_\phi = 0$ then $(\alpha^{-1}, r_0) = (0, r_0)$ is a fixed point, with $r'_0 = r_0$ arbitrary and $\alpha'^{-1} = (b^2 \alpha)^{-1}$. If the kinetic energy term vanishes, then the fields on different lattice sites decouple and is essentially an infinite temperature system. There is not much else interesting about this case.

The more interesting case is evident when looking at (7.25). If $D - 2d_\phi - 2 = 0$ then $(\alpha^{-1}, r_0) = (\alpha^{-1}, 0)$ is a fixed point where $r'_0 = b^2 r_0$ and $\alpha'^{-1} = \alpha^{-1}$ can be arbitrary. This is the **Gaussian fixed point**.

Let us restore $h \neq 0$. In the rescaled Hamiltonian $h' = b^{\frac{D}{2}+1} h$. Near the fixed point, the correlation length $\xi = 1/m$ diverges. Expand r_0 about $T = T_c$ and assume it vanishes linearly as $r_0 \sim |t|$. Then under rescaling, we can determine the scaling exponents:

$$\begin{aligned} |t'| &= b^2 |t| &\Rightarrow & y_t = 2 \\ h' &= b^{\frac{D}{2}+1} h &\Rightarrow & y_h = \frac{D}{2} + 1. \end{aligned} \quad (7.27)$$

from which we can read off the critical exponents (see Chapter 10)

$$\alpha = \frac{4-D}{2}, \quad \beta = \frac{D-2}{4}, \quad \nu = \frac{1}{2}, \quad \gamma = 1. \quad (7.28)$$

These are not the mean field exponents in general, but they do coincide with the mean field predictions when $D = 4$.

7.2 Interacting model

Having used the Gaussian model for a noninteracting scalar field to present the ideas of the renormalization group, we now turn to the physically interesting case where interactions are present. We take as the Hamiltonian

$$H = \int d^D x \left[\frac{1}{2} \alpha^{-1} (\nabla \phi)^2 + \frac{1}{2} r_0 \phi^2 + \frac{u_0}{4!} \phi^4 \right]. \quad (7.29)$$

The interaction term ϕ^4 couples different normal modes in Fourier space. This means we cannot solve the system exactly as in the pure

Gaussian model. One thing we can do is to treat the interaction term as a small perturbation from the Gaussian model

$$H = H_0 + V \quad (7.30)$$

with H_0 the Hamiltonian of the Gaussian model and $V = \frac{u_0}{4!} \int d^D x \phi^4(\vec{x})$.

As in the solution of the Gaussian model, we divide the field ϕ into long and short wavelength parts

$$\tilde{\phi}(\vec{p}) = \tilde{\phi}_<(\vec{p}) + \tilde{\phi}_>(\vec{p}) \quad (7.31)$$

with

$$\tilde{\phi}_<(\vec{p}) = \begin{cases} \tilde{\phi}(\vec{p}) & 0 \leq p \leq \Lambda/b \\ 0 & \Lambda/b < p \leq \Lambda \end{cases} \quad (7.32)$$

$$\tilde{\phi}_>(\vec{p}) = \begin{cases} 0 & 0 \leq p \leq \Lambda/b \\ \tilde{\phi}(\vec{p}) & \Lambda/b < p \leq \Lambda \end{cases} . \quad (7.33)$$

The thinned Hamiltonian H'_1 is obtained as follows

$$\begin{aligned} Z &= \int \mathcal{D}\tilde{\phi}_< \mathcal{D}\tilde{\phi}_> \exp \{-H_0[\tilde{\phi}_<] - H_0[\tilde{\phi}_>] - V[\tilde{\phi}_<, \tilde{\phi}_>]\} \\ &= \int \mathcal{D}\tilde{\phi}_< e^{-H_0[\tilde{\phi}_<]} \int \mathcal{D}\tilde{\phi}_> \exp \{-H_0[\tilde{\phi}_>] - V[\tilde{\phi}_<, \tilde{\phi}_>]\} \\ &\equiv \int \mathcal{D}\tilde{\phi}_< e^{-H'_1[\tilde{\phi}_<]} \left\{ \int \mathcal{D}\tilde{\phi}_> e^{-H_0[\tilde{\phi}_>]} \right\} \end{aligned} \quad (7.34)$$

