Examples Sheet 0

This sheet contains questions which help revise material similar to what you would have seen in an undergraduate Statistical Physics and Thermodynamics course.

1. Consider a system consisting of N spin-$\frac{1}{2}$ particles, each of which can be in one of two states, 'up' and 'down'. In a magnetic field h, the energy of a spin in the up/down state is $\pm h/2$. Show that the partition function is

$$Z = 2^N \cosh^N\left(\frac{\beta h}{2}\right)$$

Find the average energy U and entropy S. Check that your results for both quantities make sense in at $T = 0$ and $T \to \infty$.

2. The Hamiltonian for a set of N spins $\{\boldsymbol{\sigma}_n\}$ that are 3-dimensional vectors in the presence of an external magnetic field $\mathbf{h}$ is

$$H = \sum_n h \cos\theta_n$$

where θ_n is the angle between $\boldsymbol{\sigma}_n$ and $\mathbf{h}$. Show that the partition function is

$$Z = \left(\frac{4\pi \sinh \beta h}{\beta h}\right)^N .$$

Compute the free energy F, the entropy S, and the internal energy U. Find the equation of state and compute the susceptibility χ_T. Examine the behaviour at low T.

3. In standard notation, the first law of thermodynamics for a magnetic system is

$$dU = TdS - M\,dh$$

where U is the internal energy, M is the magnetization, and h is the applied magnetic field. Define other functions of state by appropriate Legendre transform, for example $F = U - TS$, and hence obtain the Maxwell relations

$$\left.\frac{\partial T}{\partial h}\right|_S = -\left.\frac{\partial M}{\partial S}\right|_h \qquad \left.\frac{\partial S}{\partial h}\right|_T = \left.\frac{\partial M}{\partial T}\right|_h$$
$$\left.\frac{\partial S}{\partial M}\right|_T = -\left.\frac{\partial h}{\partial T}\right|_M \qquad \left.\frac{\partial T}{\partial M}\right|_S = \left.\frac{\partial h}{\partial S}\right|_M$$

4. The heat capacities at constant magnetic field and at constant magnetization for the magnetic system are

$$C_h = T\left.\frac{\partial S}{\partial T}\right|_h \quad \text{and} \quad C_M = T\left.\frac{\partial S}{\partial T}\right|_M .$$

The isothermal and adiabatic susceptibilities are

$$\chi_T = \left.\frac{\partial M}{\partial h}\right|_T \quad \text{and} \quad \chi_S = \left.\frac{\partial M}{\partial h}\right|_S .$$

Also define

$$\alpha_h = \left.\frac{\partial M}{\partial T}\right|_h .$$

From the identity (which you should be able to derive)

$$\left.\frac{\partial S}{\partial T}\right|_M = \left.\frac{\partial S}{\partial T}\right|_h + \left.\frac{\partial S}{\partial h}\right|_T \left.\frac{\partial h}{\partial T}\right|_M$$

deduce that

$$\chi_T(C_h - C_M) = -T \left.\frac{\partial M}{\partial h}\right|_T \left.\frac{\partial M}{\partial T}\right|_h \left.\frac{\partial h}{\partial T}\right|_M ,$$

and hence show

$$\chi_T(C_h - C_M) = T\alpha_h^2 . \tag{1}$$

By similar means show that

$$C_h(\chi_T - \chi_S) = T\alpha_h^2$$

which proves $\chi_S C_h = \chi_T C_M$.

[Hint: you may need the identity $\left.\frac{\partial x}{\partial y}\right|_z \left.\frac{\partial y}{\partial z}\right|_x \left.\frac{\partial z}{\partial x}\right|_y = -1$.]

5. For $T \to T_c^-$ and $h = 0$ the dependence on the following observables is parametrized by

$$C_h \sim (T_c - T)^{-\alpha}$$
$$M \sim (T_c - T)^{\beta}$$
$$\chi_T \sim (T_c - T)^{-\gamma} .$$

Using the fact that $C_M > 0$ and (1) from Question 4 (also $\chi_T \geq 0$), derive Rushbrooke's inequality

$$\alpha + 2\beta + \gamma \geq 2 .$$

6. *Optional. This is a bit long for a revision question. See Tong's Part II Statistical Physics notes, §5.1.3, for the corresponding calculation for a van der Waals gas.* The Dieterici equation of state for a gas is

$$p = \frac{T}{v - b} \exp\left(-\frac{a}{Tv}\right)$$

where $v = V/N$ is the volume per particle, and a and b are constants. Find the critical point and compute the ratio $p_c v_c / T_c$. Calculate the critical exponents β, γ, and δ defined through

$$\begin{aligned} \delta v &\sim (T_c - T)^{\beta} \\ \kappa &\sim (T - T_c)^{-\gamma} \\ p - p_c &\sim (v - v_c)^{\delta} \end{aligned}$$

with $\kappa = -\frac{1}{v} \left.\frac{\partial v}{\partial p}\right|_T$.