where the last line defines H'_1 . Therefore

$$e^{-H'_1[\tilde{\phi}_<]} = e^{-H_0[\tilde{\phi}_<]} \frac{\int \mathcal{D}\tilde{\phi}_> \exp \{-H_0[\tilde{\phi}_>] - V[\tilde{\phi}_<, \tilde{\phi}_>]\}}{\int \mathcal{D}\tilde{\phi}_> e^{-H_0[\tilde{\phi}_>]}} . \quad (7.35)$$

Expanding about small V (i.e. small u_0) we find

$$\begin{aligned} H'_1[\tilde{\phi}_<] &= H_0[\tilde{\phi}_<] + \frac{\int \mathcal{D}\tilde{\phi}_> V[\tilde{\phi}_<, \tilde{\phi}_>] e^{-H_0[\tilde{\phi}_>]}}{\int \mathcal{D}\tilde{\phi}_> e^{-H_0[\tilde{\phi}_>]}} \\ &= H_0[\tilde{\phi}_<] + \langle V[\tilde{\phi}_<, \tilde{\phi}_>] \rangle_0^{\text{shell}} \end{aligned} \quad (7.36)$$

In the last term, the subscript means the integration is weighted by the free Hamiltonian H_0 and the superscript "shell" means that the integration is performed only over the short wavelength modes, those with momenta satisfying $\Lambda/b < p \leq \Lambda$.

In order to evaluate (7.36) let us recall the definition of the free propagator

$$\begin{aligned} \langle \phi(\vec{x}) \phi(\vec{y}) \rangle_0 &= G_0(r) \\ &= \int_0^\Lambda \frac{d^D p}{(2\pi)^D} \frac{\alpha e^{-i\vec{p}\cdot\vec{x}}}{p^2 + \alpha r_0} . \end{aligned} \quad (7.37)$$

Let us define a similar propagator for short wavelength modes

$$\begin{aligned} \langle \phi_>(\vec{x}) \phi_>(\vec{y}) \rangle_0^{\text{shell}} &= G_0^>(r) \\ &= \int_{\Lambda/b}^\Lambda \frac{d^D p}{(2\pi)^D} \frac{\alpha e^{-i\vec{p}\cdot\vec{x}}}{p^2 + \alpha r_0} . \end{aligned} \quad (7.38)$$

Then we can write

$$\begin{aligned} \langle V(\tilde{\phi}_<, \tilde{\phi}_>) \rangle_0^{\text{shell}} &= \frac{u_0}{4!} \langle (\phi_<(\vec{x}) + \phi_>(\vec{x}))^4 \rangle_0^{\text{shell}} \\ &= \frac{u_0}{4!} \left[\phi_<^4 + 6\phi_<^2 \langle \phi_>(\vec{x}) \phi_>(\vec{x}) \rangle_0^{\text{shell}} + \langle \phi_>^4 \rangle_0^{\text{shell}} \right]. \end{aligned} \quad (7.39)$$

The last term is just a constant and can be dropped. Also note the appearance of $G_0^>(0)$ in the penultimate term. Therefore we find

$$H'_1 = \int d^D x \left[\frac{1}{2} \alpha^{-1} (\nabla \phi_<)^2 + \frac{1}{2} r_0 \phi_<^2 + \frac{u_0}{4!} \phi_<^4 + \frac{u_0}{4} \phi_<^2 G_0^>(0) \right]. \quad (7.40)$$

We now rescale so that the momentum cutoff is again Λ :

$$\begin{aligned} \vec{x}' &= \vec{x}/b \\ \phi'(\vec{x}') &= b^{d_\phi} \phi_<(\vec{x}). \end{aligned} \quad (7.41)$$

The rescaled Hamiltonian is

$$\begin{aligned} H' &= \int d^D x' b^{D-2d_\phi-2} \left[\frac{1}{2} \alpha^{-1} (\nabla' \phi')^2 + \frac{1}{2} b^2 \left(r_0 + \frac{u_0}{2} G_0^>(0) \right) \phi'^2 \right. \\ &\quad \left. + b^{2-2d_\phi} \frac{u_0}{4!} \phi'^4 \right]. \end{aligned} \quad (7.42)$$

As we did in the free case, we consider the physically interesting case $D - 2d_\phi - 2 = 0$, *i.e.* we rescale the fields using $d_\phi = \frac{D}{2} - 1$. Now the fixed point is

$$r'_0 = b^2 \left(r_0 + \frac{u_0}{2} G_0^>(0) \right) \quad (7.43)$$

$$u'_0 = b^{4-D} u_0 = b^\epsilon u_0 \quad (7.44)$$

where $\epsilon \equiv 4 - D$. The point $(r_0, u_0) = (0, 0)$ is still a fixed point and is still called the Gaussian fixed point. Evaluating $G_0^>(0)$ near this point we find

$$\begin{aligned} G_0^>(0) &= \int_{\Lambda/b}^{\Lambda} \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + r_0} \\ &= \frac{S_{D-1}}{(2\pi)^D} \int_{\Lambda/b}^{\Lambda} \frac{p^{D-1} dp}{p^2} + \mathcal{O}(r_0) \\ &= \frac{S_{D-1}}{(2\pi)^D} \frac{\Lambda^{D-2}}{D-2} (1 - b^{2-D}) + \mathcal{O}(r_0) \\ &\equiv 2B(1 - b^{2-D}) + \mathcal{O}(r_0) \end{aligned} \quad (7.45)$$

where S_{D-1} is the area of a $(D-1)$ -sphere.¹⁶ The last line defines B . The linearised RG flow near the fixed point can then be expressed by the matrix equation

$$\begin{pmatrix} r_0 \\ u_0 \end{pmatrix}' = \begin{pmatrix} b^2 & Bb^2(1 - b^{2-D}) \\ 0 & b^\epsilon \end{pmatrix} \begin{pmatrix} r_0 \\ u_0 \end{pmatrix} \quad (7.46)$$

The 2×2 matrix on the righthand side of (7.46) is an example of $K_{ij}(u^*) = \partial R_i / \partial u_j|_{u^*}$ of Chapter 10, eqn. (72). Note that dropping

¹⁶ An earlier version of the notes used an incorrect label for the dimensionality of the sphere. A circle is a 1-sphere, etc.

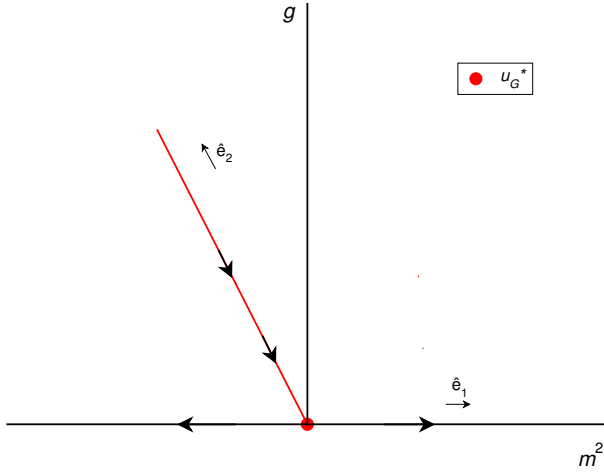


Figure 7.1: RG flow for $D > 4$. Axes labels need to be updated $m^2 \mapsto r_0$ and $g \mapsto u_0$.

the $\mathcal{O}(r_0)$ terms from $G^>(0)$ is equivalent to linearised the RG equations near the Gaussian fixed point. The eigenvalues λ and eigenvectors \vec{e} are

$$\lambda_1 = b^2, \quad y_1 = \frac{\log \lambda_1}{\log b} = 2, \quad \hat{e}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (7.47)$$

$$\lambda_2 = b^\epsilon, \quad y_2 = \frac{\log \lambda_2}{\log b} = \epsilon, \quad \hat{e}^{(2)} = \begin{pmatrix} -B \\ 1 \end{pmatrix} \quad (7.48)$$

The exponents of the eigenvectors, y are the critical exponents. The RG flow clearly depends on the sign of ϵ , *i.e.* whether the system is more or fewer than 4 dimensions.

For $D > 4$ we have 1 relevant direction, \vec{e}_1 , and 1 irrelevant direction, \vec{e}_2 . A general point in the 2-dimensional coupling constant space (but near the critical point) can be written as

$$\begin{aligned} \vec{u} &= r_0 \hat{r}_0 + u_0 \hat{u}_0 \\ &= r_0 \vec{e}_1 + u_0 (B \vec{e}_1 + \vec{e}_2) \\ &= (r_0 + B u_0) \vec{e}_1 + u_0 \vec{e}_2 \end{aligned} \quad (7.49)$$

The critical surface, the set of points which flow into the fixed point, must satisfy $r_0 + B u_0 = 0$, otherwise the flow will be carried away from the fixed point in the \vec{e}_1 -direction. Figure 7.1 shows the Gaussian fixed point u_G^* , and the critical surface for the ϕ^4 model in $D > 4$ dimensions.

Since the long wavelength physics is governed by the Gaussian fixed point for $D > 4$, we can expect the critical exponents to be given by mean field theory. Actually one needs care with α , β , and δ due to **dangerous irrelevant variables**. We do not discuss this here, and refer to §VII.4 of S-K Ma's 1976 text.

Let us now turn to the case where $D < 4$. Then both $y_1 > 0$ and $y_2 > 0$, so the Gaussian fixed point is infrared repulsive and does not govern the long wavelength physics. The critical exponents are thus not given by mean field theory.

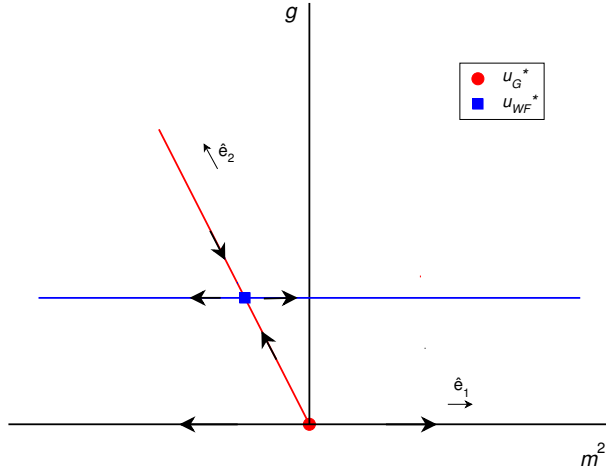


Figure 7.2: RG flow for $D < 4$. Axes labels need to be updated $m^2 \mapsto r_0$ and $g \mapsto u_0$.

All is not lost, however. Using a perturbative expansion in g to go beyond linear order, (7.44) is modified to give

$$u'_0 = b^\epsilon u_0 (1 - C u_0 \log b) \quad (7.50)$$

where C is a constant. One of the 2 conditions for a fixed point, that $u'_0 = u_0$, can be written conveniently as

$$0 = \left. \frac{\partial u'_0}{\partial \log b} \right|_{b=1} = \epsilon u_0 - C u_0^2 \quad (7.51)$$

which is satisfied by $u_0 = 0$ and $u_0 = \epsilon/C$. So, in addition to the Gaussian fixed point $u_G^* = (0, 0)$, we find another fixed point $u_{WF}^* = (0, \epsilon/C)$, called the **Wilson-Fisher fixed point**. We display the $D < 4$ situation in Figure 7.2. It turns out this is the fixed point which will control the long wavelength behaviour of the system. When $D > 4$, this fixed point was in the unphysical $u_0 < 0$ half-plane.

If ϵ is small, in some sense which is usually left vague, then u_{WF}^* is near enough to u_G^* that we can calculate small departures from mean field exponents using perturbation theory in what is called an ϵ -expansion.¹⁷ For example, we will find that the correlation length critical exponent is

$$\nu = \frac{1}{2} + \frac{\epsilon}{12} + \mathcal{O}(\epsilon^2). \quad (7.52)$$

¹⁷ See also the classic papers: Wilson & Kogut, Phys. Rep. **12**, 75 (1974) and Wilson's Nobel lecture in Rev. Mod. Phys. **55**, 583 (1983).

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